

CLASSICAL GAUGE THEORY ON QUANTUM PRINCIPAL BUNDLES

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ABSTRACT. We propose a conceptually economical and computationally tractable completion of the foundations of gauge theory on quantum principal bundles à la Brzeziński–Majid to the case of general differential calculi and strong bimodule connections. In particular, we use explicit groupoid equivalences to reframe the basic concepts of gauge theory—gauge transformation, gauge potential, and field strength—in terms of reconstruction of calculi on the total space (to second order) from given calculi on the structure quantum group and base, respectively. We therefore construct gauge-equivariant moduli spaces of all suitable first- and second-order total differential calculi, respectively, compatible with these choices. As a first illustration, we relate the gauge theory of a crossed product algebra *qua* trivial quantum principal bundle to lazy Sweedler and Hochschild cohomology with coefficients. As a second illustration, we show that a noncommutative 2-torus with real multiplication is the base space of a non-trivial $U_{q^2}(1)$ -gauge theory admitting Connes’s constant curvature connection as a q -monopole connection, where—in the spirit of Manin’s Alterstraum—one must take q to be the norm-positive fundamental unit of the corresponding real quadratic field.

CONTENTS

1. Introduction and summary of results	2
2. Gauge theory to first order	7
2.1. Deconstruction of quantum principal bundles to first order	7
2.2. Gauge transformations and gauge potentials	12
2.3. Reconstruction of quantum principal bundles to first order	15
3. Gauge theory to second order	21
3.1. Deconstruction of quantum principal bundles to second order	21
3.2. Prolongability and field strength	29
3.3. Reconstruction of quantum principal bundles to second order	35
4. Gauge theory on crossed products as lazy cohomology	47
4.1. Cohomological preliminaries	48
4.2. Gauge transformations and (relative) gauge potentials	58
4.3. Reconstruction of total calculi	65
5. q -Monopoles over real multiplication noncommutative 2-tori	69
5.1. Number-theoretic preliminaries	70
5.2. Basic Heisenberg modules over irrational noncommutative 2-tori	71
5.3. Constant curvature connections as q -monopoles	73
Appendix A. Groupoids	81
References	84

1. INTRODUCTION AND SUMMARY OF RESULTS

Noncommutative geometry permits the semiclassical modelling of quantum physics by classical physics on ‘quantum’ (noncommutative) spaces, whose underlying geometry serves as both generator of and receptacle for quantum corrections; indeed, in much of theoretical physics, this might as well be the operational definition of noncommutative geometry itself. From this perspective, it may be surprising to note that a faithful and more-or-less complete formulation of classical Yang–Mills gauge theory on ‘quantum’ principal bundles does not yet exist. Nonetheless, partial foundations have already been laid by Brzeziński–Majid [11], who make sense of principal (Ehresmann) connections on quantum principal bundles in terms of suitable first-order differential calculi (FODC) on principal comodule algebras in a manner compatible with the theory of quantum groups and quantum homogeneous spaces.

There are gaps, however, in these partial foundations. Principal connections correspond to gauge potentials—their curvature, which would then encode field strength, is typically only accessed in a piecemeal and somewhat indirect fashion through the curvature of induced module connections on quantum associated vector bundles. More significantly, gauge transformations are generally only defined in the theoretically convenient but geometrically and physically unnatural special case of universal FODC. Furthermore, to the author’s best knowledge, almost all calculations of affine spaces of principal connections on quantum principal bundles are restricted to this special case; Brzeziński–Gaunt–Schenkel’s computations on the θ -deformed complex Hopf fibration [9] provide a valuable and instructive exception.

We aim to fill these gaps in a conceptually economical and computationally tractable fashion that readily interfaces with noncommutative Riemannian geometry, whether that mean Connes’s functional-analytic theory of spectral triples [16, 17] or, e.g., noncommutative Kähler geometry as championed by Ó Buachalla [29]. We take our cue from Čačić–Mesland [13], who construct a comprehensive theory of principal connections and global gauge transformations on noncommutative Riemannian principal bundles with compact connected Lie structure group in terms of spectral triples. For our purposes, their main formal innovation is to encode principal connections in terms of horizontal covariant derivatives (cf. Đurđević [21]), which permits a workable notion of gauge transformation acting affinely on principal connections in a manner that faithfully generalises the classical case. There is an apparent drawback to this change of perspective: neither variation of the principal connection nor application of a gauge transformation generally preserves the total FODC. However, we shall have reason to embrace this as a feature, not a bug.

We begin in Section 2 by effecting the aforementioned change of perspective at the level of FODC. Let H be a Hopf $*$ -algebra and let P be a principal left H -comodule $*$ -subalgebra. Fix a horizontal calculus on P , which therefore consists of a FODC (Ω_B^1, d_B) on $B := {}^{\text{co}H}P$ and a suitable *projectable horizontal lift* $\Omega_{P,\text{hor}}^1 = P \cdot \Omega_B^1 \cdot P = P \cdot \Omega_B^1$ of $\Omega_B^1 = {}^{\text{co}H}\Omega_{P,\text{hor}}^1$. For us, a *gauge transformation* is an H -covariant $*$ -automorphism f of P satisfying $f|_B = \text{id}_B$ (i.e., a *vertical* $*$ -automorphism) that induces via $\text{id}_{\Omega_B^1}$ an H -comodule automorphism $f_{*,\text{hor}}$ of $\Omega_{P,\text{hor}}^1$, while a *gauge potential* is a lift of $d_B : B \rightarrow \Omega_B^1$ to an H -covariant $*$ -derivation $P \rightarrow \Omega_{P,\text{hor}}^1$. The group \mathfrak{G} of gauge transformations now acts affine-linearly on the real affine space $\mathfrak{A}\mathfrak{t}$ of gauge potentials, whose space of translations $\mathfrak{a}\mathfrak{t}$ consists of *relative gauge potentials*. In turn, we can define the subgroup $\text{Inn}(\mathfrak{G})$ of inner gauge transformations and the subspace $\text{Inn}(\mathfrak{a}\mathfrak{t})$ of inner relative gauge potentials and check that $\text{Out}(\mathfrak{G}) := \mathfrak{G}/\text{Inn}(\mathfrak{G})$ still acts affine-linearly on $\text{Out}(\mathfrak{A}\mathfrak{t}) := \mathfrak{A}\mathfrak{t}/\text{Inn}(\mathfrak{a}\mathfrak{t})$.

We now relate these constructions to the standard theory. Fix a bicovariant FODC (Ω_H^1, d_H) on H with respect to which P is *locally free*. We simultaneously consider all H -covariant FODC (Ω_P^1, d_P) on P that make $(P; \Omega_P^1, d_P)$ into a quantum principal $(H; \Omega_H^1, d_H)$ -bundle inducing the horizontal calculus $(\Omega_B^1, d_B; \Omega_{P,\text{hor}}^1)$ and admitting a $*$ -preserving bimodule connection—for convenience, call such an FODC *admissible*. Indeed, let $\mathcal{G}[\Omega_H^1]$ be the groupoid of *abstract gauge transformations*, whose objects are admissible FODC on P and whose arrows are vertical $*$ -automorphisms of P that are differentiable with respect to the source FODC on the domain and the target FODC on the codomain; let $\mu[\Omega_H^1] : \mathcal{G}[\Omega_H^1] \rightarrow \text{Aut}(P)$ be the forgetful homomorphism. Likewise, let $\mathcal{A}[\Omega_H^1]$ be the set of all triples $(\Omega_P^1, d_P; \Pi)$, where (Ω_P^1, d_P) is admissible and where Π is a (necessarily strong) $*$ -preserving bimodule connection on $(P; \Omega_P^1, d_P)$. Thus, the classical affine action of gauge transformations on principal Ehresmann connections generalises to the action of $\mathcal{G}[\Omega_H^1]$ on $\mathcal{A}[\Omega_H^1]$ by conjugation of bimodule connections. We can now summarise the non-trivial results of Section 2 as follows:

Theorem 1.1. *Reconstruction of FODC induces an explicit equivalence of groupoids*

$$\Sigma[\Omega_H^1] : \mathfrak{G} \ltimes \mathfrak{At} \rightarrow \mathcal{G}[\Omega_H^1] \ltimes \mathcal{A}[\Omega_H^1]$$

with explicit homotopy inverse manifesting the action groupoid $\mathfrak{G} \ltimes \mathfrak{At}$ as a deformation retraction of the action groupoid $\mathcal{G}[\Omega_H^1] \ltimes \mathcal{A}[\Omega_H^1]$. Furthermore:

- (1) the forgetful homomorphism $\mu[\Omega_H^1] : \mathcal{G}[\Omega_H^1] \rightarrow \text{Aut}(P)$ maps surjectively onto \mathfrak{G} ;
- (2) the equivalence of groupoids $\Sigma[\Omega_H^1]$ descends to a groupoid isomorphism

$$\mathfrak{G} \ltimes \mathfrak{At}/\text{at}[\Omega_H^1] \xrightarrow{\sim} \mathcal{G}[\Omega_H^1]/\ker \mu[\Omega_H^1],$$

where $\text{at}[\Omega_H^1]$ is the \mathfrak{G} -invariant subspace of (Ω_H^1, d_H) -adapted relative gauge potentials.

Hence, in particular, the quotient affine space $\mathfrak{At}/\text{at}[\Omega_H^1]$ defines a \mathfrak{G} -equivariant moduli space of admissible FODC.

This justifies our definition of gauge transformation and proves that we can work directly with the group \mathfrak{G} and real affine space \mathfrak{At} without any loss of geometrical or physical information, at least to first order.

Next, in Section 3, we turn to making sense of curvature—field strength—in a conceptually minimalistic fashion. This requires the careful refinement of the constructions and results from Section 2 to the context of second-order differential calculi (sodc), i.e., $*$ -differential calculi truncated at degree 2. Suppose that we have prolonged our horizontal calculus to a *second-order horizontal calculus* $(\Omega_B, d_B; \Omega_{P,\text{hor}})$ on P . A gauge transformation f is *prolongable* if it extends via $f_* : \Omega_{P,\text{hor}}^1 \rightarrow \Omega_{P,\text{hor}}^1$ to an H -covariant $*$ -automorphism of the graded left H -comodule $*$ -algebra $\Omega_{P,\text{hor}}$; a gauge potential ∇ is *prolongable* if it extends via $d_B : \Omega_B^1 \rightarrow \Omega_B^2$ and $0 : \Omega_{P,\text{hor}}^2 \rightarrow 0$ to an H -covariant degree 1 $*$ -derivation of $\Omega_{P,\text{hor}}$, in which case its *field strength* is $F[\nabla] := \nabla^2|_P$. The subgroup \mathfrak{G}^{pr} of prolongable gauge transformations continues to act affine-linearly on the affine subspace $\mathfrak{At}^{\text{pr}}$ of prolongable gauge transformations, whose space of translations at^{pr} consists of *prolongable* relative gauge potentials.

Now, suppose that we have prolonged the bicovariant FODC (Ω_H^1, d_H) to a bicovariant $*$ -differential calculus (Ω_H, d_H) . We distill a proposal of Beggs–Majid [5, §5.5] into a notion of *strong second-order quantum principal $(H; \Omega_H, d_H)$ -bundle* and propose a compatible notion of *prolongable* $*$ -preserving bimodule connection, which turns out to be related to Đurđević’s notion of multiplicative connection [21]. We can still define a groupoid

$\mathcal{G}[\Omega_H^{\leq 2}]$ of *prolongable* abstract gauge transformations on P , a set $\mathcal{A}[\Omega_H^{\leq 2}]$ of prolongable $*$ -preserving bimodule connections, and an action of $\mathcal{G}[\Omega_H^{\leq 2}]$ on $\mathcal{A}[\Omega_H^{\leq 2}]$ by conjugation. However, constructing an analogue $\Sigma[\Omega_H^{\leq 2}]$ of the groupoid equivalence $\Sigma[\Omega_H^1]$ turns out to be a subtle matter.

As it turns out, we can still reconstruct from $\Omega_{P,\text{hor}}$ and $(\Omega_H)^{\text{co}H}$ a canonical H -covariant graded $*$ -algebra $\Omega_{P,\oplus}$ of total differential forms on P through degree 2 (cf. Đurđević [23]). However, a prolongable gauge potential ∇ will yield an object $(\Omega_{P,\oplus}, d_{P,\nabla})$ of $\mathcal{G}[\Omega_H^{\leq 2}]$ if and only if its field strength factors as $F[\nabla] = F[\nabla] \circ d_{P,\text{ver}}$, where $(\Omega_{P,\text{ver}}^1, d_{P,\text{ver}})$ is the first-order *vertical calculus* of P induced by (Ω_H^1, d_H) and $F[\nabla] : \Omega_{P,\text{ver}}^1 \rightarrow \Omega_{P,\text{hor}}^2$ is a (necessarily unique) H -covariant morphism of P - $*$ -bimodules. We say that such a gauge potential ∇ is (Ω_H^1, d_H) -*adapted*, in which case, we call $F[\nabla]$ its *curvature 2-form*; we denote the \mathfrak{G}^{pr} -invariant quadric subset of all (Ω_H^1, d_H) -adapted prolongable gauge potentials by $\mathfrak{A}t^{\text{pr}}[\Omega_H^1]$. Reconstruction of sodc yields an explicit equivalence of groupoids

$$\Sigma[\Omega_H^{\leq 2}] : \mathfrak{G}^{\text{pr}} \times \mathfrak{A}t^{\text{pr}}[\Omega_H^1] \rightarrow \mathcal{G}[\Omega_H^{\leq 2}] \times \mathcal{A}[\Omega_H^{\leq 2}]$$

with explicit homotopy inverse manifesting the action groupoid $\mathfrak{G}^{\text{pr}} \times \mathfrak{A}t^{\text{pr}}[\Omega_H^1]$ as a deformation retraction of the action groupoid $\mathcal{G}[\Omega_H^{\leq 2}] \times \mathcal{A}[\Omega_H^{\leq 2}]$, so that one can work directly with \mathfrak{G}^{pr} and $\mathfrak{A}t^{\text{pr}}[\Omega_H^1]$ without any loss of geometric or physical information, at least to second order. Once more, it follows that the obvious forgetful homomorphism $\mu[\Omega_H^{\leq 2}] : \mathcal{G}[\Omega_H^{\leq 2}] \rightarrow \text{Aut}(P)$ maps surjectively onto \mathfrak{G}^{pr} , justifying our notion of prolongable gauge potential. However, the construction of a \mathfrak{G}^{pr} -equivariant moduli space of admissible sodc on P becomes considerably more involved:

Theorem 1.2. *Suppose that (Ω_H, d_H) is given through degree 2 by the canonical prolongation à la Woronowicz of (Ω_H^1, d_H) . Then $\Sigma[\Omega_H^{\leq 2}]$ descends to a groupoid isomorphism*

$$\mathfrak{G}^{\text{pr}} \times \mathfrak{A}t^{\text{pr}}[\Omega_H^1] / \text{at}_{\text{can}}^{\text{pr}}[\Omega_H^1] \xrightarrow{\sim} \mathcal{G}[\Omega_H^{\leq 2}] / \ker \mu[\Omega_H^{\leq 2}],$$

where $\text{at}_{\text{can}}^{\text{pr}}[\Omega_H^1]$ is the \mathfrak{G}^{pr} -invariant subspace of canonically (Ω_H^1, d_H) -adapted prolongable relative gauge potentials. Thus, the \mathfrak{G}^{pr} -invariant quadric subset $\mathfrak{A}t^{\text{pr}}[\Omega_H^1] / \text{at}_{\text{can}}^{\text{pr}}[\Omega_H^1]$ of the affine space $\mathfrak{A}t^{\text{pr}} / \text{at}_{\text{can}}^{\text{pr}}[\Omega_H^1]$ defines a \mathfrak{G}^{pr} -equivariant moduli space of admissible sodc.

Next, in Section 4, we illustrate our definitions and results in the case of crossed product algebras, which one views as trivial quantum principal bundles. As Čačić–Mesland showed in the context of spectral triples [13], the group \mathfrak{G} of gauge transformations and the affine space $\mathfrak{A}t$ of gauge potentials of a crossed product by \mathbf{Z}^n *qua* trivial quantum principal \mathbf{T}^n -bundle can be computed in terms of the degree 1 group cohomology of \mathbf{Z}^n with coefficients in a certain group of unitaries and a certain $\mathbf{R}[\mathbf{Z}^m]$ -module of noncommutative 1-forms, respectively. Generalising their calculations from the commutative and cocommutative Hopf $*$ -algebra $\mathbf{C}[\mathbf{Z}^m] \cong \mathcal{O}(\mathbf{T}^m)$ to arbitrary Hopf $*$ -algebras requires the construction of novel *ad hoc* degree 1 cohomology groups, which we term ‘lazy’ on account of their formal resemblance to a construction of Bichon–Carnovale [6]. These generalisations reduce appropriately to conventional group cohomology in the case of a group algebra and to Lie algebra cohomology in the case of the universal enveloping algebra of a real Lie algebra.

Let H be a Hopf $*$ -algebra, let B be a right H -module $*$ -algebra, and let M be an H -equivariant B - $*$ -bimodule. On the one hand, we can define the degree 1 *lazy* (B, M) -valued Sweedler cohomology $\text{HS}_\ell^1(H; B, M) := \text{ZS}_\ell^1(H; B, M) / \text{BS}_\ell^1(H; B, M)$ of H , where $\text{ZS}_\ell^1(H; B, M)$ is the group of *lazy Sweedler 1-cocycles* and $\text{BS}_\ell^1(H; B, M)$ is the central subgroup of *lazy Sweedler 1-coboundaries*; this generalises Sweedler cohomology [39] through

degree 1 to the case of not-necessarily-cocommutative Hopf $*$ -algebras and non-trivial coefficients. On the other hand, we can refine the degree 1 Hochschild cohomology of H with coefficients in M with the given right H -action and the trivial left H -action to degree 1 *lazy M -valued Hochschild cohomology* $\mathrm{HH}_\ell^1(H; M) := \mathrm{ZH}_\ell^1(H; M)/\mathrm{BH}_\ell^1(H; M)$, where $\mathrm{ZH}_\ell^1(H; M)$ is the real vector space of *lazy Hochschild 1-cocycles* and $\mathrm{BH}_\ell^1(H; M)$ is the subspace of *lazy Hochschild 1-coboundaries*. Conjugation with respect to convolution on H now yields a representation of $\mathrm{ZS}_\ell^1(H; B, M)$ on $\mathrm{ZH}_\ell^1(H; M)$ that descends to a representation of $\mathrm{HS}_\ell^1(H; B, M)$ on $\mathrm{HH}_\ell^1(H; M)$. Furthermore, any H -equivariant $*$ -derivation $\partial : B \rightarrow M$ induces a canonical group 1-cocycle $\mathrm{MC}[\partial] : \mathrm{ZS}_\ell^1(H; B, M) \rightarrow \mathrm{ZH}_\ell^1(H; M)$ that descends, in turn, to a group 1-cocycle $\widetilde{\mathrm{MC}}[\partial] : \mathrm{HS}_\ell^1(H; B, M) \rightarrow \mathrm{HH}_\ell^1(H; M)$.

We can now exemplify the use of lazy Sweedler and Hochschild cohomology to compute the group \mathfrak{G} , the real affine space $\mathfrak{A}\mathfrak{t}$, and the affine-linear action of \mathfrak{G} on $\mathfrak{A}\mathfrak{t}$ for a trivial quantum principal H -bundle $B \rtimes H$; note that none of this structure would be visible if we did not consider all possible admissible FODC simultaneously.

Proposition 1.3. *Let $P := B \rtimes H$, so that $B = {}^{\mathrm{co}H}P$. Suppose that (Ω_B^1, d_B) is an H -equivariant FODC on B , and endow P with the horizontal calculus $(\Omega_B^1, d_B; \Omega_B^1 \rtimes H)$. We have a group isomorphism $\mathrm{Op} : \mathrm{ZS}_\ell^1(H; B, \Omega_B^1) \rightarrow \mathfrak{G}$ and an isomorphism $\mathrm{Op} : \mathrm{ZH}_\ell^1(H; \Omega_B^1) \xrightarrow{\sim} \mathfrak{A}\mathfrak{t}$ of real affine spaces given, respectively, by*

$$\begin{aligned} \forall \sigma \in \mathrm{ZS}_\ell^1(H; B, \Omega_B^1), \forall h \in H, \forall b \in B, \quad \mathrm{Op}(\sigma)(hb) &:= h_{(1)}\sigma(h_{(2)})b, \\ \forall \mu \in \mathrm{ZH}_\ell^1(H; \Omega_B^1), \forall h \in H, \forall b \in B, \quad \mathrm{Op}(\mu)(hb) &:= h \cdot d_B(b) + h_{(1)} \cdot \mu(h_{(2)}) \cdot b; \end{aligned}$$

these descend, respectively, to a group isomorphism $\widetilde{\mathrm{Op}} : \mathrm{HS}_\ell^1(H; B, \Omega_B^1) \xrightarrow{\sim} \mathrm{Out}(\mathfrak{G})$ and an affine-linear isomorphism $\widetilde{\mathrm{Op}} : \mathrm{HH}_\ell^1(H; \Omega_B^1) \xrightarrow{\sim} \mathrm{Out}(\mathfrak{A}\mathfrak{t})$. Moreover for every lazy Sweedler 1-cocycle $\sigma \in \mathrm{ZS}_\ell^1(H; B, \Omega_B^1)$ and lazy Hochschild 1-cocycle $\mu \in \mathrm{ZH}_\ell^1(H; \Omega_B^1)$,

$$\begin{aligned} \mathrm{Op}(\sigma) \triangleright \mathrm{Op}(\mu) &= \mathrm{Op}(\sigma \triangleright \mu + \mathrm{MC}[d_B](\sigma)), \\ \widetilde{\mathrm{Op}}([\sigma]) \triangleright \widetilde{\mathrm{Op}}([\mu]) &= \widetilde{\mathrm{Op}}([\sigma] \triangleright [\mu] + \widetilde{\mathrm{MC}}[d_B]([\sigma])). \end{aligned}$$

We can similarly compute the second-order gauge theory of $B \rtimes H$ in terms of suitable further refinements of lazy Sweedler and lazy Hochschild cohomology. As we shall show in forthcoming work [14], one can similarly analyse general quantum principal $U(1)$ -bundles with commutative base space $C^\infty(M)$ in terms of the degree 1 group cohomology of \mathbb{Z} with coefficients in $C^\infty(M, U(1))$ and $\Omega_{\mathrm{dR}}^1(M, i\mathbb{R})$; although $\mathfrak{G} \ltimes \mathfrak{A}\mathfrak{t}$ will only depend on M , all other groupoids of interest will turn out to be sensitive to the underlying dynamics.

Finally, in Section 5, we apply our formalism to the example of the non-trivial quantum principal $\mathcal{O}(U(1))$ -bundle implicit to Manin's 'Alterstrraum' [28]. Let $\theta \in \mathbb{R}$ be a quadratic irrationality, and let \mathcal{A}_θ be the corresponding smooth noncommutative 2-torus endowed with the canonical sodc $(\Omega_{\mathcal{A}_\theta}, d_{\mathcal{A}_\theta})$. Then, by combining results of Schwarz [38], Dieng–Schwarz [20], Polishchuk–Schwarz [32], Polishchuk [31], and Vlasenko [40], we canonically assemble the self-Morita equivalence bimodules among the basic Heisenberg modules over \mathcal{A}_θ into a non-cleft principal $\mathcal{O}(U(1))$ -comodule $*$ -algebra P with base ${}^{\mathrm{co}\mathcal{O}(U(1))}P = \mathcal{A}_\theta$. Moreover, using results of Polishchuk–Schwarz [32], we canonically assemble Connes's constant curvature connections [15] on the isotypical components into a prolongable gauge potential ∇_0 on P with respect to a certain second-order horizontal calculus constructed from $(\Omega_{\mathcal{A}_\theta}, d_{\mathcal{A}_\theta})$. From there, we can show that $\mathfrak{G} = \mathfrak{G}^{\mathrm{Pr}} \cong U(1)$, where $U(1)$ acts as the structure group on P , that $\mathrm{Inn}(\mathfrak{G}) = \{1\}$, that every relative gauge potential is inner prolongable and given by supercommutation by a 1-form in $\mathbb{R}^2 \subset \mathcal{A}_\theta^{\oplus 2} = \Omega_{\mathcal{A}_\theta}^1$, and that $\mathfrak{G} = \mathfrak{G}^{\mathrm{Pr}}$ acts trivially on $\mathfrak{A}\mathfrak{t} \cong \mathbb{R}^2$. In particular, we can prove the following:

Theorem 1.4. *Let $\epsilon = c_1 \theta + d_1$ be the norm-positive fundamental unit of $\mathcal{Q}[\theta]$, where $c_1 \in \mathbf{N}$ and $d_1 \in \mathbf{Z}$ are uniquely determined. Given $q \in \mathbf{R}^\times$, let (Ω_q^1, d_q) denote the corresponding q -deformed bicovariant FODC on $\mathcal{O}(U(1))$, so that (Ω_1^1, d_1) is the de Rham calculus. Then*

$$\forall q \in \mathbf{R}^\times, \quad \mathfrak{A}t^{\text{PR}}[\Omega_q^1] = \begin{cases} \mathfrak{A}t & \text{if } q = \epsilon^2, \\ \emptyset & \text{else,} \end{cases}$$

and for every $\nabla \in \mathfrak{A}t = \mathfrak{A}t^{\text{PR}}[\Omega_{\epsilon^2}^1]$, the curvature 2-form $F[\nabla]$ is non-zero and given by

$$F[\nabla](d_{\epsilon^2} t) = -i\epsilon c_1 \text{vol}_{\mathcal{A}_\theta},$$

where $d_{\epsilon^2} t := \frac{1}{2\pi i} d_{\epsilon^2}(z) \cdot z^{-1}$ and where $\text{vol}_{\mathcal{A}_\theta} := 1_{\mathcal{A}_\theta} \in \mathcal{A}_\theta = \Omega_{\mathcal{A}_\theta}^2$. Furthermore,

$$\forall q \in \mathbf{R}^\times, \quad \mathfrak{a}t[\Omega_q^1] = \mathfrak{a}t_{\text{can}}^{\text{PR}}[\Omega_q^1] = \begin{cases} \mathfrak{a}t & \text{if } q = \epsilon, \\ 0 & \text{else,} \end{cases}$$

so that $\mathfrak{A}t/\mathfrak{a}t[\Omega_{\epsilon^2}^1] = \mathfrak{A}t^{\text{PR}}[\Omega_{\epsilon^2}^1]/\mathfrak{a}t_{\text{can}}^{\text{PR}}[\Omega_{\epsilon^2}^1] = \mathfrak{A}t \cong \mathbf{R}^2$.

Thus, when $q = \epsilon$, we obtain a q -monopole over \mathcal{A}_θ exactly analogous to the q -monopole constructed by Brzeziński–Majid [11] on the q -deformed complex Hopf fibration; furthermore, distinct gauge potentials are gauge-inequivalent and yield non-isomorphic admissible FODC. Note that the elementary algebraic number theory of the real quadratic irrational θ appears again and again in all constructions and calculations.

Notation and conventions. We shall mostly follow the notation, terminology, and conventions of Beggs–Majid [5] with certain exceptions. In this work, unless otherwise stated, all algebras are unital \mathbf{C} -algebras and all modules are unital modules.

We shall use Sweedler notation as follows. If h is an element of a bialgebra H , its coproduct will be denoted by $\Delta(h) =: h_{(1)} \otimes h_{(2)} \in H \otimes H$. If p is an element of a left H -comodule P , we denote its left H -coaction by $\delta(p) =: p_{[-1]} \otimes p_{[0]} \in H \otimes P$, and we denote the subspace of left H -coinvariants of P by ${}^{\text{co}H}P$; similarly, if q is an element of a right H -comodule Q , we denote its right H -coaction by $\delta(q) =: q_{[0]} \otimes q_{[1]} \in P \otimes H$, and we denote the subspace of right H -coinvariants of Q by $Q^{\text{co}H}$. We shall also use the corresponding higher-order Sweedler notation, e.g.,

$$\begin{aligned} \forall h \in H, \quad (\text{id}_H \otimes \Delta) \circ \Delta(h) &= (\Delta \otimes \text{id}) \circ \Delta(h) =: h_{(1)} \otimes h_{(2)} \otimes h_{(3)}, \\ \forall p \in P, \quad (\text{id} \otimes \delta) \circ \delta(p) &= (\Delta \otimes \text{id}) \circ \delta(p) =: p_{[-2]} \otimes p_{[-1]} \otimes p_{[0]}, \\ \forall q \in Q, \quad (\delta \otimes \text{id}) \circ \delta(q) &= (\text{id} \otimes \Delta) \circ \delta(q) =: q_{[0]} \otimes q_{[1]} \otimes q_{[2]}. \end{aligned}$$

We use the following conventions related to bimodules. Given a $*$ -algebra B , a B - $*$ -bimodule is a B -bimodule M with a \mathbf{C} -antilinear involution $*$: $M \rightarrow M$ satisfying

$$\forall b_1, b_2 \in B, \forall m \in M, \quad (b_1 \cdot m \cdot b_2)^* = b_2^* \cdot m^* \cdot b_1^*.$$

In this case, we set $M_{\text{sa}} := \{m \in M \mid m^* = m\}$ and define the *centre* of M with respect to B by $Z_B(M) := \{m \in M \mid \forall b \in B, b \cdot m = m \cdot b\}$; moreover, given a subset $S \subset M$, we define the *centraliser* of S in B by $C_B(S) := \{b \in B \mid \forall m \in S, b \cdot m = m \cdot b\}$. Given a $*$ -algebra B and a B - $*$ -module M , we say that a derivation $\partial : B \rightarrow M$ is a $*$ -derivation whenever

$$\forall b \in B, \quad \partial(b^*) = -\partial(b)^*,$$

and we denote by $\text{Der}_B(M)$ the \mathbf{R} -vector space of all $*$ -derivations $B \rightarrow M$; given $m \in M$, the resulting *inner* $*$ -derivation $\text{ad}_m \in \text{Der}_B(M)$ is defined by setting

$$\forall b \in B, \quad \text{ad}_m(b) := [m, b].$$

Note that Beggs–Majid use the opposite convention, where a $*$ -derivation is $*$ -preserving (i.e., intertwines \mathbb{C} -antilinear involutions). A $*$ -algebra B admits an obvious category of B - $*$ -bimodules, where a morphism is a left and right B -linear map that is $*$ -preserving.

Finally, let us set the following terminology and conventions related to differential calculi. A *graded $*$ -algebra* is a \mathbb{Z} -graded algebra $\Omega = \bigoplus_{k \in \mathbb{Z}} \Omega^k$ with $\Omega^k = 0$ for $k < 0$ together with a \mathbb{C} -linear involution $*$: $\Omega \rightarrow \Omega$ such that $*(\Omega^k) = \Omega^k$ for all $k \in \mathbb{Z}$ and

$$\forall m, n \in \mathbb{Z}_{\geq 0}, \forall \alpha \in \Omega^m, \forall \beta \in \Omega^n, \quad (\alpha \wedge \beta)^* = (-1)^{mn} \beta^* \wedge \alpha^*.$$

Given a graded $*$ -algebra Ω and $k \in \mathbb{Z}$, a *degree k $*$ -derivation* is a \mathbb{C} -linear map $\partial : \Omega \rightarrow \Omega$ satisfying $\partial(\Omega^m) \subseteq \Omega^{m+k}$ for $m \in \mathbb{Z}_{\geq 0}$ and

$$\begin{aligned} \forall m, n \in \mathbb{Z}_{\geq 0}, \forall \alpha \in \Omega^m, \forall \beta \in \Omega^n, \quad \partial(\alpha \wedge \beta) &= \partial(\alpha) \wedge \beta + (-1)^{km} \alpha \wedge \partial(\beta), \\ \forall \alpha \in \Omega, \quad \partial(\alpha^*) &= -\partial(\alpha)^*. \end{aligned}$$

Hence, given a $*$ -algebra B , a *$*$ -differential calculus* on B is a pair (Ω_B, d_B) , where Ω_B is a graded $*$ -algebra with $\Omega_B^0 = B$ and $d_B : \Omega_B \rightarrow \Omega_B$ is a degree 1 $*$ -derivation with $d_B^2 = 0$, such that Ω_B is generated over B by $d_B(B) \subseteq \Omega_B^1$; in this case, we say that (Ω_B, d_B) is a *prolongation* of the first-order differential calculus (FODC) (Ω_B^1, d_B) on B . In particular, a *second-order differential calculus* (SODC) on a $*$ -algebra B is a $*$ -differential calculus (Ω_B, d_B) on B , such that Ω_B is truncated at degree 2, i.e., $\Omega_B^k = 0$ for $k > 2$.

