OPTIMAL CONTROL OF FLOWS OF VISCOELASTIC SEMI-COMPRESSIBLE FLUIDS

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ABSTRACT. Semilinear parabolic systems with bi-linear nonlinearities cover a lot of applications and their optimal control leads to relatively simple optimality conditions. An example is the incompressible Navier-Stokes system for homogeneous fluids, which is however here modified towards a physically reasonable model of slightly (so-called "semi") compressible liquids rather than fully compressible gases. An optimal control problem optimizing also pressure on the boundary is considered and, in the simple variant, analysed as far as uniqueness of the control-to-state mapping and 1st-order optimality conditions in the 2-dimensional case and outlined in a nonsimple variant for the 3-dimensional case. Some other bi-linear parabolic systems as Cahn-Hilliard diffusion or magneto-hydrodynamics can be treated analogously.

1. INTRODUCTION

The optimal control of semilinear parabolic systems with bi-linear nonlinearities allows to exploit such special nonlinear structure, in particular to formulate optimality conditions in a relatively lucid way. An example of such bi-linear nonlinearities arises from convective time derivatives in Eulerian description of continuum-mechanical models, as properly used in fluid dynamics. The bi-linear nonlinear terms thus occurs in incompressible Newtonien homogeneous fluid flows, described by the incompressible Navier-Stokes equations.

Optimal control of such system has been vastly addressed in literature mainly in the fully incompressible variants. Standardly expected results are available especially in twodimensions, cf. [1, 13, 16]. In three dimensions, rather only steady-state situations have been scrutinized (as in [18, 24, 27, 34]) while only particular results are available in the evolutionary situations [2, 8, 12, 17] because of the well-recognized difficulties related with lack of regularity and uniqueness of the weak solutions.

Incompressible fluids are however only an idealized model and, although well applicable in many situations, it ignores various physical phenomena (most importantly the propagation of pressure waves) and, as mentioned, brings even serious analytical difficulties especially in three-dimensional cases. On the other hand, the fully compressible models involves nonlinearities of more complicated structure than only bi-linear. Actually, although most fluids (liquids or melted metals or magma) are quite compressible, they are not really "fully" compressible as gases. E.g. water is about 50-times more compressible than steel but, nevertheless, it is far from to be so compressible as gases and has a specific density even under zero pressure and even can withstand a certain negative pressure (cf. e.g. [9] and references therein), in contrast to gases.

To reflect this phenomenology, models for a class of so-called *semi-compressible fluids* have been devised in [26] as a compromise between fully compressible models with substantially

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varying mass density and the fully incompressible Navier-Stokes model. They pursue the following attributes:

- (α) propagation of longitudinal waves (i.e. pressure waves, called also P-waves) is allowed and their dispersion is controlled in a certain way,
- (β) the energy balance is preserved at least formally, but in some models even rigorously,
- (γ) the pressure is well defined also on the boundary,
- (δ) the equations are consistently written in Eulerian coordinates (i.e. the model is fully convective),
- (ϵ) in some models, uniqueness of weak solutions holds even in the physically relevant 3-dimensional cases.

For optimal control, it is important that such models still exhibit bi-linear nonlinear structure. In particular, we will avoid usage of continuity equation for mass density. This is well acceptable for fluids whose density is (nearly) constant in space and varies (nearly) negligibly with pressure, so that it can be modeled as constant. Let us remind that the involvement of density as a variable governed by the continuity equation $\dot{\varrho} + \operatorname{div}(\varrho \boldsymbol{v}) = 0$ would make the term $\varrho(\boldsymbol{v} \cdot \nabla)\boldsymbol{v}$ in (2.1a) below tri-linear, which would make optimality conditions more complicated, not mentioning many other analytical problems including uniqueness of the state response.

In Section 2, we first specify a simple semi-compressible model and an initial-boundaryvalue problem for it. Some analytical properties of it are then analyses in Section 3 for the two-dimensional case and then in Section 4 we consider a simple optimal-control problem with a quadratic cost and formulate first-order optimality conditions. Eventually, in Section 5, we outline some modifications towards three-dimensional case and enhancements of the basic model to obtain some better properties or applicability to other phenomena coupled with the fluid flows.

2. Semi-compressible fluids

We consider a fixed bounded domain $\Omega \subset \mathbb{R}^n$ with a Lipschitz boundary Γ and a finite time interval I = [0, T]. We will use the dot-notation means the partial derivative in time. The basic scenario will use the velocity \boldsymbol{v} and pressure p, and consider a model devised in [26]. It consists of the system of two parabolic equations epressing momentum equation and convective transport and diffusion of pressure:

(2.1a)
$$\varrho \dot{\boldsymbol{v}} + \varrho (\boldsymbol{v} \cdot \nabla) \boldsymbol{v} - \operatorname{div}(\nu \boldsymbol{e}(\boldsymbol{v})) + \frac{\varrho}{2} (\operatorname{div} \boldsymbol{v}) \boldsymbol{v} + \nabla \left(p + \frac{\beta}{2} p^2 \right) = \boldsymbol{u} \quad \text{on } I \times \Omega ,$$

(2.1b)
$$\beta(\dot{p} + \boldsymbol{v} \cdot \nabla p) + \operatorname{div} \boldsymbol{v} = \gamma \Delta p$$
 on $I \times \Omega$

completed by the boundary and initial conditions

(2.1c)
$$\left[\nu \boldsymbol{e}(\boldsymbol{v})\boldsymbol{n}\right]_{\mathrm{T}} + b\boldsymbol{v}_{\mathrm{T}} = 0, \quad \boldsymbol{n}\cdot\boldsymbol{v} = 0, \text{ and } \boldsymbol{n}\cdot\nabla p = 0 \quad \text{on } I \times \Gamma,$$

(2.1d)
$$\boldsymbol{v}(0,\cdot) = \boldsymbol{v}_0$$
 and $p(0,\cdot) = p_0$ on Ω

with mass density $\rho > 0$ assumed constant (in particular independent of pressure), with some constants β and $\gamma > 0$ commented below, viscosity $\nu > 0$, and $\boldsymbol{e}(\boldsymbol{v}) = \frac{1}{2} \nabla \boldsymbol{v} + \frac{1}{2} (\nabla \boldsymbol{v})^{\top}$. The bulk force \boldsymbol{u} is here prescribed and later in Section 4 will be used as a distributed control. Let us note that, beside the usual "hydrostatic" pressure p, there is also the pressure contribution $\frac{1}{2}\beta p^2$ in (2.1a) due to the elastic internal energy of the fluid. The system (2.1) gets a good physical consistency in Eulerian description with $\beta > 0$ being the impressibility. In terms of the bulk elastic modulus K in physical units $Pa=J/m^3$, the impressibility is $\beta = 1/K$. This modulus determines the velocity of P-waves (namely $\sqrt{K/\rho}$ = sound speed, provided $\gamma = 0$) which can propagate through such fluids, in contrast to ideally incompressible fluids. Moreover, $\gamma > 0$ allows for modeling a certain dispersion of pressure waves and is motivated by a mass diffusion in the continuity equation in the full compressible model, advocated by H. Brenner [6, 7]. The physical dimension of γ/β is m²/s and, vaguely speaking, dividing it by a "characteristic velocity" of the flow and a "characteristic size" of the system, one gets a dimensionless Péclet number (or, in fluid dynamics, also called Brenner's number) expressing dominance of either the convective or the diffusive transport phenomena. The mentioned compromise between fully incompressible models (where P-waves cannot propagate at all) and fully compressible models (where density can substantially vary) is well legitimate when pressure variations (and thus density variations in compressible situations) are much smaller than the elastic bulk modulus K and simultaneously the shear elastic modulus is zero, which is relevant in most situation in liquids (but usually not in gases neither in solids). E.g. water has the elastic bulk modulus about $2 \,\mathrm{GPa}$ (which is much less than e.g. rocks with $> 10 \,\mathrm{GPa}$ or steel with >100 GPa) but still it is much larger than usual pressure variations in most practical situations, and the speed of P-waves is about 1.5 km/s. These numbers are known with 5-digit (or more) accuracy (including its dependence on temperature, salinity, and pressure itself, cf. IAPWS-standard [36]) and putting $K = \infty$ (which is, in fact, done in incompressible models) is a simplification which might be often not well acceptable.

The (so-called Navier) boundary conditions (2.1c) involves the tangential velocity $\boldsymbol{v}_{\mathrm{T}} = \boldsymbol{v} - (\boldsymbol{n} \cdot \boldsymbol{v})\boldsymbol{n}$.