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2. GAUGE THEORY TO FIRST ORDER

In this section, we use explicit groupoid equivalences to [re]formulate the notions of gauge transformation and principal connection on quantum principal bundles in terms of horizontal covariant derivatives, thereby yielding a computationally tractable generalisation of the gauge action on principal connections to quantum principal bundles with general first-order differential calculi (FODC); in the process, we obtain a gauge-equivariant moduli space of all relevant FODC inducing the same vertical and horizontal calculi.

2.1. Deconstruction of quantum principal bundles to first order. Let H be a Hopf $*$ -algebra over \mathbb{C} , and let (Ω_H^1, d_H) be a bicovariant FODC on H . We give a novel review of the theory of quantum principal H -bundles à la Brzeziński–Majid [11] incorporating recent insights of Beggs–Majid [5] while recovering earlier insights of Đurđević [21]. In particular, we shall see how a strong bimodule connection decomposes the total FODC of a quantum principal H -bundle into the direct sum of independent vertical and horizontal calculi, where the latter can be viewed as a horizontal lift of the induced FODC on the base.

Let us first recall the relevant notion of topological quantum principal H -bundle.

Definition 2.1 (Brzeziński–Hajac [10]). A left H -comodule $*$ -algebra P is *principal* if and only if both of the following conditions hold:

- (1) the canonical map $P \otimes P \rightarrow H \otimes P$ defined by $p \otimes p' \mapsto p_{[-1]} \otimes p'_{[0]}$ descends to a bijection $P \otimes_B P \rightarrow H \otimes P$;
- (2) there exists a unital bicovariant map $\omega : H \rightarrow P \otimes P$, such that $m_{P \circ \omega} = \epsilon(\cdot)1_P$, where $m_P : P \otimes P \rightarrow P$ is multiplication in P and ϵ is the counit of H .

Example 2.2 (Schneider [36]). If the Hopf $*$ -algebra H is cosemisimple, e.g., if $H = \mathcal{O}(G)$ for G a compact Lie group or $H = \mathbb{C}[\Gamma]$ for Γ a discrete group, then P is principal if and only if its canonical map is surjective.

Remark 2.3 (Baum–De Commer–Hajac [3]). One can interpret principality of a left H -comodule $*$ -algebra P as topological freeness of the H -coaction on P together with vestigial local triviality of the topological quantum principal H -bundle P to the extent that quantum associated vector bundles are finitely generated and projective as ${}^{\text{co}}H P$ -modules [18, Cor. 2.6].

We now consider differentiable quantum principal H -bundles compatible with the given bicovariant FODC (Ω_H^1, d_H) on H . To this end, we will find it convenient to encode (Ω_H^1, d_H) in terms of the $*$ -closed left H -subcomodule

$$(2.1) \quad \Lambda_H^1 := (\Omega_H^1)^{\text{co}H}$$

of right H -covariant 1-forms and *quantum Maurer–Cartan form* $\omega_H : H \rightarrow \Lambda_H^1$ given by

$$(2.2) \quad \forall h \in H, \quad \omega_H(h) := d_H(h_{(1)}) \cdot S(h_{(2)});$$

the relevant properties of Λ_H^1 and ω_H are given by the following definition.

Definition 2.4. A *left crossed H - $*$ -module* is a left H -module and comodule V over \mathbb{C} together with a conjugate-linear involution $*$: $V \rightarrow V$, such that

$$\begin{aligned} \forall h \in H, \forall v \in V, \quad \delta(h \triangleright v) &= h_{(1)}v_{[-1]}S(h_{(3)}) \otimes h_{(2)} \triangleright v_{[0]}, \\ \forall h \in H, \forall v \in V, \quad (h \triangleright v)^* &= S(h)^* \triangleright v^*, \\ \forall v \in V, \quad \delta(v^*) &= (v_{[-1]})^* \otimes (v_{[0]})^*; \end{aligned}$$

in this case, a V -valued 1-cocycle is a \mathbb{C} -linear map $\omega : H \rightarrow V$ satisfying

$$\begin{aligned} \forall h, k \in H, \quad \omega(hk) &= h \triangleright \omega(k) + \omega(h)\epsilon(k), \\ \forall h \in H, \quad \omega(h)^* &= \omega(S(h)^*), \end{aligned}$$

which we call *Ad-covariant* whenever it also satisfies

$$\forall h \in H, \quad \delta(\omega(h)) = h_{(1)}S(h_{(3)}) \otimes \omega(h_{(2)}),$$

One can now show that Λ_H^1 defines a left crossed H - $*$ -module with respect to the left *adjoint* action of H given by

$$(2.3) \quad \forall h \in H, \forall \omega \in \Lambda_H^1, \quad h \triangleright \omega := h_{(1)} \cdot \omega \cdot S(h_{(2)})$$

and that ω_H defines a surjective Λ_H^1 -valued Ad-covariant 1-cocycle. Furthermore, one can show that the bicovariant FODC (Ω_H^1, d_H) can be recovered from the data (Λ_H^1, ω_H) up to isomorphism [5, Proof of Thm. 2.26]. The proof of this fact, *mutatis mutandis*, permits the following noncommutative generalisation—essentially due to Đurđević—of a locally free action of a connected Lie group and its orbitwise differential calculus.

Definition 2.5 (cf. Đurđević [21, Lemma 3.1]). The *vertical calculus* of a left H -comodule $*$ -algebra P with respect to the bicovariant FODC (Ω_H^1, d_H) is the pair $(\Omega_{P, \text{ver}}^1, d_{P, \text{ver}})$, where:

- (1) $\Omega_{P, \text{ver}}^1 := \Lambda_H^1 \otimes P$ is a left H -comodule P - $*$ -bimodule with respect to the left H -coaction, P -bimodule structure, and $*$ -structure given respectively by

$$(2.4) \quad \forall p \in P, \forall \omega \in \Lambda_H^1, \quad \delta(\omega \otimes p) := \omega_{[-1]}p_{[-1]} \otimes \omega_{[0]} \otimes p_{[0]},$$

$$(2.5) \quad \forall p, q, q' \in P, \forall \omega \in \Lambda_H^1, \quad q \cdot (\omega \otimes p) \cdot q' := q_{[-1]} \triangleright \omega \otimes q_{[0]}p q',$$

$$(2.6) \quad \forall \omega \in \Lambda_H^1, \forall p \in P, \quad (\omega \otimes p)^* := p_{[-1]}^* \triangleright \omega^* \otimes p_{[0]}^*;$$

(2) $d_{P,\text{ver}} : P \rightarrow \Omega_{P,\text{ver}}^1$ is the left H -covariant $*$ -derivation defined by

$$(2.7) \quad \forall p \in P, \quad d_{P,\text{ver}}(p) := (\varpi_H \otimes \text{id}) \circ \delta(p) = \varpi_H(p_{[-1]}) \otimes p_{[0]}.$$

We say that the bicovariant FODC (Ω_H^1, d_H) is *locally freeing* for P whenever

$$(2.8) \quad \Omega_{P,\text{ver}}^1 = P \cdot d_{P,\text{ver}}(P),$$

so that $(\Omega_{P,\text{ver}}^1, d_{P,\text{ver}})$ defines a left H -covariant FODC on P .

Example 2.6. If P is a principal H -comodule $*$ -algebra, then the universal FODC on H is locally freeing for P .

Remark 2.7. If (Ω_H^1, d_H) is locally freeing for P , we can interpret $(\Omega_{P,\text{ver}}^1, d_{P,\text{ver}})$ as the orbitwise differential calculus of the locally free H -coaction on P ; in particular, we can interpret $d_{P,\text{ver}} : P \rightarrow \Omega_{P,\text{ver}}^1$ as the dualised “infinitesimal coaction” of H on P with respect to (Ω_H^1, d_H) .

We can now give a suitable formulation of the standard notion of differentiable quantum principal H -bundle. From now on, let P be a principal left H -comodule $*$ -algebra with $*$ -subalgebra of coinvariants $B := {}^{\text{co}}H P$.

Definition 2.8 (Brzeziński–Majid [11, Def. 4.9], cf. Beggs–Majid [5, §5.4]). Let (Ω_P^1, d_P) be a left H -covariant FODC on P . Then $(P; \Omega_P^1, d_P)$ defines a *quantum principal* $(H; \Omega_H^1, d_H)$ -*bundle* if and only if the *vertical map* $\text{ver}[d_P] : \Omega_P^1 \rightarrow \Omega_{P,\text{ver}}^1$ given by

$$(2.9) \quad \forall p, p' \in P, \quad \text{ver}[d_P](p \cdot d_P(p')) := p \cdot d_{P,\text{ver}}(p')$$

is well-defined and surjective with kernel $P \cdot d_P(B) \cdot P$, where $B := {}^{\text{co}}H P$; in this case, we say that (Ω_P^1, d_P) is $(H; \Omega_H^1, d_H)$ -*principal*.

Example 2.9 (Brzeziński–Majid [11, §4]). Let $(\Omega_{H,u}^1, d_{H,u})$ be the universal FODC on H . Then the universal FODC on P is $(H; \Omega_{H,u}^1, d_{H,u})$ -principal.

Remark 2.10 (Brzeziński–Majid [11, §4.1]). Suppose that (Ω_P^1, d_P) on P is an $(H; \Omega_H^1, d_H)$ -principal FODC on P . We view its vertical map $\text{ver}[d_P] : \Omega_P^1 \rightarrow \Omega_{P,\text{ver}}^1$ as encoding contraction of 1-forms with fundamental vector fields, so that $\ker \text{ver}[d_P] = P \cdot d_P(B) \cdot P$ is the P - $*$ -bimodule of horizontal 1-forms; note that $\ker[d_P]$ is correctly generated as a P -bimodule by the basic 1-forms $B \cdot d_P(B)$.

Remark 2.11. If P admits an $(H; \Omega_H^1, d_H)$ -principal FODC (Ω_P^1, d_P) , then (Ω_H^1, d_H) is locally freeing for P . Indeed, since $d_{P,\text{ver}} = \text{ver}[d_P] \circ d_P$, it follows that

$$\Omega_{P,\text{ver}}^1 = \text{ver}[d_P](\Omega_P^1) = \text{ver}[d_P](P \cdot d_P) = P \cdot d_{P,\text{ver}}(P).$$

Thus, following Đurđević [21, §3], we can also view $\text{ver}[d_P]$ as encoding restriction of 1-forms to orbitwise 1-forms.

The following now gives the most commonly used notion of principal connection on a differentiable quantum principal H -bundle.

Definition 2.12 (Brzeziński–Majid [11, §4.2], Hajac [24, Def. 2.1], Beggs–Majid [5, §5.4]). Suppose that (Ω_P^1, d_P) is an $(H; \Omega_H^1, d_H)$ -principal FODC on the principal left H -comodule $*$ -algebra P . A *connection* on the quantum principal $(H; \Omega_H^1, d_H)$ -bundle $(P; \Omega_P^1, d_P)$ is a left H -covariant left P -linear map $\Pi : \Omega_P^1 \rightarrow \Omega_P^1$ satisfying

$$\Pi^2 = \Pi, \quad \ker \Pi = \ker \text{ver}[d_P];$$

in this case, Π is a *bimodule connection* if and only if it is right P -linear and $*$ -preserving, and it is *strong* if and only if

$$(2.10) \quad (\text{id} - \Pi) \circ d_P(P) \subseteq P \cdot d_P(B).$$

Remark 2.13 (cf. Atiyah [2]). Suppose that (Ω_P^1, d_P) is an $(H; \Omega_H^1, d_H)$ -principal FODC on P , so that we have a short exact sequence

$$(2.11) \quad 0 \rightarrow P \cdot d_P(B) \cdot P \rightarrow \Omega_P^1 \xrightarrow{\text{ver}[d_P]} \Omega_{P,\text{ver}}^1 \rightarrow 0$$

of left H -comodule P - $*$ -bimodules generalising the Atiyah sequence of a smooth principal bundle [2]. Then a connection Π corresponds to a splitting of (2.11) in the category of left H -comodule left P -modules, which is a splitting in the category of left H -comodule P - $*$ -bimodules if and only if Π is a bimodule connection. In particular, a connection Π induces the right splitting $(\text{ver}[d_P] \circ \Pi)^{-1}$ and the left splitting $\text{id} - \Pi$.

Remark 2.14. Suppose that (Ω_P^1, d_P) is an $(H; \Omega_H^1, d_H)$ -principal FODC on P . If the quantum principal $(H; \Omega_H^1, d_H)$ -bundle $(P; \Omega_P^1, d_P)$ admits a strong connection, then every connection on $(P; \Omega_P^1, d_P)$ is strong.

Remark 2.15. Suppose that (Ω_P^1, d_P) is an $(H; \Omega_H^1, d_H)$ -principal FODC on P . If Π is a connection on $(P; \Omega_P^1, d_P)$, then the restriction of

$$(\text{ver}[d_P] \circ \Pi|_{\text{ran} \Pi})^{-1} : \Omega_{P,\text{ver}}^1 \rightarrow \Omega_P^1$$

to Λ_H^1 is its (absolute) connection 1-form in the sense of Brzeziński–Majid [11, Prop. 4.10] and Đurđević [21, Def. 4.1]. Thus, by a result of Beggs–Majid [5, Prop. 5.54], bimodule connections correspond to regular connections in the sense of Đurđević [21, Def. 4.3].

The algebraic significance of the strong connection condition (2.10) was already noted by Hajac [24], while its functional-analytic significance has recently been demonstrated by Čačić–Mesland [13, Appx. A]. However, a conceptual understanding of this condition has only recently been provided by Beggs–Majid [5, §5.4.2]: under standard hypotheses, (2.10) is equivalent to requiring that ${}^{\text{co}H}\ker(\text{ver}[d_P]) = B \cdot d_P(B)$, i.e., that a 1-form is basic if and only if it is horizontal and H -coinvariant. We shall repeatedly use the following abstract restatement of this reformulation.

Definition 2.16. Recall that $B := {}^{\text{co}H}P$ for P a principal left H -comodule $*$ -subalgebra. Let E be a B - $*$ -bimodule. A *horizontal lift* of E is a pair (\tilde{E}, ι) , where \tilde{E} is a left H -covariant P - $*$ -bimodule and $\iota : E \hookrightarrow {}^{\text{co}H}\tilde{E}$ is an injective morphism of B - $*$ -bimodules, such that $\tilde{E} = P \cdot \iota(E) \cdot P$; we say that (\tilde{E}, ι) is *projectable* whenever ${}^{\text{co}H}\tilde{E} = \iota(E)$.

Proposition 2.17 (Beggs–Majid [5, Cor. 5.53]). *Let E be a B - $*$ -bimodule and let (\tilde{E}, ι) be a horizontal lift of E . Then (\tilde{E}, ι) is projectable if and only if $\tilde{E} = P \cdot \iota(E)$.*

We now expand upon Beggs–Majid’s characterisation of strong connections to reinterpret a strong bimodule connection on a regular quantum principal $(H; \Omega_H^1, d_H)$ -bundle as a splitting of the total differential calculus into the direct sum of the vertical calculus and a projectible horizontal lift of the basic calculus.

Proposition 2.18 (cf. Đurđević [21, Prop. 4.6, Thm. 4.12], Beggs–Majid [5, Prop. 5.54]). *Suppose that (Ω_P^1, d_P) is an $(H; \Omega_H^1, d_H)$ -principal FODC on the principal left H -comodule $*$ -algebra P , such that quantum principal $(H; \Omega_H^1, d_H)$ -bundle $(P; \Omega_P^1, d_P)$ admits a bimodule connection. Define the restriction of (Ω_P^1, d_P) to an FODC on $B := {}^{\text{co}H}P$ by*

$$(2.12) \quad (\Omega_B^1, d_B) := (B \cdot d_P(B), d_P|_B),$$

Then the left H -covariant P -*-submodule

$$(2.13) \quad \Omega_{P,\text{hor}}^1 := \ker \text{ver}[d_P] = P \cdot d_P(B) \cdot P$$

of Ω_P^1 together with the inclusion $\Omega_B^1 \hookrightarrow {}^{\text{co}H}\Omega_{P,\text{hor}}^1$ defines a horizontal lift of Ω_B^1 , which is projectable if and only if every bimodule connection on $(P; \Omega_P^1, d_P)$ is strong. In this case, for every strong bimodule connection Π on $(P; \Omega_P^1, d_P)$:

(1) $\nabla_\Pi := (\text{id} - \Pi) \circ d_P : P \rightarrow \Omega_{P,\text{hor}}^1$ is a left H -covariant *-derivation satisfying

$$\Omega_{P,\text{hor}}^1 = P \cdot \nabla_\Pi(P), \quad \nabla_\Pi|_B = d_B;$$

(2) the map $\psi_\Pi : \Omega_P^1 \rightarrow \Omega_{P,\text{ver}}^1 \oplus \Omega_{P,\text{hor}}^1$ given by

$$\forall \omega \in \Omega_P^1, \quad \psi_\Pi(\omega) := (\text{ver}[d_P](\omega), (\text{id} - \Pi)(\omega))$$

defines a left H -covariant isomorphism of P -*-bimodules, such that

$$\forall p \in P, \quad \psi_\Pi \circ d_P(p) = (d_{P,\text{ver}}(p), \nabla_\Pi(p)).$$

Proof. By Proposition 2.17, $\Omega_{P,\text{hor}}^1$ is a projectable horizontal lift of Ω_B^1 if and only if $\Omega_{P,\text{hor}}^1 = P \cdot \Omega_B^1$. On the one hand, if $(P; \Omega_P^1, d_P)$ admits a strong connection Π , then

$$\Omega_{P,\text{hor}}^1 = (\text{id} - \Pi)(\Omega_P^1) = (\text{id} - \Pi)(P \cdot d_P(P)) = P \cdot (\text{id} - \Pi) \circ d_P(P) = P \cdot P \cdot d_P(B) = P \cdot \Omega_B^1;$$

on the other hand, if $\Omega_{P,\text{hor}}^1 = P \cdot \Omega_B^1$, then, for every connection Π on $(P; \Omega_P^1, d_P)$,

$$(\text{id} - \Pi) \circ d_P(B) \subset \Omega_{P,\text{hor}}^1 = P \cdot \Omega_B^1 = P \cdot d_P(B).$$

Now, suppose that Π is a strong bimodule connection on $(P; \Omega_P^1, d_P)$. First, since Π is a left H -covariant morphism of P -*-bimodules, $\nabla_\Pi := (\text{id} - \Pi) \circ d_P$ is a left H -covariant *-derivation; since $\ker \Pi = \Omega_{P,\text{hor}}^1$ and $\Pi^2 = \Pi$, it follows that $\nabla_\Pi|_B = d_B$ and

$$\Omega_{P,\text{hor}}^1 = (\text{id} - \Pi)(P \cdot d_P(P)) = P \cdot (\text{id} - \Pi) \circ d_P(P) = P \cdot \nabla_\Pi(P).$$

Next, in light of Remark 2.13, the map $\psi_\Pi : \Omega_P^1 \rightarrow \Omega_{P,\text{ver}}^1 \oplus \Omega_{P,\text{hor}}^1$ is simply the isomorphism of left H -comodule P -*-bimodules induced by splitting of the short exact sequence (2.11) of left H -comodule P -*-bimodules defined by Π as a bimodule connection. Finally, for all $p \in P$,

$$\psi_\Pi \circ d_P(p) = (\text{ver}[d_P] \circ d_P(p), (\text{id} - \Pi) \circ d_P(p)) = (d_{P,\text{ver}}(p), \nabla_\Pi(p)). \quad \square$$

Thus, if a quantum principal $(H; \Omega_H^1, d_H)$ -bundle $(P; \Omega_P^1, d_P)$ admits strong bimodule connection, then a choice of strong bimodule connection Π is equivalent to a decomposition (up to isomorphism) of (Ω_P^1, d_P) as a direct sum of left H -covariant FODC

$$(2.14) \quad (\Omega_P^1, d_P) \cong (\Omega_{P,\text{ver}}^1, d_{P,\text{ver}}) \oplus (\Omega_{P,\text{hor}}^1, \nabla_\Pi).$$

Here, the left H -covariant FODC $(\Omega_{P,\text{ver}}^1, d_{P,\text{ver}})$ on P is completely determined by the left H -coaction on P and by the bicovariant FODC (Ω_H^1, d_H) on H , while the left H -covariant FODC $(\Omega_{P,\text{hor}}^1, \nabla_\Pi)$ on P is a projectable horizontal lift of (Ω_B^1, d_B) in the sense that

$$\left({}^{\text{co}H}\Omega_{P,\text{hor}}^1, \nabla_\Pi|_B \right) = (\Omega_B^1, d_B).$$

This strongly suggests that variation of the (principal) connection Π can be viewed as tantamount to variation of the lift $\nabla_\Pi : P \rightarrow \Omega_{P,\text{hor}}^1$ of $d_B : B \rightarrow \Omega_B^1$, a change of perspective that we shall explore in the next section and justify in §2.3.

2.2. Gauge transformations and gauge potentials. Let H be a Hopf $*$ -algebra, and let P be a principal left H -comodule $*$ -algebra with algebra of coinvariants $B := {}^{\text{co}H}P$. We shall now will study the problem of lifting a FODC (Ω_B^1, d_B) on B to a left H -covariant FODC $(\Omega_{P,\text{hor}}^1, \nabla)$ on P , such that $({}^{\text{co}H}\Omega_{P,\text{hor}}^1, \nabla|_B)$ recovers (Ω_B^1, d_B) up to isomorphism. As Đurđević observed [22], this suggests encoding principal connections through their induced horizontal covariant derivatives; we shall see that this also yields a tentative definition of gauge transformation for quantum principal bundles with general FODC in the spirit of Čačić–Mesland [13, §3]. Our definitions will be rigorously justified in §2.3.

We begin with the following definition, which encodes a choice of FODC on the base B together with compatible left H -covariant P - $*$ -bimodule of horizontal 1-forms.

Definition 2.19 (cf. Đurđević [23, § 3.1]). A (first-order) horizontal calculus on the principal left H -comodule $*$ -algebra P is a quadruple $(\Omega_B^1, d_B; \Omega_{P,\text{hor}}^1, \iota)$, where:

- (1) (Ω_B^1, d_B) is a (trivially left H -covariant) FODC on $B := {}^{\text{co}H}P$;
- (2) $(\Omega_{P,\text{hor}}^1, \iota)$ is a projectable horizontal lift of the B - $*$ -bimodule Ω_B^1 .

Example 2.20. Let (Ω_H^1, d_H) be a bicovariant FODC for H , and suppose that (Ω_P^1, d_P) is an $(H; \Omega_H^1, d_H)$ -principal FODC on P , such that the quantum principal $(H; \Omega_H^1, d_H)$ -bundle $(P; \Omega_P^1, d_P)$ admits a strong bimodule connection. Then, by Proposition 2.18, the triple

$$(2.15) \quad (\Omega_B^1, d_B; \Omega_{P,\text{hor}}^1, \iota) := \left(B \cdot d_P(B), d_P|_B; \ker \text{ver}[d_P], \text{id}_{\Omega_P^1}|_{B \cdot d_P(B)} \right)$$

defines a first-order horizontal calculus on P , which we view as the *canonical* first-order horizontal calculus induced by the $(H; \Omega_H^1, d_H)$ -principal FODC (Ω_P^1, d_P) .

Assume, therefore, that P admits a first-order horizontal calculus $(\Omega_B^1, d_B; \Omega_{P,\text{hor}}^1, \iota)$, which we now fix. Let us also assume that $d_B : B \rightarrow \Omega_B^1$ admits a lift to a left H -covariant $*$ -derivation $P \rightarrow \Omega_{P,\text{hor}}^1$. To simplify notation, we suppress the inclusion map ι and identify Ω_B^1 with its image in $\Omega_{P,\text{hor}}^1$; hence, where convenient, we denote the first-order horizontal calculus $(\Omega_B^1, d_B; \Omega_{P,\text{hor}}^1, \iota)$ by the triple $(\Omega_B^1, d_B; \Omega_{P,\text{hor}}^1)$.

We begin by considering automorphisms of horizontal lifts; this will permit us to define gauge transformations as automorphisms of $\Omega_{P,\text{hor}}^1$ *qua* horizontal lift of Ω_B^1 .

Definition 2.21. Let E be a B - $*$ -bimodule, where $B := {}^{\text{co}H}P$, and let (\tilde{E}, ι) be a horizontal lift of E . An *automorphism* of (\tilde{E}, ι) is a pair (ϕ, ϕ_*) , where $\phi : P \rightarrow P$ is a left H -covariant $*$ -automorphism satisfying $\phi|_B = \text{id}_B$ and $\phi_* : \tilde{E} \rightarrow \tilde{E}$ is a left H -covariant $*$ -preserving \mathbb{C} -linear bijection satisfying $\phi_* \circ \iota = \iota$ and

$$(2.16) \quad \forall p, p' \in P, \forall \eta \in \tilde{E}, \quad \phi_*(p \cdot \eta \cdot p') = \phi(p) \cdot \phi_*(\eta) \cdot \phi(p').$$

We denote the group of all automorphisms of (\tilde{E}, ι) by $\text{Aut}(\tilde{E}, \iota)$.

Remark 2.22. Suppose that E is a B - $*$ -bimodule with horizontal lift (\tilde{E}, ι) . Since $\iota(E)$ generates E as a P -bimodule, an automorphism $(\phi, \phi_*) \in \text{Aut}(\tilde{E}, \iota)$ is uniquely determined by ϕ . By mild abuse of notation, we can therefore identify $\text{Aut}(\tilde{E}, \iota)$ with the subgroup of $\text{Aut}(P)$ consisting of left H -covariant $\phi \in \text{Aut}(P)$, such that $\phi|_B = \text{id}_B$ and the map

$$\phi_* : \tilde{E} \rightarrow \tilde{E}, \quad p \cdot \iota(e) \cdot p' \mapsto \phi(p) \cdot \iota(e) \cdot \phi(p')$$

is well-defined and bijective; with this convention, it follows that

$$(\phi \mapsto \phi_*) : \text{Aut}(\tilde{E}, \iota) \rightarrow \text{GL}(\tilde{E})$$

is a group homomorphism.

If $B := {}^{\text{co}}H P$ is not central in P (or even if B is central in P but does not centralise the lifted B - $*$ -bimodule), then the inner automorphisms will form a non-trivial central subgroup of the group of automorphisms of a horizontal lift. More broadly, the significance of inner automorphisms in noncommutative gauge theory has been pointed out by Connes [17, p. 165] in the context of the spectral action on spectral triples.

Definition 2.23. Let E be a B - $*$ -bimodule, and let (\tilde{E}, ι) be a horizontal lift of E . We say that an automorphism $\phi \in \text{Aut}(\tilde{E}, \iota)$ is *inner* if and only if

$$\phi = \text{Ad}_v := (p \mapsto v p v^*), \quad \phi_* = \text{Ad}_v := (\eta \mapsto v \cdot \eta \cdot v^*)$$

for some $v \in U(B)$. We denote the subset of all inner automorphisms of (\tilde{E}, ι) by $\text{Inn}(\tilde{E}, \iota)$.

Proposition 2.24. Let E be a B - $*$ -bimodule, and let (\tilde{E}, ι) be a horizontal lift of E . For every $v \in U(B)$, the automorphism Ad_v of P defines an inner automorphism of (\tilde{E}, ι) if and only if $v \in C_B(B \oplus E)$. Hence, $\text{Inn}(\tilde{E}, \iota)$ defines a central subgroup of $\text{Aut}(\tilde{E}, \iota)$.

Proof. First, Ad_v restricts to the identity on B if and only if $v \in Z(B)$. Now, given $v \in U(Z(B))$, for every $p, p' \in P$ and $e \in E$,

$$\begin{aligned} \text{Ad}_v(p \cdot e \cdot p') - \text{Ad}_v(p) \cdot e \cdot \text{Ad}_v(p') &= v \cdot (p \cdot e \cdot p') \cdot v^* - v p v^* \cdot e \cdot v p' v^* \\ &= v p \cdot (e - v^* \cdot e \cdot v) \cdot p' v^*, \end{aligned}$$

so that $\text{Ad}_v \in \text{Inn}(\tilde{E}, \iota)$ if and only if $v \in C_B(\Omega_B^1)$. Thus, in particular, the group homomorphism $(v \mapsto \text{Ad}_v) : U(P) \rightarrow \text{Aut}(P)$ restricts to a surjection $U(C_B(B \oplus E)) \twoheadrightarrow \text{Inn}(\tilde{E}, \iota)$, so that $\text{Inn}(\tilde{E}, \iota) \leq \text{Aut}(\tilde{E}, \iota)$ since $U(C_B(B \oplus E)) \leq U(P)$.

Let us now show that $\text{Inn}(\tilde{E}, \iota)$ is central. Let $\phi \in \text{Aut}(\tilde{E}, \iota)$ and let $v \in U(C_B(B \oplus E))$. Then, for all $p \in P$, since $\phi|_B = \text{id}_B$,

$$\phi \circ \text{Ad}_v \circ \phi^{-1}(p) = \phi(v \phi^{-1}(p) v^*) = v \phi(\phi^{-1}(p)) v^* = \text{Ad}_v(p). \quad \square$$

At last, we can tentatively define a gauge transformation to be an automorphism of $\Omega_{P, \text{hor}}^1$ as a horizontal lift of Ω_B^1 and check its differentiability with respect to any vertical calculus on P . This definition, which is a non-trivial refinement of Brzeziński's notion of vertical automorphism [8, §5], will be justified by Proposition 2.40 and Corollary 2.41.

Definition 2.25. A *gauge transformation* of the principal left H -comodule $*$ -algebra P with respect to the horizontal calculus $(\Omega_B^1, d_B; \Omega_{P, \text{hor}}^1, \iota)$ is an automorphism ϕ of the projectable horizontal lift $(\Omega_{P, \text{hor}}^1, \iota)$ of Ω_B^1 ; hence, the *gauge group*, *inner gauge group*, and *outer gauge group* of P with respect to $(\Omega_B^1, d_B; \Omega_{P, \text{hor}}^1, \iota)$ are given by

$$\mathfrak{G} := \text{Aut}(\Omega_{P, \text{hor}}^1, \iota), \quad \text{Inn}(\mathfrak{G}) := \text{Inn}(\Omega_{P, \text{hor}}^1, \iota), \quad \text{Out}(\mathfrak{G}) := \mathfrak{G} / \text{Inn}(\mathfrak{G}),$$

respectively. When clarity requires, for $\phi \in \mathfrak{G}$, we will denote ϕ_* by $\phi_{*, \text{hor}}$.

Remark 2.26. Thus, by Proposition 2.24, the map $(v \mapsto \text{Ad}_v) : U(C_B(B \oplus \Omega_B^1)) \twoheadrightarrow \text{Inn}(\mathfrak{G})$ yields the short exact sequence of Abelian groups

$$1 \rightarrow U(C_B(\Omega_B^1) \cap Z(P)) \rightarrow U(C_B(B \oplus \Omega_B^1)) \rightarrow \text{Inn}(\mathfrak{G}) \rightarrow 1,$$

where $\text{Inn}(\mathfrak{G})$ is central in \mathfrak{G} .

We will find it useful to record the following straightforward observation, which guarantees that gauge transformations are differentiable with respect to the vertical calculus induced by any locally freeing bicovariant FODC on H .

Proposition 2.27. *Suppose that (Ω_H^1, d_H) is a bicovariant FODC on H that is locally freeing for P ; let $(\Omega_{P,\text{ver}}^1, d_{P,\text{ver}})$ be the resulting vertical calculus. For every $f \in \mathfrak{G}$, the map*

$$(2.17) \quad f_{*,\text{ver}} := \text{id}_{\Lambda_H^1} \otimes f$$

defines a left H -covariant $$ -preserving \mathbb{C} -linear endomorphism of $\Omega_{P,\text{ver}}^1$ satisfying*

$$(2.18) \quad \forall p, p' \in P, \forall \omega \in \Omega_{P,\text{ver}}^1, \quad f_{*,\text{ver}}(p \cdot \omega \cdot p') = f(p) \cdot f_{*,\text{ver}}(\omega) \cdot f(p'),$$

$$(2.19) \quad d_{P,\text{ver}} \circ f = f_{*,\text{ver}} \circ d_{P,\text{ver}}.$$

Furthermore, the map $(f \mapsto f_{,\text{ver}}) : \mathfrak{G} \rightarrow \text{GL}(\Omega_{P,\text{ver}}^1)$ is a group homomorphism.*

Having tentatively defined gauge transformations to be symmetries of the projectable horizontal lift $\Omega_{P,\text{hor}}^1$ of Ω_B^1 , we now tentatively define a gauge potential to be a lift of $d_B : B \rightarrow \Omega_B^1$ to a left H -covariant $*$ -derivation $P \rightarrow \Omega_{P,\text{hor}}^1$, which, like Đurđević [22], we view as the horizontal covariant derivative of a principal connection. The justification for this definition, which builds crucially on the role of (strong bimodule) connections in splitting the noncommutative Atiyah sequence (2.11), will be provided by Proposition 2.40.

Definition 2.28 (cf. Đurđević [22, Def. 1]). *A gauge potential on the principal left H -comodule $*$ -algebra P with respect to the first-order horizontal calculus $(\Omega_B^1, d_B; \Omega_{P,\text{hor}}^1)$ is a left H -covariant $*$ -derivation $\nabla : P \rightarrow \Omega_{P,\text{hor}}^1$, such that $\nabla|_B = d_B$; we define the Atiyah space of P with respect to $(\Omega_B^1, d_B; \Omega_{P,\text{hor}}^1)$ to be the set \mathfrak{At} of all gauge potentials.*

Example 2.29. *Suppose that the FODC (Ω_B^1, d_B) is inner, so that $d_B = \text{ad}_\alpha$ for some 1-form $\alpha \in (\Omega_B^1)_{\text{sa}}$. Then $\nabla := \text{ad}_{i(\alpha)}$ is a gauge potential on P with respect to $(\Omega_B^1, d_B; \Omega_{P,\text{hor}}^1)$.*

Our assumption on the horizontal calculus $(\Omega_B^1, d_B; \Omega_{P,\text{hor}}^1)$ is that $\mathfrak{At} \neq \emptyset$. Since $\Omega_{P,\text{hor}}^1$ is projectable as a horizontal lift of Ω_B^1 , it follows that $\Omega_{P,\text{hor}}^1 = P \cdot \Omega_B^1 = P \cdot d_B(B)$. Thus for every $\nabla \in \mathfrak{At}$, the pair $(\Omega_{P,\text{hor}}^1, \nabla)$ defines a left H -covariant FODC on P satisfying

$$(\Omega_B^1, d_B) = ({}^{\text{co}H}\Omega_{P,\text{hor}}^1, \nabla|_{\text{co}H}\Omega_{P,\text{hor}}^1);$$

in other words, it defines a projectable horizontal lift of (Ω_B^1, d_B) .

It is now straightforward to check that the Atiyah space \mathfrak{At} is an \mathfrak{G} -invariant \mathbb{R} -affine subspace of the \mathbb{R} -vector space $\text{Der}_P(\Omega_{P,\text{hor}}^1)$ of $*$ -derivations on the P - $*$ -bimodule $\Omega_{P,\text{hor}}^1$.

Definition 2.30 (cf. Đurđević [22, p. 98]). *A relative gauge potential on P with respect to the horizontal calculus $(\Omega_B^1, d_B; \Omega_{P,\text{hor}}^1)$ is a left H -covariant $*$ -derivation $\mathbf{A} : P \rightarrow \Omega_{P,\text{hor}}^1$, such that $\mathbf{A}|_B = 0$; we denote by at the \mathbb{R} -vector space of all relative gauge potentials on P with respect to $(\Omega_B^1, d_B; \Omega_{P,\text{hor}}^1)$.*

Proposition 2.31. *The Atiyah space \mathfrak{At} of P with respect to $(\Omega_B^1, d_B; \Omega_{P,\text{hor}}^1)$ defines a \mathbb{R} -affine space modelled on the \mathbb{R} -vector space at with respect to the subtraction map $\mathfrak{At} \times \mathfrak{At} \rightarrow \text{at}$ defined by $(\nabla, \nabla') \mapsto \nabla - \nabla'$. Moreover, \mathfrak{At} admits an affine action of the gauge group \mathfrak{G} defined by*

$$(2.20) \quad \forall \phi \in \mathfrak{G}, \forall \nabla \in \mathfrak{At}, \quad \phi \triangleright \nabla := \phi_* \circ \nabla \circ \phi^{-1},$$

whose linear part is the linear action of \mathfrak{G} on at defined by

$$(2.21) \quad \forall \phi \in \mathfrak{G}, \forall \mathbf{A} \in \text{at}, \quad \phi \triangleright \mathbf{A} := \phi_* \circ \mathbf{A} \circ \phi^{-1}.$$

If $B := {}^{\text{co}H}P$ is not central in P (or if B is central in P but does not centralise $\Omega_{P,\text{hor}}^1$), one obtains a non-trivial \mathfrak{G} -invariant \mathbf{R} -subspace of inner relative gauge potentials. More broadly, the significance of inner derivations in noncommutative gauge theory has been pointed out by Connes [17, p. 166] in the context of the spectral action on spectral triples.

Definition 2.32. A relative gauge potential $\mathbf{A} \in \mathfrak{at}$ on P with respect to the horizontal calculus $(\Omega_B^1, d_B; \Omega_{P,\text{hor}}^1)$ is *inner* whenever $\mathbf{A} = \text{ad}_\alpha$ for some $\alpha \in (\Omega_B^1)_{\text{sa}}$; we denote the subspace of all inner relative gauge potentials by $\text{Inn}(\mathfrak{at})$. Thus, we define the *outer Atiyah space* of P with respect to $(\Omega_B^1, d_B; \Omega_{P,\text{hor}}^1)$ to be the affine space $\text{Out}(\mathfrak{At}) := \mathfrak{At}/\text{Inn}(\mathfrak{at})$ with space of translations $\text{Out}(\mathfrak{at}) := \mathfrak{at}/\text{Inn}(\mathfrak{at})$.

Proposition 2.33. A map $\mathbf{A} : P \rightarrow \Omega_{P,\text{hor}}^1$ is an inner relative gauge potential if and only if $\mathbf{A} = \text{ad}_\alpha$ for some $\alpha \in Z_B(\Omega_B^1)_{\text{sa}}$. Thus, the map $(\alpha \mapsto \text{ad}_\alpha) : Z_B(\Omega_B^1)_{\text{sa}} \rightarrow \text{Inn}(\mathfrak{at})$ yields the short exact sequence

$$(2.22) \quad 0 \rightarrow Z_B(\Omega_B^1)_{\text{sa}} \cap Z_P(\Omega_{P,\text{hor}}^1) \rightarrow Z_B(\Omega_B^1)_{\text{sa}} \rightarrow \text{Inn}(\mathfrak{at}) \rightarrow 0.$$

Moreover, the subspace $\text{Inn}(\mathfrak{at})$ of inner relative gauge potentials consists of \mathfrak{G} -invariant vectors, so that the affine action of \mathfrak{G} on \mathfrak{At} descends to an affine action of \mathfrak{G} on $\text{Out}(\mathfrak{At})$.

Proof. First, given $\alpha \in (\Omega_B^1)_{\text{sa}}$, the left H -covariant $*$ -derivation $\mathbf{A} := \text{ad}_\alpha$ defines a relative gauge potential if and only if $\ker \mathbf{A} \supseteq B$, if and only if $\alpha \in Z_B(\Omega_B^1)$; this immediately yields the short exact sequence (2.22). \square

In fact, it turns out that the affine action of the gauge group \mathfrak{G} on the Atiyah space \mathfrak{At} descends further to an affine action of the outer gauge group $\text{Out}(\mathfrak{G})$ on the outer Atiyah space $\text{Out}(\mathfrak{At})$, which, as we shall see, will yield a non-trivial invariant of the principal left H -comodule $*$ -algebra P endowed with the horizontal calculus $(\Omega_B^1, d_B; \Omega_{P,\text{hor}}^1)$.

Proposition 2.34. The inner gauge group $\text{Inn}(\mathfrak{G})$ acts trivially on $\text{Out}(\mathfrak{at})$, so that the induced action of \mathfrak{G} on $\text{Out}(\mathfrak{At})$ descends further to an affine action of $\text{Out}(\mathfrak{G})$ on $\text{Out}(\mathfrak{At})$.

Proof. Let $\phi \in \text{Inn}(\mathfrak{G})$, so that by Proposition 2.24, $\phi = \text{Ad}_v$ and $\phi_* = \text{Ad}_v$ for some unitary $v \in U(Z(B) \cap C_B(\Omega_B^1))$; let $\nabla \in \mathfrak{At}$. Then, for all $p \in P$,

$$\begin{aligned} (\phi \triangleright \nabla - \nabla)(p) &= v \cdot \nabla(v^* p v) \cdot v^* - \nabla(p) \\ &= v \cdot (d_B(v^*) \cdot p v + v^* \cdot \nabla(p) \cdot v + v^* p \cdot d_B(v)) \cdot v^* - \nabla(p) \\ &= [-v^* \cdot d_B(v), p] \end{aligned}$$

so that $\phi \triangleright \nabla - \nabla = \text{ad}_{-v^* d_B(v)} \in \text{Inn}(\mathfrak{at})$. Hence, $\text{Inn}(\mathfrak{G})$ acts trivially on $\text{Out}(\mathfrak{At})$. \square

2.3. Reconstruction of quantum principal bundles to first order. Let H be a Hopf $*$ -algebra and let P be a principal left H -comodule $*$ -algebra with $*$ -subalgebra of coinvariants $B := {}^{\text{co}H}P$. Given a horizontal calculus $(\Omega_B^1, d_B; \Omega_{P,\text{hor}}^1)$ for P and a bicovariant FODC (Ω_H^1, d_H) on H that is locally freeing for P , we consider *all* $(H; \Omega_H^1, d_H)$ -principal FODC on P inducing the first-order horizontal calculus $(\Omega_B^1, d_B; \Omega_{P,\text{hor}}^1)$. This will allow us to justify our notions of gauge transformation, gauge potential, and gauge action relative to the theory of Brzeziński–Majid [11] by means of a conceptually transparent equivalence of relevant groupoids. Furthermore, this equivalence will yield a gauge-equivariant affine moduli space of $(H; \Omega_H^1, d_H)$ -principal FODC on P inducing $(\Omega_B^1, d_B; \Omega_{P,\text{hor}}^1)$. For relevant definitions from the basic theory of groupoids, see Appendix A.

From now on, fix a horizontal calculus $(\Omega_B^1, d_B; \Omega_{P,\text{hor}}^1, \iota)$ on the principal left H -comodule $*$ -algebra P . Given a bicovariant FODC (Ω_H^1, d_H) on H that is locally freeing for P ,

we construct a groupoid whose objects are $(H; \Omega_H^1, d_H)$ -principal FODC on P inducing the horizontal calculus $(\Omega_B^1, d_B; \Omega_{P, \text{hor}}^1, \iota)$ on P and whose arrows will give an abstract notion of gauge transformation adapted to general FODC; note that we do not require an abstract gauge transformation to be differentiable with respect to the same FODC on P as both domain and codomain. In what follows, let $(\Omega_{P, \text{ver}}^1, d_{P, \text{ver}})$ denote the vertical calculus on P induced by a locally freeing bicovariant FODC (Ω_H^1, d_H) .

Definition 2.35. Let (Ω_H^1, d_H) be a bicovariant FODC on H that is locally freeing for P . We define the *abstract gauge groupoid* with respect to (Ω_H^1, d_H) to be the groupoid $\mathcal{G}[\Omega_H^1]$ defined as follows:

- (1) an object is an $(H; \Omega_H^1, d_H)$ -principal FODC (Ω_P^1, d_P) on P , such that the quantum principal $(H; \Omega_H^1, d_H)$ -bundle $(P; \Omega_P^1, d_P)$ admits bimodule connections, and

$$(\ker \text{ver}[d_P], d_B(b) \mapsto d_P(b))$$

defines a horizontal lift of Ω_B^1 admitting a (necessarily unique) left H -covariant isomorphism $C[\Omega_P^1] : P \cdot d_B(P) \rightarrow \Omega_{P, \text{hor}}^1$ of P - $*$ -bimodules satisfying

$$(2.23) \quad C[\Omega_P^1] \circ d_P|_B = \iota \circ d_B;$$

- (2) given objects (Ω_1, d_1) and (Ω_2, d_2) , an arrow $f : (\Omega_1, d_1) \rightarrow (\Omega_2, d_2)$ consists of a left H -covariant $*$ -automorphism $f : P \rightarrow P$, such that $f|_B = \text{id}$, and such that

$$(2.24) \quad f_* : \Omega_1 \rightarrow \Omega_2, \quad p \cdot d_1(p') \cdot p'' \mapsto f(p) \cdot d_2(f(p')) \cdot f(p'')$$

is a well-defined bijection;

- (3) composition of arrows is induced by composition of $*$ -automorphisms of P , and the identity of an object (Ω, d) is given by $\text{id}_{(\Omega, d)} := (\text{id}_P : (\Omega, d) \rightarrow (\Omega, d))$.

Moreover, we define the star-injective homomorphism $\mu[\Omega_H^1] : \mathcal{G}[\Omega_H^1] \rightarrow \text{Aut}(P)$ by

$$(2.25) \quad \forall (f : (\Omega_1, d_1) \rightarrow (\Omega_2, d_2)) \in \mathcal{G}[\Omega_H^1], \quad \mu[\Omega_H^1](f : (\Omega_1, d_1) \rightarrow (\Omega_2, d_2)) := f.$$

Remark 2.36. Thus, the canonical horizontal calculus of an object (Ω_P, d_P) of $\mathcal{G}[\Omega_H^1]$ induces the given horizontal calculus $(\Omega_B^1, d_B; \Omega_{P, \text{hor}}^1)$ up to the canonical isomorphism $C[\Omega_P]$ of left H -covariant P - $*$ -bimodules.

Remark 2.37. Suppose that $f : (\Omega_1, d_1) \rightarrow (\Omega_2, d_2)$ is an arrow in $\mathcal{G}[\Omega_H^1]$. Then Proposition 2.27 applies unchanged to the vertical automorphism f , i.e., there exists a unique left H -covariant $*$ -preserving bijection $f_{*, \text{ver}} : \Omega_{P, \text{ver}}^1 \rightarrow \Omega_{P, \text{ver}}^1$, such that

$$\forall p, q \in P, \forall \omega \in \Omega_{P, \text{ver}}^1, \quad f_{*, \text{ver}}(p \cdot \omega \cdot q) = f(p) \cdot f_{*, \text{ver}}(\omega) \cdot f(q)$$

and $f_{*, \text{ver}} \circ d_{P, \text{ver}} = d_{P, \text{ver}} \circ f_{*, \text{ver}}$. A straightforward calculation now shows that

$$\text{ver}[d_2] \circ f_* = f_{*, \text{ver}} \circ \text{ver}[d_1].$$

In keeping with our stated philosophy, given a bicovariant FODC (Ω_H^1, d_H) on H , we consider bimodule connections on all quantum principal $(H; \Omega_H^1, d_H)$ -bundles induced from P by FODC in $\text{Ob}(\mathcal{G}[\Omega_H^1])$. It is straightforward to check that the abstract gauge groupoid $\mathcal{G}[\Omega_H^1]$ admits a canonical action on this set of bimodule connections.

Proposition-Definition 2.38. Let (Ω_H^1, d_H) be a bicovariant FODC on H that is locally freeing for P . Let $\mathcal{A}[\Omega_H^1]$ to be the set of all triples $(\Omega_P^1, d_P; \Pi)$, where $(\Omega_P^1, d_P) \in \text{Ob}(\mathcal{G}[\Omega_H^1])$

and Π is a bimodule connection on the quantum principal (Ω_H^1, d_H) -bundle $(P; \Omega_P^1, d_P)$; hence, let $p[\Omega_H^1] : \mathcal{A}[\Omega_H^1] \rightarrow \text{Ob}(\mathcal{G}[\Omega_H^1])$ be the canonical surjection given by

$$\forall (\Omega_P^1, d_P; \Pi) \in \mathcal{A}[\Omega_H^1], \quad p[\Omega_H^1](\Omega_P^1, d_P; \Pi) := (\Omega_P^1, d_P).$$

Then the *abstract gauge action* is the action of $\mathcal{G}[\Omega_H^1]$ on $\mathcal{A}[\Omega_H^1]$ via $p[\Omega_H^1]$ defined by

$$(2.26) \quad \forall (f : (\Omega_1, d_1) \rightarrow (\Omega_2, d_2)) \in \mathcal{G}[\Omega_H^1], \forall (\Omega_1, d_1; \Pi) \in p[\Omega_H^1]^{-1}(\Omega_1, d_1), \\ (f : (\Omega_1, d_1) \rightarrow (\Omega_2, d_2)) \triangleright (\Omega_1, d_1; \Pi) := (\Omega_2, d_2; f_* \circ \Pi \circ f_*^{-1}).$$

Hence, the canonical covering $\pi[\Omega_H^1] : \mathcal{G}[\Omega_H^1] \ltimes \mathcal{A}[\Omega_H^1] \rightarrow \mathcal{G}[\Omega_H^1]$ is given by

$$(2.27) \quad \forall ((f : (\Omega_1, d_1) \rightarrow (\Omega_2, d_2)), (\Omega_1, d_1; \Pi)) \in \mathcal{G}[\Omega_H^1] \ltimes \mathcal{A}[\Omega_H^1], \\ \pi[\Omega_H^1]((f : (\Omega_1, d_1) \rightarrow (\Omega_2, d_2)), (\Omega_1, d_1; \Pi)) := (f : (\Omega_1, d_1) \rightarrow (\Omega_2, d_2)).$$

As a convenient abuse of notation, we will denote an arrow

$$((f : (\Omega_1, d_1) \rightarrow (\Omega_2, d_2)), (\Omega_1, d_1; \Pi))$$

of the action groupoid $\mathcal{G}[\Omega_H^1] \ltimes \mathcal{A}[\Omega_H^1]$ by

$$f : (\Omega_1, d_1; \Pi_1) \rightarrow (\Omega_2, d_2; \Pi_2),$$

where $\Pi_2 := f_* \circ \Pi_1 \circ f_*^{-1}$, so that, in particular

$$\pi[\Omega_H^1](f : (\Omega_1, d_1; \Pi_1) \rightarrow (\Omega_2, d_2; \Pi_2)) := (f : (\Omega_1, d_1) \rightarrow (\Omega_2, d_2)).$$

Example 2.39. Let $\Omega_{P, \oplus}^1 := \Omega_{P, \text{ver}}^1 \oplus \Omega_{P, \text{hor}}^1$, and let $\Pi_{\oplus} : \Omega_{P, \oplus}^1 \rightarrow \Omega_{P, \oplus}^1$ denote the projection onto $\Omega_{P, \text{ver}}^1$ along $\Omega_{P, \text{hor}}^1$. For every gauge potential $\nabla \in \mathfrak{A}t$ on P with respect to $(\Omega_B^1, d_B; \Omega_{P, \text{hor}}^1)$, the left H -covariant $*$ -derivation $d_{P, \nabla} : P \rightarrow \Omega_{P, \oplus}^1$ given by

$$(2.28) \quad \forall p \in P, \quad d_{P, \nabla}(p) := (d_{P, \text{ver}}(p), \nabla(p))$$

makes $(\Omega_{P, \oplus}^1, d_{P, \nabla}; \Pi_{\oplus})$ into an element of $\mathcal{A}[\Omega_H^1]$, such that $\nabla_{\Pi_{\oplus}} = \nabla$; in particular, the resulting vertical map $\text{ver}[d_{P, \nabla}]$ is simply the projection onto $\Omega_{P, \text{ver}}^1$ along $\Omega_{P, \text{hor}}^1$.

Let \mathfrak{G} and $\mathfrak{A}t$ respectively denote the gauge group and Atiyah space of the principal left H -comodule $*$ -algebra P with respect to the horizontal calculus $(\Omega_B^1, d_B; \Omega_{P, \text{hor}}^1)$, where, by the usual abuse of notation, we suppress the inclusion $\iota : \Omega_B^1 \hookrightarrow \Omega_{P, \text{hor}}^1$. We now promote the decomposition (2.14) given by Proposition 2.18 to an explicit equivalence of categories that realises the action groupoid $\mathfrak{G} \ltimes \mathfrak{A}t$, which is independent of the choice of bicovariant FODC (Ω_H^1, d_H) on H , as a deformation retraction of the action groupoid $\mathcal{G}[\Omega_H^1] \ltimes \mathcal{A}[\Omega_H^1]$ of the abstract gauge action. This rigorously justifies identifying the action of the gauge group \mathfrak{G} on the Atiyah space $\mathfrak{A}t$ as the affine action of global gauge transformations on principal connections for the quantum principal H -bundle P with respect to the bicovariant FODC (Ω_H^1, d_H) on H and the horizontal calculus $(\Omega_B^1, d_B; \Omega_{P, \text{hor}}^1)$.

Proposition 2.40. *Let (Ω_H^1, d_H) be a bicovariant FODC on H that is locally free for P . The groupoid homomorphism $\Sigma[\Omega_H^1] : \mathfrak{G} \ltimes \mathfrak{A}t \rightarrow \mathcal{G}[\Omega_H^1] \ltimes \mathcal{A}[\Omega_H^1]$ given by*

$$(2.29) \quad \forall (\phi, \nabla) \in \mathfrak{G} \ltimes \mathfrak{A}t, \quad \Sigma[\Omega_H^1](\phi, \nabla) := (\phi : (\Omega_{P, \oplus}^1, d_{P, \nabla}; \Pi_{\oplus}) \rightarrow (\Omega_{P, \oplus}^1, d_{P, \phi \triangleright \nabla}; \Pi_{\oplus}))$$

is an equivalence of groupoids with left inverse and homotopy inverse $A[\Omega_H^1]$ given by

$$(2.30) \quad \forall (f : (\Omega_1, d_1; \Pi_1) \rightarrow (\Omega_2, d_2; \Pi_2)) \in \mathcal{G}[\Omega_H^1] \ltimes \mathcal{A}[\Omega_H^1], \\ A[\Omega_H^1](f : (\Omega_1, d_1; \Pi_1) \rightarrow (\Omega_2, d_2; \Pi_2)) := (f, C[\Omega_1] \circ (\text{id} - \Pi_1) \circ d_1).$$

In particular, there exists a homotopy $\eta[\Omega_H^1] : \text{id}_{\mathcal{G}[\Omega_H^1] \times \mathcal{A}[\Omega_H^1]} \Rightarrow \Sigma[\Omega_H^1] \circ A[\Omega_H^1]$, which is necessarily unique, such that

$$(2.31) \quad \forall (\Omega_P^1, d_P; \Pi) \in \mathcal{A}[\Omega_H^1], \quad \mu[\Omega_H^1] \circ \pi[\Omega_H^1] \left(\eta[\Omega_H^1]_{(\Omega_P^1, d_P; \Pi)} \right) = \text{id}_P.$$

Proof. First, let us check that $\Sigma[\Omega_H^1]$ is well-defined. Let $(\phi, \nabla) \in \mathfrak{G} \times \mathfrak{A}t$. Then, by Remark 2.22 and Proposition 2.27, for all $p, p', p'' \in P$,

$$\begin{aligned} \phi(p) \cdot d_{P, \phi \triangleright \nabla}(\phi(p')) \cdot \phi(p'') &= \phi(p) \cdot (d_{P, \text{ver}}(\phi(p')), (\phi \triangleright \nabla)(\phi(p'))) \cdot \phi(p'') \\ &= \phi(p) \cdot (\phi_{*, \text{ver}}(d_{P, \text{ver}}(p')), \phi_{*, \text{hor}}(\nabla(p'))) \cdot \phi(p'') \\ &= (\phi_{*, \text{ver}}(p \cdot d_{P, \text{ver}}(p')) \cdot p'', \phi_{*, \text{hor}}(p \cdot \nabla(p')) \cdot p'') \\ &= (\phi_{*, \text{ver}} \oplus \phi_{*, \text{hor}})(p \cdot d_{P, \nabla}(p') \cdot p'') \end{aligned}$$

so that $\phi_* = \phi_{*, \text{ver}} \oplus \phi_{*, \text{hor}}$ is a well-defined bijection that satisfies

$$\begin{aligned} \phi_{*, \text{ver}} \circ \text{ver}[d_{P, \nabla}] &= \phi_{*, \text{ver}} \circ \text{Proj}_1 = \text{Proj}_1 \circ (\phi_{*, \text{ver}} \oplus \phi_{*, \text{hor}}) = \text{ver}[d_{P, \phi \triangleright \nabla}] \circ \phi_*, \\ \phi_* \circ \Pi_{\oplus} &= (\phi_{*, \text{ver}} \oplus \phi_{*, \text{hor}}) \circ \Pi_{\oplus} = \Pi_{\oplus} \circ (\phi_{*, \text{ver}} \oplus \phi_{*, \text{hor}}) = \Pi_{\oplus} \circ \phi_*. \end{aligned}$$

Hence, $\phi : (\Omega_{P, \oplus}^1, d_{P, \nabla}; \Pi_{\oplus}) \rightarrow (\Omega_{P, \oplus}^1, d_{P, \phi \triangleright \nabla}; \Pi_{\oplus})$ is an arrow in $\mathcal{G}[\Omega_H^1] \times \mathcal{A}[\Omega_H^1]$. Thus, $\Sigma[\Omega_H^1]$ is well-defined as a function between sets of arrows. That $\Sigma[\Omega_H^1]$ is a groupoid homomorphism now follows from the fact that both of

$$(\phi \mapsto \phi_{*, \text{ver}}) : \mathfrak{G} \rightarrow \text{GL}(\Omega_{P, \text{ver}}^1), \quad (\phi \mapsto \phi_{*, \text{hor}}) : \mathfrak{G} \rightarrow \text{GL}(\Omega_{P, \text{hor}}^1)$$

are group homomorphisms.

Next, let us check that $A[\Omega_H^1]$ is well-defined. Let $f : (\Omega_1, d_1; \Pi_1) \rightarrow (\Omega_2, d_2; \Pi_2)$ be an arrow in $\mathcal{G}[\Omega_H^1] \times \mathcal{A}[\Omega_H^1]$, so that $f_* : \Omega_1 \rightarrow \Omega_2$ is a well-defined bijection satisfying

$$\text{ver}[d_2] \circ f_* = f_{*, \text{ver}} \circ \text{ver}[d_1], \quad \Pi_2 \circ f_* = f_* \circ \Pi_1.$$

For $i = 1, 2$, let $\nabla_i := C[\Omega_i] \circ (\text{id} - \Pi_i) \circ d_P \in \mathfrak{A}t$; we need to show that $f \in \mathfrak{G}$ and that $\nabla_2 = f \triangleright \nabla_1$. Now, for every $p, p' \in P$ and $b \in B$,

$$\begin{aligned} f(p) \cdot d_B(b) \cdot f(p') &= C[\Omega_2](f(p) \cdot d_2(b) \cdot f(p')) \\ &= C[\Omega_2] \circ f_* (p \cdot d_1(b) \cdot p') \\ &= C[\Omega_2] \circ f_* \circ C[\Omega_1]^{-1} (p \cdot d_B(b) \cdot p'), \end{aligned}$$

so that $f_{*, \text{hor}} = C[\Omega_2] \circ f_* \circ C[\Omega_1]^{-1}$ is well-defined and bijective, hence $f \in \mathfrak{G}$, with

$$\begin{aligned} f \triangleright \nabla_1 &= f_{*, \text{hor}} \circ \nabla_1 \circ f^{-1} = C[\Omega_2] \circ f_* \circ C[\Omega_1]^{-1} \circ C[\Omega_1] \circ (\text{id} - \Pi_1) \circ d_1 \circ f^{-1} \\ &= C[\Omega_2] \circ f_* \circ (\text{id} - \Pi_1) \circ d_1 \circ f^{-1} = C[\Omega_2] \circ (\text{id} - \Pi_2) \circ d_2 = \nabla_2. \end{aligned}$$

As a result, $(f, \nabla_1) : \nabla_1 \rightarrow \nabla_2$ is a well-defined arrow in $\mathfrak{G} \times \mathfrak{A}t$. Thus, $A[\Omega_H^1]$ is well-defined as a function between sets of arrows. That $A[\Omega_H^1]$ is a groupoid homomorphism now follows since $(\phi \mapsto \phi_{*, \text{hor}}) : \mathfrak{G} \rightarrow \text{GL}(\Omega_{P, \text{hor}}^1)$ is a group homomorphism.

On the one hand, $C[\Omega_{P, \oplus}^1] = \text{id}_{\Omega_{P, \oplus}^1}$ by uniqueness of $C[\Omega_{P, \oplus}^1]$; hence, it follows that $A[\Omega_H^1] \circ \Sigma[\Omega_H^1] = \text{id}_{\mathfrak{G} \times \mathfrak{A}t}$. On the other hand, by the proof of Proposition 2.18, *mutatis mutandis*, we can define a homotopy $\eta[\Omega_H^1] : \text{id}_{\mathcal{G}[\Omega_H^1] \times \mathcal{A}[\Omega_H^1]} \Rightarrow \Sigma[\Omega_H^1] \circ A[\Omega_H^1]$ by

$$\begin{aligned} \forall (\Omega_P^1, d_P; \Pi) \in \mathcal{A}[\Omega_H^1], \\ \eta[\Omega_H^1]_{(\Omega_P^1, d_P; \Pi)} := (\text{id}_P : (\Omega_P^1, d_P; \Pi) \rightarrow A[\Omega_H^1] \circ \Sigma[\Omega_H^1]_{(\Omega_P^1, d_P; \Pi)}), \end{aligned}$$

which, by construction, satisfies (2.31). \square

In particular, this justifies the identification of \mathfrak{G} as the group of global gauge transformations on the quantum principal H -bundle P with respect to the bicovariant FODC (Ω_H^1, d_H) on H and horizontal calculus $(\Omega_B^1, d_B; \Omega_{P,\text{hor}}^1)$.

Corollary 2.41. *Let (Ω_H^1, d_H) be a bicovariant FODC on H that is locally freeing for P . The range of the star-injective groupoid homomorphism $\mu[\Omega_H^1] : \mathcal{G}[\Omega_H^1] \rightarrow \text{Aut}(P)$ is \mathfrak{G} , so that, after restriction of codomain,*

$$\mu[\Omega_H^1] : \mathcal{G}[\Omega_H^1] \rightarrow \mathfrak{G}, \quad \mu[\Omega_H^1] \circ \pi[\Omega_H^1] : \mathcal{G}[\Omega_H^1] \times \mathcal{A}[\Omega_H^1] \rightarrow \mathfrak{G}$$

both define coverings of groupoids.

Given a bicovariant FODC (Ω_H^1, d_H) on H , we now use the groupoid equivalence $\Sigma[\Omega_H^1]$ of Proposition 2.40 to construct a \mathfrak{G} -equivariant moduli space of $(H; \Omega_H^1, d_H)$ -principal FODC on P inducing the horizontal calculus $(\Omega_B^1, d_B; \Omega_{P,\text{hor}}^1)$. Indeed, this moduli space will turn out to be a quotient of the Atiyah space \mathfrak{At} by the space of all relative gauge potentials of the following form.

Definition 2.42. Let (Ω_H^1, d_H) be a bicovariant FODC on H that is locally freeing for P . We say that a relative gauge potential $\mathbf{A} \in \text{at}$ for P with respect to $(\Omega_B^1, d_B; \Omega_{P,\text{hor}}^1)$ is (Ω_H^1, d_H) -adapted whenever

$$(2.32) \quad \mathbf{A} = \omega[\mathbf{A}] \circ d_{P,\text{ver}}$$

for some (necessarily unique) left H -covariant morphism $\omega[\mathbf{A}] : \Omega_{P,\text{ver}}^1 \rightarrow \Omega_{P,\text{hor}}^1$ of P - $*$ -bimodules; in this case, we call $\omega[\mathbf{A}]$ the *relative connection 1-form* of \mathbf{A} . We denote by $\text{at}[\Omega_H^1]$ the subspace of all (Ω_H^1, d_H) -adapted relative gauge potentials on P with respect to the horizontal calculus $(\Omega_B^1, d_B; \Omega_{P,\text{hor}}^1)$.

Remark 2.43 (cf. Brzeziński–Majid [11, §4.2], Đurđević [21, §4], Beggs–Majid [5, Prop. 5.54]). Using uniqueness, one can now check that the map

$$\omega : \text{at}[\Omega_H^1] \rightarrow \text{Hom}_P(\Omega_{P,\text{ver}}^1, \Omega_{P,\text{hor}}^1), \quad \mathbf{A} \mapsto \omega[\mathbf{A}]$$

is \mathbf{R} -linear; moreover, given $\mathbf{A} \in \text{at}[\Omega_H^1]$, the relative connection 1-form $\omega[\mathbf{A}]$ is completely determined by its restriction to a map $\Lambda_H^1 \cong \Lambda_H^1 \otimes 1_P \rightarrow \Omega_{P,\text{hor}}^1$, which can be viewed as a noncommutative Lie-valued 1-form.