This semi-compressible modification now contributes also to the stored energy due to the calculus

(2.2)
$$\int_{\Gamma} p(\boldsymbol{v} \cdot \boldsymbol{n}) \, \mathrm{d}S - \int_{\Omega} \nabla p \cdot \boldsymbol{v} \, \mathrm{d}x = \int_{\Omega} p \, \mathrm{div} \, \boldsymbol{v} \, \mathrm{d}x$$
$$= \int_{\Omega} p \left(\beta \dot{p} + \beta \boldsymbol{v} \cdot \nabla p - \gamma \Delta p\right) \mathrm{d}x$$
$$= \int_{\Omega} \gamma |\nabla p|^2 - \frac{\beta}{2} p^2 \mathrm{div} \, \boldsymbol{v} \, \mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} \frac{\beta}{2} p^2 \, \mathrm{d}x$$

if employing also the boundary condition $\boldsymbol{n} \cdot \boldsymbol{v} = 0$ and $\nabla p \cdot \boldsymbol{n} = 0$. More specifically, for the term $\beta p \boldsymbol{v} \cdot \nabla p$, we have used the Green formula for

$$\int_{\Omega} p\boldsymbol{v} \cdot \nabla p \, \mathrm{d}x = \int_{\Gamma} p^2 \boldsymbol{v} \cdot \boldsymbol{n} \, \mathrm{d}S - \int_{\Omega} \operatorname{div}(p\boldsymbol{v}) p \, \mathrm{d}x$$
$$= \int_{\Gamma} p^2 \boldsymbol{v} \cdot \boldsymbol{n} \, \mathrm{d}S - \int_{\Omega} (\nabla p \cdot \boldsymbol{v}) p + p^2 \operatorname{div} \boldsymbol{v} \, \mathrm{d}x$$
$$= \int_{\Gamma} \frac{1}{2} p^2 \boldsymbol{v} \cdot \boldsymbol{n} \, \mathrm{d}S - \int_{\Omega} \frac{1}{2} p^2 \operatorname{div} \boldsymbol{v} \, \mathrm{d}x \, .$$

This storage-energy mechanism together with the kinetic energy just facilitates wave propagation. The extra bulk force $\frac{\varrho}{2}(\operatorname{div} \boldsymbol{v}) \boldsymbol{v}$ in (2.1a), proposed by R. Temam [29], arises by (slight) compressibility and is presumably small as div \boldsymbol{v} is presumably very small but (slightly) violates Galilean invariance of the model, as pointed out in [31]. This is the price payed for omitting the continuity equation for ϱ and simplifying considerably the analysis of the model, which also gets bi-linear semi-linear structure with still keeping a lot of physically relevant features. This extra "structural" force balances the energetics due to the calculus

(2.3)
$$\int_{\Omega} \varrho(\widehat{\boldsymbol{v}} \cdot \nabla) \boldsymbol{v} \cdot \boldsymbol{v} + \frac{1}{2} \int_{\Omega} \varrho |\boldsymbol{v}|^2 (\operatorname{div} \widehat{\boldsymbol{v}}) \, \mathrm{d}x = \frac{1}{2} \int_{\Gamma} \varrho |\boldsymbol{v}|^2 (\widehat{\boldsymbol{v}} \cdot \boldsymbol{n}) \, \mathrm{d}S$$

to be used for $\hat{v} = v$. The overall energy balance looks as

(2.4)
$$\int_{\Omega} \underbrace{\frac{\varrho}{2} |\boldsymbol{v}(t)|^{2}}_{\text{kinetic energy}} + \underbrace{\frac{\beta}{2} p(t)^{2} dx}_{\text{elastic energy}} + \int_{0}^{t} \int_{\Omega} \underbrace{\nu |\boldsymbol{e}(\boldsymbol{v})|^{2} + \gamma |\nabla p|^{2}}_{\text{dissipation rate in the bulk}} dx dt + \int_{0}^{t} \underbrace{\int_{\Gamma} b |\boldsymbol{v}_{\mathrm{T}}|^{2} dS dt}_{\text{dissipation rate on the boundary}} = \int_{0}^{t} \int_{\Omega} \underbrace{\boldsymbol{u} \cdot \boldsymbol{v}}_{\text{power of the control}} dx dt + \int_{\Omega} \underbrace{\frac{\varrho}{2} |\boldsymbol{v}_{0}|^{2} + \frac{\beta}{2} p_{0}^{2} dx}_{\text{initial kinetic and stored energy}}.$$

In fact, (5.4) holds for weak solutions rigorously only for n = 2 while in higher dimensions, it might hold only as an inequality unless the solution is enough regular.

Remark 2.1 (Quasi-incompressible fluids). Without the convective term $\beta \boldsymbol{v} \cdot \nabla p$ in (2.1b) and the corresponding pressure contribution $\frac{1}{2}\beta p^2$ in (2.1a), the system (2.1a,b) models so-called quasi-incompressible fluids, considered as an artificial regularization of the incompressible model rather for numerical purposes, cf. e.g. R. Temam [30, Ch. III, Sect 8] or, with $\gamma > 0$, A. Prohl [23], which can also be understood as a singularly perturbed variant (mathematically understood as a regularization) of the incompressible model, i.e. $\beta = 0$ and $\gamma = 0$. For analytical purposes, the quasi-incompressible regularization with $\beta > 0$ and $\gamma > 0$ was devised by A.P. Oskolkov [22].

3. The state problem in two-dimensions

Throughout this article, we will use the standard notation for the function spaces: the Lebesgue and the Sobolev spaces, namely $L^p(\Omega; \mathbb{R}^n)$ for Lebesgue measurable functions $\Omega \to \mathbb{R}^n$ whose Euclidean norm is integrable with p-power, and $W^{k,p}(\Omega;\mathbb{R}^n)$ for functions from $L^p(\Omega; \mathbb{R}^n)$ whose all derivative up to the order k have their Euclidean norm integrable with p-power. We also write briefly $H^k = W^{k,2}$. Moreover, for a Banach space X and for I = [0, T], we will use the notation $L^{p}(I; X)$ for the Bochner space of Bochner measurable functions $I \to X$ whose norm $\|\cdot\|_X$ is in $L^p(I)$, and $H^1(I;X)$ for functions $I \to X$ whose distributional derivative is in $L^2(I; X)$. Furthermore, C(I; X) will denote the Banach space of continuous functions $I \to X$. We will use the notation $(\cdot)^*$ for the dual space and define specifically

(3.1a)
$$\mathcal{V} = \left\{ \boldsymbol{v} \in L^2(I; H^1(\Omega; \mathbb{R}^n)) \cap H^1(I; H^1(\Omega; \mathbb{R}^n)^*); \; \boldsymbol{n} \cdot \boldsymbol{v} \right|_{I \times \Gamma} = 0 \right\},$$

(3.1a)
$$\mathcal{V} = \left\{ \boldsymbol{v} \in L^2(I; H^1(\Omega; \mathbb{R}^n)) \cap H^1(I; H^1(\Omega; \mathbb{R}^n)^*); \right\}$$

(3.1b)
$$\mathcal{P} = \left\{ p \in L^2(I; H^1(\Omega)) \cap H^1(I; H^1(\Omega)^*) \right\}, \text{ and }$$

(3.1c)
$$\mathcal{U} = L^2(I \times \Omega; \mathbb{R}^n).$$

In any case, under the assumptions

(3.2)
$$\boldsymbol{v}_0 \in L^2(\Omega; \mathbb{R}^n) \text{ and } p_0 \in L^2(\Omega)$$

with $\boldsymbol{u} \in \mathcal{U}$, from (5.4) we can read a-priori estimates $\boldsymbol{v}_i, p \in L^{\infty}(I; L^2(\Omega)) \cap L^2(I; H^1(\Omega))$, i = 1, ..., n, for any dimension. In this and the following section, we will restrict ourselves on n = 2.

The definition of a weak solution to (2.1) will be based on the state-equation mapping $\Pi: \mathcal{U} \times \mathcal{V} \times \mathcal{P} \to \mathcal{V}^* \times \mathcal{P}^*$ defined by

$$\begin{split} \langle \Pi(\boldsymbol{u},\boldsymbol{v},p),(\widetilde{\boldsymbol{v}},\widetilde{p})\rangle &= \int_{0}^{T}\!\!\!\int_{\Omega} \nu \boldsymbol{e}(\boldsymbol{v}): \boldsymbol{e}(\widetilde{\boldsymbol{v}}) + \left(\varrho(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \frac{\varrho}{2}(\operatorname{div}\boldsymbol{v})\,\boldsymbol{v} - \boldsymbol{u}\right)\cdot\widetilde{\boldsymbol{v}} \\ &+ \left(p + \frac{\beta}{2}p^{2}\right)\operatorname{div}\widetilde{\boldsymbol{v}} + \gamma\nabla p\cdot\nabla\widetilde{p} + \left(\beta\boldsymbol{v}\cdot\nabla p + \operatorname{div}\boldsymbol{v}\right)\widetilde{p}\,\mathrm{d}x\mathrm{d}t \\ &+ \int_{0}^{T}\!\!\!\int_{\Gamma} b\boldsymbol{v}_{\mathrm{T}}\cdot\widetilde{\boldsymbol{v}}_{\mathrm{T}}\,\mathrm{d}S\mathrm{d}t + \int_{\Omega} \varrho\boldsymbol{v}(T)\cdot\widetilde{\boldsymbol{v}}(T) + \beta p(T)\widetilde{p}(T) \\ &- \varrho\boldsymbol{v}_{0}\cdot\widetilde{\boldsymbol{v}}(0) - \beta p_{0}\widetilde{p}(0)\,\mathrm{d}x \end{split}$$

for any $(\tilde{\boldsymbol{v}}, \tilde{p}) \in \mathcal{V}^* \times \mathcal{P}^*$. Here we also used that $\mathcal{V} \subset C(I; L^2(\Omega; \mathbb{R}^n))$ and $\mathcal{P} \subset C(I; L^2(\Omega))$ so that the values $\boldsymbol{v}(t), \tilde{\boldsymbol{v}}(t), p(t)$, and $\tilde{p}(t)$ are well defined in L^2 -spaces for t = T or t = 0. For $\boldsymbol{u} \in \mathcal{U}$ given, we say that $(\boldsymbol{v}, p) \in \mathcal{V} \times \mathcal{P}$ is a weak solution to (2.1) if $\Pi(\boldsymbol{u}, \boldsymbol{v}, p) = 0$.