Now, let $(\Omega_1, d_1), (\Omega_2, d_2) \in \text{Ob}(\mathcal{G}[\Omega_H^1])$, where (Ω_H^1, d_H) is a locally freeing bicovariant FODC on H . Observe that (Ω_1, d_1) and (Ω_2, d_2) admit an isomorphism of left H -covariant FODC for P if and only if $\text{id}_P : (\Omega_1, d_1) \rightarrow (\Omega_2, d_2)$ is an arrow in $\mathcal{G}[\Omega_H^1]$. Since the subgroupoid of all such arrows is precisely $\ker \mu[\Omega_H^1]$, it follows that (Ω_1, d_1) and (Ω_2, d_2) are isomorphic if and only if they define the same object in the quotient groupoid $\mathcal{G}[\Omega_H^1] / \ker \mu[\Omega_H^1]$, which will turn out to be well-defined and canonically isomorphic to $\mathfrak{G} \times \mathfrak{At} / \text{at}[\Omega_H^1]$. Thus, the quotient affine space $\mathfrak{At} / \text{at}[\Omega_H^1]$ yields the desired \mathfrak{G} -equivariant affine moduli space of (Ω_H^1, d_H) -principal FODC on P inducing $(\Omega_B^1, d_B; \Omega_{P,\text{hor}}^1)$.

Theorem 2.44. *Let (Ω_H^1, d_H) be a bicovariant FODC on H . Suppose that the Atiyah space \mathfrak{At} is non-empty. The subspace $\text{at}[\Omega_H^1]$ is \mathfrak{G} -invariant; the subgroupoid $\ker \mu[\Omega_H^1]$ of $\mathcal{G}[\Omega_H^1]$ is wide and has trivial isotropy groups, so that the quotient groupoid $\mathcal{G}[\Omega_H^1] / \ker \mu[\Omega_H^1]$ is well-defined; and there exists a unique groupoid isomorphism*

$$\tilde{\Sigma}[\Omega_H^1] : \mathfrak{G} \times (\mathfrak{At} / \text{at}[\Omega_H^1]) \xrightarrow{\sim} \mathcal{G}[\Omega_H^1] / \ker \mu[\Omega_H^1],$$

such that

$$(2.33) \quad \forall(\phi, \nabla) \in \mathfrak{G} \ltimes \mathfrak{A}t, \quad \tilde{\Sigma}[\Omega_H^1](\phi, \nabla + \text{at}[\Omega_H^1]) = [\pi[\Omega_H^1] \circ \Sigma[\Omega_H^1](\phi, \nabla)]_{\ker \mu[\Omega_H^1]}.$$

Proof. Let us first show that $\text{at}[\Omega_H^1]$ is \mathfrak{G} -invariant. Let $\mathbf{A} \in \text{at}[\Omega_H^1]$, so that $\mathbf{A} = \omega[\mathbf{A}] \circ d_{P, \text{ver}}$. Then, for any $\phi \in \mathfrak{G}$,

$$\phi \triangleright \mathbf{A} = \phi_{*, \text{hor}} \circ \omega[\mathbf{A}] \circ d_{P, \text{ver}} \circ \phi^{-1} = (\phi_* \circ \omega[\mathbf{A}] \circ \phi_{*, \text{ver}}^{-1}) \circ d_{P, \text{ver}},$$

where, by properties of ϕ_* and ϕ^{-1} , the map $\phi_{*, \text{hor}} \circ \omega[\mathbf{A}] \circ \phi_{*, \text{ver}}^{-1} : \Omega_{P, \text{ver}}^1 \rightarrow \Omega_{P, \text{hor}}^1$ remains a left H -covariant morphism of P -*-bimodules, so that $\phi \triangleright \mathbf{A} \in \text{at}[\Omega_H^1]$, with

$$\omega[\phi \triangleright \mathbf{A}] = \phi_{*, \text{hor}} \circ \omega[\mathbf{A}] \circ \phi_{*, \text{ver}}^{-1}.$$

Now, by Proposition 2.40, the surjective covering $\pi[\Omega_H^1] : \mathcal{G}[\Omega_H^1] \ltimes \mathcal{A}[\Omega_H^1] \rightarrow \mathcal{G}[\Omega_H^1]$, the star-injective groupoid homomorphism $\mu[\Omega_H^1] : \mathcal{G}[\Omega_H^1] \rightarrow \text{Aut}(P)$, the injective groupoid homomorphism $\Sigma[\Omega_H^1] : \mathfrak{G} \ltimes \mathfrak{A}t \rightarrow \mathcal{G}[\Omega_H^1] \ltimes \mathcal{A}[\Omega_H^1]$, and the left inverse $A[\Omega_H^1]$ of $\Sigma[\Omega_H^1]$ combine to satisfy the hypotheses of Lemma A.1. Thus, the subgroupoid $\ker \mu[\Omega_H^1]$ is wide and has trivial isotropy groups, the quotient groupoid $\mathcal{G}[\Omega_H^1] / \ker \mu[\Omega_H^1]$ is well-defined, the equivalence kernel \sim of the map

$$(\nabla \mapsto [\pi[\Omega_H^1] \circ \Sigma[\Omega_H^1](\text{id}_P, \nabla)]_{\ker \mu[\Omega_H^1]}) : \mathfrak{A}t \rightarrow \text{Ob}(\mathcal{G}[\Omega_H^1] / \ker \mu[\Omega_H^1])$$

is a \mathfrak{G} -invariant equivalence relation on $\mathfrak{A}t$, and there exists a unique groupoid isomorphism $\tilde{\Sigma}[\Omega_H^1] : \mathfrak{G} \ltimes \mathfrak{A}t / \sim \xrightarrow{\sim} \mathcal{G}[\Omega_H^1] / \ker \mu[\Omega_H^1]$, such that

$$\forall(\phi, \nabla) \in \mathfrak{G} \ltimes \mathfrak{A}t, \quad \tilde{\Sigma}[\Omega_H^1](\phi, [\nabla]_{\sim}) = [\Sigma[\Omega_H^1](\phi, \nabla)]_{\ker \mu[\Omega_H^1]}.$$

Thus, it remains to show that \sim is the orbit equivalence with respect to the translation action of $\text{at}[\Omega_H^1] \leq \text{at}$ on $\mathfrak{A}t$.

On the one hand, suppose that $\nabla_1, \nabla_2 \in \mathfrak{A}t$ satisfy $\nabla_1 \sim \nabla_2$, so that id_P defines an arrow

$$(\text{id}_P : (\Omega_{P, \oplus}^1, d_{P, \nabla_1}) \rightarrow (\Omega_{P, \oplus}^1, d_{P, \nabla_2})) \in \ker \mu[\Omega_H^1] \subset \mathcal{G}[\Omega_H^1].$$

Thus, $(\text{id}_P)_* : \Omega_{P, \oplus}^1 \rightarrow \Omega_{P, \oplus}^1$ is a left H -covariant automorphism of the P -*-bimodule $\Omega_{P, \oplus}^1$, such that $d_{P, \nabla_2} = (\text{id}_P)_* \circ d_{P, \nabla_1}$ and

$$\text{Proj}_1 \circ (\text{id}_P)_* = \text{ver}[d_{P, \nabla_2}] \circ (\text{id}_P)_* = (\text{id}_P)_{*, \text{ver}} \circ \text{ver}[d_{P, \nabla_1}] = \text{Proj}_1.$$

Let $\mathcal{N} := (\text{id}_P)_* - \text{id}$. Now, for all $p \in P$,

$$0 = d_{P, \text{ver}}(p) - d_{P, \text{ver}}(p) = \text{Proj}_1 \circ d_{P, \nabla_2} - \text{Proj}_1 \circ d_{P, \nabla_1} = \text{Proj}_1 \circ \mathcal{N} \circ d_{P, \nabla_1},$$

so that $\text{ran } \mathcal{N} \subset \Omega_{P, \text{hor}}^1$, while for all $p \in P$ and $b \in B$,

$$(\text{id}_P)_*(p \cdot d_B(b)) = p \cdot d_B(b) = \text{id}(p \cdot d_B(b)),$$

so that $\Omega_{P, \text{hor}}^1 \subset \ker \mathcal{N}$, and hence $\mathcal{N} \circ \Pi_{\oplus} = \mathcal{N}$. Thus, for all $p \in P$,

$$(\nabla_2 - \nabla_1)(p) = \text{Proj}_2 \circ (d_{P, \nabla_2} - d_{P, \nabla_1})(p) = \text{Proj}_2 \circ \mathcal{N} \circ \Pi_{\oplus} \circ d_{P, \nabla_1} = \text{Proj}_2 \circ \mathcal{N} \circ d_{P, \text{ver}}(p),$$

so that $\nabla_2 - \nabla_1 \in \text{at}[\Omega_H^1]$ with $\omega[\nabla_2 - \nabla_1] = \text{Proj}_2 \circ \mathcal{N} \Big|_{\Omega_{P, \text{ver}}^1}$.

On the other hand, suppose that $\nabla_1, \nabla_2 \in \mathfrak{A}t$ satisfy $\nabla_2 - \nabla_1 \in \text{at}[\Omega_H^1]$. Let

$$N := \omega[\nabla_2 - \nabla_1],$$

and observe that $(N \circ \Pi_{\oplus})^2 = 0$. Then, for all $p, p', p'' \in P$,

$$\begin{aligned} p \cdot d_{P, \nabla_2}(p') \cdot p'' &= p \cdot (d_{P, \text{ver}}(p'), \nabla_2(p')) \cdot p'' \\ &= p \cdot (d_{P, \text{ver}}(p'), \nabla_1(p') + N \circ d_{P, \text{ver}}(p')) \cdot p'' \end{aligned}$$

$$\begin{aligned}
 &= p \cdot \left(d_{P, \nabla_1}(p') + N \circ \Pi_{\mathfrak{G}} \circ d_{P, \nabla_1}(p') \right) \cdot p'' \\
 &= (\text{id} + (N \circ \Pi_{\mathfrak{G}})) (p \cdot d_{P, \nabla_1}(p') \cdot p''),
 \end{aligned}$$

so that $(\text{id}_P)_* = \text{id} + (N \circ \Pi_{\mathfrak{G}})$ is a well-defined bijection satisfying

$$\text{ver}[d_{P, \nabla_2}] \circ (\text{id}_P)_* = \text{Proj}_1 \circ (\text{id} + (N \circ \Pi_{\mathfrak{G}})) = \text{Proj}_1 = (\text{id}_P)_{*, \text{ver}} \circ \text{ver}[d_{P, \nabla_1}].$$

Hence, $\text{id}_P : (\Omega_{P, \mathfrak{G}}^1, d_{P, \nabla_1}) \rightarrow (\Omega_{P, \mathfrak{G}}^1, d_{P, \nabla_2})$ defines an arrow in $\ker \mu[\Omega_H^1] \subset \mathcal{G}[\Omega_H^1]$. \square

Remark 2.45 (cf. Zucca [42, §8.4]). Let $(\Omega_P^1, d_P) \in \text{Ob}(\mathcal{G}[\Omega_H^1])$. We can view the stabilizer subgroup of $\tilde{\Sigma}[\Omega_H^1]^{-1}([\Omega_P^1, d_P]_{\ker \mu[\Omega_H^1]})$ in \mathfrak{G} as the gauge group of the quantum principal $(H; \Omega_H^1, d_H)$ -bundle $(P; \Omega_P^1, d_P)$. Moreover, $\Sigma[\Omega_H^1]$ restricts to a gauge-equivariant bijection from $\text{at}[\Omega_H^1]$ to the set of all strong bimodule connections on $(P; \Omega_P^1, d_P)$.

3. GAUGE THEORY TO SECOND ORDER

Our goal for this section is to develop and justify a conceptually economical notion of curvature of a principal connection on a quantum principal bundle that only involves differential calculus to second order and remains simultaneously compatible with both the theory of quantum principal H -bundles and strong bimodule connections à la Brzeziński–Majid and our notions of gauge transformations and (relative) gauge potentials. Our guiding principle will be prolongability of principal FODC to second order in a manner compatible with decomposition into vertical and horizontal calculi.

3.1. Deconstruction of quantum principal bundles to second order. We begin by prolonging the standard theory of quantum principal H -bundles and strong bimodule connections à la Brzeziński–Majid to second order in the spirit of Beggs–Brzeziński [4] and Beggs–Majid [5, §5.5]; in the process, we recover certain insights of Đurđević [21, §4].

Let (Ω_H^1, d_H) be a bicovariant FODC on H with corresponding left crossed H - $*$ -module Λ_H^1 of right coinvariant 1-forms and quantum Maurer–Cartan form ω_H ; let (Ω_H, d_H) be a prolongation of (Ω_H^1, d_H) to a bicovariant $*$ -differential calculus on H . We wish to consider differentiable quantum principal H -bundles compatible with (Ω_H, d_H) through degree 2. To this end, we will find it convenient to encode (Ω_H, d_H) in terms of the graded left crossed H -module $*$ -algebra of right H -coinvariant forms

$$(3.1) \quad \Lambda_H := (\Omega_H)^{\text{co}H}$$

and the restricted differential $d_H|_{\Lambda_H} : \Lambda_H \rightarrow \Lambda_H$. By results of Majid–Tao [27, Prop. 3.3, 3.4], it follows that $d_H|_{\Lambda_H^1} : \Lambda_H^1 \rightarrow \Lambda_H^2$ is given by the Maurer–Cartan equation

$$(3.2) \quad \forall h \in H, \quad d_H(\omega_H(h)) := \omega_H(h_{(1)}) \wedge \omega_H(h_{(2)}).$$

and satisfies

$$(3.3) \quad \forall h, k \in H, \\ d_H(h \triangleright \omega_H(k)) - h \triangleright d_H \omega(k) = h_{(1)} \triangleright \omega_H(k) \wedge \omega_H(h_{(2)}) + \omega_H(h_{(1)}) \wedge h_{(2)} \triangleright \omega_H(k).$$

The proof that Ω_H can be recovered from Λ_H up to isomorphism [27, Prop. 3.3], *mutatis mutandis*, lets us make the following definition, which we shall use to reconstruct graded $*$ -algebras of (total) differential forms on a quantum principal H -bundle from relevant graded $*$ -algebras of vertical and horizontal forms, respectively. Note that we shall only be concerned with $*$ -differential calculi and graded $*$ -algebras through degree 2.

Definition 3.1 (cf. Đurđević [21, Eqq. 4.49, 4.50]). Let Ω be a graded left H -comodule $*$ -algebra. We define the graded left H -comodule $*$ -algebra $\Lambda_H \widehat{\otimes}^{\leq 2} \Omega$ as follows:

- (1) $(\Lambda_H \widehat{\otimes}^{\leq 2} \Omega)^0 := \Omega^0$ as a left H -comodule $*$ -algebra;
- (2) $(\Lambda_H \widehat{\otimes}^{\leq 2} \Omega)^1 := (\Lambda_H^1 \otimes \Omega^0) \oplus \Omega^1$ as a left H -comodule right Ω^0 -module together with the left Ω^0 -module structure defined by

$$(3.4) \quad \forall p, q \in \Omega^0, \forall \omega \in \Lambda_H^1, \forall \alpha \in \Omega^1, \quad q \cdot (\omega \otimes p, \alpha) := (q_{[-1]} \triangleright \omega \otimes q_{[0]} p, q \cdot \alpha)$$

and the $*$ -structure defined by

$$(3.5) \quad \forall p \in \Omega^0, \forall \omega \in \Lambda_H^1, \forall \alpha \in \Omega^1, \quad (\omega \otimes p, \alpha)^* := (p_{[-1]}^* \triangleright \omega^* \otimes p_{[0]}^*, \alpha^*);$$

- (3) $(\Lambda_H \widehat{\otimes}^{\leq 2} \Omega)^1 := (\Lambda_H^2 \otimes \Omega^0) \oplus (\Lambda_H^1 \otimes \Omega^1) \oplus \Omega^2$ as a left H -comodule right Ω^0 -module together with the left Ω^0 -structure defined by

$$(3.6) \quad \forall \omega \in \Lambda_H^1, \forall \mu \in \Lambda_H^2, \forall p, q \in \Omega^0, \forall \alpha \in \Omega^1, \forall \beta \in \Omega^2, \\ q \cdot (\mu \otimes p, \omega \otimes \alpha, \beta) := (q_{[-1]} \triangleright \mu \otimes q_{[0]} p, q_{[-1]} \triangleright \omega \otimes q_{[0]} \cdot \alpha, q \cdot \beta)$$

and the $*$ -structure defined by

$$(3.7) \quad \forall \omega \in \Lambda_H^1, \forall \mu \in \Lambda_H^2, \forall p \in \Omega^0, \forall \alpha \in \Omega^1, \forall \beta \in \Omega^2, \\ (\mu \otimes p, \omega \otimes \alpha, \beta)^* := (p_{[-1]}^* \triangleright \mu^* \otimes p_{[0]}^*, p_{[-1]}^* \triangleright \omega^* \otimes p_{[0]}^*, \beta^*);$$

- (4) the wedge product $\wedge : (\Lambda_H \widehat{\otimes}^{\leq 2} \Omega)^1 \otimes_{\Omega^0} (\Lambda_H \widehat{\otimes}^{\leq 2} \Omega)^1 \rightarrow (\Lambda_H \widehat{\otimes}^{\leq 2} \Omega)^2$ is defined by

$$(3.8) \quad \forall \omega, \omega' \in \Lambda_H^1, \forall p, p' \in \Omega^0, \forall \alpha, \alpha' \in \Omega^1, \\ (\omega \otimes p, \alpha) \wedge (\omega' \otimes p', \alpha') := (\omega \wedge p_{[-1]} \triangleright \omega' \otimes p_{[0]} p', \omega \otimes p \cdot \alpha' + \alpha_{[-1]} \triangleright \omega' \otimes \omega_{[0]} \cdot p', \alpha \wedge \alpha');$$

- (5) $(\Lambda_H \widehat{\otimes}^{\leq 2} \Omega)^k := 0$ for $k > 2$.

For greater notational simplicity, given a graded left H -comodule $*$ -algebra Ω , we shall view $\Lambda_H \widehat{\otimes}^{\leq 2} \Omega$ as the graded left H -comodule $*$ -algebra, truncated at degree 2, generated by the graded left H -subcomodule $*$ -subalgebras Λ_H and Ω subject the relation $1_{\Lambda_H} = 1_{\Omega}$ and the braided graded commutation relations

$$(3.9) \quad \forall \omega \in \Lambda_H, \forall \alpha \in \Omega, \quad \alpha \wedge \omega = (-1)^{|\alpha||\omega|} \alpha_{[-1]} \triangleright \omega \wedge \alpha_{[0]}.$$

Now, let P once more be a principal left H -comodule $*$ -algebra over \mathbb{C} with $B := {}^{\text{co}}H P$. If the bicovariant FODC (Ω_H^1, d_H) on H is locally freeing for P , then the proof [27, §3.1] that (Ω_H, d_H) can be recovered from the data $(\Lambda_H, d_H|_{\Lambda_H})$, *mutatis mutandis*, lets us extend the induced (first-order) vertical calculus on P to a left H -covariant SODC as follows.

Definition 3.2 (Đurđević [21, Lemma 3.1]). Suppose that the bicovariant FODC (Ω_H^1, d_H) is locally freeing for P . The *second-order vertical calculus* of P is the extension of the vertical calculus $(\Omega_{P, \text{ver}}^1, d_{P, \text{ver}})$ to a left H -covariant SODC $(\Omega_{P, \text{ver}}, d_{P, \text{ver}})$ on P defined as follows:

- (1) $\Omega_{P, \text{ver}} := \Lambda_H \widehat{\otimes}^{\leq 2} P$ as a graded left H -comodule $*$ -algebra, where P is trivially extended to a graded left H -comodule $*$ -algebra;
- (2) the derivative $d_{P, \text{ver}} : \Omega_{P, \text{ver}} \rightarrow \Omega_{P, \text{ver}}$ is given by

$$(3.10) \quad \forall \omega \in \Lambda_H, \forall p \in P, \quad d_{P, \text{ver}}(\omega \cdot p) := d_H(\omega) \cdot p + (-1)^{|\omega|} \omega \cdot d_{P, \text{ver}}(p).$$

We can now refine the standard definition of differentiable quantum principal H -bundle to account for differential calculus through degree 2. This can be viewed as a distillation of recent results of Beggs–Majid [5, §5.5] that specialise Beggs–Brzeziński's

theory of noncommutative fibrations [4] to the case of quantum principal bundles; it also echoes Đurđević's notion of differentiable quantum principal bundle [21, §3].

Definition 3.3 (cf. Beggs–Majid [5, §5.5]). Suppose that the bicovariant FODC (Ω_H^1, d_H) on H is locally freeing for the principal left H -comodule $*$ -algebra P ; hence, let $(\Omega_{P,\text{ver}}, d_{P,\text{ver}})$ be the second-order vertical calculus of P with respect to the bicovariant prolongation (Ω_H, d_H) of (Ω_H^1, d_H) . Let (Ω_P, d_P) be a left H -covariant SODC on P . Define the restriction (Ω_B, d_B) of (Ω_P, d_P) to a SODC on $B := {}^{\text{co}}H_P$ by

$$\Omega_B^1 := B \cdot d_P(B), \quad \Omega_B^2 := \Omega_B^1 \wedge \Omega_B^1, \quad d_B := d_P|_{\Omega_B}.$$

Then $(P; \Omega, d_P)$ defines a *strong (second-order) quantum principal $(H; \Omega_H, d_H)$ -bundle* if and only if the following all hold:

- (1) the pair (Ω_P^1, d_P) defines an $(H; \Omega_H^1, d_H)$ -principal FODC on P whose vertical map $\text{ver}[d_P] : \Omega_P^1 \rightarrow \Omega_{P,\text{ver}}^1$ satisfies

$$(3.11) \quad \ker \text{ver}[d_P] = P \cdot \Omega_B^1;$$

- (2) the map $\text{ver}^{2,2}[d_P] : \Omega_P^2 \rightarrow \Omega_{P,\text{ver}}^2$ given by

$$(3.12) \quad \forall \alpha, \beta \in \Omega_P^1, \quad \text{ver}^{2,2}[d_P](\alpha \wedge \beta) := \text{ver}[d_P](\alpha) \wedge \text{ver}[d_P](\beta)$$

is well-defined, is surjective, and satisfies

$$(3.13) \quad \ker(\text{ver}^{2,2}[d_P]) = \Omega_P^1 \wedge \Omega_B^1;$$

- (3) the map $\text{ver}^{2,1}[d_P] : \Omega_P^2 \rightarrow \Lambda_H^1 \otimes \Omega_P^1 \subset (\Lambda_H \widehat{\otimes}^{\leq 2} \Omega_P)^2$ given by

$$(3.14) \quad \forall \alpha, \beta \in \Omega_P^1, \quad \text{ver}^{2,1}[d_P](\alpha \wedge \beta) := \text{ver}[d_P](\alpha) \wedge \beta + \alpha \wedge \text{ver}[d_P](\beta)$$

is well-defined and satisfies

$$(3.15) \quad \ker(\text{ver}^{2,1}[d_P]) \cap \ker(\text{ver}^{2,2}[d_P]) = P \cdot \Omega_B^2,$$

$$(3.16) \quad \text{ver}^{2,1}[d_P](\ker(\text{ver}^{2,2}[d_P])) = \Lambda_H^1 \otimes P \cdot \Omega_B^1.$$

In this case, we call (Ω_P, d_P) a *strongly $(H; \Omega_H, d_H)$ -principal SODC* on P , and we define the left H -covariant graded $*$ -subalgebra $\Omega_{P,\text{hor}}$ of *horizontal forms* in Ω_P by

$$\Omega_{P,\text{hor}}^0 := P, \quad \Omega_{P,\text{hor}}^1 := P \cdot \Omega_B^1, \quad \Omega_{P,\text{hor}}^2 := P \cdot \Omega_B^2.$$

Remark 3.4. We can provide the following conceptual interpretation of Definition 3.3. On the one hand, by condition 2, the map $\text{ver}[d_P] : \Omega_P^1 \rightarrow \Omega_{P,\text{ver}}^1$ extends via $\text{ver}^{2,2}[d_P]$ to a left H -covariant graded $*$ -epimorphism $\text{ver}[d_P] : \Omega_P \rightarrow \Omega_{P,\text{ver}}$, such that

$$\text{ver}[d_P]|_P = \text{id}_P, \quad \ker \text{ver}[d_P] = \Omega_P \wedge \Omega_{P,\text{hor}}^1;$$

following Đurđević [21, §3], we interpret this extension of $\text{ver}[d_P]$ as encoding restriction of differential forms to orbitwise differential forms. On the other hand, by condition 3, the rescaled vertical map $-i \text{ver}[d_P] : \Omega_P^1 \rightarrow \Omega_{P,\text{ver}}^1$ extends via $-i \text{ver}^{2,1}[d_P]$ to a degree 0 left H -covariant $*$ -derivation $\text{int}[d_P] : \Omega_P \rightarrow \Lambda_H \widehat{\otimes}^{\leq 2} \Omega_P$, such that

$$\text{int}[d_P]|_P = 0, \quad \ker \text{int}[d_P] \cap \ker \text{ver}[d_P] = \Omega_{P,\text{hor}}^2, \quad \text{int}[d_P](\ker \text{ver}[d_P]) = \Lambda_H^1 \otimes \Omega_{P,\text{hor}}^1;$$

we interpret the map $\text{int}[d_P] : \Omega_P \rightarrow \Lambda_H \widehat{\otimes}^{\leq 2} \Omega_P$ as encoding contraction of differential forms with fundamental vector fields. Proposition 2.17 now implies that

$$\ker \text{int}[d_P] \cap \ker \text{ver}[d_P] = \Omega_{P,\text{hor}}^1 \oplus \Omega_{P,\text{hor}}^2$$

recovers the graded $*$ -ideal of horizontal forms of positive degree, so that the equality

$${}^{\text{co } H}(\ker \text{int}[d_P] \cap \ker \text{ver}[d_P]) = \Omega_B^1 \oplus \Omega_B^2$$

recovers the basic differential forms as the H -coinvariant horizontal differential forms.

Remark 3.5. Suppose that (Ω_H, d_H) is Woronowicz's canonical prolongation [41, §§3–4] of the bicovariant FODC (Ω_H^1, d_H) on H , so that Ω_H defines a graded super-Hopf $*$ -algebra [7]. Suppose that (Ω_P, d_P) is a $*$ -differential calculus on P , such that the left H -coaction of H on P extends to a differentiable left Ω_H -coaction $\hat{\delta}_{\Omega_P}$ of the graded super-Hopf $*$ -algebra Ω_H on Ω_P (cf. Beggs–Majid [5, §5.5]), so that $(P; \Omega_P, d_P)$ is a quantum principal $(H; \Omega_H, d_H)$ -bundle à la Đurđević [21, §3], and hence the map $\text{ver}[d_P] : \Omega_P \rightarrow \Omega_{P, \text{ver}}$ of Remark 3.4 is a well-defined surjective morphism of left H - and Ω_H -covariant $*$ -DGA [21, Prop. 3.9]. Let Ω_B be the graded $*$ -subalgebra of left Ω_H -coinvariants generated by B and $d_P(B)$. Then the degree 2 truncation $(\Omega_P^{\leq 2}, d_P)$ of (Ω_P, d_P) is a strongly $(H; \Omega_H, d_H)$ -principal sodc on P whenever

$$\ker \text{ver}[d_P] = \Omega_P \wedge \Omega_B, \quad \{\omega \in \Omega_P \mid \hat{\delta}_{\Omega_P}(\omega) = \delta_{\Omega_P}(\omega)\} = P \cdot \Omega_B,$$

in which case, ${}^{\text{co } \Omega_H} \Omega_P = \Omega_B$ by Proposition 2.17.

Remark 3.6 (Beggs–Majid [5, Lemma 5.60, Theorem 5.61]). In the context of Definition 3.3, suppose that the B - $*$ -bimodule of basic 1-forms $\Omega_B^1 := B \cdot d_P(B)$ is flat as a left B -module (e.g., finitely generated and projective), that conditions 1 and 2 are satisfied, and that the map $\text{ver}^{2,1}[d_P] : \Omega_P^2 \rightarrow \Lambda_H^1 \otimes \Omega_P^1$ of condition 3 is well-defined. Then (Ω_P, d_P) is a strongly $(H; \Omega_H, d_H)$ -principal sodc on the principal left H -comodule $*$ -algebra P , and the inclusion $\Omega_B \hookrightarrow \Omega_P$ defines (to second order) a noncommutative fibration in the sense of Beggs–Brzeziński [4].

We now refine the definition of strong bimodule connection to the context of strong second-order quantum principal bundles, thereby providing a straightforward notion of noncommutative principal connection compatible with non-universal sodc. We shall soon see that this refinement is related to Đurđević's notion of multiplicative connection [21, Def. 4.2], and as such resolves (through degree 2) the following question implicitly flagged by Hajac [24, §4]: when does a connection extend from Ω_P^1 to all of Ω_P ?

Definition 3.7. Suppose that the bicovariant FODC (Ω_H^1, d_H) on H is locally freeing for P and that (Ω_P, d_P) is a strongly $(H; \Omega_H, d_H)$ -principal sodc on the principal left H -comodule $*$ -algebra P . A connection Π on the quantum principal $(H; \Omega_H^1, d_H)$ -bundle $(P; \Omega_P^1, d_P)$ is called *prolongable* if and only if it is a bimodule connection that satisfies both of the following:

(1) the map $\Pi \wedge \Pi : \Omega_P^2 \rightarrow \Omega_P^2$ given by

$$(3.17) \quad \forall \alpha, \beta \in \Omega_P^1, \quad \Pi \wedge \Pi(\alpha \wedge \beta) := \Pi(\alpha) \wedge \Pi(\beta)$$

is well-defined and satisfies $\ker(\Pi \wedge \Pi) = \Omega_P^1 \wedge \Omega_{P, \text{hor}}^1$.

(2) the map $\Pi \wedge \text{id} + \text{id} \wedge \Pi : \Omega_P^2 \rightarrow \Omega_P^2$ given by

$$(3.18) \quad \forall \alpha, \beta \in \Omega_P^1, \quad (\Pi \wedge \text{id} + \text{id} \wedge \Pi)(\alpha \wedge \beta) := \Pi(\alpha) \wedge \beta + \alpha \wedge \Pi(\beta)$$

is well-defined and satisfies $\ker((\Pi \wedge \text{id} + \text{id} \wedge \Pi) - \Pi \wedge \Pi) = \Omega_{P, \text{hor}}^2$.

Remark 3.8. Suppose that the bimodule connection Π satisfies Condition 1. Then Π satisfies condition 2 if and only if the map $(\text{id} - \Pi) \wedge (\text{id} - \Pi) : \Omega_P^2 \rightarrow \Omega_P^2$ given by

$$\forall \alpha, \beta \in \Omega_P^1, \quad (\text{id} - \Pi) \wedge (\text{id} - \Pi)(\alpha \wedge \beta) := (\text{id} - \Pi)(\alpha) \wedge (\text{id} - \Pi)(\beta)$$

is well-defined and satisfies $\ker(\text{id} - (\text{id} - \Pi) \wedge (\text{id} - \Pi)) = \Omega_{P,\text{hor}}^2$.

Suppose that (Ω_P, d_P) is a strongly $(H; \Omega_H, d_H)$ -principal sODC on the principal left H -comodule $*$ -algebra P . Recall that a bimodule connection Π on the quantum principal $(H; \Omega_H^1, d_H)$ -bundle $(P; \Omega_P^1, d_P)$ yields a resolution of the identity $\{\Pi, \text{id} - \Pi\}$ on Ω_P^1 realising $\ker \text{ver}[d_P] = \Omega_{P,\text{hor}}^1$ as a complementable left H -subcomodule P - $*$ -subbimodule of Ω_P^1 . Straightforward calculations now show that a prolongable bimodule connection Π yields a compatible resolution of the identity $\{\Pi_{2,2}, \Pi_{2,1}, \Pi_{2,0}\}$ on Ω_P^2 that realises

$$\ker(\text{ver}^{2,2}[d_P]) = \Omega_P^1 \wedge \Omega_{P,\text{hor}}^1, \quad \ker(\text{ver}^{2,1}[d_P]) \cap \ker(\text{ver}^{2,2}[d_P]) = \Omega_{P,\text{hor}}^2$$

as complementable left H -subcomodule P - $*$ -subbimodules of Ω_P^2 .

Proposition 3.9. *Suppose that the bicovariant FODC (Ω_H^1, d_H) on H is locally freeing for P , that (Ω_P, d_P) is a strongly $(H; \Omega_H, d_H)$ -principal sODC on P , and that Π is a prolongable bimodule connection on the strong quantum principal $(H; \Omega_H, d_H)$ -bundle $(P; \Omega_P, d_P)$.*

- (1) *The maps $\Pi_{1,1}, \Pi_{1,0} : \Omega_P^1 \rightarrow \Omega_P^1$ given by*

$$\Pi_{1,1} := \Pi, \quad \Pi_{1,0} := \text{id} - \Pi,$$

respectively, define an orthogonal pair of idempotent left H -covariant morphisms of P - $$ -bimodules, such that $\Pi_{1,1} + \Pi_{1,0} = \text{id}$ and*

$$\text{ran } \Pi_{1,0} = \ker \Pi_{1,1} = P \cdot \Omega_B^1.$$

- (2) *The maps $\Pi_{2,2}, \Pi_{2,1}, \Pi_{2,0} : \Omega_P^2 \rightarrow \Omega_P^2$ given by*

$$\Pi_{2,2} := \Pi \wedge \Pi, \quad \Pi_{2,1} := (\Pi \wedge \text{id} + \text{id} \wedge \Pi) - 2(\Pi \wedge \Pi),$$

$$\Pi_{2,0} := \text{id} - (\Pi \wedge \text{id} + \text{id} \wedge \Pi) + \Pi \wedge \Pi,$$

respectively, define a pairwise orthogonal triple of idempotent left H -covariant morphisms of P - $$ -bimodules, such that $\Pi_{2,2} + \Pi_{2,1} + \Pi_{2,0} = \text{id}$ and*

$$\text{ran } \Pi_{2,0} = \ker(\Pi_{2,2} + \Pi_{2,1}) = \Omega_{P,\text{hor}}^2, \quad \text{ran}(\Pi_{2,1} + \Pi_{2,0}) = \ker(\Pi_{2,2}) = \Omega_P^1 \wedge \Omega_{P,\text{hor}}^1;$$

- (3) *For all $\alpha, \beta \in \Omega_P^1$,*

$$\Pi_{2,2}(\alpha \wedge \beta) = \Pi_{1,1}(\alpha) \wedge \Pi_{1,1}(\beta),$$

$$\Pi_{2,1}(\alpha \wedge \beta) = \Pi_{1,1}(\alpha) \wedge \Pi_{1,0}(\beta) + \Pi_{1,0}(\alpha) \wedge \Pi_{1,1}(\beta),$$

$$\Pi_{2,0}(\alpha \wedge \beta) = \Pi_{1,0}(\alpha) \wedge \Pi_{1,0}(\beta).$$

Remark 3.10. Recall the maps $\text{ver}[d_P] : \Omega_P \rightarrow \Omega_{P,\text{ver}}$ and $\text{int}[d_P] : \Omega_P \rightarrow \Lambda_H^{\widehat{\otimes} \leq 2} \Omega_P$ from Remark 3.4 induced by the vertical map $\text{ver}[d_P] : \Omega_P^1 \rightarrow \Omega_{P,\text{ver}}^1$. On the one hand, the strong bimodule connection $\Pi = \Pi_{1,1}$ extends via $\Pi_{2,2}$ to a left H -covariant graded $*$ -homomorphism $\text{ver}_\Pi : \Omega_P \rightarrow \Omega_P$, such that

$$\text{ver}_\Pi|_P = \text{id}, \quad (\text{ver}_\Pi)^2 = \text{ver}_\Pi, \quad \ker \text{ver}_\Pi = \Omega_P^1 \wedge \Omega_{P,\text{hor}}^1 = \ker \text{ver}[d_P];$$

in particular, it follows that $\text{ver}[d_P] \circ \text{ver}_\Pi = \text{ver}[d_P]$. On the other, following an observation of Đurđević [21, Eq. 4.58], we see that $\text{id} - \Pi = \Pi_{1,0}$ extends via $\Pi_{2,0}$ to a left H -covariant graded $*$ -homomorphism $\text{hor}_\Pi : \Omega_P \rightarrow \Omega_P$, such that

$$\text{hor}_\Pi|_P = \text{id}, \quad (\text{hor}_\Pi)^2 = \text{hor}_\Pi, \quad \text{ran } \text{hor}_\Pi = \Omega_{P,\text{hor}} = P \oplus \ker \text{int}[d_P] \cap \ker \text{ver}[d_P];$$

in particular, it follows that $\text{hor}_\Pi|_{\Omega_B} = \text{id}$. Thus, in particular, the map ver_Π yields the extension from Ω_P^1 to Ω_P of the strong bimodule connection Π required by Hajac for the discussion of curvature [24, §4].

We now see that if Π is a prolongable bimodule projection on a strong quantum principal $(H; \Omega_H, d_H)$ -bundle $(P; \Omega_P, d_P)$, then $\nabla_\Pi := \Pi \circ d_P$ extends to a lift of $d_B : \Omega_B \rightarrow \Omega_B$ to a degree 1 left H -covariant $*$ -derivation on $\Omega_{P,\text{hor}}$, thereby addressing—if only through degree 2—an open issue for connections à la Brzeziński–Majid that was first flagged by Hajac [24, §4]. Furthermore, we shall also see that ∇_Π^2 directly yields a well-defined curvature 2-form for Π *qua* principal connection without invoking (absolute) connection 1-forms in the sense of Brzeziński–Majid [11, Prop. 4.10] or Đurđević [21, Def. 4.1].

Proposition-Definition 3.11 (cf. Brzeziński–Majid [11, Appx. A and §3], Hajac [24, §4], Đurđević [21, Def. 4.5 and Prop. 4.6]). Suppose that the bicovariant FODC (Ω_H^1, d_H) on H is locally freeing for the principal left H -comodule $*$ -algebra P , that (Ω_P, d_P) is a strongly $(H; \Omega_H, d_H)$ -principal sodc on P , and that Π is a prolongable bimodule connection on the strong quantum principal $(H; \Omega_H, d_H)$ -bundle $(P; \Omega_P, d_P)$.

- (1) The left H -covariant map $\nabla_\Pi : \Omega_P \rightarrow \Omega_{P,\text{hor}}$ given by

$$\nabla_\Pi|_P := \Pi_{1,0} \circ d_P, \quad \nabla_\Pi|_{\Omega_P^1} := \Pi_{2,0} \circ d_P$$

restricts to a degree 1 left H -covariant $*$ -derivation $\Omega_{P,\text{hor}} \rightarrow \Omega_{P,\text{hor}}$, such that

$$\nabla_\Pi|_{\Omega_B} = d_B;$$

we call ∇_Π the *exterior covariant derivative* induced by Π .

- (2) The left H -covariant right P -linear map $F_\Pi : \Omega_{P,\text{ver}}^1 \rightarrow \Omega_{P,\text{hor}}^2$ given by

$$(3.19) \quad F_\Pi|_{\Lambda_H^1} := i \Pi_{2,0} \circ d_P \circ (\text{ver}[d_P]|_{\text{ran} \Pi_{1,1}})^{-1}|_{\Lambda_H^1}$$

defines the unique left H -covariant morphism of P - $*$ -bimodules, such that

$$(3.20) \quad \nabla_\Pi^2|_P = i F_\Pi \circ d_{P,\text{ver}};$$

we call F_Π the *curvature* of Π .

Proof. Let us first check that the map ∇_Π , which is left H -covariant and satisfies

$$\forall \alpha \in \Omega_P, \quad \nabla_\Pi(\alpha^*) = -\nabla(\alpha)^*,$$

restricts to a degree 1 $*$ -derivation on $\Omega_{P,\text{hor}}$. Since $\nabla_\Pi|_P$ is the horizontal derivative of Proposition 2.18, it remains to show that $\nabla_\Pi|_{\Omega_P^1}$ satisfies the correct graded Leibniz rules with respect to the products $P \otimes \Omega_{P,\text{hor}}^1 \rightarrow \Omega_{P,\text{hor}}^1$ and $\Omega_{P,\text{hor}}^1 \otimes P \rightarrow \Omega_{P,\text{hor}}^1$. Indeed, for all $p \in P$ and $\alpha \in \Omega_{P,\text{hor}}^1$,

$$\begin{aligned} \nabla_\Pi(p \cdot \alpha) &= \Pi_{2,0} \circ d_P(p \cdot \alpha) = \Pi_{2,0}(d_P(p) \wedge \alpha + p \cdot d_P(\alpha)) \\ &= \Pi_{1,0}(d_P(p)) \wedge \Pi_{1,0}(\alpha) + p \cdot \Pi_{2,0}(d_P(\alpha)) = \nabla_\Pi(p) \wedge \alpha + p \cdot \nabla_\Pi(\alpha), \\ \nabla_\Pi(\alpha \cdot p) &= \Pi_{2,0} \circ d_P(\alpha \cdot p) = \Pi_{2,0}(d_P(\alpha) \cdot p - \alpha \wedge d_P(p)) \\ &= \Pi_{2,0}(d_P(\alpha)) \cdot p - \Pi_{1,0}(\alpha) \wedge \Pi_{1,0}(d_P(p)) = \nabla_\Pi(\alpha) \cdot p - \alpha \wedge \nabla_\Pi(p) \end{aligned}$$

as required. Note that $\nabla_\Pi|_{\Omega_B} = d_B$ because $\Pi_{1,0}|_{\Omega_B^1} = \text{id}$ and $\Pi_{2,0}|_{\Omega_B^2} = \text{id}$.

Let us now check that F_Π is left P -linear and $*$ -preserving. Set $\theta := (\text{ver}[d_P]|_{\text{ran} \Pi_{1,1}})^{-1}$. Then, for all $p \in P$ and $\omega \in \Lambda_H^1$,

$$\begin{aligned} p \cdot F_\Pi(\omega) &= p \cdot (i \Pi_{2,0} \circ d_P \circ \theta(\omega)) \\ &= i \Pi_{2,0}(d_P(\theta(p \cdot \omega)) - d_P(p) \wedge \theta(\omega)) \\ &= i \Pi_{2,0}(d_P(\theta(p_{[-1]} \triangleright \omega) \cdot p_{[0]})) \\ &= i \Pi_{2,0}(d_P(\theta(p_{[-1]} \triangleright \omega) \cdot p_{[0]} - \theta(p_{[-1]} \triangleright \omega) \wedge d_P(p_{[0]})) \end{aligned}$$

$$\begin{aligned}
 &= i \Pi_{2,0} \circ d_P (\theta(p_{[-1]} \triangleright \omega) \cdot p_{[0]}) \\
 &= F_\Pi ((p_{[-1]} \triangleright \omega) \cdot p_{[0]}),
 \end{aligned}$$

so that F_Π is left P -linear; that F_Π is $*$ -preserving now follows from the observation that $F_\Pi|_{\Lambda_H^1} = -i \Pi_{2,0} \circ d_P \circ \theta|_{\Lambda_H^1}$ is $*$ -preserving.

Finally, let us check the relation between F_Π and ∇^2 . For every $p \in P$,

$$\begin{aligned}
 \nabla_\Pi^2(p) &= \nabla_\Pi(d_P(p) - \theta \circ d_{P,\text{ver}}(p)) \\
 &= \Pi_{2,0} \circ d_P (d_P(p) - \theta(\omega_H(p_{[-1]})) \cdot p_{[0]}) \\
 &= \Pi_{2,0} (-d_P \circ \theta(\omega_H(p_{[-1]})) \cdot p_{[0]} + \theta(\omega_H(p_{[-1]})) \wedge d_P(p_{[0]})) \\
 &= -\Pi_{2,0} \circ d_P \circ \theta(\omega_H(p_{[-1]})) \cdot p_{[0]} \\
 &= i F_\Pi \circ d_{P,\text{ver}}(p),
 \end{aligned}$$

so that $\nabla_\Pi^2|_P = i F_\Pi \circ d_{P,\text{ver}}$, as was claimed; since $\Omega_{P,\text{ver}}^1 = P \cdot d_{P,\text{ver}}(P)$, it now follows that $F_\Pi : \Omega_{P,\text{ver}}^1 \rightarrow \Omega_{P,\text{hor}}^2$ is the unique such left H -covariant morphism of P - $*$ -bimodules. \square

Remark 3.12 (cf. Brzeziński–Majid [11, Appx. A], Đurđević [21, Prop. 4.16]). In terms of the map $\text{hor}_\Pi : \Omega_P \rightarrow \Omega_P$ defined in Remark 3.10,

$$\nabla_\Pi = \text{hor}_\Pi \circ d_P, \quad F_\Pi|_{\Lambda_H^1} = \text{hor}_\Pi \circ d_P \circ (\text{ver}[d_P]|_{\text{ran } \Pi})^{-1}|_{\Lambda_H^1}.$$

Given a prolongable connection Π on a strong quantum principal $(H; \Omega_H, d_H)$ -bundle $(P; \Omega_P, d_P)$, we can extend the decomposition of the $(H; \Omega_H^1, d_H)$ -principal FODC (Ω_P^1, d_P) on P given by Proposition 2.18 to a left H -covariant isomorphism of graded $*$ -algebras $\psi_\Pi : \Omega_P \rightarrow \Lambda_H \widehat{\otimes} \Omega_{P,\text{hor}}$, such that $\psi_\Pi \circ d_P \circ \psi_\Pi^{-1}$ can be explicitly expressed in terms of the quantum Maurer–Cartan form ω_H of the bicovariant FODC (Ω_H^1, d_H) on H , the gauge potential ∇_Π induced by Π , and the curvature F_Π of Π . This will provide a roadmap for replacing prolongable connections with sufficiently well-behaved gauge potentials by the appropriate refinement of Proposition 2.40.

Proposition 3.13 (cf. Đurđević [21, Thm. 4.12]). *Suppose that the bicovariant first-order differential calculus (Ω_H^1, d_H) on H is locally freeing for the principal left H -comodule $*$ -algebra P , that (Ω_P, d_P) is a strongly $(H; \Omega_H, d_H)$ -principal SODC on P , and that Π is a prolongable bimodule connection on the strong quantum principal $(H; \Omega_H, d_H)$ -bundle $(P; \Omega_P, d_P)$. The map $\psi_\Pi : \Omega_P \rightarrow \Lambda_H \widehat{\otimes} \Omega_{P,\text{hor}}$ given by*

$$\psi_\Pi|_P := \text{id}, \quad \psi_\Pi|_{\Omega_P^1} := \text{ver}[d_P] + \Pi_{1,0}, \quad \psi_\Pi|_{\Omega_P^2} := \text{ver}^{2,2}[d_P] + \text{ver}^{2,1}[d_P] \circ \Pi_{2,1} + \Pi_{2,0}$$

defines a left H -covariant isomorphism of graded $*$ -algebras, such that

$$(3.21) \quad \forall p \in P, \quad \psi_\Pi \circ d_P \circ \psi_\Pi^{-1}(p) = d_{P,\text{ver}}(p) + \nabla_\Pi(p),$$

$$(3.22) \quad \forall \omega \in \Lambda_H^1, \quad \psi_\Pi \circ d_P \circ \psi_\Pi^{-1}(\omega) = d_H(\omega) - i F_\Pi(\omega),$$

$$(3.23) \quad \forall \alpha \in \Omega_{P,\text{hor}}^1, \quad \psi_\Pi \circ d_P \circ \psi_\Pi^{-1}(\alpha) = \omega_H(\alpha_{[-1]}) \wedge \alpha_{[0]} + \nabla_\Pi(\alpha),$$

and hence, in particular, $\psi_\Pi \circ d_P \circ \psi_\Pi^{-1}|_{\Omega_B} = d_B$.

Proof. Before continuing, recall that the restriction $\psi_\Pi|_{\Omega_P^1}$ is a left H -covariant isomorphism of P - $*$ -bimodules by Proposition 2.18. First, observe the the left H -covariant graded

\mathbb{C} -linear map $\psi_\Pi : \Omega_P \rightarrow \Lambda_H \widehat{\otimes}^{\leq 2} \Omega_{P,\text{hor}}$ is bijective: since $(P; \Omega_P, d_P)$ is a strong second-order quantum principal $(H; \Omega_H, d_H)$ -bundle and since Π is prolongable, it follows that

$$\text{ver}^{2,2}[d_P] \Big|_{\text{ran } \Pi_{2,2}} : \text{ran } \Pi_{2,2} \rightarrow \Omega_{P,\text{ver}}^2, \quad \text{ver}^{2,1} \Big|_{\text{ran } \Pi_{2,1}} : \text{ran } \Pi_{2,1} \rightarrow \Lambda_H^1 \otimes \Omega_{P,\text{hor}}^1,$$

are both bijective, so that, in turn, the restriction

$$\psi_\Pi|_{\Omega_P^2} := \text{ver}^{2,2}[d_P] + \text{ver}^{2,1}[d_P] \circ \Pi_{2,1} + \Pi_{2,0} = \text{ver}^{2,2}[d_P] \circ \Pi_{2,2} + \text{ver}^{2,1}[d_P] \circ \Pi_{2,1} + \Pi_{2,0}$$

is also bijective. Next, to show that the map ψ_Π is a homomorphism, it suffices to show that it is multiplicative with respect to the product $\Omega_P^1 \otimes \Omega_P^1 \rightarrow \Omega_P^2$; indeed, for all $\alpha, \beta \in \Omega_P^1$,

$$\begin{aligned} \psi_\Pi(\alpha \wedge \beta) &= \text{ver}^{2,2}[d_P](\alpha \wedge \beta) + \text{ver}^{2,1}[d_P] \circ \Pi_{2,1}(\alpha \wedge \beta) + \Pi_{2,0}(\alpha \wedge \beta) \\ &= \text{ver}^{2,2}[d_P](\alpha \wedge \beta) + \text{ver}^{2,1}[d_P](\Pi_{1,1}(\alpha) \wedge \Pi_{1,0}(\beta) + \Pi_{1,0}(\alpha) \wedge \Pi_{1,1}(\beta)) \\ &\quad + \Pi_{1,0}(\alpha) \wedge \Pi_{1,0}(\beta) \\ &= \text{ver}[d_P](\alpha) \wedge \text{ver}[d_P](\beta) + \text{ver}[d_P](\alpha) \wedge \Pi_{1,0}(\beta) + \Pi_{1,0}(\alpha) \wedge \text{ver}[d_P](\beta) \\ &\quad + \Pi_{1,0}(\alpha) \wedge \Pi_{1,0}(\beta) \\ &= (\text{ver}[d_P](\alpha) + \Pi_{1,0}(\alpha)) \wedge (\text{ver}[d_P](\beta) + \Pi_{1,0}(\beta)) \\ &= \psi_\Pi(\alpha) \wedge \psi_\Pi(\beta). \end{aligned}$$

Since $\psi_\Pi|_{\Omega_P^1}$ is $*$ -preserving and $\Omega_P^2 = \Omega_P^1 \wedge \Omega_P^1$, it now follows that $\psi_\Pi|_{\Omega_P^2}$ is $*$ -preserving. Hence, the map ψ_Π is indeed a left H -covariant isomorphism of graded $*$ -algebras.

Let us now compute $\psi_\Pi \circ d_P \circ \psi_\Pi^{-1}$. By Proposition 2.18, we know that

$$\psi_\Pi \circ d_P \circ \psi_\Pi^{-1} \Big|_P = \psi_\Pi \circ d_P = d_{P,\text{ver}} + \nabla,$$

so it remains to compute $\psi_\Pi \circ d_P \circ \psi_\Pi^{-1} \Big|_{\Omega_P^1}$. Observe that

$$\begin{aligned} \psi_\Pi \circ d_P &= (\text{ver}^{2,2}[d_P] + \text{ver}^{2,1}[d_P] \circ \Pi_{2,1} + \Pi_{2,0}) \circ d_P \\ &= \text{ver}^{2,2}[d_P] \circ d_P + \text{ver}^{2,1}[d_P] \circ \Pi_{2,1} \circ d_P + \nabla_\Pi. \end{aligned}$$

On the one hand, for all $p, q \in P$,

$$\text{ver}^{2,2}[d_P] \circ d_P(p \cdot d_P q) = \text{ver}^{2,2}[d_P](d_P(p) \wedge d_P(q)) = d_{P,\text{ver}}(p) \wedge d_{P,\text{ver}}(q) = d_{P,\text{ver}}(p \cdot d_{P,\text{ver}}(q)),$$

so that $\text{ver}^{2,2}[d_P] \circ d_P = d_{P,\text{ver}} \circ \text{ver}[d_P]$. On the other, for all $p, q \in P$,

$$\begin{aligned} &\text{ver}^{2,1}[d_P] \circ \Pi_{2,1} \circ d_P(p \cdot d_P(q)) \\ &= \text{ver}^{2,1}[d_P] \circ \Pi_{2,1}(d_P(p) \wedge d_P(q)) \\ &= \text{ver}^{2,1}[d_P](\Pi_{1,1}(d_P(p)) \wedge \Pi_{1,0}(d_P(q)) + \Pi_{1,0}(d_P(p)) \wedge \Pi_{1,1}(d_P(q))) \\ &= (\text{ver}[d_P] \circ \Pi_{1,1})(d_P(p)) \wedge \nabla_\Pi(q) + \nabla_\Pi(p) \wedge (\text{ver}[d_P] \circ \Pi_{1,1})(d_P(q)) \\ &= d_{P,\text{ver}}(p) \wedge \nabla_\Pi(q) + \nabla_\Pi(p) \wedge d_{P,\text{ver}}(q) \\ &= \omega_H(p_{[-1]}) \epsilon(q_{[-1]}) \otimes p_{[0]} \cdot \nabla_\Pi(q_{[0]}) - p_{[-1]} \triangleright \omega_H(q_{[-1]}) \otimes \nabla_\Pi(p_{[0]}) \cdot q_{[0]} \\ &= \omega_H(p_{[-1]} q_{[-1]}) \otimes p_{[0]} \cdot \nabla_\Pi(p_{[0]}) - p_{[-1]} \triangleright \omega_H(q_{[-1]}) \otimes \nabla_\Pi(p_{[0]} q_{[0]}) \\ &= \omega_H(\Pi_{1,0}(p \cdot d_P(q))_{[-1]}) \otimes \Pi_{1,0}(p \cdot d_P(q))_{[0]} + (\text{id} \widehat{\otimes} \nabla_\Pi) \circ \text{ver}[d_P](p \cdot d_P(q)), \end{aligned}$$

where $\text{id} \widehat{\otimes} \nabla_\Pi : \Lambda_H^1 \otimes P \rightarrow \Lambda_H^1 \otimes \Omega_{P,\text{hor}}^1$, yielding the projected Cartan's magic formula

$$\forall \alpha \in \Omega_P^1, \quad \text{ver}^{2,1}[d_P] \circ \Pi_{2,1} \circ d_P(\alpha) = \omega_H(\Pi_{1,0}(\alpha)_{[-1]}) \otimes \Pi_{1,0}(\alpha)_{[0]} + (\text{id} \widehat{\otimes} \nabla_\Pi) \circ \text{ver}[d_P](\alpha).$$

It therefore follows that

$$\begin{aligned} \forall \alpha \in \Omega_P^1, \quad \psi_{\Pi} \circ d_P(\alpha) &= d_{P, \text{ver}} \circ \text{ver}[d_P](\alpha) + \varpi_H(\Pi_{1,0}(\alpha)_{[-1]}) \otimes \Pi_{1,0}(\alpha)_{[0]} \\ &\quad + (\text{id} \widehat{\otimes} \nabla_{\Pi}) \circ \text{ver}[d_P](\alpha) + \nabla_{\Pi}(\alpha). \end{aligned}$$

On the one hand, for all $\omega \in \Lambda_H^1$, since $\text{ver}[d_P] \circ \theta(\omega) = \omega$ and $\Pi_{1,0} \circ \theta(\omega) = 0$, it follows that

$$\psi_{\Pi} \circ d_P \circ \psi_{\Pi}^{-1}(\omega) = d_H(\omega) + \nabla_{\Pi} \circ \theta(\omega) = d_H(\omega) - i F_{\Pi}(\omega),$$

while on the other, for all $\alpha \in \Omega_{P, \text{hor}}^1$, since $\text{ver}[d_P](\alpha) = 0$ and $\Pi_{1,0}(\alpha) = \alpha$, it follows that

$$\psi_{\Pi} \circ d_P \circ \psi_{\Pi}^{-1}(\alpha) = \varpi_H(\alpha_{[-1]}) \otimes \alpha_{[0]} + \nabla_{\Pi}(\alpha) = \varpi_H(\alpha_{[-1]}) \wedge \alpha_{[0]} + \nabla_{\Pi}(\alpha).$$

Finally, since $\nabla_{\Pi}|_{\Omega_B} = d_B$ and $\Omega_B = {}^{\text{co}H}\Omega_{P, \text{hor}}$, it follows that $\psi_{\Pi} \circ d_P \circ \psi_{\Pi}^{-1}|_{\Omega_B} = d_B$. \square

Remark 3.14. The isomorphism ψ_{Π} was first constructed Đurđević [21, Thm. 4.12] for multiplicative regular connections in his sense [21, Def. 4.2 and 4.3]. Since regular connections à la Đurđević yield bimodule connections à la Beggs–Majid, this suggests that our notion of prolongable connection can be viewed as a variant of Đurđević’s notion of multiplicative connection. Our computation of $\psi_{\Pi}^{-1} \circ d_P \circ \psi_{\Pi}$, however, seems to be novel.

Remark 3.15. *Mutatis mutandis*, one can also prove the following (unprojected) Cartan’s magic formula for a strong quantum principal $(H; \Omega_H, d_H)$ -bundle $(P; \Omega_P, d_P)$:

$$(3.24) \quad \forall \alpha \in \Omega_P^1, \quad \text{ver}^{2,1}[d_P] \circ d_P(\alpha) = \varpi(\alpha_{[-1]}) \wedge \alpha_{[0]} + (\text{id} \widehat{\otimes} d_P) \circ \text{ver}[d_P](\alpha).$$

3.2. Prolongability and field strength. Having adapted the notions of quantum principal bundle and strong bimodule connection to the setting of second-order differential calculi, we now turn to the notions of gauge transformation and gauge potential with respect to a fixed horizontal calculus. In particular, we see that the standard noncommutative-geometric notion of curvature of a module connection can be adapted to gauge potentials *qua* horizontal covariant derivatives. In contrast to Đurđević [22, 23], we are explicitly concerned with the problem of extending constructions from degree 1 to degree 2.

From now on, let P be a principal H -comodule $*$ -algebra, and let $B := {}^{\text{co}H}P$. We begin with the following refinement of the notion of horizontal calculus on P , which encodes a choice of basic differential calculus (through degree 2) together with compatible left H -covariant graded $*$ -algebra over P of horizontal forms (through degree 2).

Definition 3.16 (cf. Đurđević [23, §3.1]). A *second-order horizontal calculus* on the principal left H -comodule $*$ -algebra P is a quadruple $(\Omega_B, d_B; \Omega_{P, \text{hor}}, \iota)$, where:

- (1) (Ω_B, d_B) is a sodc on B ;
- (2) $\Omega_{P, \text{hor}}$ is a left H -covariant graded $*$ -algebra generated by $\Omega_{P, \text{hor}}^1$ over $\Omega_P^0 = P$ and truncated at degree 2;
- (3) $\iota : \Omega_B \hookrightarrow {}^{\text{co}H}\Omega_{P, \text{hor}}$ is an injective morphism of graded $*$ -algebras, such that the pair $(\Omega_{P, \text{hor}}^1, \iota|_{\Omega_{P, \text{hor}}^1})$ defines a projectable horizontal lift of the B - $*$ -bimodule Ω_B^1 .

Example 3.17. Let (Ω_H, d_H) be a bicovariant sodc for H , and suppose that the bicovariant fodc (Ω_H^1, d_H) on H is locally freeing for P . Suppose that (Ω_P, d_P) is a strongly $(H; \Omega_H, d_H)$ -principal sodc on P admitting a prolongable bimodule connection; recall that (Ω_P, d_P) therefore restricts to the sodc (Ω_B, d_B) on $B := {}^{\text{co}H}P$ given by

$$\Omega_B^1 := B \cdot d_P(B), \quad \Omega_B^2 := \Omega_B^1 \wedge \Omega_B^1, \quad d_B := d_P|_{\Omega_B}.$$

Finally, let $\Omega_{P, \text{hor}}$ be the left H -subcomodule graded $*$ -subalgebra of Ω_P generated by Ω_B^1 over P , and let $\iota : \Omega_B \hookrightarrow \Omega_{P, \text{hor}}$ be the inclusion map. Then, by Proposition 2.18 applied

to the quantum principal $(H; \Omega_H^1, d_H)$ -bundle $(P; \Omega_P^1, d_P)$, the data $(\Omega_B, d_B; \Omega_{P,\text{hor}}, \iota)$ define a second-order horizontal calculus on P , which we can view as the *canonical* second-order horizontal calculus on P induced by the strongly $(H; \Omega_H, d_H)$ -principal SODC (Ω_P, d_P) on P .

Remark 3.18. If $(\Omega_B, d_B; \Omega_{P,\text{hor}}, \iota)$ is a second-order horizontal calculus on P , then the data $(\Omega_B^1, d_B; \Omega_{P,\text{hor}}^1, \iota|_{\Omega_B^1})$ define a first-order horizontal calculus on P .

Although it is not obvious, in a second-order horizontal calculus $(\Omega_B, d_B; \Omega_{P,\text{hor}}, \iota)$ on P , the left H -covariant graded $*$ -algebra $\Omega_{P,\text{hor}}$ of horizontal forms defines a projectable horizontal lift of the entire graded $*$ -algebra Ω_B of basic forms (through degree 2) on P .

Proposition 3.19. *Suppose that $(\Omega_B, d_B; \Omega_{P,\text{hor}}, \iota)$ is a second-order horizontal calculus on P ; recall that $B := {}^{\text{co}H}P$. Then $(\Omega_{P,\text{hor}}^2, \iota|_{\Omega_{P,\text{hor}}^2})$ defines a projectable horizontal lift of the B - $*$ -bimodule Ω_B^2 , and hence $\iota : \Omega_B \hookrightarrow {}^{\text{co}H}\Omega_{P,\text{hor}}$ is an isomorphism of graded $*$ -algebras.*

Proof. First, by applying Proposition 2.17 to the projectable horizontal lift $(\Omega_{P,\text{hor}}^1, \iota|_{\Omega_{P,\text{hor}}^1})$ of the B - $*$ -bimodule Ω_B^1 , we find that

$$\Omega_{P,\text{hor}}^2 = \Omega_{P,\text{hor}}^1 \wedge \Omega_{P,\text{hor}}^1 = P \cdot \iota(\Omega_B^1) \wedge \iota(\Omega_B^1) \cdot P = P \cdot \iota(\Omega_B^2) \cdot P,$$

so that $(\Omega_{P,\text{hor}}^2, \iota|_{\Omega_{P,\text{hor}}^2})$ is a horizontal lift of Ω_B^2 .

Now, let $p, p' \in P$ and $\omega, \omega' \in \Omega_B^1$. Then $\iota(\omega') \cdot p' = \sum_i q_i \cdot \iota(\omega'_i)$ for some $q_i \in P$ and $\omega'_i \in \Omega_B^1$, and for each i , $\iota(\omega) \cdot q_i = \sum_j q_{ij} \cdot \iota(\omega_{ij})$ for some $q_{ij} \in P$ and $\omega_{ij} \in \Omega_B^1$, so that

$$p \cdot \iota(\omega) \wedge \iota(\omega') \cdot p' = \sum_i p \cdot \iota(\omega) \wedge q_i \cdot \iota(\omega'_i) = \sum_{i,j} p q_{ij} \cdot \iota(\omega_{ij} \wedge \omega'_i) \in P \cdot \iota(\Omega_B^2).$$

Hence, by Proposition 2.17, the horizontal lift $(\Omega_{P,\text{hor}}^2, \iota|_{\Omega_{P,\text{hor}}^2})$ of Ω_B^2 is projectable. In particular, it now follows that $\iota(\Omega_B) = {}^{\text{co}H}\Omega_{P,\text{hor}}$. \square

Assume, therefore, that P admits a second-order horizontal calculus $(\Omega_B, d_B; \Omega_{P,\text{hor}}, \iota)$, which we now fix. To simplify notation, we suppress the inclusion map ι and identify Ω_B with its image in $\Omega_{P,\text{hor}}$; hence, where convenient, we denote the second-order horizontal calculus $(\Omega_B, d_B; \Omega_{P,\text{hor}}, \iota)$ by the triple $(\Omega_B, d_B; \Omega_{P,\text{hor}})$. Since $(\Omega_B^1, d_B; \Omega_{P,\text{hor}}^1)$ is a first-order horizontal calculus on P , we can define its gauge group \mathfrak{G} , its inner gauge group $\text{Inn}(\mathfrak{G})$, and its Atiyah space \mathfrak{At} with corresponding space of translations at .

We begin by characterizing those gauge transformations $f \in \mathfrak{G}$ that extend via the induced map $f_* : \Omega_{P,\text{hor}}^1 \rightarrow \Omega_{P,\text{hor}}^1$ to automorphisms of $\Omega_{P,\text{hor}}$.