Existence of weak solutions is by standard arguments: an approximation e.g. by a Galerkin method, usage of apriori estimates to be read from (5.4) written for the approximates solutions, then passage to the limit by weak convergence and Aubin-Lions compact embedding theorem for the nonlinear terms. It is important that, if n = 2,

(3.3a)
$$\| (\boldsymbol{v} \cdot \nabla) \boldsymbol{v} \cdot \widetilde{\boldsymbol{v}} \|_{L^{1}(I \times \Omega)} \leq \| \boldsymbol{v} \|_{L^{4}(I \times \Omega; \mathbb{R}^{2})} \| \nabla \boldsymbol{v} \|_{L^{2}(I \times \Omega; \mathbb{R}^{2 \times 2})} \| \widetilde{\boldsymbol{v}} \|_{L^{4}(I \times \Omega; \mathbb{R}^{2})}$$
$$\leq C_{GN}^{2} \| \boldsymbol{v} \|_{L^{\infty}(I; L^{2}(\Omega; \mathbb{R}^{2}))}^{1/2} \| \boldsymbol{v} \|_{L^{2}(I; H^{1}(\Omega; \mathbb{R}^{2}))}^{1/2} \times$$
$$\times \| \nabla \boldsymbol{v} \|_{L^{2}(I \times \Omega; \mathbb{R}^{2 \times 2})} \| \widetilde{\boldsymbol{v}} \|_{L^{\infty}(I; L^{2}(\Omega; \mathbb{R}^{2}))}^{1/2} \| \widetilde{\boldsymbol{v}} \|_{L^{2}(I; H^{1}(\Omega; \mathbb{R}^{2}))}^{1/2} ,$$

where the Gagliardo-Nirenberg inequality $\|\boldsymbol{v}_{12}\|_{L^4(\Omega;\mathbb{R}^2)} \leq C_{\text{GN}} \|\boldsymbol{v}_{12}\|_{L^2(\Omega;\mathbb{R}^2)}^{1/2} \|\boldsymbol{v}_{12}\|_{H^1(\Omega;\mathbb{R}^2)}^{1/2}$ has been used. Analogously, one can estimate the term (div $\boldsymbol{v})\boldsymbol{v}\cdot\tilde{\boldsymbol{v}}$. Also

(3.3b)
$$\|p^{2}\operatorname{div}\widetilde{\boldsymbol{v}}\|_{L^{1}(I\times\Omega)} \leq \|p\|_{L^{4}(I\times\Omega)}^{2} \|\operatorname{div}\widetilde{\boldsymbol{v}}\|_{L^{2}(I\times\Omega)} \\ \leq C_{\mathrm{GN}}^{2}\|p\|_{L^{\infty}(I;L^{2}(\Omega))}\|p\|_{L^{2}(I;H^{1}(\Omega))}\|\operatorname{div}\widetilde{\boldsymbol{v}}\|_{L^{2}(I\times\Omega)} .$$

Similar estimate hold for the nonlinear term $(\boldsymbol{v} \cdot \nabla p) \widetilde{p}$:

$$(3.3c) \qquad \|(\boldsymbol{v}\cdot\nabla)p\cdot\widetilde{p}\|_{L^{1}(I\times\Omega)} \leq \|\boldsymbol{v}\|_{L^{4}(I\times\Omega;\mathbb{R}^{2})}\|\nabla p\|_{L^{2}(I\times\Omega;\mathbb{R}^{2})}\|\widetilde{p}\|_{L^{4}(I\times\Omega)}$$
$$\leq C_{GN}^{2}\|\boldsymbol{v}\|_{L^{\infty}(I;L^{2}(\Omega;\mathbb{R}^{2}))}^{1/2}\|\boldsymbol{v}\|_{L^{2}(I;H^{1}(\Omega;\mathbb{R}^{2}))}^{1/2} \times$$
$$\times \|\nabla p\|_{L^{2}(I\times\Omega;\mathbb{R}^{2})}\|\widetilde{p}\|_{L^{\infty}(I;L^{2}(\Omega))}^{1/2}\|\widetilde{p}\|_{L^{2}(I;H^{1}(\Omega))}^{1/2}$$

This shows that indeed $\Pi(\boldsymbol{u},\cdot,\cdot): \mathcal{V} \times \mathcal{P} \to \mathcal{V}^* \times \mathcal{P}^*$.

Lemma 3.1 (Well-posedness of the controlled system). Let $\varrho, \nu, \beta, \gamma > 0$, (3.2) hold, and n = 2. For any $\boldsymbol{u} \in \mathcal{U}$, there is a unique weak solution $(\boldsymbol{v}, p) \in \mathcal{V} \times \mathcal{P}$ and the mapping $\boldsymbol{u} \mapsto (\boldsymbol{v}, p) : \mathcal{U} \to \mathcal{V} \times \mathcal{P}$ is locally Lipschitz continuous and also (weak, weak)-continuous.

Proof. An important attribute especially in the context of control is uniqueness of the response (\boldsymbol{v}, p) for a given control \boldsymbol{u} . As for the incompressible Navier-Stokes model, the uniqueness holds unfortunately only for two-dimensional problems. Denoting $\boldsymbol{v}_{12} = \boldsymbol{v}_1 - \boldsymbol{v}_2$ and $p_{12} = p_1 - p_2$ for two weak solutions (\boldsymbol{u}_1, p_1) and (\boldsymbol{u}_2, p_2) and analysing the identity $\langle \Pi(\boldsymbol{u}, \boldsymbol{v}_1, p_1) - \Pi(\boldsymbol{u}, \boldsymbol{v}_2, p_2), (\boldsymbol{v}_{12}, p_{12}) \rangle = 0$, we have for a.a. time instants $t \in I$ (with tomitted in the following formulas for notational simplicity) that

$$(3.4) \qquad \nu \|\boldsymbol{e}(\boldsymbol{v}_{12})\|_{L^{2}(\Omega;\mathbb{R}^{2\times2})}^{2} + \gamma \|\nabla p_{12}\|_{L^{2}(\Omega;\mathbb{R}^{2})}^{2} + b \|\boldsymbol{v}_{12}\|_{L^{2}(\Gamma;\mathbb{R}^{2})}^{2} \\ \qquad + \frac{\mathrm{d}}{\mathrm{d}t} \Big(\frac{\varrho}{2} \|\boldsymbol{v}_{12}\|_{L^{2}(\Omega;\mathbb{R}^{2})}^{2} + \frac{\beta}{2} \|p_{12}\|_{L^{2}(\Omega)}^{2} \Big) \\ = \int_{\Omega} \Big(\varrho \big((\boldsymbol{v}_{2} \cdot \nabla) \boldsymbol{v}_{2} - \boldsymbol{v}_{1} \cdot \nabla) \boldsymbol{v}_{1} \big) \cdot \boldsymbol{v}_{12} + \frac{\beta}{2} \big(p_{1}^{2} - p_{2}^{2} \big) \mathrm{div} \, \boldsymbol{v}_{12} \\ \qquad + \frac{\varrho}{2} \big((\mathrm{div} \, \boldsymbol{v}_{2}) \boldsymbol{v}_{2} - (\mathrm{div} \, \boldsymbol{v}_{1}) \boldsymbol{v}_{1} \big) \cdot \boldsymbol{v}_{12} + \beta \big(\boldsymbol{v}_{1} \cdot \nabla p_{1} - \boldsymbol{v}_{2} \cdot \nabla p_{2} \big) p_{12} \big) \mathrm{d}x \\ = \int_{\Omega} \varrho \Big((\boldsymbol{v}_{12} \cdot \nabla) \boldsymbol{v}_{1} + \frac{1}{2} (\mathrm{div} \, \boldsymbol{v}_{1}) \, \boldsymbol{v}_{12} + (\boldsymbol{v}_{2} \cdot \nabla) \boldsymbol{v}_{12} + \frac{1}{2} (\mathrm{div} \, \boldsymbol{v}_{12}) \, \boldsymbol{v}_{2} \big) \cdot \boldsymbol{v}_{12} \, \mathrm{d}x \\ \qquad + \int_{\Omega} \beta \Big(\frac{p_{1} + p_{2}}{2} \mathrm{div} \, \boldsymbol{v}_{12} + \boldsymbol{v}_{12} \cdot \nabla p_{1} + \boldsymbol{v}_{2} \cdot \nabla p_{12} \Big) p_{12} \, \mathrm{d}x. \end{aligned}$$

The first right-hand integral can be estimated standardly as for the incompressible Navier-Stokes equation by Hölder's and Young's inequalities and by the Gagliardo-Nirenberg inequality combined with Korn's inequality $\|\boldsymbol{v}_{12}\|_{L^4(\Omega;\mathbb{R}^2)} \leq C_{\text{GNK}} \|\boldsymbol{v}_{12}\|_{L^2(\Omega;\mathbb{R}^2)}^{1/2} \|\boldsymbol{e}(\boldsymbol{v}_{12})\|_{L^2(\Omega;\mathbb{R}^{2\times 2})}^{1/2}$; here we again rely on n = 2. The same Gagliardo-Nirenberg inequality holds for p_{12} and can be exploited for estimating the particular terms in the last integral in (3.4) as:

$$(3.5a) \qquad \int_{\Omega} \frac{\beta}{2} (p_{1}+p_{2}) (\operatorname{div} \mathbf{v}_{12}) p_{12} \, \mathrm{d}x \\ \leq \frac{\beta}{2} \|p_{1}+p_{2}\|_{L^{4}(\Omega)} \|\mathbf{v}_{12}\|_{H^{1}(\Omega;\mathbb{R}^{2})} \|p_{12}\|_{L^{2}(\Omega)} \|\nabla p_{12}\|_{L^{2}(\Omega;\mathbb{R}^{2})}^{1/2} \\ \leq C_{\mathrm{GNK}} \frac{\beta}{2} \|p_{1}+p_{2}\|_{L^{4}(\Omega)} \|\mathbf{v}_{12}\|_{H^{1}(\Omega;\mathbb{R}^{2})} \|p_{12}\|_{L^{2}(\Omega)}^{1/2} \|\nabla p_{12}\|_{L^{2}(\Omega;\mathbb{R}^{2})}^{1/2} \\ \leq \epsilon \|\mathbf{v}_{12}\|_{H^{1}(\Omega;\mathbb{R}^{2})}^{2} + C_{\mathrm{GNK}}^{2} \frac{\beta^{2}}{16\epsilon} \|p_{1}+p_{2}\|_{L^{4}(\Omega)}^{2} \|p_{12}\|_{L^{2}(\Omega)} \|\nabla p_{12}\|_{L^{2}(\Omega;\mathbb{R}^{2})}^{1/2} \\ \leq \epsilon \|\mathbf{v}_{12}\|_{H^{1}(\Omega;\mathbb{R}^{2})}^{2} + \epsilon \|\nabla p_{12}\|_{L^{2}(\Omega;\mathbb{R}^{2})}^{2} + C\frac{\beta^{4}}{\epsilon^{2}} \|p_{1}+p_{2}\|_{L^{4}(\Omega)}^{4} \|p_{12}\|_{L^{2}(\Omega;\mathbb{R}^{2})}^{2} \\ \leq \epsilon \|\mathbf{v}_{12}\|_{L^{1}(\Omega;\mathbb{R}^{2})}^{1/2} + \epsilon \|\nabla p_{12}\|_{L^{2}(\Omega;\mathbb{R}^{2})}^{1/2} \|\nabla p_{1}\|_{L^{2}(\Omega;\mathbb{R}^{2})}^{1/2} \|p_{12}\|_{L^{2}(\Omega;\mathbb{R}^{2})}^{1/2} \\ \leq \beta C_{\mathrm{GNK}}^{2} \|\mathbf{v}_{12}\|_{L^{2}(\Omega;\mathbb{R}^{2})}^{1/2} \|\mathbf{v}_{12}\|_{H^{1}(\Omega;\mathbb{R}^{2})}^{1/2} \\ \leq \beta C_{\mathrm{GNK}}^{2} \|\mathbf{v}_{12}\|_{L^{2}(\Omega;\mathbb{R}^{2})}^{1/2} \|\mathbf{v}_{12}\|_{L^{2}(\Omega;\mathbb{R}^{2})}^{1/2} \|p_{12}\|_{L^{2}(\Omega;\mathbb{R}^{2})}^{1/2} \\ \leq \beta C_{\mathrm{GNK}}^{2} \|\mathbf{v}_{12}\|_{L^{2}(\Omega;\mathbb{R}^{2})}^{1/2} \|\nabla p_{11}\|_{L^{2}(\Omega;\mathbb{R}^{2})}^{1/2} \|p_{12}\|_{L^{2}(\Omega;\mathbb{R}^{2})}^{1/2} \|p_{12}\|_{L^{2}(\Omega;\mathbb{R}^{2})}^{1/2} \\ \leq 2\epsilon \|\mathbf{v}_{12}\|_{H^{1}(\Omega;\mathbb{R}^{2})}^{2} + \epsilon \|\nabla p_{12}\|_{L^{2}(\Omega;\mathbb{R}^{2})}^{2} \|\mathbf{v}_{12}\|_{L^{2}(\Omega;\mathbb{R}^{2})}^{1/2} \|p_{12}\|_{L^{2}(\Omega;\mathbb{R}^{2})}^{1/2} \|p_{12}\|_{L^{2}(\Omega;\mathbb{R}^{2})}^{1/2} \\ \leq \epsilon \|\mathbf{v}_{12}\|_{H^{1}(\Omega;\mathbb{R}^{2})}^{2} + \epsilon \|\nabla p_{12}\|_{L^{2}(\Omega;\mathbb{R}^{2})}^{2} \|\mathbf{v}_{12}\|_{L^{2}(\Omega;\mathbb{R}^{2})}^{2} \|p_{12}\|_{L^{2}(\Omega;\mathbb{R}^{2})}^{2} + C_{\mathrm{GNK}}^{4} \frac{\beta}{4} \|\nabla p_{11}\|_{L^{2}(\Omega;\mathbb{R}^{2})}^{2} \|p_{12}\|_{L^{2}(\Omega;\mathbb{R}^{2})}^{2} \|p_{12}\|_{L^{2}(\Omega)}^{2} \\ \leq \epsilon \|\nabla p_{12}\|_{L^{2}(\Omega;\mathbb{R}^{2})}^{2} \|\nabla p_{12}\|_{L^{2}(\Omega;\mathbb{R}^{2})}^{3/2} \|p_{12}\|_{L^{2}(\Omega)}^{2} \\ \leq \epsilon \|\nabla p_{12}\|_{L^{2}(\Omega;\mathbb{R}^{2})}^{2} + C\frac{\beta^{4}}{\epsilon} \|\mathbf{v}_{2}\|_{L^{4}(\Omega;\mathbb{R}^{2})}^{2} \|p_{12}\|_{L^{2}(\Omega)}^{2} .$$

Taking $\epsilon > 0$ small enough, the ϵ -terms on the right-hand sides of (3.5) can be absorbed in the left-hand side of (3.4) while the others can be treated by Gronwall's inequality, using that $t \mapsto \|p_1(t) + p_2(t)\|_{L^4(\Omega)}^4$, $t \mapsto \|\nabla p_1(t)\|_{L^2(\Omega;\mathbb{R}^2)}^2$ and $t \mapsto \|\boldsymbol{e}(\boldsymbol{v}_2(t))\|_{L^4(\Omega;\mathbb{R}^{2\times 2})}^4$ are $L^1(I)$.

Considering the above two solutions (\boldsymbol{v}_1, p_1) and (\boldsymbol{v}_2, p_2) for two different controls \boldsymbol{u}_1 and \boldsymbol{u}_2 , respectively, the right-hand side of (3.4) would augment by the term $\int_{\Omega} \boldsymbol{u}_{12} \cdot \boldsymbol{v}_{12} \, dx$ with $\boldsymbol{u}_{12} = \boldsymbol{u}_1 - \boldsymbol{u}_2$. Then the above estimates in fact show the local Lipschitz continuity of the control-to-state mapping $\boldsymbol{u} \mapsto (\boldsymbol{v}, p)$ from \mathcal{U} to $(L^2(I; H^1(\Omega; \mathbb{R}^n)) \cap L^{\infty}(I; L^2(\Omega; \mathbb{R}^n))) \times (L^2(I; H^1(\Omega)) \cap L^{\infty}(I; L^2(\Omega)))$. By a slight modification of these estimates, we can also see the local Lipschitz continuity $\boldsymbol{u} \mapsto (\dot{\boldsymbol{v}}, \dot{p})$ from \mathcal{U} to $(L^2(I; H^1(\Omega; \mathbb{R}^n)) \cap L^{\infty}(I; L^2(\Omega; \mathbb{R}^n)))^* \times (L^2(I; H^1(\Omega)) \cap L^{\infty}(I; L^2(\Omega)))^*$.

When u_1 is fixed and $u_2 \to u_1$ converges weakly in \mathcal{U} , we can prove the (weak,weak)continuity by using the compact embedding of $\mathcal{V} \times \mathcal{P}$ into $L^2(I \times \Omega; \mathbb{R}^d \times \mathbb{R})$ by the Aubin-Lions theorem. This allows to pass to the limit in all nonlinear terms. At this point, it is also important that u occurs linearly in (2.1a).

Let us end this section by noting that the energy balance (5.4) holds rigorously for any weak solution (\boldsymbol{v}, p) because $\dot{\boldsymbol{v}}$ is in duality with \boldsymbol{v} and also \dot{p} is in duality with p, so that the tests of (2.1a) by \boldsymbol{v} and of (2.1b) by p are legitimate.

Notably, all these estimates are exact without any "reserve". Let us point out that, in the three-dimensional case, this semi-compressible model admits only a very weak solution and also the uniqueness and continuity of the control-to-state mapping analogous to Lemma 3.1 is not granted. Cf. also Remark 5.1 below for a modification of the model working for n = 3.

4. Optimal control in the two-dimensional case

Beside facilitating pressure waves and their dispersion, the benefit of this semi-compressible model is that pressure is well defined even with the traces on the boundary. Also the values of p at particular time instants are well defined in the sense of $L^2(\Omega)$. This allows us to involve pressure on the boundary and in the terminal time into the cost functional which will be considered quadratic for simplicity. We thus consider the optimal-control problem

$$(4.1) \qquad \begin{cases} \text{Minimize } \Phi(\boldsymbol{u}, \boldsymbol{v}, p) \coloneqq \int_{0}^{T} \int_{\Omega} \frac{\kappa_{1}}{2} |\boldsymbol{v} - \boldsymbol{v}_{d}|^{2} + \frac{\kappa_{2}}{2} |p - p_{d}|^{2} + \frac{\kappa_{3}}{2} |\boldsymbol{u}|^{2} \, \mathrm{d}x \, \mathrm{d}t \\ + \int_{0}^{T} \int_{\Gamma} \frac{\varkappa_{1}}{2} |\boldsymbol{v} - \boldsymbol{v}_{d1}|^{2} + \frac{\varkappa_{2}}{2} |p - p_{d1}|^{2} \, \mathrm{d}S \, \mathrm{d}t \\ + \int_{\Omega} \frac{\lambda_{1}}{2} |\boldsymbol{v}(T) - \boldsymbol{v}_{dT}|^{2} + \frac{\lambda_{2}}{2} |p(T) - p_{dT}|^{2} \, \mathrm{d}x \\ \text{subject to } (\boldsymbol{v}, p) \text{ satisfying } (2.1) \text{ in the weak sense,} \\ \boldsymbol{u} \in \mathcal{U}, \quad \boldsymbol{v} \in \mathcal{V}, \quad p \in \mathcal{P}, \end{cases}$$

where κ 's, \varkappa 's, and λ 's are nonnegative constants, $\kappa_3 > 0$. Here, rather to avoid technicalities, the control \boldsymbol{u} is in the bulk, which makes the problem a bit academical, although some boundary control or control through initial conditions might be considered, too.

The optimality conditions involves the multipliers $(\vartheta, \pi) \in \mathcal{V} \times \mathcal{P}$ and the Lagrangian

$$\mathscr{L}(\boldsymbol{u}, \boldsymbol{v}, p, \boldsymbol{\vartheta}, \pi) := \langle \Pi(\boldsymbol{u}, \boldsymbol{v}, p), (\boldsymbol{\vartheta}, \pi) \rangle - \Phi(\boldsymbol{u}, \boldsymbol{v}, p)$$

We denote $S : \mathbf{u} \mapsto (\mathbf{v}, p) : \mathcal{U} \to \mathcal{V} \times \mathcal{P}$ the control-to-state mapping (which was shown in Section 3 single-valued and continuous) and the composed cost $J(\mathbf{u}) := \Phi(\mathbf{u}, S(\mathbf{u}))$. If J has the Gâteaux differential $J' : \mathcal{U} \to \mathcal{U}^*$, the first-order optimality conditions for our unconstrained problem looks simply as $J'(\boldsymbol{u}) = 0$.