Definition 3.20. We say that a gauge transformation $\phi \in \mathfrak{G}$ is *prolongable* with respect to the second-order horizontal calculus $(\Omega_B, d_B; \Omega_{P,\text{hor}}, \iota)$ on P whenever ϕ is also an automorphism of the projectable horizontal lift $(\Omega_{P,\text{hor}}^2, \iota|_{\Omega_B^2})$ of Ω_B^2 . Hence, we define the *prolongable gauge group* of P with respect to $(\Omega_B, d_B; \Omega_{P,\text{hor}}, \iota)$ by

$$\mathfrak{G}^{\text{pr}} := \mathfrak{G} \cap \text{Aut}\left(\Omega_{P,\text{hor}}^2, \iota|_{\Omega_B^2}\right),$$

where \mathfrak{G} is the gauge group of P with respect to $(\Omega_B^1, d_B; \Omega_{P,\text{hor}}^1, \iota|_{\Omega_B^1})$ and $\text{Aut}(\Omega_{P,\text{hor}}^2, \iota|_{\Omega_B^2})$ is the automorphism group of $(\Omega_{P,\text{hor}}^2, \iota|_{\Omega_B^2})$. By abuse of notation, given $\phi \in \mathfrak{G}^{\text{pr}}$, we denote by ϕ_* or by $\phi_{*,\text{hor}}$ the induced automorphism of the left H -comodule graded $*$ -algebra $\Omega_{P,\text{hor}}$.

Remark 3.21. That a prolongable gauge transformation $\phi \in \mathfrak{G}^{\text{Pr}}$ induces an automorphism of the entire left H -covariant graded $*$ -algebra $\Omega_{P,\text{hor}}$, viz, that

$$\forall \omega, \omega' \in \Omega_{P,\text{hor}}^1, \quad \phi_*(\omega \wedge \omega') = \phi_*(\omega) \wedge \phi_*(\omega'),$$

is a consequence of the proof of Proposition 3.19.

As an example, all inner gauge transformations are automatically prolongable.

Proposition-Definition 3.22. The prolongable gauge group \mathfrak{G}^{Pr} of P with respect to the second-order horizontal calculus $(\Omega_B, d_B; \Omega_{P,\text{hor}})$ contains the inner gauge group $\text{Inn}(\mathfrak{G})$ of P with respect to the first-order horizontal calculus $(\Omega_B^1, d_B; \Omega_{P,\text{hor}}^1)$ as a central subgroup. Hence, the *outer prolongable gauge group* of P with respect to $(\Omega_B, d_B; \Omega_{P,\text{hor}})$ is

$$\text{Out}(\mathfrak{G}^{\text{Pr}}) := \mathfrak{G}^{\text{Pr}} / \text{Inn}(\mathfrak{G}).$$

Proof. Since $\Omega_{P,\text{hor}}^2$ is a horizontal lift of $\Omega_B^2 = \Omega_B^1 \wedge \Omega_B^1$, it follows that $C_B(\Omega_B^2) \supseteq C_B(\Omega_B^1)$, so that, by Proposition 2.24,

$$\text{Inn}(\mathfrak{G}) = \{\text{Ad}_v \mid v \in \text{U}(Z(B) \cap C_B(\Omega_B^1))\} \leq \{\text{Ad}_v \mid v \in \text{U}(Z(B) \cap C_B(\Omega_B^2))\} = \text{Inn}(\Omega_{P,\text{hor}}^2, \iota),$$

and hence, $\text{Inn}(\mathfrak{G}) \leq \mathfrak{G} \cap \text{Aut}(\Omega_{P,\text{hor}}^2, \iota) =: \mathfrak{G}^{\text{Pr}}$. \square

We can now characterise those gauge potentials on P with respect to the first-order horizontal calculus $(\Omega_B^1, d_B; \Omega_{P,\text{hor}}^1)$ that extend to lifts of $d_B : \Omega_B \rightarrow \Omega_B$ to degree 1 left H -covariant $*$ -derivations of $\Omega_{P,\text{hor}}$. Because we only work through degree 2, such extensions are completely determined by maps $\Omega_{P,\text{hor}}^1 \rightarrow \Omega_{P,\text{hor}}^2$ of the following form.

Definition 3.23. Let Ω be a left H -comodule graded $*$ -algebra truncated at degree 2. Let $\partial : \Omega^0 \rightarrow \Omega^1$ be a left H -comodule $*$ -derivation on the Ω^0 - $*$ -bimodule Ω^1 . A *second-order prolongation* of ∂ is a left H -covariant \mathbb{C} -linear map $\partial' : \Omega^1 \rightarrow \Omega^2$ satisfying

$$\begin{aligned} \forall a, b \in \Omega^0, \forall \omega \in \Omega^1, \quad \partial'(a \cdot \omega \cdot b) &= \nabla(a) \wedge \omega \cdot b + a \cdot \partial'(\omega) \cdot b - a \cdot \omega \wedge \partial(b), \\ \forall \alpha \in \Omega^1, \quad \partial'(a)^* &= -\partial'(\alpha^*), \end{aligned}$$

so that $\partial : \Omega^0 \rightarrow \Omega^1$ extends via $\partial' : \Omega^1 \rightarrow \Omega^2$ and $0 : \Omega^2 \rightarrow 0$ to a degree 1 left H -covariant $*$ -derivation on the left H -comodule graded $*$ -algebra Ω .

We can characterise those gauge potentials on P with respect to $(\Omega_B^1, d_B; \Omega_{P,\text{hor}}^1)$ that suitably extend to all of $\Omega_{P,\text{hor}}$; this, in turn, yields a conceptually minimalistic notion of curvature compatible with the standard notion of curvature for module connections.

Proposition-Definition 3.24. We say that a gauge potential $\nabla \in \mathfrak{A}\mathfrak{t}$ is *prolongable* with respect to the second-order horizontal calculus $(\Omega_B, d_B; \Omega_{P,\text{hor}})$ whenever its *canonical prolongation*

$$\nabla^{\text{Pr}} : \Omega_{P,\text{hor}}^1 \rightarrow \Omega_{P,\text{hor}}^2, \quad p \cdot d_B(b) \cdot p' \mapsto \nabla(p) \wedge d_B(b) \cdot p' - p \cdot d_B(b) \wedge \nabla(p')$$

is well-defined, in which case:

- (1) ∇^{Pr} is the unique second-order prolongation of ∇ , such that $\nabla^{\text{Pr}}|_{\Omega_B^1} = d_B$;
- (2) the *field strength* $\mathbf{F}[\nabla] : P \rightarrow \Omega_{P,\text{hor}}^2$ of ∇ defined by

$$\mathbf{F}[\nabla] := -i \nabla^{\text{Pr}} \circ \nabla.$$

is a left H -covariant $*$ -derivation, such that $\mathbf{F}[\nabla]|_B = 0$.

Hence, we define the *prolongable Atiyah space* $\mathfrak{A}\mathfrak{t}^{\text{Pr}}$ of P with respect to $(\Omega_B, d_B; \Omega_{P,\text{hor}})$ to be the subset of all prolongable gauge potentials on P with respect to $(\Omega_B, d_B; \Omega_{P,\text{hor}})$.

Proof. Let us first show that ∇^{Pr} is a second-order prolongation of ∇ . On the one hand, ∇^{Pr} is left H -covariant since $\Omega_B^1 = {}^{\text{co}H}\Omega_{P,\text{hor}}^1$ and ∇ is left H -covariant. On the other, for all $p, q_1, q_2 \in P$ and $b \in d_B(b)$, we have

$$\begin{aligned} \nabla^{\text{Pr}}(q_1 \cdot (p \cdot d_B(b)) \cdot q_2) &= \nabla(q_1 \cdot p) \wedge d_B(b) \cdot q_2 - q_1 \cdot p \cdot d_B(b) \wedge \nabla(q_2) \\ &= (\nabla(q_1) \cdot p + q_1 \cdot \nabla(p)) \wedge d_B(b) \cdot q_2 - q_1 \cdot p \cdot d_B(b) \wedge \nabla(q_2) \\ &= \nabla(q_1) \wedge (p \cdot d_B(b)) \cdot q_2 + q_1 \cdot \nabla^{\text{Pr}}(p \cdot d_B(b)) \cdot q_2 \\ &\quad - q_1 \cdot (p \cdot d_B(b)) \wedge \nabla(q_2), \end{aligned}$$

while for all $p \in P$ and $b \in d_B(b)$,

$$\nabla^{\text{Pr}}(p \cdot d_B(b))^* = (\nabla(p) \wedge d_B(b))^* = -d_B(b^*) \wedge \nabla(p^*) = \nabla^{\text{Pr}}(d_B(b^*) \cdot p^*) = -\nabla^{\text{Pr}}((p \cdot d_B(b))^*).$$

Next, observe that $\nabla^{\text{Pr}}|_{\Omega_B^1} = d_B$, since for all $b_1, b_2 \in B$,

$$\nabla^{\text{Pr}}(b_1 \cdot d_B(b_2)) = \nabla(b_1) \wedge d_B(b_2) = d_B(b_1) \wedge d_B(b_2) = d_B(b_1 \cdot d_B(b_2));$$

in particular, it now follows that $\mathbf{F}[\nabla]$ vanishes on B ; thus, if ∇' is any second-order prolongation of ∇ , then for all $p, q \in P$ and $\beta \in \Omega_B^1$,

$$\nabla'(p \cdot \beta \cdot q) = \nabla(p) \wedge d_B(b) \cdot q + p \cdot d_B(\beta) \cdot q - p \cdot d_B(b) \wedge \nabla(q) = \nabla^{\text{Pr}}(p \cdot \beta \cdot q),$$

so that $\nabla' = \nabla^{\text{Pr}}$.

Finally, observe that the left H -covariant map $\mathbf{F}[\nabla] : P \rightarrow \Omega_{P,\text{hor}}^2$ is a $*$ -derivation, since for all $p, q \in P$,

$$\begin{aligned} \mathbf{F}[\nabla](pq) &= -i^{\nabla^{\text{Pr}}}(\nabla(p) \cdot q + p \cdot \nabla(q)) \\ &= \mathbf{F}[\nabla](p) \cdot q + i^{\nabla^{\text{Pr}}}(\nabla(p) \wedge \nabla(q)) - i^{\nabla^{\text{Pr}}}(\nabla(p) \wedge \nabla(q)) + \mathbf{F}[\nabla](q) \\ &= \mathbf{F}[\nabla](p) \cdot q + p \cdot \mathbf{F}[\nabla](q), \end{aligned}$$

while for all $p \in P$,

$$\mathbf{F}[\nabla](p)^* = (-i^{\nabla^{\text{Pr}}}(\nabla(p)))^* = -i^{\nabla^{\text{Pr}}}(\nabla(p)^*) = i^{\nabla^{\text{Pr}}}(\nabla(p^*)) = -\mathbf{F}[\nabla](p^*). \quad \square$$

Remark 3.25. That the field strength of a prolongable gauge potential defines a left H -covariant $*$ -derivation $P \rightarrow \Omega_{P,\text{hor}}^2$ was essentially first observed by Đurđević [22, p. 101].

We can now similarly characterise those relative gauge potentials on P with respect to $(\Omega_B^1, d_B; \Omega_{P,\text{hor}}^1)$ that suitably extend to all of $\Omega_{P,\text{hor}}$; this follows, *mutatis mutandis*, from the proof of Proposition-Definition 3.24.

Proposition-Definition 3.26. We say that $\mathbf{A} \in \mathfrak{at}$ is *prolongable* with respect to the second-order horizontal calculus $(\Omega_B, d_B; \Omega_{P,\text{hor}})$ whenever its *canonical prolongation*

$$\mathbf{A}^{\text{Pr}} : \Omega_{P,\text{hor}}^1 \rightarrow \Omega_{P,\text{hor}}^2, \quad p \cdot d_B(b) \cdot p' \mapsto \mathbf{A}(p) \wedge d_B(b) \cdot p' - p \cdot d_B(b) \wedge \mathbf{A}(p')$$

is well-defined, in which case, the canonical prolongation \mathbf{A}^{Pr} is the unique second-order prolongation of \mathbf{A} , such that $\mathbf{A}^{\text{Pr}}|_{\Omega_B^1} = 0$. We denote by $\mathfrak{at}^{\text{Pr}}$ the subspace of all prolongable relative gauge potentials on P with respect to $(\Omega_B, d_B; \Omega_{P,\text{hor}})$.

It now follows that the affine action of the gauge group \mathfrak{G} on the Atiyah space \mathfrak{At} restricts to an affine action of the subgroup \mathfrak{G}^{Pr} on the affine subspace $\mathfrak{At}^{\text{Pr}}$ that is compatible with canonical prolongation.

Proposition 3.27. *Suppose that the prolongable Atiyah space $\mathfrak{At}^{\text{Pr}}$ of P with respect to the second-order horizontal calculus $(\Omega_B, d_B; \Omega_{P,\text{hor}}^1)$ is non-empty. Then $\mathfrak{At}^{\text{Pr}}$ is a \mathfrak{G}^{Pr} -invariant affine subspace of the Atiyah space \mathfrak{At} with space of translations $\mathfrak{at}^{\text{Pr}}$. Moreover:*

- (1) $(\nabla \mapsto \nabla^{\text{Pr}}) : \mathfrak{A}t^{\text{Pr}} \rightarrow \text{Hom}_{\mathbb{C}}(\Omega_{P,\text{hor}}^1, \Omega_{P,\text{hor}}^2)$ is \mathfrak{G}^{Pr} -equivariant and affine linear with \mathfrak{G}^{Pr} -equivariant linear part $(\mathbf{A} \mapsto \mathbf{A}^{\text{Pr}}) : \mathfrak{a}t^{\text{Pr}} \rightarrow \text{Hom}_{\mathbb{C}}(\Omega_{P,\text{hor}}^1, \Omega_{P,\text{hor}}^2)$;
- (2) $\mathbf{F} := (\nabla \mapsto \mathbf{F}[\nabla]) : \mathfrak{A}t^{\text{Pr}} \rightarrow \text{Der}_P(\Omega_{P,\text{hor}}^2)$ is \mathfrak{G}^{Pr} -equivariant and affine quadratic, satisfying

$$\forall \nabla \in \mathfrak{A}t^{\text{Pr}}, \forall \mathbf{A} \in \mathfrak{a}t^{\text{Pr}}, \quad \mathbf{F}[\nabla + \mathbf{A}] - \mathbf{F}[\nabla] = -i(\nabla^{\text{Pr}} \circ \mathbf{A} + \mathbf{A}^{\text{Pr}} \circ \nabla + \mathbf{A}^{\text{Pr}} \circ \mathbf{A}).$$

Proof. First, that $\mathfrak{A}t^{\text{Pr}}$ is an affine subspace of $\mathfrak{A}t$ with space of translations $\mathfrak{a}t$ follows from the observation that $\mathfrak{A}t^{\text{Pr}}$ is defined by an affine-linear condition whose linear part is defines $\mathfrak{a}t^{\text{Pr}}$. Next, let $\phi \in \mathfrak{G}^{\text{Pr}}, \nabla \in \mathfrak{A}t^{\text{Pr}}$. Then, for all $p, p' \in P$ and $b \in B$,

$$\begin{aligned} & (\phi \triangleright \nabla)(p) \wedge d_B(b) \cdot p' - p \cdot d_B(b) \wedge (\phi \triangleright \nabla)(p') \\ &= (\phi_* \circ \nabla \circ \phi^{-1})(p) \wedge d_B(b) \cdot p' - p \cdot d_B(b) \wedge (\phi_* \circ \nabla \circ \phi^{-1})(p') \\ &= \phi_* (\nabla(\phi^{-1}(p)) \wedge \phi_*^{-1}(d_B(b) \cdot p') - \phi_*^{-1}(p \cdot d_B(b)) \wedge \nabla(\phi^{-1}(p'))) \\ &= \phi_* \circ \nabla^{\text{Pr}} \circ \phi_*^{-1}(p \cdot d_B(b) \cdot p'), \end{aligned}$$

so that $\phi \triangleright \nabla \in \mathfrak{A}t^{\text{Pr}}$ with $(\phi \triangleright \nabla)^{\text{Pr}} = \phi_* \circ \nabla^{\text{Pr}} \circ \phi_*^{-1}$. Thus, $\mathfrak{A}t^{\text{Pr}}$ is \mathfrak{G}^{Pr} -invariant and the map $(\nabla \mapsto \nabla^{\text{Pr}}) : \mathfrak{A}t^{\text{Pr}} \rightarrow \text{Hom}_{\mathbb{C}}(\Omega_{P,\text{hor}}^1, \Omega_{P,\text{hor}}^2)$ is \mathfrak{G}^{Pr} -equivariant. A similiary calculation shows that $\mathfrak{a}t^{\text{Pr}}$ is \mathfrak{G}^{Pr} -invariant and that $(\mathbf{A} \mapsto \mathbf{A}^{\text{Pr}}) : \mathfrak{a}t^{\text{Pr}} \rightarrow \text{Hom}_{\mathbb{C}}(\Omega_{P,\text{hor}}^1, \Omega_{P,\text{hor}}^2)$ is \mathfrak{G}^{Pr} -equivariant. Finally, our claims about $\mathbf{F} : \mathfrak{A}t^{\text{Pr}} \rightarrow \text{Der}_P^H(P, \Omega_{P,\text{hor}}^2)$ follow by straightforward calculation. \square

While inner gauge transformations are automatically prolongable and yield inner automorphisms of the left H -comodule graded $*$ -algebra $\Omega_{P,\text{hor}}$, the analogous statement is no longer generally true for inner relative gauge potentials. Instead, we shall consider those prolongable inner relative gauge potentials that yield inner derivations of $\Omega_{P,\text{hor}}$.

Definition 3.28. Let $\mathbf{A} \in \mathfrak{a}t^{\text{Pr}}$ be a prolongable gauge potential on P with respect to the second-order horizontal calculus $(\Omega_B, d_B; \Omega_{P,\text{hor}})$. We say that \mathbf{A} is *inner* if and only if there exists a 1-form $\alpha \in (\Omega_B^1)_{\text{sa}}$, such that

$$\mathbf{A} = \text{ad}_{\alpha} := (p \mapsto [\alpha, p]), \quad \mathbf{A}^{\text{Pr}} = \text{ad}_{\alpha} := (\omega \mapsto [\alpha, \omega]).$$

We denote by $\text{Inn}(\mathfrak{a}t^{\text{Pr}})$ the subspace of all inner prolongable relative gauge potentials on P with respect to $(\Omega_B, d_B; \Omega_{P,\text{hor}})$. Thus, we define the *outer prolongable Atiyah space* of P with respect to $(\Omega_B, d_B; \Omega_{P,\text{hor}})$ to be the quotient affine space $\text{Out}(\mathfrak{A}t^{\text{Pr}}) := \mathfrak{A}t^{\text{Pr}} / \text{Inn}(\mathfrak{a}t^{\text{Pr}})$ with space of translations $\text{Out}(\mathfrak{a}t^{\text{Pr}}) := \mathfrak{a}t^{\text{Pr}} / \text{Inn}(\mathfrak{a}t^{\text{Pr}})$.

We can now characterize those elements of Ω_B^1 that yield inner prolongable relative gauge potentials on P with respect to $(\Omega_B, d_B; \Omega_{P,\text{hor}})$; as an upshot, we find that the field strength of a prolongable gauge potential varies *linearly*—not quadratically—under translation by inner prolongable relative gauge potentials.

Proposition 3.29. *Let $\alpha \in (\Omega_B^1)_{\text{sa}}$. Then the inner derivation ad_{α} defines an inner prolongable relative gauge potential on P with respect to $(\Omega_B, d_B; \Omega_{P,\text{hor}})$ if and only if the 1-form α is central in Ω_B , in which case*

$$(3.25) \quad \forall \nabla \in \mathfrak{A}t^{\text{Pr}}, \forall p \in P, \quad i(\mathbf{F}[\nabla + \text{ad}_{\alpha}] - \mathbf{F}[\nabla])(p) = \text{ad}_{d_B(\alpha)}(p) := [d_B(\alpha), p].$$

Thus, the map $(\alpha \mapsto \text{ad}_{\alpha}) : (\Omega_B^1)_{\text{sa}} \cap Z(\Omega_B) \rightarrow \text{Inn}(\mathfrak{a}t^{\text{Pr}})$ yields a short exact sequence

$$0 \rightarrow (\Omega_B^1)_{\text{sa}} \cap Z(\Omega_{P,\text{hor}}) \rightarrow (\Omega_B^1)_{\text{sa}} \cap Z(\Omega_B) \rightarrow \text{Inn}(\mathfrak{a}t^{\text{Pr}}) \rightarrow 0.$$

Proof. The first part of the claim follows from observing that for all $p, p' \in P$ and $\beta \in \Omega_B^1$,

$$[\alpha, p] \wedge \beta \cdot p' - p \cdot \beta \wedge [\alpha, p'] = [\alpha, p \cdot \beta \cdot p'] - p \cdot [\alpha, \beta] \cdot p'.$$

Now, if $[\alpha, \Omega_B^1] = \{0\}$, then, by the above calculation, ad_α is prolongable with canonical prolongation $(\text{ad}_\alpha)^{\text{Pr}} = [\alpha, \cdot]$; moreover, if $\nabla \in \mathfrak{A}t^{\text{Pr}}$, then for any $p \in P$,

$$\begin{aligned} i(\mathbf{F}[\nabla + \mathbf{A}] - \mathbf{F}[\nabla])(p) &= (\nabla^{\text{Pr}} \circ \text{ad}_\alpha + (\text{ad}_\alpha)^{\text{Pr}} \circ \nabla)(p) + (\text{ad}_\alpha)^{\text{Pr}} \circ \text{ad}_\alpha(p) \\ &= \nabla^{\text{Pr}}([\alpha, p]) + [\alpha, \nabla(p)] + [\alpha, [\alpha, p]] \\ &= [d_B(\alpha) + \alpha \wedge \alpha, p] \\ &= [d_B(\alpha), p]. \end{aligned} \quad \square$$

Just as in the first-order case, the affine action of the prolongable gauge group \mathfrak{G} on the prolongable Atiyah space $\mathfrak{A}t^{\text{Pr}}$ descends further to an affine action of the outer prolongable gauge group $\text{Out}(\mathfrak{G}^{\text{Pr}})$ on the outer prolongable Atiyah space $\text{Out}(\mathfrak{A}t^{\text{Pr}})$, which will yield a non-trivial invariant of the principal left H -comodule $*$ -algebra P endowed with the second-order horizontal calculus $(\Omega_B, d_B; \Omega_{P, \text{hor}})$.

Proposition 3.30. *The subspace $\text{Inn}(\mathfrak{at}^{\text{Pr}})$ of inner prolongable gauge potentials on P consists of \mathfrak{G}^{Pr} -invariant vectors, so that the affine action of \mathfrak{G}^{Pr} on $\mathfrak{A}t^{\text{Pr}}$ descends to an affine action of \mathfrak{G}^{Pr} on $\text{Out}(\mathfrak{A}t^{\text{Pr}}) := \mathfrak{A}t^{\text{Pr}} / \text{Inn}(\mathfrak{at}^{\text{Pr}})$. Moreover, the inner gauge group $\text{Inn}(\mathfrak{G})$ acts trivially on $\text{Out}(\mathfrak{at}^{\text{Pr}})$, so that the affine action of \mathfrak{G}^{Pr} on $\mathfrak{A}t^{\text{Pr}}$ descends further to an affine action of $\text{Out}(\mathfrak{G}^{\text{Pr}})$ on $\text{Out}(\mathfrak{A}t^{\text{Pr}})$.*

Proof. Since $\text{Inn}(\mathfrak{at})$ consists of \mathfrak{G} -invariant vectors, it follows *a fortiori* that $\text{Inn}(\mathfrak{at}^{\text{Pr}})$ consists of \mathfrak{G}^{Pr} -invariant vectors. Now, let $\phi \in \text{Inn}(\mathfrak{G})$, so that by Proposition 2.24, $\phi = \text{Ad}_v$ and $\phi_* = \text{Ad}_v$ for some $v \in \text{U}(Z(B) \cap C_B(\Omega_B^1))$; let $\nabla \in \mathfrak{A}t^{\text{Pr}}$. By the proof of Proposition 2.34, we find that $\phi \triangleright \nabla - \nabla = \text{ad}_\alpha$ for $\alpha := -v^* \cdot d_B(v)$, so that $\phi \triangleright \nabla - \nabla \in \text{Inn}(\mathfrak{at}^{\text{Pr}})$ if and only if $-v^* \cdot d_B(v) \in Z(\Omega_B)$, if and only if $[d_B(v), d_B(B)] = \{0\}$. However, for all $b \in B$,

$$[d_B(v), d_B(b)] = d_B(v) \wedge d_B(b) + d_B(b) \wedge d_B(v) = d_B(v \cdot d_B(b)) - d_B(d_B(b) \cdot v) = 0,$$

so that, indeed, $\phi \triangleright \nabla - \nabla \in \text{Inn}(\mathfrak{at}^{\text{Pr}})$. Hence, $\text{Inn}(\mathfrak{G})$ acts trivially on $\text{Out}(\mathfrak{A}t^{\text{Pr}})$. \square

Finally, by Proposition 3.30, we can characterize the variation of field strength under translation by inner prolongable gauge potentials as follows—note that field strength, *a priori*, is affine quadratic, not linear.

Definition 3.31. Let \mathbf{A} be an inner prolongable gauge potential on P with respect to the second-order horizontal calculus $(\Omega_B, d_B; \Omega_{P, \text{hor}})$. The *relative field strength* of \mathbf{A} is the unique H -covariant $*$ -derivation $\mathbf{F}_{\text{rel}}[\mathbf{A}] : P \rightarrow \Omega_{P, \text{hor}}^2$ vanishing on B , such that

$$\forall \nabla \in \mathfrak{A}t^{\text{Pr}}, \quad \mathbf{F}[\nabla + \mathbf{A}] - \mathbf{F}[\nabla] = \mathbf{F}_{\text{rel}}[\mathbf{A}].$$

Corollary 3.32. *Let $\alpha \in (\Omega_B^1)_{\text{sa}} \cap Z(\Omega_B)$, so that ad_α defines an inner prolongable gauge potential on P with respect to $(\Omega_B, d_B; \Omega_{P, \text{hor}})$. Then*

$$(3.26) \quad \mathbf{F}_{\text{rel}}[\text{ad}_\alpha] = \text{ad}_{-i d_B(\alpha)}.$$

Hence, the map $\mathbf{F}_{\text{rel}} := (\mathbf{A} \mapsto \mathbf{F}_{\text{rel}}[\mathbf{A}]) : \text{Inn}(\mathfrak{at}^{\text{Pr}}) \rightarrow \text{Der}^H(P, \Omega_{P, \text{hor}}^2)$ is \mathfrak{G}^{Pr} -equivariant, $\text{Inn}(\mathfrak{G})$ -invariant, and \mathbf{R} -linear.

Example 3.33. Let $\phi \in \text{Inn}(\mathfrak{G})$; we claim that

$$\forall \nabla \in \mathfrak{A}t^{\text{Pr}}, \quad \mathbf{F}_{\text{rel}}[\phi \triangleright \nabla - \nabla] = 0.$$

Indeed, by Proposition 2.24, choose $v \in U(C_B(B \oplus \Omega_B^1))$, such that $\phi = \text{Ad}_v$ and $\phi_* = \text{Ad}_v$, and set $\alpha := -v^* \cdot d_B(v)$; by Proposition 2.34, it follows that

$$\forall \nabla \in \mathfrak{A}^{\text{Pr}}, \quad \phi \triangleright \nabla - \nabla = \text{ad}_\alpha.$$

Hence, by Corollary 3.32, it suffices to show that $d_B(\alpha) = 0$. Since $v \in U(C_B(B \oplus \Omega_B^1))$,

$$d_B(\alpha) = d_B(v^* \cdot d_B(v)) = d_B(v^*) \wedge d_B(v) = -v^* \cdot d_B(v) \cdot v^* \wedge d_B(v) = 0.$$

3.3. Reconstruction of quantum principal bundles to second order. Recall that H is a Hopf $*$ -algebra; let P be a principal left H -comodule $*$ -algebra with $*$ -subalgebra of coinvariants $B := {}^{\text{co}H}P$. Given a second-order horizontal calculus $(\Omega_B, d_B; \Omega_{P, \text{hor}})$ on P and a bicovariant $*$ -differential calculus (Ω_H, d_H) on H whose FODC is locally freeing for P , we consider *all* possible strongly $(H; \Omega_H, d_H)$ -principal SODC on P inducing the second-order horizontal calculus $(\Omega_B, d_B; \Omega_{P, \text{hor}})$. This will permit us to justify our notions of prolongable gauge transformation, prolongable gauge potential, and field strength by refining the equivalence of groupoids of Proposition 2.40. Furthermore, when (Ω_H, d_H) is Woronowicz's canonical prolongation [41], this will yield a gauge-equivariant moduli space of strongly $(H; \Omega_H, d_H)$ -principal SODC on P inducing $(\Omega_B, d_B; \Omega_{P, \text{hor}})$. Again, for relevant definitions from the basic theory of groupoids, see Appendix A.

Let us now fix a second-order horizontal calculus $(\Omega_B, d_B; \Omega_{P, \text{hor}}, \iota)$ on the principal left H -comodule $*$ -algebra P . Given a bicovariant $*$ -differential calculus (Ω_H, d_H) on H whose FODC is locally freeing for P , we construct the groupoid analogous to the groupoid $\mathcal{G}[\Omega_H^1]$ of Definition 2.35 whose objects are strongly $(H; \Omega_H, d_H)$ -principal SODC on P and whose arrows give an abstract notion of gauge transformation adapted to general SODC. In what follows, let $(\Omega_H^{\leq 2}, d_H)$ denote the truncation of (Ω_H, d_H) to a bicovariant SODC on H .

Definition 3.34. Let (Ω_H, d_H) be a bicovariant $*$ -differential calculus on H whose FODC is locally freeing for P . We define the *prolongable abstract gauge groupoid* $\mathcal{G}[\Omega_H^{\leq 2}]$ with respect to (Ω_H, d_H) as follows:

- (1) an object is a strongly $(H; \Omega_H, d_H)$ -principal SODC (Ω_P, d_P) on P , such that the resulting strong quantum principal $H; \Omega_H, d_H)$ -principal bundle $(P; \Omega_P, d_P)$ admits prolongable bimodule connections, and

$$(\ker \text{ver}[d_P], d_B(b) \mapsto d_P(b))$$

defines a horizontal lift of Ω_B^1 admitting a (necessarily unique) left H -covariant isomorphism

$$C[\Omega_P] : P \oplus \ker \text{ver}[d_P] \oplus (\ker(\text{ver}^{2,2}[d_P]) \cap \ker(\text{ver}^{2,1}[d_P])) \rightarrow \Omega_{P, \text{hor}}$$

of graded $*$ -algebras satisfying $C[\Omega_P]|_P = \text{id}$ and $C[\Omega_P] \circ d_P|_B = \iota \circ d_B$.

- (2) given objects (Ω_1, d_1) and (Ω_2, d_2) , an arrow $f : (\Omega_1, d_1) \rightarrow (\Omega_2, d_2)$ consists of a left H -covariant $*$ -automorphism $f : P \rightarrow P$, such that $f|_B = \text{id}$ and

$$f_* : \Omega_1^1 \rightarrow \Omega_2^1, \quad p \cdot d_1(p') \cdot p'' \mapsto f(p) \cdot d_2(f(p')) \cdot f(p'')$$

is well-defined and extends multiplicatively to a bijection $f_* : \Omega_1 \rightarrow \Omega_2$;

- (3) composition of arrows is induced by composition of $*$ -automorphisms of P , and the identity of an object (Ω, d) is given by $\text{id}_{(\Omega, d)} := (\text{id}_P : (\Omega, d) \rightarrow (\Omega, d))$.

Moreover, we define the star-injective homomorphism $\mu[\Omega_H^{\leq 2}] : \mathcal{G}[\Omega_H^{\leq 2}] \rightarrow \text{Aut}(P)$ by

$$\forall (f : (\Omega_1, d_1) \rightarrow (\Omega_2, d_2)) \in \mathcal{G}[\Omega_H^{\leq 2}], \quad \mu[\Omega_H^{\leq 2}](f : (\Omega_1, d_1) \rightarrow (\Omega_2, d_2)) := f.$$

Remark 3.35. Thus, the canonical second-order horizontal calculus (Ω_P, d_P) of $\mathcal{G}[\Omega_H^{\leq 2}]$ induces the second-order horizontal calculus $(\Omega_B, d_B; \Omega_{P, \text{hor}})$ up to the canonical isomorphism $C[\Omega_P]$ of graded left H -comodule $*$ -algebras.

Just as in the first-order case, given a bicovariant $*$ -differential calculus (Ω_H, d_H) on H , we simultaneously consider prolongable bimodule connections on all strong quantum principal $(H; \Omega_H^1, d_H)$ -bundles induced from P by sodc in $\text{Ob}(\mathcal{G}[\Omega_H^{\leq 2}])$. Once more, it is straightforward to check that the prolongable abstract gauge groupoid admits a canonical action on this set of bimodule connections.

Proposition-Definition 3.36. Let (Ω_H, d_H) be a bicovariant $*$ -differential calculus on H whose FODC is locally freeing for P . Let $\mathcal{A}[\Omega_H^{\leq 2}]$ be the set of all triples $(\Omega_P, d_P; \Pi)$, where $(\Omega_P, d_P) \in \text{Ob}(\mathcal{G}[\Omega_H^{\leq 2}])$ and Π is a prolongable bimodule connection on the strong quantum principal $(H; \Omega_H, d_H)$ -bundle $(P; \Omega_P, d_P)$; hence, let $p[\Omega_H^{\leq 2}] : \mathcal{A}[\Omega_H^{\leq 2}] \rightarrow \text{Ob}(\mathcal{G}[\Omega_H^{\leq 2}])$ be the canonical surjection given by

$$\forall (\Omega_P, d_P; \Pi) \in \mathcal{A}[\Omega_H^{\leq 2}], \quad p[\Omega_H^{\leq 2}](\Omega_P, d_P; \Pi) := (\Omega_P, d_P).$$

Then the *abstract gauge action* is the action of $\mathcal{G}[\Omega_H^{\leq 2}]$ on $\mathcal{A}[\Omega_H^{\leq 2}]$ via $p[\Omega_H^{\leq 2}]$ defined by

$$(3.27) \quad \forall (f : (\Omega_1, d_1) \rightarrow (\Omega_2, d_2)) \in \mathcal{G}[\Omega_H^{\leq 2}], \forall (\Omega_1, d_1; \Pi) \in p[\Omega_H^1]^{-1}(\Omega_1, d_1), \\ (f : (\Omega_1, d_1) \rightarrow (\Omega_2, d_2)) \triangleright (\Omega_1, d_1; \Pi) := (\Omega_2, d_2; f_* \circ \Pi \circ f_*^{-1}).$$

Hence, the canonical covering $\pi[\Omega_H^{\leq 2}] : \mathcal{G}[\Omega_H^{\leq 2}] \ltimes \mathcal{A}[\Omega_H^{\leq 2}] \rightarrow \mathcal{G}[\Omega_H^{\leq 2}]$ is given by

$$(3.28) \quad \forall ((f : (\Omega_1, d_1) \rightarrow (\Omega_2, d_2)), (\Omega_1, d_1; \Pi)) \in \mathcal{G}[\Omega_H^{\leq 2}] \ltimes \mathcal{A}[\Omega_H^{\leq 2}], \\ \pi[\Omega_H^{\leq 2}]((f : (\Omega_1, d_1) \rightarrow (\Omega_2, d_2)), (\Omega_1, d_1; \Pi)) := (f : (\Omega_1, d_1) \rightarrow (\Omega_2, d_2)).$$

Once more, as a convenient abuse of notation, we will denote an arrow

$$((f : (\Omega_1, d_1) \rightarrow (\Omega_2, d_2)), (\Omega_1, d_1; \Pi))$$

of the action groupoid $\mathcal{G}[\Omega_H^{\leq 2}] \ltimes \mathcal{A}[\Omega_H^{\leq 2}]$ by

$$f : (\Omega_1, d_1; \Pi_1) \rightarrow (\Omega_2, d_2; \Pi_2),$$

where $\Pi_2 := f_* \circ \Pi \circ f_*^{-1}|_{\Omega_2^1}$, so that, in particular,

$$\pi[\Omega_H^{\leq 2}](f : (\Omega_1, d_1; \Pi_1) \rightarrow (\Omega_2, d_2; \Pi_2)) := (f : (\Omega_1, d_1) \rightarrow (\Omega_2, d_2)).$$

Remark 3.37. There are an obvious star-injective homomorphism $\mathcal{G}[\Omega_H^{\leq 2}] \rightarrow \mathcal{G}[\Omega_H^1]$ and an obvious surjection $\mathcal{A}[\Omega_H^{\leq 2}] \rightarrow \mathcal{A}[\Omega_H^1]$ defined by restricting sodc to FODC, which, in turn, yield the following commutative diagram of groupoid homomorphisms:

$$\begin{array}{ccc} \mathcal{G}[\Omega_H^{\leq 2}] \ltimes \mathcal{A}[\Omega_H^{\leq 2}] & \longrightarrow & \mathcal{G}[\Omega_H^1] \ltimes \mathcal{A}[\Omega_H^1] \\ \pi[\Omega_H^{\leq 2}] \downarrow & & \downarrow \pi[\Omega_H^1] \\ \mathcal{G}[\Omega_H^{\leq 2}] & \longrightarrow & \mathcal{G}[\Omega_H^1] \\ \mu[\Omega_H^{\leq 2}] \downarrow & & \downarrow \mu[\Omega_H^1] \\ \text{Aut}(P) & \longleftarrow & \mathfrak{G} \end{array}$$

here, \mathfrak{G} is the gauge group of the principal left H -comodule $*$ -algebra P with respect to the first-order horizontal calculus $(\Omega_B^1, d_B; \Omega_{P, \text{hor}}^1)$. Hence, it follows that $\text{ran } \mu[\Omega_H^{\leq 2}] \subseteq \mathfrak{G}$.

Now, let \mathfrak{G}^{PF} and $\mathfrak{A}t^{\text{PF}}$ respectively denote the prolongable gauge group and prolongable Atiyah space of the principal left H -comodule $*$ -algebra P with respect to the second-order horizontal calculus $(\Omega_B, d_B; \Omega_{P,\text{hor}})$. Our goal is to promote the isomorphism of Proposition 3.13 to an explicit equivalence of categories that suitably refines the equivalence of Proposition 2.40. This will first require a characterisation of those prolongable gauge potentials on P with respect to $(\Omega_B, d_B; \Omega_{P,\text{hor}})$ that correctly induce elements of $\mathcal{A}[\Omega_H^{\leq 2}]$. In what follows, let $(\Omega_{P,\text{ver}}, d_{P,\text{ver}})$ denote the vertical calculus on P induced by a bicovariant $*$ -differential calculus on H whose FODC is locally freeing for P .

Definition 3.38. Let (Ω_H^1, d_H) be a bicovariant FODC on H that is locally freeing for P . We say that a prolongable gauge potential $\nabla \in \mathfrak{A}t^{\text{PF}}$ on P with respect to $(\Omega_B, d_B; \Omega_{P,\text{hor}})$ is (Ω_H^1, d_H) -adapted whenever its field strength $F[\nabla]$ is given by

$$F[\nabla] = F[\nabla] \circ d_{P,\text{ver}}$$

for a (necessarily unique) left H -covariant morphism $F[\nabla] : \Omega_{P,\text{ver}}^1 \rightarrow \Omega_{P,\text{hor}}^2$ of P - $*$ -bimodules; in this case, we call $F[\nabla]$ the *curvature 2-form* of ∇ . We define the (Ω_H^1, d_H) -adapted prolongable Atiyah space of P with respect to $(\Omega_B, d_B; \Omega_{P,\text{hor}})$ to be the subset $\mathfrak{A}t^{\text{PF}}[\Omega_H^1]$ of all (Ω_H^1, d_H) -adapted prolongable gauge potentials on P .

Proposition 3.39 (cf. Đurđević [23, Prop. 26 and 27]). *Let (Ω_H, d_H) be a bicovariant $*$ -differential calculus on H whose FODC is locally freeing for P ; let Λ_H be the resulting graded left crossed H -module $*$ -algebra of right H -coinvariant forms, and set $\Omega_{P,\oplus} := \Lambda_H \widehat{\otimes}^{\leq 2} \Omega_{P,\text{hor}}$. Let ∇ be a gauge potential on P with respect to $(\Omega_B^1, d_B; \Omega_{P,\text{hor}}^1)$. The following are equivalent:*

- (1) *the left H -covariant $*$ -derivation $d_{P,\nabla} : P \rightarrow \Omega_{P,\oplus}^1$ given by*

$$\forall p \in P, \quad d_{P,\nabla}(p) := d_{P,\text{ver}}(p) + \nabla(p)$$

admits a (necessarily unique) extension to a left H -covariant degree 1 $$ -derivation*

$$d_{P,\nabla} : \Omega_{P,\oplus} \rightarrow \Omega_{P,\oplus},$$

such that $(\Omega_{P,\oplus}, d_{P,\nabla}, \Pi_{\oplus}) \in \text{Ob}(\mathcal{G}[\Omega_H^{\leq 2}] \ltimes \mathcal{A}[\Omega_H^{\leq 2}])$, where $\Pi_{\oplus} : \Omega_{P,\oplus}^1 \rightarrow \Omega_{P,\oplus}^1$ is the projection onto $\Omega_{P,\text{ver}}^1$ along $\Omega_{P,\text{hor}}^1$;

- (2) *the gauge potential ∇ is prolongable with respect to $(\Omega_B, d_B; \Omega_{P,\text{hor}})$ and (Ω_H^1, d_H) -adapted.*

If either (and hence both) of the above conditions are satisfied, the prolongable bimodule connection Π_{\oplus} on the strong quantum principal $(H; \Omega_H, d_H)$ -bundle $(P; \Omega_{P,\oplus}, d_{P,\oplus})$ satisfies

$$\nabla_{\Pi_{\oplus}} = \nabla, \quad F_{\Pi_{\oplus}} = F[\nabla].$$

Proof. First, suppose that $d_{P,\nabla}$ extends to $\Omega_{P,\oplus}$ and that $(\Omega_{P,\oplus}, d_{P,\nabla}; \Pi_{\oplus})$ defines an object of $\mathcal{G}[\Omega_H^1] \ltimes \mathcal{A}[\Omega_H^1]$, so that $\nabla_{\Pi} = \nabla$ and

$$\text{ver}[d_{P,\nabla}] = \text{Proj}_1 : \Omega_{P,\oplus}^1 = \Omega_{P,\text{ver}}^1 \oplus \Omega_{P,\text{hor}}^1 \rightarrow \Omega_{P,\text{ver}}^1.$$

For every $p, q \in P$ and $b \in B$,

$$\begin{aligned} \nabla_{\Pi_{\oplus}}(p \cdot d_B(b) \cdot q) &= \nabla_{\Pi_{\oplus}}(p) \wedge d_B(b) \cdot q + p \cdot \nabla_{\Pi_{\oplus}}(d_B(b)) \cdot q - p \cdot d_B(b) \wedge \nabla_{\text{Proj}_1}(q) \\ &= \nabla(p) \wedge d_B(b) \cdot q - p \cdot d_B(b) \wedge \nabla(q) \end{aligned}$$

so that ∇ is prolongable with $\nabla^{\text{PF}} = \nabla_{\Pi_{\oplus}}|_{\Omega_{P,\text{hor}}^1}$ and (Ω_H^1, d_H) -adapted with $F[\nabla] = F_{\Pi_{\oplus}}$.

Now, suppose that ∇ is prolongable and (Ω_H^1, d_H) -adapted. We know that $(P; \Omega_{P,\oplus}^1; \Pi_\oplus)$ defines an element of $\mathcal{A}[\Omega_H^1]$ with

$$\text{ver}[d_{P,\nabla}] = \text{Proj}_1 : \Omega_{P,\oplus}^1 = \Omega_{P,\text{ver}}^1 \oplus \Omega_{P,\text{hor}}^1 \rightarrow \Omega_{P,\text{ver}}^1.$$

By mild abuse of notation, let $d_{P,\nabla} : \Omega_{P,\oplus}^1 \rightarrow \Omega_{P,\oplus}^2$ be the left H -covariant degree 1 map defined by

$$\begin{aligned} \forall \omega \in \Lambda_H^1, \forall p \in P, \forall \alpha \in \Omega_{P,\text{hor}}^1, \\ d_{P,\nabla}(\omega \cdot p + \alpha) := d_{P,\text{ver}}(\omega \cdot p) - iF[\nabla](\omega \cdot p) - \omega \wedge \nabla(p) + \omega(\alpha_{[-1]}) \wedge \alpha_{[0]} + \nabla^{\text{Pr}}(\alpha). \end{aligned}$$

While a routine calculation using the properties of $d_{P,\text{ver}}$, ∇ and ω shows that $d_{P,\nabla}$ as defined on P and $\Omega_{P,\oplus}^1$, respectively, yields a $*$ -derivation $d_{P,\nabla} : \Omega_{P,\oplus} \rightarrow \Omega_{P,\oplus}$ of degree 1, but it is less obvious that $d_{P,\nabla}^2 = 0$. However, for all $p \in P$,

$$\begin{aligned} d_{P,\nabla}^2(p) &= d_{P,\nabla}(\omega(p_{[-1]}) \cdot p_{[0]} + \nabla(p)) \\ &= d_{P,\text{ver}}(\omega(p_{[-1]}) \cdot p_{[0]}) - iF[\nabla](\omega(p_{[-1]}) \cdot p) - \omega(p_{[-1]}) \wedge \nabla(p_{[0]}) \\ &\quad + \omega(\nabla(p)_{[-1]}) \wedge \nabla(p)_{[0]} + \nabla^{\text{Pr}} \circ \nabla(p) \\ &= d_{P,\text{ver}}(d_{P,\text{ver}}(p)) - iF[\nabla](d_{P,\text{ver}}(p)) + iF[\nabla](d_{P,\text{ver}}(p)) = 0, \end{aligned}$$

as required. Given the explicit form of $\text{ver}[d_{P,\nabla}]$ and $d_{P,\nabla}|_{\Omega_{P,\oplus}^1}$, one can now check that

$$\begin{aligned} \text{ver}^{2,2}[d_{P,\nabla}] &= \text{Proj}_1 : \Omega_{P,\oplus}^2 = \Omega_{P,\text{ver}}^2 \oplus \Lambda_H^1 \otimes \Omega_{P,\text{hor}}^1 \oplus \Omega_{P,\text{hor}}^2 \rightarrow \Omega_{P,\text{ver}}^2, \\ \text{ver}^{2,1}[d_{P,\nabla}] &= \text{Proj}_2 : \Omega_{P,\oplus}^2 = \Omega_{P,\text{ver}}^2 \oplus \Lambda_H^1 \otimes \Omega_{P,\text{hor}}^1 \oplus \Omega_{P,\text{hor}}^2 \rightarrow \Lambda_H^1 \otimes \Omega_{P,\text{hor}}^1, \end{aligned}$$

so that $(\Omega_{P,\oplus}, d_{P,\nabla})$ defines an object of $\mathcal{C}[\Omega_H^2]$; given the explicit form of Π_\oplus , it now follows that the connection Π is totally prolongable with

$$\Pi_\oplus \wedge \Pi_\oplus = \text{Proj}_1 \oplus 0 \oplus 0, \quad \Pi_\oplus \wedge \text{id} + \text{id} \wedge \Pi_\oplus = \text{Proj}_1 \oplus \text{Proj}_2 \oplus 0$$

with respect to the decomposition $\Omega_{P,\oplus}^2 = \Omega_{P,\text{ver}}^2 \oplus \Lambda_H^1 \otimes \Omega_{P,\text{hor}}^1 \oplus \Omega_{P,\text{hor}}^2$. \square

Given a bicovariant FODC (Ω_H^1, d_H) on H that is locally freeing for P , Propositions 3.27 and 2.27 now guarantee that the (Ω_H^1, d_H) -adapted prolongable Atiyah space $\mathfrak{A}t^{\text{Pr}}[\Omega_H^1]$ is a \mathfrak{G}^{Pr} -invariant subset of the prolongable Atiyah space $\mathfrak{A}t^{\text{Pr}}$ on which the assignment of curvature 2-forms defines a \mathfrak{G}^{Pr} -equivariant map.

Proposition 3.40. *Let (Ω_H^1, d_H) be a bicovariant FODC on H that is locally freeing for P . The (Ω_H^1, d_H) -adapted prolongable Atiyah space $\mathfrak{A}t^{\text{Pr}}[\Omega_H^1]$ is a \mathfrak{G}^{Pr} -invariant subset of the prolongable Atiyah space $\mathfrak{A}t^{\text{Pr}}$, and the assignment*

$$F := (\nabla \mapsto F[\nabla]) : \text{at}_{\text{can}}^{\text{Pr}}[\Omega_H^1] \rightarrow \text{Hom}_P^H(\Omega_{P,\text{ver}}^1, \Omega_{P,\text{hor}}^2)$$

defines a \mathfrak{G}^{Pr} -equivariant map.

Remark 3.41. The \mathfrak{G}^{Pr} -invariant subset $\mathfrak{A}t^{\text{Pr}}[\Omega_H^1]$ of the affine space $\mathfrak{A}t^{\text{Pr}}$ is defined by an affine-quadratic constraint, and as such can be viewed as a \mathfrak{G}^{Pr} -invariant affine quadric subset of $\mathfrak{A}t^{\text{Pr}}$. In general, one should not expect $\mathfrak{A}t^{\text{Pr}}[\Omega_H^1]$ to be an affine-linear subspace of $\mathfrak{A}t^{\text{Pr}}$.

Remark 3.42. It follows that the resulting action groupoid $\mathfrak{G}^{\text{Pr}} \ltimes \mathfrak{A}t^{\text{Pr}}[\Omega_H^1]$ defines a subgroupoid of the action groupoid $\mathfrak{G} \ltimes \mathfrak{A}t$, where \mathfrak{G} and $\mathfrak{A}t$ are, respectively, the gauge group and Atiyah space of P with respect to the first-order horizontal calculus $(\Omega_B^1, d_B; \Omega_{P,\text{hor}}^1)$.

Suppose that (Ω_H^1, d_H) is a bicovariant FODC on H that is locally freeing for P , and let (Ω_H, d_H) be any bicovariant prolongation of (Ω_H^1, d_H) . We now promote Proposition 3.13 to an explicit equivalence of categories—refining the equivalence of Proposition 2.40—that realises the concrete action groupoid $\mathfrak{G}^{\text{Pr}} \ltimes \mathfrak{A}^{\text{tPr}}[\Omega_H^1]$, which is independent of the choice of bicovariant prolongation (Ω_H, d_H) , as a deformation retraction of the action groupoid $\mathcal{G}[\Omega_H^{\leq 2}] \ltimes \mathcal{A}[\Omega_H^{\leq 2}]$ of the abstract gauge action on prolongable bimodule connections. This rigorously justifies identifying the action of \mathfrak{G}^{Pr} on $\mathfrak{A}^{\text{tPr}}[\Omega_H^1]$ as the affine action of global gauge transformations on principal connections for the quantum principal H -bundle P with respect to the bicovariant $*$ -differential calculus (Ω_H, d_H) and the second-order horizontal calculus $(\Omega_B, d_B; \Omega_{P, \text{hor}})$. This follows from the proof of Proposition 2.40, *mutatis mutandis*, together with Proposition 3.39 and the proof of Proposition 3.13.

Proposition 3.43. *Let (Ω_H, d_H) be a bicovariant $*$ -differential calculus on H whole FODC is locally freeing for P . The groupoid homomorphism*

$$\Sigma[\Omega_H^{\leq 2}] : \mathfrak{G}^{\text{Pr}} \ltimes \mathfrak{A}^{\text{tPr}}[\Omega_H^1] \rightarrow \mathcal{G}[\Omega_H^{\leq 2}] \ltimes \mathcal{A}[\Omega_H^{\leq 2}]$$

given by

$$(3.29) \quad \forall (\phi, \nabla) \in \mathfrak{G}^{\text{Pr}} \ltimes \mathfrak{A}^{\text{tPr}}[\Omega_H^1], \\ \Sigma[\Omega_H^{\leq 2}](\phi, \nabla) := (\phi : (\Omega_{P, \mathfrak{G}}, d_{P, \nabla}, \Pi_{\mathfrak{G}}) \rightarrow (\Omega_{P, \mathfrak{G}}, d_{P, \phi \circ \nabla}, \Pi_{\mathfrak{G}}))$$

is an equivalence of groupoids with left inverse and homotopy inverse $A[\Omega_H^{\leq 2}]$ given by

$$(3.30) \quad \forall (f : (\Omega_1, d_1; \Pi_1) \rightarrow (\Omega_2, d_2; \Pi_2)) \in \mathcal{G}[\Omega_H^{\leq 2}] \ltimes \mathcal{A}[\Omega_H^{\leq 2}], \\ A[\Omega_H^{\leq 2}](f : (\Omega_1, d_1; \Pi_1) \rightarrow (\Omega_2, d_2; \Pi_2)) := (f, C[\Omega_1] \circ \nabla_{\Pi_1}).$$

In particular, there exists a homotopy $\eta[\Omega_H^{\leq 2}] : \text{id}_{\mathcal{G}[\Omega_H^{\leq 2}] \ltimes \mathcal{A}[\Omega_H^{\leq 2}]} \Rightarrow \Sigma[\Omega_H^{\leq 2}] \circ A[\Omega_H^{\leq 2}]$, which is necessarily unique, such that

$$(3.31) \quad \forall (\Omega_P, d_P; \Pi) \in \text{Ob}(\mathcal{G}[\Omega_H^{\leq 2}] \ltimes \mathcal{A}[\Omega_H^{\leq 2}]), \quad \mu[\Omega_H^{\leq 2}] \circ \pi[\Omega_H^{\leq 2}](\eta[\Omega_H^{\leq 2}]_{(\Omega_P, d_P; \Pi)}) = \text{id}_P.$$

This justifies the identification of the prolongable gauge group \mathfrak{G}^{Pr} as the gauge group of the quantum principal H -bundle P with respect to the bicovariant $*$ -differential calculus (Ω_H, d_H) on H and the second-order horizontal calculus $(\Omega_B, d_B; \Omega_{P, \text{hor}})$.

Corollary 3.44. *Let (Ω_H, d_H) be a bicovariant $*$ -differential calculus on H whole FODC is locally freeing for P . The star-injective groupoid homomorphism $\mu[\Omega_H^{\leq 2}] : \mathcal{G}[\Omega_H^{\leq 2}] \rightarrow \text{Aut}(P)$ has range \mathfrak{G}^{Pr} , so that, after restriction of codomain,*

$$\mu[\Omega_H^{\leq 2}] : \mathcal{G}[\Omega_H^{\leq 2}] \rightarrow \mathfrak{G}^{\text{Pr}}, \quad \mu[\Omega_H^{\leq 2}] \circ \pi[\Omega_H^{\leq 2}] : \mathcal{G}[\Omega_H^{\leq 2}] \ltimes \mathcal{A}[\Omega_H^{\leq 2}] \rightarrow \mathfrak{G}^{\text{Pr}}$$

both define coverings of groupoids.

Remark 3.45. The groupoid equivalences $\Sigma[\Omega_H^1]$, $\Sigma[\Omega_H^{\leq 2}]$, $A[\Omega_H^1]$, and $A[\Omega_H^{\leq 2}]$, the subgroupoid inclusion $\mathfrak{G}^{\text{Pr}} \ltimes \mathfrak{A}^{\text{tPr}}[\Omega_H^1] \hookrightarrow \mathfrak{G} \ltimes \mathfrak{A}^{\text{t}}$, and the canonical star-injective groupoid homomorphisms $\mathcal{G}[\Omega_H^{\leq 2}] \ltimes \mathcal{A}[\Omega_H^{\leq 2}] \rightarrow \mathcal{G}[\Omega_H^1] \ltimes \mathcal{A}[\Omega_H^1]$ and $\mathcal{G}[\Omega_H^{\leq 2}] \rightarrow \mathcal{G}[\Omega_H^1]$ fit into the following commutative diagrams in the category of groupoids:

$$\begin{array}{ccc} \mathfrak{G}^{\text{Pr}} \ltimes \mathfrak{A}^{\text{tPr}}[\Omega_H^1] & \hookrightarrow & \mathfrak{G} \ltimes \mathfrak{A}^{\text{t}} & & \mathfrak{G}^{\text{Pr}} \ltimes \mathfrak{A}^{\text{tPr}}[\Omega_H^1] & \hookrightarrow & \mathfrak{G} \ltimes \mathfrak{A}^{\text{t}} \\ \Sigma[\Omega_H^{\leq 2}] \downarrow & & \downarrow \Sigma[\Omega_H^1] & & A[\Omega_H^{\leq 2}] \uparrow & & \uparrow A[\Omega_H^1] \\ \mathcal{G}[\Omega_H^{\leq 2}] \ltimes \mathcal{A}[\Omega_H^{\leq 2}] & \rightarrow & \mathcal{G}[\Omega_H^1] \ltimes \mathcal{A}[\Omega_H^1] & & \mathcal{G}[\Omega_H^{\leq 2}] \ltimes \mathcal{A}[\Omega_H^{\leq 2}] & \rightarrow & \mathcal{G}[\Omega_H^1] \ltimes \mathcal{A}[\Omega_H^1] \end{array}$$

Moreover, the homotopies

$$\eta[\Omega_H^1] : \text{id}_{\mathcal{G}[\Omega_H^1] \ltimes \mathcal{A}[\Omega_H^1]} \Rightarrow \Sigma[\Omega_H^1] \circ A[\Omega_H^1], \quad \eta[\Omega_H^{\leq 2}] : \text{id}_{\mathcal{G}[\Omega_H^{\leq 2}] \ltimes \mathcal{A}[\Omega_H^{\leq 2}]} \Rightarrow \Sigma[\Omega_H^{\leq 2}] \circ A[\Omega_H^{\leq 2}]$$

and the natural transformations

$$\text{id}_{\mathcal{G}[\Omega_H^{\leq 2}] \ltimes \mathcal{A}[\Omega_H^{\leq 2}]} \Rightarrow \text{id}_{\mathcal{G}[\Omega_H^1] \ltimes \mathcal{A}[\Omega_H^1]}, \quad \Sigma[\Omega_H^{\leq 2}] \circ A[\Omega_H^{\leq 2}] \Rightarrow \Sigma[\Omega_H^1] \circ A[\Omega_H^1]$$

induced by the subgroupoid inclusion $\mathfrak{G}^{\text{pr}} \ltimes \mathfrak{A}^{\text{pr}}[\Omega_H^1] \hookrightarrow \mathfrak{G} \ltimes \mathfrak{A}$ and the canonical star-injective groupoid homomorphisms $\mathcal{G}[\Omega_H^{\leq 2}] \ltimes \mathcal{A}[\Omega_H^{\leq 2}] \rightarrow \mathcal{G}[\Omega_H^1] \ltimes \mathcal{A}[\Omega_H^1]$ fit into the following commutative diagram in the category of functors:

$$\begin{array}{ccc} \text{id}_{\mathcal{G}[\Omega_H^{\leq 2}] \ltimes \mathcal{A}[\Omega_H^{\leq 2}]} & \Longrightarrow & \text{id}_{\mathcal{G}[\Omega_H^1] \ltimes \mathcal{A}[\Omega_H^1]} \\ \eta[\Omega_H^{\leq 2}] \Downarrow & & \Downarrow \eta[\Omega_H^1] \\ \Sigma[\Omega_H^{\leq 2}] \circ A[\Omega_H^{\leq 2}] & \Longrightarrow & \Sigma[\Omega_H^1] \circ A[\Omega_H^1] \end{array}$$

Given a bicovariant $*$ -differential calculus (Ω_H, d_H) on H whose FODC is locally free-ing for P , we now use the weak equivalence $\Sigma[\Omega_H^{\leq 2}]$ of Proposition 3.43 to construct a \mathfrak{G}^{pr} -equivariant moduli space of strongly $(H; \Omega_H, d_H)$ -principal sODC on P inducing the second-order horizontal calculus $(\Omega_B, d_B; \Omega_{P, \text{hor}})$ on P . To do so, we shall need (Ω_H, d_H) to be Woronowicz's *canonical* prolongation [41, §§3–4] of its FODC (Ω_H^1, d_H) , so that

$$(3.32) \quad \Lambda_H^2 := (\Omega_H^2)^{\text{co}H} = \frac{\Lambda_H^1 \otimes_{\mathbb{C}} \Lambda_H^1}{\text{Span}\{\mu \otimes \nu + \mu_{[-1]} \triangleright \nu \otimes \mu_{[0]} \mid \mu, \nu \in \Lambda_H^1\}}$$

by [26, §2]. For us, the distinguishing feature of this particular prolongation (as opposed to, e.g., the maximal prolongation) will be the H -equivariance of d_H when restricted to $\Lambda := \Omega^{\text{co}H}$, which one can view as Ad-equivariance of the dualised Lie bracket.

Proposition 3.46. *Let (Ω_H^1, d_H) be a bicovariant FODC on H , let (Ω, d) be a bicovariant prolongation of (Ω_H^1, d_H) , and let $\Lambda := \Omega^{\text{co}H}$ denote the corresponding graded left crossed H -module $*$ -algebra of right coinvariant forms, where $\Lambda^0 = \mathbb{C}$. The following are equivalent:*

- (1) *the restriction of d to $\Lambda^1 = \Lambda_H^1$ satisfies*

$$\forall h \in H, \forall \mu \in \Lambda_H^1, \quad d_H(h \triangleright \mu) = h \triangleright d_H(\mu);$$

- (2) *the product $\Lambda^1 \otimes_{\mathbb{C}} \Lambda^1 \rightarrow \Lambda^2$ satisfies the braided commutation relation*

$$\forall \mu, \nu \in \Lambda_H^1, \quad \mu \wedge \nu + \mu_{[-1]} \triangleright \nu \wedge \mu_{[0]} = 0.$$

Proof. On the one hand, suppose that Condition 1 holds. Then for all $h, k \in H$, by (3.3),

$$\begin{aligned} \omega(h)_{[-1]} \triangleright \omega(k) \wedge \omega(h)_{[0]} &= h_{(1)} S(h_{(3)}) \triangleright \omega(k) \wedge \omega(h_{(2)}) \\ &= -h_{(1)} S(h_{(4)}) \triangleright \omega(k) \wedge h_{(2)} \triangleright \omega(S(h_{(3)})) \\ &= -h_{(1)} \triangleright (S(h_{(3)}) \triangleright \omega(k) \wedge \omega(S(h_{(2)}))) \\ &= h_{(1)} \triangleright (\omega(S(h_{(3)})) \wedge S(h_{(2)}) \triangleright \omega(k)) \\ &= h_{(1)} \triangleright \omega(S(h_{(4)})) \wedge h_{(2)} S(h_{(3)}) \triangleright \omega(k) \\ &= -\omega(h) \wedge \omega(k), \end{aligned}$$

so that Condition 2 is satisfied. On the other hand, suppose that Condition 2 is satisfied. Then for all $h, k \in H$, by (3.3),

$$d_H(h \triangleright \omega(k)) - h \triangleright d_H \omega(k) = h_{(1)} \triangleright \omega(k) \wedge \omega(h_{(2)}) + \omega(h_{(1)}) \wedge h_{(2)} \triangleright \omega(k)$$

$$\begin{aligned}
 &= h_{(1)} \triangleright \omega(k) \wedge \omega(h_{(2)}) - h_{(1)} S(h_{(3)}) h_{(4)} \triangleright \omega(k) \wedge \omega(h_{(2)}) \\
 &= 0
 \end{aligned}$$

so that Condition 1 is satisfied. \square

In the case that (Ω_H, d_H) is the canonical prolongation of (Ω_H^1, d_H) , we shall construct the aforementioned \mathfrak{G}^{pr} -equivariant moduli space using the following distinguished set of relative gauge potentials, which will indeed turn out to be a \mathfrak{G}^{pr} -invariant subspace of prolongable relative gauge potentials.

Definition 3.47. Let (Ω_H^1, d_H) be a bicovariant FODC on H that is locally freeing for P . Let \mathbf{A} be a prolongable relative gauge potential on P with respect to the second-order horizontal calculus $(\Omega_B, d_B; \Omega_{P,\text{hor}})$. We say that \mathbf{A} is *canonically (Ω_H^1, d_H) -adapted* if and only if it is (Ω_H^1, d_H) -adapted and satisfies both of the following:

$$(3.33) \quad \forall \mu \in \Lambda_H^1, \forall \alpha \in \Omega_B^1, \quad [\omega[\mathbf{A}](\mu), \alpha] = 0,$$

$$(3.34) \quad \forall \mu, \nu \in \Lambda_H^1, \quad \omega[\mathbf{A}](\mu) \wedge \omega[\mathbf{A}](\nu) + \omega[\mathbf{A}](\mu_{[-1]} \triangleright \nu) \wedge \omega[\mathbf{A}](\mu_{[0]}) = 0.$$

We denote by $\text{at}_{\text{can}}^{\text{pr}}[\Omega_H^1]$ the subset of all canonically (Ω_H^1, d_H) -adapted relative gauge potentials on P with respect to $(\Omega_B, d_B; \Omega_{P,\text{hor}})$.