The Gâteaux differential (if exists) can be calculated by the so-called adjoint-equation technique. More specifically, $J'(\boldsymbol{u}) = \mathscr{L}'_{\boldsymbol{u}}(\boldsymbol{u}, \boldsymbol{v}, p, \boldsymbol{\vartheta}, \pi)$ provided the adjoint velocity $\boldsymbol{\vartheta}$ and the adjoint pressure π satisfied

(4.2a)
$$\langle \Pi'_{(\boldsymbol{v},p)}(\boldsymbol{u},\boldsymbol{v},p),(\boldsymbol{\vartheta},\pi)\rangle = \Phi'_{(\boldsymbol{v},p)}(\boldsymbol{u},\boldsymbol{v},p)$$

Thus, the mentioned first-order optimality conditions $J'(\boldsymbol{u}) = 0$ then result to

(4.2b)
$$\langle \Pi'_{\boldsymbol{u}}(\boldsymbol{u},\boldsymbol{v},p),(\boldsymbol{\vartheta},\pi)\rangle = \Phi'_{\boldsymbol{u}}(\boldsymbol{u},\boldsymbol{v},p)$$

More specifically, realizing the partial Gâteaux derivatives $\Pi'_{(\boldsymbol{v},p)}(\boldsymbol{u},\boldsymbol{v},p) \in \operatorname{Lin}(\mathcal{V}\times\mathcal{P},\mathcal{V}^*\times\mathcal{P}^*)$ and $\Phi'_{(\boldsymbol{v},p)}(\boldsymbol{u},\boldsymbol{v},p) \in \operatorname{Lin}(\mathcal{V}\times\mathcal{P},\mathbb{R}) \cong \mathcal{V}^*\times\mathcal{P}^*$, the adjoint equation (4.2a) means

$$\forall (\widetilde{\boldsymbol{\vartheta}}, \widetilde{\pi}) \in \mathcal{V} \times \mathcal{P} : \left\langle [\Pi'_{(\boldsymbol{v}, p)}(\boldsymbol{u}, \boldsymbol{v}, p)](\widetilde{\boldsymbol{\vartheta}}, \widetilde{\pi}), (\boldsymbol{\vartheta}, \pi) \right\rangle = \left\langle \varPhi'_{(\boldsymbol{v}, p)}(\boldsymbol{u}, \boldsymbol{v}, p), (\widetilde{\boldsymbol{\vartheta}}, \widetilde{\pi}) \right\rangle$$

which further means

(4.3a)
$$\left[\Pi'_{(\boldsymbol{v},p)}(\boldsymbol{u},\boldsymbol{v},p)\right]^*(\boldsymbol{\vartheta},\pi) = \varPhi'_{(\boldsymbol{v},p)}(\boldsymbol{u},\boldsymbol{v},p)$$

where $[\cdot]^*$ denotes the adjoint operator, and similarly (4.2b) reads as

(4.3b)
$$\left[\Pi'_{\boldsymbol{u}}(\boldsymbol{u},\boldsymbol{v},p)\right]^*(\boldsymbol{\vartheta},\pi) = \varPhi'_{\boldsymbol{u}}(\boldsymbol{u},\boldsymbol{v},p) \,.$$

We note that the control-to-state mapping S is in our two-dimensional case even continuously differentiable. It is important that the adjoint equation (4.3a) has a solution, for which it suffices to show that $\Pi'_{(\boldsymbol{v},p)}(\boldsymbol{u},\cdot,\cdot)$ is surjective because then, by open-mapping theorem, there exists a continuous inverse operator. Then, from the state equation $\Pi(\boldsymbol{u}, S(\boldsymbol{u})) = 0$, we get $\Pi'_{\boldsymbol{u}}(\boldsymbol{u}, S(\boldsymbol{u})) + \Pi'_{(\boldsymbol{v},p)}(\boldsymbol{u}, S(\boldsymbol{u})) \circ S'(\boldsymbol{u}) = 0$ so that the Gâteaux differential of Sis given by the explicit formula $S'(\boldsymbol{u}) = [\Pi'_{(\boldsymbol{v},p)}(\boldsymbol{u}, S(\boldsymbol{u}))]^{-1}\Pi'_{\boldsymbol{u}}(\boldsymbol{u}, S(\boldsymbol{u}))$ and it depends continuously on \boldsymbol{u} . By this surjectivity, also the adjoint equation (4.3a) has a solution $(\boldsymbol{\vartheta}, \pi) = [\Pi'_{(\boldsymbol{v},p)}(\boldsymbol{u}, \boldsymbol{v}, p)]^{-1}(\Phi'_{(\boldsymbol{v},p)}(\boldsymbol{u}, \boldsymbol{v}, p))$. As it is the only solution of (4.3a), the adjoint state is determined uniquely for a current \boldsymbol{u} .

The semi-compressible system contains three bi-linear and one quadratic terms in (2.1), namely $\varrho(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}, \frac{\varrho}{2}(\operatorname{div}\boldsymbol{v})\boldsymbol{v}, \frac{\beta}{2}\nabla(p^2)$, and $\beta\boldsymbol{v}\cdot\nabla p$. These terms give rise to seven bilinear terms in the adjoint system, mixing the state and the adjoint variables. These seven bilinear terms arise by the Green formula through the following detailed componentwise calculus

$$(4.4) \quad \int_{\Omega} \left(\vartheta \cdot \left(\varrho \widetilde{\boldsymbol{v}} \cdot \nabla \boldsymbol{v} + \varrho \boldsymbol{v} \cdot \nabla \widetilde{\boldsymbol{v}} + \frac{\varrho}{2} (\operatorname{div} \boldsymbol{v}) \widetilde{\boldsymbol{v}} + \frac{\varrho}{2} (\operatorname{div} \widetilde{\boldsymbol{v}}) \boldsymbol{v} + \beta \nabla (p \widetilde{p}) \right) \\ + \pi \left(\widetilde{\boldsymbol{v}} \cdot \nabla p + \boldsymbol{v} \cdot \nabla \widetilde{p} \right) \right) \mathrm{d}x \\ = \sum_{i,j=1}^{n} \int_{\Omega} \left(\vartheta_{j} \left(\varrho \widetilde{\boldsymbol{v}}_{i} \frac{\partial \boldsymbol{v}_{j}}{\partial x_{i}} + \varrho \boldsymbol{v}_{i} \frac{\partial \widetilde{\boldsymbol{v}}_{j}}{\partial x_{i}} + \frac{\varrho}{2} \frac{\partial \boldsymbol{v}_{i}}{\partial x_{i}} \widetilde{\boldsymbol{v}}_{j} + \frac{\varrho}{2} \frac{\partial \widetilde{\boldsymbol{v}}_{i}}{\partial x_{i}} \boldsymbol{v}_{j} + \beta \frac{\partial}{\partial x_{j}} (p \widetilde{p}) \right) \\ + \beta \pi \left(\widetilde{\boldsymbol{v}}_{i} \frac{\partial p}{\partial x_{i}} + \boldsymbol{v}_{i} \frac{\partial \widetilde{p}}{\partial x_{i}} \right) \right) \mathrm{d}x \\ = \sum_{i,j=1}^{n} \int_{\Omega} \left(\left(\varrho \vartheta_{i} \frac{\partial \boldsymbol{v}_{i}}{\partial x_{j}} - \varrho \frac{\partial}{\partial x_{i}} (\boldsymbol{v}_{i} \vartheta_{j}) + \frac{\varrho}{2} \frac{\partial \boldsymbol{v}_{i}}{\partial x_{i}} \vartheta_{j} - \frac{\varrho}{2} \frac{\partial}{\partial x_{j}} (\boldsymbol{v}_{i} \vartheta_{i}) + \beta \pi \frac{\partial p}{\partial x_{j}} \right) \widetilde{\boldsymbol{v}}_{j} \right)$$

$$-\beta \Big(\frac{\partial(\pi \boldsymbol{v}_{i})}{\partial x_{i}} + p\frac{\partial\boldsymbol{\vartheta}_{i}}{\partial x_{i}}\Big)\widetilde{p}\Big) \mathrm{d}x + \sum_{i,j=1}^{n} \int_{\Gamma} \rho \boldsymbol{v}_{i} \boldsymbol{\vartheta}_{j} \widetilde{\boldsymbol{v}}_{j} \boldsymbol{n}_{i} + \frac{\rho}{2} \boldsymbol{v}_{i} \boldsymbol{\vartheta}_{i} \widetilde{\boldsymbol{v}}_{j} \boldsymbol{n}_{j} + \beta(\pi \boldsymbol{v}_{i} + p\boldsymbol{\vartheta}_{i}) \widetilde{\rho} \boldsymbol{n}_{i} \mathrm{d}S$$
$$= \int_{\Omega} \Big(\Big(\rho(\nabla \boldsymbol{v})^{\top} \boldsymbol{\vartheta} - \rho \mathrm{div}(\boldsymbol{v} \otimes \boldsymbol{\vartheta}) + \frac{\rho}{2} (\mathrm{div} \, \boldsymbol{v}) \boldsymbol{\vartheta} - \frac{\rho}{2} \nabla(\boldsymbol{v} \cdot \boldsymbol{\vartheta}) + \beta \pi \nabla p \Big) \cdot \widetilde{\boldsymbol{v}}$$
$$-\beta \Big(\mathrm{div}(\pi \boldsymbol{v}) + p \, \mathrm{div} \, \boldsymbol{\vartheta} \Big) \widetilde{p} \Big) \mathrm{d}x + \int_{\Gamma} \Big(\frac{3\rho}{2} \boldsymbol{\vartheta} \cdot \widetilde{\boldsymbol{v}} + \beta \widetilde{p} \pi \Big) (\boldsymbol{v} \cdot \boldsymbol{n}) + \beta p \widetilde{p}(\boldsymbol{\vartheta} \cdot \boldsymbol{n}) \, \mathrm{d}S \,,$$

where the boundary integral actually vanishes due to the boundary conditions $\boldsymbol{v}\cdot\boldsymbol{n} = 0$ and $\boldsymbol{\vartheta}\cdot\boldsymbol{n} = 0$. From this, we can read the five bi-linear terms in (4.5a) when varying the test function $\tilde{\boldsymbol{v}}$ and two bi-linear terms in (4.5b) when varying the test function $\tilde{\boldsymbol{p}}$. By straightforward modifications of the estimates from Sect. 3, cf. in particular (3.3a)–(3.3b), we can see integrability of these tri-linear terms in this two-dimensional case. The resting linear parabolic terms contributing to the linear operator $\Pi'_{(\boldsymbol{v},\boldsymbol{p})}(\boldsymbol{u},\cdot,\cdot)$ are standard.

More specifically, (4.3a) leads to the adjoint terminal-boundary-value for a linear parabolic system for the multipliers $\boldsymbol{\vartheta}$ and π in the classical formulations reads as

(4.5a)
$$-\varrho \dot{\boldsymbol{\vartheta}} + \varrho (\nabla \boldsymbol{v})^{\top} \boldsymbol{\vartheta} - \varrho \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{\vartheta}) + \beta \pi \nabla p - \operatorname{div}(\nu \boldsymbol{e}(\boldsymbol{\vartheta})) \\ + \frac{\varrho}{2} (\operatorname{div} \boldsymbol{v}) \boldsymbol{\vartheta} - \frac{\varrho}{2} \nabla (\boldsymbol{v} \cdot \boldsymbol{\vartheta}) + \nabla \pi = \kappa_1 (\boldsymbol{v} - \boldsymbol{v}_{\mathrm{d}}) \quad \text{in } I \times \Omega,$$

(4.5b)
$$-\beta \dot{\pi} - \beta \operatorname{div}(\pi \boldsymbol{v}) - \gamma \Delta \pi - (1 + \beta p) \operatorname{div} \boldsymbol{\vartheta} = \kappa_2 (p - p_d) \quad \text{in } I \times \Omega$$

(4.5c)
$$[\boldsymbol{\nu}\boldsymbol{e}(\boldsymbol{\vartheta})\boldsymbol{n}]_{\mathrm{T}} + b\boldsymbol{v}_{\mathrm{T}} = \boldsymbol{\varkappa}_{1}(\boldsymbol{v}-\boldsymbol{v}_{\mathrm{d}1}) \text{ and } \boldsymbol{n}\cdot\boldsymbol{\vartheta} = 0 \text{ on } I \times \Gamma$$

(4.5d)
$$\gamma \nabla \pi \cdot \boldsymbol{n} = \varkappa_2 (p - p_{\mathrm{d1}})$$
 on $I \times \Gamma$,

(4.5e)
$$\boldsymbol{\vartheta}(T,\cdot) = \lambda_1(\boldsymbol{v}(T) - \boldsymbol{v}_{\mathrm{d}T}) \text{ and } \pi(T,\cdot) = \lambda_2(p(T) - p_{\mathrm{d}T}) \text{ on } \Omega.$$

The second condition (4.2b) leads here simply to $\boldsymbol{\vartheta} = \kappa_3 \boldsymbol{u}$.

Since $\kappa_3 > 0$, the functional $\boldsymbol{u} \mapsto \Phi(\boldsymbol{u}, S(\boldsymbol{u}))$ is coercive on \mathcal{U} . Existence of optimal controls, i.e. solutions to (4.1), can then be shown by the classical direct-method argument. Here we use the (weak,weak)-continuity of S from Lemma 3.1 and also the (weak,weak)-continuity of the trace operator $(\boldsymbol{v}, p) \mapsto (\boldsymbol{v}|_{I \times \Gamma}, p|_{I \times \Gamma}) : \mathcal{V} \times \mathcal{P} \to L^2(I \times \Gamma; \mathbb{R}^d \times \mathbb{R})$ and also of the operator $(\boldsymbol{v}, p) \mapsto (\boldsymbol{v}(T), p(T)) : \mathcal{V} \times \mathcal{P} \to L^2(\Omega; \mathbb{R}^d \times \mathbb{R})$. The weak lower semicontinuity of $\boldsymbol{u} \mapsto \Phi(\boldsymbol{u}, S(\boldsymbol{u}))$ is then implied by convexity of Φ .

The existence and uniqueness of the solution to the adjoint problem has been already discussed on the abstract level, based on the solvability of the linearized state equation for all right-hand sides, i.e. surjectivity of the Gâteaux derivative of the state equation with respect to (\boldsymbol{v}, p) . At a given (\boldsymbol{v}, p) , this linearization in the direction $(\tilde{\boldsymbol{v}}, \tilde{p})$ gives seven terms originated from the bilinear/quadratic nonlinearities in (2.1a,b). The a-priori estimates for this linearized system can be performed by a test by $(\tilde{\boldsymbol{v}}, \tilde{p})$, which gives the trilinear terms of the type div $\boldsymbol{v}|\tilde{\boldsymbol{v}}|^2$, $(\boldsymbol{v}\cdot\nabla)\tilde{\boldsymbol{v}}\cdot\tilde{\boldsymbol{v}}$, $(\boldsymbol{v}\cdot\nabla p)p$, and $(\text{div}\boldsymbol{v})\tilde{p}^2$, cf. (4.4) for $\boldsymbol{\vartheta} = \tilde{\boldsymbol{v}}$ and $\pi = \tilde{p}$. These terms can be estimated by Gagliardo-Nirenberg, Korn, Hölder, Young, and Gronwall inequalities quite similarly as we did in (3.5). Again, it works only for the two-dimensional situation.

Let us briefly summarize the above arguments and calculations:

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Proposition 4.1 (Existence and optimality conditions). Let $\rho, \nu, \beta, \gamma > 0$, (3.2) hold, and n = 2. Then:

- (i) The optimal-control problem (4.1) possesses at least one solution $(\boldsymbol{u}, \boldsymbol{v}, p)$.
- (ii) For any such solution (u, v, p), there exists the unique adjoint state (ϑ, π) ∈ V×P satisfying the terminal-boundary-value problem (4.5) in the weak sense and u = ϑ/κ₃. In particular, any optimal control u ∈ U must be more regular, belonging also to V from (3.1a).

5. Concluding Remarks

We end this article by several remarks suggesting modifications or enhancements of the controlled semilinear parabolic system still keeping the bilinear character of all involved nonlinearities and widening applicability towards three-dimensional situations or coupling with other phenomena.

Remark 5.1 (Multipolar fluids for three-dimensional problems). The semi-compressible model (similarly as fully compressible for $\beta = 0$ and $\gamma = 0$) works only in two-dimensional situations because the needed uniqueness of the response is not granted in higher dimensions. To extend the above results towards 3-dimensional problems, a concept of so-called multipolar fluids can be used. This introduces a higher-order viscosity. Under the name 2nd-grade nonsimple fluids, (5.1) was devised by E. Fried and M. Gurtin [14] and earlier, even more generally and nonlinearly as multipolar fluids, by J. Nečas at al. [3, 20, 21]. Here it means that (2.1a,b) modifies as

 \sim

(5.1a)
$$\varrho \dot{\boldsymbol{v}} + \varrho (\boldsymbol{v} \cdot \nabla) \boldsymbol{v} - \operatorname{div} \left(\nu \boldsymbol{e}(\boldsymbol{v}) - \operatorname{div}(\nu_1 \nabla \boldsymbol{e}(\boldsymbol{v})) \right) + \nabla \left(p + \frac{\beta}{2} p^2 \right) = \boldsymbol{u}$$

and $\beta (\dot{\boldsymbol{p}} + \boldsymbol{v} \cdot \nabla p) + \operatorname{div} \boldsymbol{v} = \gamma \Delta p \quad \text{on } I \times \Omega ,$

(5.1b)
$$\begin{bmatrix} \nu \boldsymbol{e}(\boldsymbol{v})\boldsymbol{n} - \operatorname{div}_{s}(\nu_{1}\nabla \boldsymbol{e}(\boldsymbol{v}))\boldsymbol{n} \end{bmatrix}_{T} + b\boldsymbol{v}_{T} = 0, \quad \boldsymbol{n} \cdot \boldsymbol{v} = 0 \\ \text{and} \quad \nabla^{2}\boldsymbol{v} \vdots (\boldsymbol{n} \otimes \boldsymbol{n}) = 0, \quad \boldsymbol{n} \cdot \nabla p = 0 \quad \text{on } I \times \Gamma, \end{cases}$$

with a (presumably small) "hyper"-viscosity coefficient $\nu_1 > 0$ and with the initial conditions (2.1d). In (5.1b), "div_s" denotes the surface divergence defined as $\operatorname{div}_{s}(\cdot) = \operatorname{tr}(\nabla_{s}(\cdot))$ with $\operatorname{tr}(\cdot)$ denoting the trace and ∇_{s} denoting the surface gradient given by $\nabla_{s}v = (\mathbb{I} - \boldsymbol{n} \otimes \boldsymbol{n})\nabla v =$ $\nabla v - \frac{\partial v}{\partial \boldsymbol{n}}\boldsymbol{n}$. The energetics (5.4) expands by the dissipation rate $\nu_1 |\nabla^2 \boldsymbol{v}|^2$ so that \mathcal{V} from (3.1a) takes $H^2(\Omega; \mathbb{R}^n)$ instead of $H^1(\Omega; \mathbb{R}^n)$. The above uniqueness arguments can now be modified for n = 3 too, cf. [26, Prop. 4], and again $\Pi(\boldsymbol{u}, \cdot, \cdot) : \mathcal{V} \times \mathcal{P} \to \mathcal{V}^* \times \mathcal{P}^*$. Here one uses that, by interpolation, $L^{\infty}(I; L^2(\Omega)) \cap L^2(I; H^2(\Omega)) \subset L^4(I; H^1(\Omega)) \subset L^4(I; L^6(\Omega))$ so that (3.3a) holds in the modification

$$\|(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\cdot\widetilde{\boldsymbol{v}}\|_{L^{1}(I\times\Omega)} \leq C\|\boldsymbol{v}\|_{L^{4}(I;L^{6}(\Omega;\mathbb{R}^{3}))}\|\nabla\boldsymbol{v}\|_{L^{4}(I;L^{2}(\Omega;\mathbb{R}^{3\times3}))}\|\widetilde{\boldsymbol{v}}\|_{L^{4}(I;L^{6}(\Omega;\mathbb{R}^{3}))}$$

while the nonlinear term $p^2 \operatorname{div} \widetilde{\boldsymbol{v}}$ bears the estimation

$$\begin{aligned} \|p^{2} \operatorname{div} \widetilde{\boldsymbol{v}}\|_{L^{1}(I \times \Omega)} &\leq \|p\|_{L^{8/3}(I;L^{4}(\Omega))}^{2} \|\operatorname{div} \widetilde{\boldsymbol{v}}\|_{L^{4}(I;L^{2}(\Omega))} \\ &\leq C \|p\|_{L^{\infty}(I;L^{2}(\Omega))}^{1/2} \|p\|_{L^{2}(I;H^{1}(\Omega))}^{3/2} \|\widetilde{\boldsymbol{v}}\|_{L^{\infty}(I;L^{2}(\Omega;\mathbb{R}^{3}))}^{1/2} \|\widetilde{\boldsymbol{v}}\|_{L^{2}(I;H^{2}(\Omega;\mathbb{R}^{3}))}^{1/2} \end{aligned}$$

when using twice the Gagliardo-Nirenberg interpolation; note that here the estimation is exact without any "reserve". Similar estimate hold for the nonlinear term $(\boldsymbol{v} \cdot \nabla p)\tilde{p}$, namely

 $\|(\boldsymbol{v}\cdot\nabla p)\widetilde{p}\|_{L^1(I\times\Omega)} \le \|\boldsymbol{v}\|_{L^4(I;L^6(\Omega;\mathbb{R}^3))} \|\nabla p\|_{L^2(I\times\Omega)} \|\widetilde{p}\|_{L^4(I;L^3(\Omega))}$

$$\leq C \|\boldsymbol{v}\|_{L^{\infty}(I;L^{2}(\Omega;\mathbb{R}^{3}))}^{1/2} \|\boldsymbol{v}\|_{L^{2}(I;H^{2}(\Omega;\mathbb{R}^{3}))}^{1/2} \times \\ \times \|\nabla p\|_{L^{2}(I\times\Omega)} \|\widetilde{p}\|_{L^{\infty}(I;L^{2}(\Omega))}^{1/2} \|\widetilde{p}\|_{L^{2}(I;L^{6}(\Omega))}^{1/2}$$

due to the Gagliardo-Nirenberg interpolation; note that again the estimation is exact without any "reserve". Thus Proposition 4.1 holds for n = 3 for this 2nd-grade nonsimple semicompressible system (5.1).

Remark 5.2 (*Multipolar fluids for pressure constraints*). One can think about 3rd-grade nonsimple semi-compressible fluids governed by the system:

(5.2a) $\rho \dot{\boldsymbol{v}} + \rho(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} - \operatorname{div}\left(\nu \boldsymbol{e}(\boldsymbol{v}) + \operatorname{div}^2(\nu_1 \nabla^2 \boldsymbol{e}(\boldsymbol{v}))\right) + \nabla\left(p + \frac{\beta}{2}p^2\right) = \boldsymbol{u},$

(5.2b)
$$\beta(\dot{p} + \boldsymbol{v} \cdot \nabla p) + \operatorname{div} \boldsymbol{v} = 0$$

on $I \times \Omega$ with the corresponding three boundary conditions for \boldsymbol{v} on $I \times \Gamma$; these conditions are rather technical and we refer to [25]. On the other hand, a reasonable analysis can be performed without pressure diffusion $\gamma \Delta p$ in (5.2b) and thus no boundary condition is prescribed for pressure. Now $\boldsymbol{v} \in C_{\mathbf{w}}(I; H^1(\Omega; \mathbb{R}^n)) \cap L^2(I; H^3(\Omega; \mathbb{R}^n))$, so that even for n = 3, we now have $\nabla \boldsymbol{v} \in L^2(I; L^{\infty}(\Omega; \mathbb{R}^{3\times 3}))$. Then, if the initial pressure is enough regular, this regularity now will be transported along the evolution. Namely, if $p_0 \in W^{1,q}(\Omega)$, we obtain $p \in L^{\infty}(I; W^{1,q}(\Omega))$ by applying ∇ to (5.2b) and then testing it by $|\nabla p|^{q-2} \nabla p$. Here we use the calculus

$$\begin{split} \frac{1}{q} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} |\nabla p|^{q} \,\mathrm{d}x &= \int_{\Omega} |\nabla p|^{q-2} \nabla p \cdot \nabla \dot{p} \,\mathrm{d}x \\ &= -\int_{\Omega} |\nabla p|^{q-2} \nabla p \cdot \left(\nabla (\boldsymbol{v} \cdot \nabla p) + \nabla \mathrm{div} \,\boldsymbol{v} \right) \,\mathrm{d}x \\ &= \int_{\Omega} \boldsymbol{e}(\boldsymbol{v}) : \left(|\nabla p|^{q-2} \nabla p \otimes \nabla p - \frac{1}{q} |\nabla p|^{q} \mathbb{I} \right) - |\nabla p|^{q-2} \nabla p \cdot \nabla \mathrm{div} \,\boldsymbol{v} \,\mathrm{d}x \\ &\leq 2 \| \boldsymbol{e}(\boldsymbol{v}) \|_{L^{\infty}(\Omega; \mathbb{R}^{n \times n})} \|\nabla p\|_{L^{q}(\Omega; \mathbb{R}^{n})}^{q} \\ &+ \|\nabla \boldsymbol{e}(\boldsymbol{v})\|_{L^{q}(\Omega; \mathbb{R}^{n \times n \times n})}^{q} + \|\nabla p\|_{L^{q}(\Omega; \mathbb{R}^{n})}^{q} \end{split}$$

and then Gronwall's inequality. The penultimate term needs $q \leq 2n/(n-2)$ or $q < \infty$ for n = 2. Also, one can see the estimate $\dot{p} \in L^2(I \times \Omega)$. Notably, for q > n, we have pressure p in $C(I \times \overline{\Omega})$ and, in particular, also traces on Γ are in $C(I \times \Gamma)$. Therefore, one can consider also the pointwise state constraints on pressure even on the boundary, e.g. a technologically relevant condition on the local pressure on the wall Γ of the container Ω of the type $|p|_{\Gamma}| \leq p_{\text{max}}$. This set of p's has a nonempty interior in $C(I \times \Gamma)$ and thus the corresponding multiplier in the 1st-order condition is well determined as a measure on $I \times \Gamma$.

Remark 5.3 (Enhancement: Cahn-Hilliard diffusion). Liquids can contain some other constituent (e.g. salt in sea water or Nickel in molten Iron outer Earth core). This consistent with a concentration χ can diffuse according the gradient of a chemical potential μ . Then the original semi-compressible initial-boundary-value system (2.1) expands as:

(5.3a)
$$\rho \dot{\boldsymbol{v}} + \rho (\boldsymbol{v} \cdot \nabla) \boldsymbol{v} - \operatorname{div} \left(\nu \boldsymbol{e}(\boldsymbol{v}) + \alpha \nabla \chi \otimes \nabla \chi \right)$$
$$+ \frac{\rho}{2} (\operatorname{div} \boldsymbol{v}) \boldsymbol{v} + \nabla \left(p + \frac{\beta}{2} p^2 + (\chi - \chi_{eq})^2 + \frac{\alpha}{2} |\nabla \chi|^2 \right) = \boldsymbol{u} \qquad \text{on } I \times \Omega \,,$$

(5.3b)
$$\beta(\dot{p} + \boldsymbol{v} \cdot \nabla p) + \operatorname{div} \boldsymbol{v} = \gamma \Delta p$$
 on $I \times \Omega$

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(5.3c)
$$\dot{\chi} + \boldsymbol{v} \cdot \nabla \chi = \operatorname{div}(m\nabla \mu)$$
 with $\mu = (\chi - \chi_{eq})^2 - \alpha \Delta \chi$ on $I \times \Omega$

(5.3d)
$$\left[\nu \boldsymbol{e}(\boldsymbol{v})\boldsymbol{n}\right]_{\mathrm{T}} + b\boldsymbol{v}_{\mathrm{T}} = 0, \quad \boldsymbol{n}\cdot\boldsymbol{v} = 0, \quad \boldsymbol{n}\cdot\nabla p = 0, \quad \boldsymbol{n}\cdot\nabla\chi = 0 \quad \text{on } I \times \Gamma$$

(5.3e)
$$v(0, \cdot) = v_0, \quad p(0, \cdot) = p_0, \text{ and } \chi(0, \cdot) = \chi_0 \quad \text{on } \Omega,$$

where χ_{eq} is an equilibrium concentration, m > 0 a mobility and $\alpha > 0$ a capillarity coefficient. The equation (5.3c) is called the Cahn-Hilliard one, and the extra symmetric stress $\alpha \nabla \chi \otimes \nabla \chi - \frac{\alpha}{2} |\nabla \chi|^2 \mathbb{I}$ occurring in (5.3a) is called Korteweg's stress. The energy balance (5.4) now augments as

(5.4)
$$\int_{\Omega} \underbrace{\frac{\varrho}{2} |\boldsymbol{v}(t)|^{2}}_{\text{kinetic}} + \underbrace{\frac{\beta}{2} p(t)^{2} + (\chi - \chi_{eq})^{2} + \frac{\alpha}{2} |\nabla \chi|^{2}}_{\text{stored}} dx \\ + \int_{0}^{t} \int_{\Omega} \underbrace{\nu |\boldsymbol{e}(\boldsymbol{v})|^{2} + \gamma |\nabla p|^{2} + m |\nabla \mu|^{2}}_{\text{dissipation rate}} dx dt + \int_{0}^{t} \int_{\Gamma} \underbrace{b |\boldsymbol{v}_{T}|^{2}}_{\text{dissipation rate}} dS dt \\ = \int_{0}^{t} \int_{\Omega} \underbrace{\boldsymbol{u} \cdot \boldsymbol{v}}_{\text{dx}dt} dx dt + \int_{\Omega} \frac{\varrho}{2} |\boldsymbol{v}_{0}|^{2} + \frac{\beta}{2} p_{0}^{2} + (\chi_{0} - \chi_{eq})^{2} + \frac{\alpha}{2} |\nabla \chi_{0}|^{2} dx.$$

From this, a similar analysis of the controlled system and an optimal control problem can be casted similarly as in Sections 2 and 4. Optimal control for the incompressible variant in two dimensions has been treated in [19] and in a certain nonlocal variant [5, 15].

Remark 5.4 (Enhancement: magneto-hydrodynamics). Some fluids are electrically conductive. Typically it concerns molten metals, like hot Iron with Nickel in the outer core of Earth or metallic hydrogen in Jupiter and Saturn. It maybe also plasma especially in stellar applications, which was the original motivation for this model. This brings an interesting coupling of semi-compressible fluids with the magnetic induction **b**. The other parameter is electric conductivity σ . The so-called induction equation merges Faraday's law and Ohm's law:

$$\dot{\boldsymbol{b}} = \operatorname{rot}(\boldsymbol{v} \times \boldsymbol{b}) + \operatorname{rot}\left(\frac{1}{\mu_0 \sigma} \operatorname{rot} \boldsymbol{b}\right) \text{ and } \operatorname{div} \boldsymbol{b} = 0,$$

where μ_0 is the vacuum permeability or, using the calculus $\operatorname{rot}(\boldsymbol{v} \times \boldsymbol{b}) = (\boldsymbol{b} \cdot \nabla)\boldsymbol{v} - (\boldsymbol{v} \cdot \nabla)\boldsymbol{b}$, also

$$\dot{\boldsymbol{b}} + (\boldsymbol{v} \cdot \nabla) \boldsymbol{b} - (\boldsymbol{b} \cdot \nabla) \boldsymbol{v} = \operatorname{rot} \left(\frac{1}{\mu_0 \sigma} \operatorname{rot} \boldsymbol{b} \right) \quad \text{and} \quad \operatorname{div} \boldsymbol{b} = 0.$$

When σ is constant, then we can further use the calculus rot rot $\boldsymbol{b} = \nabla(\operatorname{div} \boldsymbol{b}) - \Delta \boldsymbol{b} = -\Delta \boldsymbol{b}$, so that

$$\dot{oldsymbol{b}} + (oldsymbol{v} \cdot
abla) oldsymbol{b} - (oldsymbol{b} \cdot
abla) oldsymbol{v} = rac{1}{\mu_0 \sigma} \Delta oldsymbol{b}$$
 and $\operatorname{div} oldsymbol{b} = 0$.

The magnetic field influences the mechanical part through Lorenz' force which, under electroneutrality, is $\boldsymbol{f} = \boldsymbol{j} \times \boldsymbol{b}$ with low-frequency Ampére's law neglecting the displacement current so that $\mu_0 \boldsymbol{j} = \operatorname{rot} \boldsymbol{b}$. Using the calculus $\frac{1}{2}\nabla(\boldsymbol{b} \cdot \boldsymbol{b}) = (\boldsymbol{b} \cdot \nabla)\boldsymbol{b} + \boldsymbol{b} \times (\operatorname{rot} \boldsymbol{b})$, we eventually have

$$oldsymbol{f} = oldsymbol{j} imes oldsymbol{b} = rac{(oldsymbol{b} \cdot
abla) oldsymbol{b}}{\mu_0} -
abla rac{|oldsymbol{b}|^2}{2\mu_0} \,.$$

The semi-compressible system (2.1) now expands as:

(5.5a)
$$\rho \dot{\boldsymbol{v}} + \rho (\boldsymbol{v} \cdot \nabla) \boldsymbol{v} - \operatorname{div} \left(\nu \boldsymbol{e}(\boldsymbol{v}) \right) + \frac{\rho}{2} (\operatorname{div} \boldsymbol{v}) \boldsymbol{v} + \nabla \left(p + \frac{\beta}{2} p^2 + \frac{|\boldsymbol{b}|^2}{2\mu_0} \right) = \frac{(\boldsymbol{b} \cdot \nabla) \boldsymbol{b}}{\mu_0} + \boldsymbol{u} \qquad \text{on } I \times \Omega ,$$

(5.5b)
$$\beta(\dot{p} + \boldsymbol{v} \cdot \nabla p) + \operatorname{div} \boldsymbol{v} = \gamma \Delta p$$
 on $I \times \Omega$

(5.5c)
$$\dot{\boldsymbol{b}} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{b} - (\boldsymbol{b} \cdot \nabla)\boldsymbol{v} = \frac{1}{\mu_0 \sigma} \Delta \boldsymbol{b}$$
 and div $\boldsymbol{b} = 0$ on $I \times \Omega$,

(5.5d)
$$\left[\nu \boldsymbol{e}(\boldsymbol{v})\boldsymbol{n}\right]_{\mathrm{T}} + b\boldsymbol{v}_{\mathrm{T}} = 0, \quad \boldsymbol{n}\cdot\boldsymbol{v} = 0, \quad \boldsymbol{n}\cdot\nabla p = 0, \quad \boldsymbol{n}\cdot\nabla \boldsymbol{b} = 0 \quad \text{on } I \times \Gamma$$

(5.5e)
$$\boldsymbol{v}(0,\cdot) = \boldsymbol{v}_0, \quad p(0,\cdot) = p_0, \text{ and } \boldsymbol{b}(0,\cdot) = \boldsymbol{b}_0 \quad \text{on } \Omega.$$

The magneto-hydrodynamic system is the basic model for a magnetic dynamo effect, and its usage in planetary physics explains the magnetic field generation in particular in our planet Earth. Mostly this system is considered incompressible but sometimes the semi-compressible variant (but without the pressure diffusion term $\gamma \Delta p$) can be found in literature, too, viz [4, 28]. Again, the departure point for analysis is the energy balance like (5.4) which now involves also the induction equation tested by the intensity of magnetic field $\mathbf{h} = \mathbf{b}/\mu_0$, so it augments as

$$\int_{\Omega} \underbrace{\frac{\varrho}{2} |\boldsymbol{v}(t)|^{2}}_{\text{kinetic}} + \underbrace{\frac{\beta}{2} p(t)^{2} + \frac{|\boldsymbol{b}|^{2}}{2\mu_{0}}}_{\text{stored energy by pressure}} \, \mathrm{d}x$$

$$+ \int_{0}^{t} \int_{\Omega} \underbrace{\nu |\boldsymbol{e}(\boldsymbol{v})|^{2} + \gamma |\nabla p|^{2} + \frac{|\nabla \boldsymbol{b}|^{2}}{\mu_{0}\sigma}}_{\text{dissipation rate}} \, \mathrm{d}x \, \mathrm{d}t + \int_{0}^{t} \int_{\Gamma} \underbrace{b |\boldsymbol{v}_{\mathrm{T}}|^{2} \, \mathrm{d}S \, \mathrm{d}t}_{\text{dissipation rate}} = \int_{0}^{t} \int_{\Omega} \underbrace{\boldsymbol{u} \cdot \boldsymbol{v} \, \mathrm{d}x \, \mathrm{d}t}_{\text{power of}} + \int_{\Omega} \frac{\varrho}{2} |\boldsymbol{v}_{0}|^{2} + \frac{\beta}{2} p_{0}^{2} + \frac{|\boldsymbol{b}_{0}|^{2}}{2\mu_{0}} \, \mathrm{d}x.$$

From this, we now also read the apriori bound for $\mathbf{b} \in L^{\infty}(I; L^{2}(\Omega; \mathbb{R}^{n})) \cap L^{2}(I; H^{1}(\Omega; \mathbb{R}^{n}))$. The convergence of an (unspecified) approximate solutions in bi-linear terms $(\mathbf{v} \cdot \nabla) \mathbf{v}$, $(\operatorname{div} \mathbf{v})\mathbf{v}, (\mathbf{v} \cdot \nabla) \pi, \nabla \pi^{2}, \nabla |\mathbf{b}|^{2}, (\mathbf{b} \cdot \nabla)\mathbf{b}, (\mathbf{v} \cdot \nabla)\mathbf{b}$, and $(\mathbf{v} \cdot \nabla)\mathbf{b}$ is then easy by the Aubin-Lions compact-embedding theorem. In the nonsimple variant as in Remarks 5.1 and 5.2, even we have a rigorous energy conservation because $\dot{\mathbf{b}} \in L^{2}(I; H^{1}(\Omega; \mathbb{R}^{n})^{*}) + L^{1}(I; L^{2}(\Omega; \mathbb{R}^{n}))$ is in duality with $\mathbf{b} \in L^{2}(I; H^{1}(\Omega; \mathbb{R}^{n})) \cap L^{\infty}(I; L^{2}(\Omega; \mathbb{R}^{n}))$. The application of the optimization is in optimal control of plasma in tokamaks or in identification of existing flows in the planetary or stellar astrophysics.

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