Now, let $(\Omega_1, d_1), (\Omega_2, d_2) \in \text{Ob}(\mathcal{G}[\Omega_H^{\leq 2}])$, where (Ω_H, d_H) is the canonical prolongation of a bicovariant FODC (Ω_H^1, d_H) on H that is locally freeing for P . Observe that the left H -covariant sodc (Ω_1, d_1) and (Ω_2, d_2) on the principal left H -comodule $*$ -algebra P are isomorphic if and only if $\text{id}_P : (\Omega_1, d_1) \rightarrow (\Omega_2, d_2)$ is an arrow in $\mathcal{G}[\Omega_H^{\leq 2}]$. Since the subgroupoid of such arrows is precisely $\ker \mu[\Omega_H^{\leq 2}]$, it follows that (Ω_1, d_1) and (Ω_2, d_2) are isomorphic if and only if they define the same object in the quotient $\mathcal{G}[\Omega_H^{\leq 2}] / \ker \mu[\Omega_H^{\leq 2}]$, which will turn out to be well-defined and canonically isomorphic to the action groupoid $\mathfrak{G}^{\text{pr}} \ltimes (\mathfrak{A}^{\text{pr}}[\Omega_H^1] / \text{at}_{\text{can}}^{\text{pr}}[\Omega_H^1])$. Thus, the quadric subset $\mathfrak{A}^{\text{pr}}[\Omega_H^1] / \text{at}_{\text{can}}^{\text{pr}}[\Omega_H^1]$ of the quotient affine space $\mathfrak{A}^{\text{pr}} / \text{at}_{\text{can}}^{\text{pr}}[\Omega_H^1]$ yields the desired \mathfrak{G}^{pr} -equivariant affine moduli space of strongly (Ω_H, d_H) -principal sodc on P inducing $(\Omega_B, d_B; \Omega_{P,\text{hor}})$.

Theorem 3.48. Let (Ω_H^1, d_H) be a bicovariant FODC on H that is locally freeing for P , and let (Ω_H, d_H) be its canonical prolongation. Suppose that $\mathfrak{A}^{\text{pr}}[\Omega_H^1]$ is non-empty. Let at^{pr} be the space of prolongable gauge potentials on P with respect to $(\Omega_B, d_B; \Omega_{P,\text{hor}})$, and let $\text{at}[\Omega_H^1]$ be subspace of (Ω_H^1, d_H) -adapted elements. The set $\text{at}_{\text{can}}^{\text{pr}}[\Omega_H^1]$ defines a \mathfrak{G}^{pr} -invariant subspace of $\text{at}[\Omega_H^1] \cap \text{at}^{\text{pr}}$, such that $\mathfrak{A}^{\text{pr}}[\Omega_H^1]$ is invariant under translation by $\text{at}_{\text{can}}^{\text{pr}}[\Omega_H^1]$; the subgroupoid $\ker \mu[\Omega_H^{\leq 2}]$ of $\mathcal{G}[\Omega_H^{\leq 2}]$ is wide and has trivial isotropy groups, so that the quotient groupoid $\mathcal{G}[\Omega_H^{\leq 2}] / \ker \mu[\Omega_H^{\leq 2}]$ is well-defined; and there exists a unique isomorphism

$$\tilde{\Sigma}^{\text{pr}}[\Omega_H^{\leq 2}] : \mathfrak{G}^{\text{pr}} \ltimes (\mathfrak{A}^{\text{pr}}[\Omega_H^1] / \text{at}_{\text{can}}^{\text{pr}}[\Omega_H^1]) \rightarrow \mathcal{G}[\Omega_H^{\leq 2}] / \ker \mu[\Omega_H^{\leq 2}],$$

such that

$$(3.35) \quad \forall (\phi, \nabla) \in \mathfrak{G}^{\text{pr}} \ltimes \mathfrak{A}^{\text{pr}}[\Omega_H^1],$$

$$\tilde{\Sigma}^{\text{pr}}[\Omega_H^{\leq 2}](\phi, \nabla + \text{at}_{\text{can}}^{\text{pr}}[\Omega_H^1]) = [\pi[\Omega_H^{\leq 2}] \circ \Sigma[\Omega_H^{\leq 2}](\phi, \nabla)]_{\ker \mu[\Omega_H^{\leq 2}]}$$

As a preliminary to the proof of this theorem, we shall prove the following sequence of lemmata. In what follows, we shall assume the hypotheses of Theorem 3.48; in particular, $(\Omega_{P,\text{ver}}, d_{P,\text{ver}})$ will denote the second-order vertical calculus induced by (Ω_H, d_H) .

Lemma 3.49. *Let $N : \Omega_{P,\text{ver}}^1 \rightarrow \Omega_{P,\text{hor}}^1$ be a left H -covariant morphism of P - $*$ -bimodules satisfying*

$$\forall \mu \in \Lambda_H^1, \forall \alpha \in \Omega_B^1, \quad [N(\mu), \alpha] = 0.$$

Then $N \circ d_{P,\text{ver}} \in \mathfrak{at}^{\text{Pr}}$. Moreover, for all $\nabla \in \mathfrak{At}^{\text{Pr}}$, the map $\nabla N : \Omega_{P,\text{ver}}^1 \rightarrow \Omega_{P,\text{hor}}^2$ given by

$$\forall \mu \in \Lambda_H^1, \forall p \in P, \quad (\nabla N)(\mu \cdot p) := \nabla(N(\mu)) \cdot p$$

defines a left H -covariant morphism of P -bimodules, such that $-i\nabla N$ is $$ -preserving.*

Proof. Let us first show that $\mathbf{A} := N \circ d_{P,\text{ver}} \in \mathfrak{at}[\Omega_H^1]$ is prolongable. Indeed, for all $p, q \in P$ and $b \in B$, we have

$$\begin{aligned} \mathbf{A}(p) \cdot d_B(b) \cdot q - p \cdot d_B \cdot \mathbf{A}(q) &= N(\omega(p_{[-1]})) \wedge p_{[0]} d_B(b) \cdot q - p \cdot d_B(b) \cdot \wedge \omega(q_{[-1]}) \cdot q_{[0]} \\ &= N(\omega(p_{[-1]})\epsilon(q_{[-1]}) + p_{[-1]} \triangleright \omega(q_{[-1]})) \wedge p_{[0]} d_B(b) q_{[0]} \\ &= N(\omega(p_{[-1]}q_{[-1]})) \wedge p_{[0]} \cdot d_B(b) \cdot q_{[0]} \\ &= N(\omega((p \cdot d_B(b) \cdot q)_{[-1]})) \wedge (p \cdot d_B(b) \cdot q)_{[0]}, \end{aligned}$$

so that \mathbf{A} is prolongable with \mathbf{A}^{Pr} given by

$$\forall \alpha \in \Omega_{P,\text{hor}}^1, \quad \mathbf{A}^{\text{Pr}}(\alpha) = N(\omega(\alpha_{[-1]})) \wedge \alpha_{[0]}.$$

Now, given $\nabla \in \mathfrak{At}^{\text{Pr}}$, let us show that the left H -covariant right P -module map ∇N is left P -linear and $*$ -preserving. First, since $[N(\Lambda_H^1), \Omega_B^1] = \{0\}$, it follows that for all $\mu \in \Lambda_H^1$, $p \in P$, and $\beta \in \Omega_B^1$,

$$\begin{aligned} p \cdot d_B(b) \wedge N(\mu) &= -p \cdot N(\mu) \wedge d_B(b) = -N(p_{[-1]} \triangleright \mu) \cdot p_{[0]} \wedge d_B(b) \\ &= -N((p \cdot d_B(b))_{[-1]} \triangleright \mu) \wedge (p \cdot d_B(b))_{[0]}, \end{aligned}$$

so that

$$\forall \mu \in \Lambda_H^1, \forall \alpha \in \Omega_{P,\text{hor}}^1, \quad \alpha \wedge N(\mu) + N(\alpha_{[-1]} \triangleright \mu) \wedge \alpha_{[0]} = 0;$$

Hence, for all $p \in P$ and $\mu \in \Lambda_H^1$,

$$\begin{aligned} p \cdot (\nabla N)(\mu) &= \nabla^{\text{Pr}}(p \cdot N(\mu)) - \nabla(p) \wedge N(\mu) \\ &= \nabla^{\text{Pr}}(N(p_{[-1]} \triangleright \mu)) \cdot p_{[0]} - N(p_{[-1]} \triangleright \mu) \wedge \nabla(p_{[0]}) - \nabla(p) \wedge N(\mu) \\ &= (\nabla N)(p \cdot \mu) - N(p_{[-1]} \triangleright \mu) \wedge \nabla(p_{[0]}) + N(\nabla(p)_{[-1]} \triangleright \mu) \wedge \nabla(p)_{[0]} \\ &= (\nabla N)(p \cdot \mu), \end{aligned}$$

which shows that ∇N is left P -linear; a similar calculation then shows that the map $-i\nabla N$ is also $*$ -preserving. \square

Lemma 3.50. *Recall that (Ω_H, d_H) is the canonical prolongation of (Ω_H^1, d_H) . Suppose that $N : \Omega_{P,\text{ver}}^1 \rightarrow \Omega_{P,\text{hor}}^1$ be a left H -covariant morphism of P - $*$ -bimodules satisfying*

$$(3.36) \quad \forall \mu, \nu \in \Lambda_H^1, \quad N(\mu) \wedge N(\nu) + N(\mu_{[-1]} \triangleright \nu) \wedge N(\mu_{[0]}) = 0,$$

Then, the map $[N, N] : \Omega_{P,\text{ver}}^1 \rightarrow \Omega_{P,\text{hor}}^2$ defined by

$$\forall h \in H, \forall p \in P, \quad [N, N](\omega(h) \cdot p) := N(\omega(h_{(1)})) \wedge N(\omega(h_{(2)})) \cdot p$$

is a left H -covariant morphism of P -bimodules, such that $-i[N, N]$ is $$ -preserving.*

Proof. First, by (3.36) together with (3.32), the map

$$(\mu \otimes \nu \mapsto N(\mu) \wedge N(\nu)) : \Lambda_H^1 \otimes_{\mathbb{C}} \Lambda_H^1 \rightarrow \Omega_{P,\text{hor}}^2$$

descends to a map $\tilde{N} : \Lambda_H^2 \rightarrow \Omega_{P,\text{hor}}^2$, so that $[N, N] : \Omega_{P,\text{ver}}^1 \rightarrow \Omega_{P,\text{hor}}^2$ is well-defined as left H -covariant right P -linear map and given by

$$\forall \mu \in \Lambda_H^1, \forall p \in P, \quad [N, N](\mu \cdot p) := \tilde{N}(d_H \mu) \cdot p.$$

Now, for all $p \in P$ and $h \in H$,

$$\begin{aligned} p \cdot [N, N](\omega(h)) &= p \cdot N(\omega(h_{(1)})) \wedge N(\omega(h_{(2)})) \\ &= N(p_{[-2]} \triangleright \omega(h_{(1)})) \wedge N(p_{[-1]} \triangleright \omega(h_{(2)})) \cdot p_{[0]} \\ &= \tilde{N}(p_{[-2]} \triangleright \omega(h_{(1)})) \wedge p_{[-1]} \triangleright \omega(h_{(2)}) \cdot p_{[0]} \\ &= \tilde{N}(p_{[-1]} \triangleright d_H \omega(h)) \cdot p_{[0]} \\ &= \tilde{N}(d_H(p_{[-1]} \triangleright \omega(h))) \cdot p_{[0]} \\ &= [N, N](p \cdot \omega(h)) \end{aligned}$$

by Proposition 3.32, so that $[N, N]$ is left P -linear. A qualitatively identical calculation now shows that $-i[N, N]$ is $*$ -preserving. \square

Lemma 3.51. *Recall that (Ω_H, d_H) is the canonical prolongation of (Ω_H^1, d_H) and that the subset $\mathfrak{A}^{\text{pr}}[\Omega_H^1]$ of \mathfrak{A}^{pr} is assumed to be non-empty. Let \mathbf{A} be a gauge potential on P with respect to the first-order horizontal calculus $(\Omega_B^1, d_B; \Omega_{P,\text{hor}}^1)$. The following are equivalent:*

- (1) *the relative gauge potential \mathbf{A} is prolongable and canonically (Ω_H^1, d_H) -adapted;*
- (2) *for every $\nabla \in \mathfrak{A}^{\text{pr}}[\Omega_H^1]$, the gauge potential $\nabla + \mathbf{A}$ is prolongable and (Ω_H^1, d_H) -adapted, and id_P induces an arrow*

$$(\text{id}_P : \pi[\Omega_H^{\leq 2}] \circ \Sigma[\Omega_H^{\leq 2}](\nabla) \rightarrow \pi[\Omega_H^{\leq 2}] \circ \Sigma[\Omega_H^{\leq 2}](\nabla + \mathbf{A})) \in \mathcal{G}[\Omega_H^{\leq 2}];$$

- (3) *there exists $\nabla \in \mathfrak{A}^{\text{pr}}[\Omega_H^1]$, such that $\nabla + \mathbf{A} \in \mathfrak{A}^{\text{pr}}[\Omega_H^1]$ and id_P induces an arrow*

$$(\text{id}_P : \pi[\Omega_H^{\leq 2}] \circ \Sigma[\Omega_H^{\leq 2}](\nabla) \rightarrow \pi[\Omega_H^{\leq 2}] \circ \Sigma[\Omega_H^{\leq 2}](\nabla + \mathbf{A})) \in \mathcal{G}[\Omega_H^{\leq 2}].$$

Proof. First, suppose that condition 1 holds; let $N := \omega[\mathbf{A}]$, so that $\mathbf{A} = N \circ d_{P,\text{ver}}$. Let $\nabla \in \mathfrak{A}^{\text{pr}}[\Omega_H^1]$ be given. First, by Lemma 3.49, $\mathbf{A} \in \mathfrak{at}^{\text{pr}}$, so that $\nabla + \mathbf{A} \in \mathfrak{A}^{\text{pr}}$. Next, by Lemma 3.49, \mathbf{A}^{pr} is given by

$$\forall \alpha \in \Omega_{P,\text{hor}}^1, \quad \mathbf{A}^{\text{pr}}(\alpha) = N(\omega(\alpha_{[-1]})) \wedge \alpha_{[0]},$$

while by Lemmata 3.49 and 3.50, respectively, $-i\nabla N$ and $-i[N, N]$ are well-defined left H -covariant morphisms of P - $*$ -algebras; hence, for all $p \in P$,

$$\begin{aligned} i(\mathbf{F}[\nabla + \mathbf{A}] - \mathbf{F}[\nabla])(p) &= (\nabla^{\text{pr}} \circ \mathbf{A} + \mathbf{A}^{\text{pr}} \circ \nabla + \mathbf{A}^{\text{pr}} \circ \mathbf{A})(p) \\ &= \nabla^{\text{pr}}(N(\omega(p_{[-1]})) \cdot p_{[0]}) + \mathbf{A}^{\text{pr}}(\nabla(p)) + \mathbf{A}^{\text{pr}}(\mathbf{A}(p)) \\ &= \nabla^{\text{pr}} \circ N(\omega(p_{[-1]}) \cdot p_{[0]}) - N(\omega(p_{[-1]})) \wedge \nabla(p_{[0]}) \\ &\quad + N(\omega(\nabla(p)_{[-1]})) \wedge \nabla(p)_{[0]} + N(\omega(\mathbf{A}(p)_{[-1]})) \wedge \mathbf{A}(p)_{[0]} \\ &= \nabla N(\omega(p_{[0]})) \cdot p_{[0]} + N(\omega(p_{[-2]}) \wedge \omega(p_{[-1]})) \cdot p_{[0]} \\ &= (\nabla N + [N, N]) \circ d_{P,\text{ver}}(p), \end{aligned}$$

so that $\nabla + \mathbf{A} \in \mathfrak{A}^{\text{pr}}[\Omega_H^1]$ with curvature 2-form

$$F[\nabla + \mathbf{A}] = F[\nabla] - i(\nabla N + [N, N]).$$

Let us now show that id_P induces a morphism

$$(\text{id}_P : \pi[\Omega_H^{\leq 2}] \circ \Sigma[\Omega_H^{\leq 2}](\nabla) \rightarrow \pi[\Omega_H^{\leq 2}] \circ \Sigma[\Omega_H^{\leq 2}](\nabla + \mathbf{A})) \in \mathcal{G}[\Omega_H^{\leq 2}].$$

Since $\Omega_{P,\oplus} := \Lambda_H \widehat{\otimes}^{\leq 2} \Omega_{P,\text{hor}}$ is generated as a left H -covariant graded $*$ -algebra over P by Λ_H^1 and $\Omega_{P,\text{hor}}^1$ subject to the relations

$$\begin{aligned} \forall \mu \in \Lambda_H^1, \forall \alpha \in \Omega_{P,\text{hor}}^1, \quad \alpha \wedge \mu + \alpha_{[-1]} \triangleright \mu \wedge \alpha_{[0]} &= 0, \\ \forall \mu, \nu \in \Lambda_H^1, \quad \forall \mu \wedge \nu + \mu_{[-1]} \triangleright \nu \wedge \mu_{[0]}, \end{aligned}$$

and since, by the proof of Lemma 3.50, the map $N : \Omega_{P,\text{ver}}^1 \rightarrow \Omega_{P,\text{hor}}^1$ satisfies

$$\begin{aligned} \forall \mu \in \Lambda_H^1, \forall \alpha \in \Omega_{P,\text{hor}}^1, \quad \alpha \wedge N(\mu) + N(\alpha_{[-1]} \triangleright \mu) \wedge \alpha_{[0]} &= 0, \\ \forall \mu, \nu \in \Lambda_H^1, \quad N(\mu) \wedge N(\nu) + N(\mu_{[-1]} \triangleright \nu) \wedge N(\mu_{[0]}), \end{aligned}$$

the left H -covariant morphisms $\phi : \Omega_{P,\oplus}^1 \rightarrow \Omega_{P,\oplus}^1$ and $\psi : \Omega_{P,\oplus}^1 \rightarrow \Omega_{P,\oplus}^1$ given by

$$\begin{aligned} \forall \omega \in \Omega_{P,\text{ver}}^1, \forall \alpha \in \Omega_{P,\text{hor}}^1, \quad \phi(\omega + \alpha) &:= \omega + N(\omega) + \alpha, \\ \forall \omega \in \Omega_{P,\text{ver}}^1, \forall \alpha \in \Omega_{P,\text{hor}}^1, \quad \psi(\omega + \alpha) &:= \omega - N(\omega) + \alpha \end{aligned}$$

respectively, extend uniquely to left H -covariant graded $*$ -endomorphisms of $\Omega_{P,\oplus}$, such that $\phi|_P = \psi|_P = \text{id}_P$. Moreover, for all $\omega \in \Omega_{P,\text{ver}}^1$ and $\alpha \in \Omega_{P,\text{hor}}^1$,

$$\begin{aligned} \psi \circ \phi(\omega + \alpha) &= \psi(\omega + N(\omega) + \alpha) = \omega - N(\omega) + N(\omega) + \alpha = \omega + \alpha, \\ \phi \circ \psi(\omega + \alpha) &= \phi(\omega - N(\omega) + \alpha) = \omega + N(\omega) - N(\omega) + \alpha = \omega + \alpha, \end{aligned}$$

so that ϕ and ψ are automorphisms with $\phi^{-1} = \psi$. Finally, on the one hand,

$$\text{ver}[d_{P,\nabla+\mathbf{A}}] \circ \phi = \text{Proj}_1 \circ \phi = \text{Proj}_1 = \text{ver}[d_{P,\nabla}],$$

where $\text{Proj}_1 : \Omega_{P,\oplus}^1 \rightarrow \Omega_{P,\text{ver}}^1$ is the projection onto $\Omega_{P,\text{ver}}^1$ along $\Omega_{P,\text{hor}}^1$, while on the other, for all $p \in P$,

$$\begin{aligned} \phi \circ d_{P,\nabla}(p) &= \phi(d_{P,\text{ver}} + \nabla(p)) = d_{P,\text{ver}}(p) + N(d_{P,\text{ver}}(p)) + \nabla(p) \\ &= d_{P,\text{ver}}(p) + \mathbf{A}(p) + \nabla(p) = d_{P,\nabla+\mathbf{A}} \circ \phi(p). \end{aligned}$$

Hence, id_P induces an arrow $\text{id}_P : \pi[\Omega_H^{\leq 2}] \circ \Sigma[\Omega_H^{\leq 2}](\nabla) \rightarrow \pi[\Omega_H^{\leq 2}] \circ \Sigma[\Omega_H^{\leq 2}](\nabla + \mathbf{A})$ in $\mathcal{G}[\Omega_H^{\leq 2}]$ with $(\text{id}_P)_* = \phi$, as required. Since $\nabla \in \mathfrak{A}^{\text{tr}}[\Omega_H^1]$ was arbitrary, condition 2 is satisfied.

Now, condition 2 trivially implies condition 3, so suppose that condition 3 is satisfied. Fix $\nabla \in \mathfrak{A}^{\text{tr}}[\Omega_H^1]$, such that id_P induces an arrow

$$(\text{id}_P : \pi[\Omega_H^{\leq 2}] \circ \Sigma[\Omega_H^{\leq 2}](\nabla) \rightarrow \pi[\Omega_H^{\leq 2}] \circ \Sigma[\Omega_H^{\leq 2}](\nabla + \mathbf{A})) \in \mathcal{G}[\Omega_H^{\leq 2}].$$

Hence, let $\Phi := (\text{id}_P)_* : \Omega_{P,\oplus} \rightarrow \Omega_{P,\oplus}$, so that Φ is an automorphism of the graded H -comodule $*$ -algebra $\Omega_{P,\oplus}$, such that $\Phi|_P = \text{id}_P$ and

$$\begin{aligned} \text{Proj}_1 \circ \Phi|_{\Omega_{P,\oplus}^1} &= \text{ver}[d_{P,\nabla+\mathbf{A}}] \circ (\text{id}_P)_* = (\text{id}_P)_{*,\text{ver}} \circ \text{ver}[d_{P,\nabla}] = \text{Proj}_1, \\ d_{P,\nabla+\mathbf{A}} &= d_{P,\nabla+\mathbf{A}} \circ \text{id}_P = (\text{id}_P)_* \circ d_{P,\nabla} = \Phi \circ d_{P,\nabla}, \end{aligned}$$

where $\text{Proj}_1 : \Omega_{P,\oplus}^1 \rightarrow \Omega_{P,\text{ver}}^1$ is the projection onto $\Omega_{P,\text{ver}}^1$ along $\Omega_{P,\text{hor}}^1$. On the one hand, since $\text{Proj}_1 \circ \Phi|_{\Omega_{P,\oplus}^1} = \text{Proj}_1$, it follows that $(\text{id} - \Phi)(\Omega_{P,\text{ver}}^1) \subseteq \ker \text{Proj}_1 = \Omega_{P,\text{hor}}^1$. On the other hand, for all $b \in B$, we see that

$$\Phi(d_B(b)) = \Phi \circ d_{P,\nabla}(b) = d_{P,\nabla+\mathbf{A}}(b) = d_B(b),$$

so that, in turn, $\Phi|_{\Omega_{P,\text{hor}}^1} = \text{id}_{\Omega_{P,\text{hor}}^1}$. Thus we can define a left H -covariant morphism $N : \Omega_{P,\text{ver}}^1 \rightarrow \Omega_{P,\text{hor}}^1$ of P -*-bimodules by

$$\forall \omega \in \Omega_{P,\text{ver}}^1, \quad N(\omega) := \omega - \Phi(\omega);$$

we claim that $\mathbf{A} = N \circ d_{P,\text{ver}} \in \text{at}^{\text{PR}}[\Omega_H]$. First, for every $p \in P$,

$$\begin{aligned} N \circ d_{P,\text{ver}}(p) &= d_{P,\text{ver}}(p) - \Phi(d_{P,\text{ver}}(p)) = d_{P,\text{ver}}(p) - \Phi(d_{P,\nabla}(p) - \nabla(p)) \\ &= d_{P,\text{ver}}(p) - d_{P,\nabla+\mathbf{A}}(p) - \nabla(p) = \mathbf{A}(p), \end{aligned}$$

so that $\mathbf{A} = N \circ d_{P,\text{ver}} \in \text{at}[\Omega_H^1]$. Next, for all $\mu \in \Lambda_H^1$ and $\alpha \in \Omega_B^1$,

$$N(\mu) \wedge \alpha = (\mu - \Phi(\mu)) \wedge \alpha = \mu \wedge \alpha - \Phi(\mu \wedge \alpha) = -\alpha \wedge \mu + \Phi(\alpha \wedge \mu) = -\alpha \wedge (\text{id} - \Phi)(\mu) = -\alpha \wedge N(\mu).$$

Finally, for all $\mu, \nu \in \Lambda_H^1$,

$$\begin{aligned} N(\mu) \wedge N(\nu) &= \mu \wedge \nu - \Phi(\mu) \wedge \nu - \mu \wedge \Phi(\nu) + \Phi(\mu) \wedge \Phi(\nu) \\ &= \mu \wedge \nu - \Phi(\mu) \wedge \nu - \Phi(\Phi^{-1}(\mu) \wedge \nu) + \Phi(\mu \wedge \nu) \\ &= -\mu_{[-1]} \triangleright \nu \wedge \mu_{[0]} + \Phi(\mu)_{[-1]} \triangleright \nu \wedge \Phi(\mu)_{[0]} + \Phi(\Phi^{-1}(\mu)_{[-1]} \triangleright \nu \wedge \Phi^{-1}(\mu)_{[0]}) \\ &\quad + \Phi(\mu_{[-1]} \triangleright \nu \wedge \mu_{[0]}) \\ &= -\mu_{[-1]} \triangleright \nu \wedge \mu_{[0]} + \mu_{[-1]} \triangleright \nu \wedge \Phi(\mu_{[0]}) + \Phi(\mu_{[-1]} \triangleright \nu \wedge \Phi^{-1}(\mu_{[0]})) \\ &\quad + \Phi(\mu_{[-1]} \triangleright \nu \wedge \mu_{[0]}) \\ &= -N(\mu_{[-1]} \triangleright \nu) \wedge N(\mu_{[0]}). \end{aligned}$$

Hence, $\mathbf{A} = N \circ d_{P,\text{ver}} \in \text{at}_{\text{can}}^{\text{PR}}[\Omega_H^1]$, as required. Thus, condition 1 holds. \square

Proof of Theorem 3.48. Let us first show that the subset $\text{at}_{\text{can}}^{\text{PR}}[\Omega_H^1]$ is \mathfrak{G}^{PR} -invariant. Let $\mathbf{A} \in \text{at}_{\text{can}}^{\text{PR}}[\Omega_H^1]$ and $N := \omega[\mathbf{A}]$; let $\phi \in \mathfrak{G}^{\text{PR}}$, so that $\phi \triangleright \mathbf{A} = (\phi \triangleright N) \circ d_{P,\text{ver}} \in \text{at}[\Omega_H^1]$ for

$$\phi \triangleright N := \phi_* \circ N \circ (\text{id} \otimes \phi^{-1}).$$

Then, for all $\mu \in \Lambda_H^1$ and $\alpha \in \Omega_B^1$,

$$(\phi \triangleright N)(\mu) \wedge \alpha + \alpha \wedge (\phi \triangleright N)(\nu) = \phi_*(N(\mu) \wedge \alpha + \alpha \wedge N(\nu)) = 0,$$

while for all $\mu, \nu \in \Lambda_H^1$,

$$\begin{aligned} (\phi \triangleright N)(\mu) \wedge (\phi \triangleright N)(\nu) + (\phi \triangleright N)(\mu_{[-1]} \triangleright \nu) \wedge (\phi \triangleright N)(\mu_{[0]}) \\ = \phi_*(N(\mu) \wedge N(\nu) + N(\mu_{[-1]} \triangleright \nu) \wedge N(\mu_{[0]})) = 0, \end{aligned}$$

so that $\phi \triangleright \mathbf{A} \in \text{at}_{\text{can}}^{\text{PR}}[\Omega_H^1]$.

Next, let us show that $\text{at}_{\text{can}}^{\text{PR}}[\Omega_H^1]$ is a subspace of $\text{at}[\Omega_H^1] \cap \text{at}^{\text{PR}}$. First, by Lemma 3.49, $\text{at}_{\text{can}}^{\text{PR}}[\Omega_H^1] \subset \text{at}[\Omega_H^1] \cap \text{at}^{\text{PR}}$. Next, since $0 \in \text{at}_{\text{can}}^{\text{PR}}[\Omega_H^1]$ and since equations (3.33) and (3.34) are homogeneous of degree 1 and 2 respectively, it follows that $\text{at}_{\text{can}}^{\text{PR}}[\Omega_H^1]$ is non-empty and closed under scalar multiplication by \mathbf{R} . Finally, let $\mathbf{A}_1, \mathbf{A}_2 \in \text{at}_{\text{can}}^{\text{PR}}[\Omega_H^1]$. Let $\nabla \in \mathfrak{A}^{\text{PR}}[\Omega_H^1]$ be given. By Lemma 3.51, $\nabla + \mathbf{A}_1 \in \mathfrak{A}^{\text{PR}}[\Omega_H^1]$, so that

$$(\text{id}_P : \pi[\Omega_H^{\leq 2}] \circ \Sigma[\Omega_H^{\leq 2}](\nabla) \rightarrow \pi[\Omega_H^{\leq 2}] \circ \Sigma[\Omega_H^{\leq 2}](\nabla + \mathbf{A}_1)) \in \mathcal{G}[\Omega_H^{\leq 2}];$$

let $\Phi := (\text{id}_P)_*$ be the unique left H -covariant automorphism of the P -*-bimodule $\Omega_{P,\mathfrak{G}}^1$, such that $\Phi \circ d_{P,\nabla} = d_{P,\nabla+\mathbf{A}_1}$. Hence, by Lemma 3.51, $\nabla + \mathbf{A}_1 + \mathbf{A}_2 \in \mathfrak{A}^{\text{PR}}[\Omega_H^1]$, so that

$$(\text{id}_P : \pi[\Omega_H^{\leq 2}] \circ \Sigma[\Omega_H^{\leq 2}](\nabla + \mathbf{A}_1) \rightarrow \pi[\Omega_H^{\leq 2}] \circ \Sigma[\Omega_H^{\leq 2}](\nabla + \mathbf{A}_1 + \mathbf{A}_2)) \in \mathcal{G}[\Omega_H^{\leq 2}];$$

let $\Psi := (\text{id}_P)_*$ be the unique left H -covariant automorphism of the P -*-bimodule $\Omega_{P,\mathfrak{G}}^1$, such that $\Psi \circ d_{P,\nabla+A_1} = d_{P,\nabla+A_1+A_2}$. Then $\Psi \circ \Phi$ is a left H -covariant automorphism of the P -*-bimodule $\Omega_{P,\mathfrak{G}}^1$ satisfying

$$\begin{aligned} (\Psi \circ \Phi) \circ d_{P,\nabla} &= \Phi \circ d_{P,\nabla+A_1} = d_{P,\nabla+A_1+A_2}, \\ \text{ver}[d_{P,\nabla+A_1+A_2}] \circ (\Psi \circ \Phi) &= \text{ver}[d_{P,\nabla+A_1}] \circ \Phi = \text{ver}[d_{P,\nabla}], \end{aligned}$$

so that $\text{id}_P : \pi[\Omega_H^{\leq 2}] \circ \Sigma[\Omega_H^{\leq 2}](\nabla) \rightarrow \pi[\Omega_H^{\leq 2}] \circ \Sigma[\Omega_H^{\leq 2}](\nabla + A_1 + A_2)$ defines an arrow in $\mathcal{G}[\Omega_H^{\leq 2}]$. Hence, by Lemma 3.51, $A_1 + A_2 \in \text{at}_{\text{can}}^{\text{pr}}[\Omega_H^1]$. Thus, $\text{at}_{\text{can}}^{\text{pr}}[\Omega_H^1]$ is a subspace of $\text{at}^{\text{pr}} \cap \text{at}[\Omega_H^1]$. Note that $\mathfrak{A}t^{\text{pr}}[\Omega_H^1] \subset \mathfrak{A}t^{\text{pr}}$ is invariant under translation by the subspace $\text{at}_{\text{can}}^{\text{pr}}[\Omega_H^1] \subset \text{at}^{\text{pr}}$ by Lemma 3.51.

Finally, by Proposition 3.43, the covering $\pi[\Omega_H^{\leq 2}] : \mathcal{G}[\Omega_H^{\leq 2}] \times \mathcal{A}[\Omega_H^{\leq 2}] \rightarrow \mathcal{G}[\Omega_H^{\leq 2}]$, the star-injective groupoid homomorphism $\mu[\Omega_H^{\leq 2}] : \mathcal{G}[\Omega_H^{\leq 2}] \rightarrow \text{Aut}(P)$, the injective groupoid homomorphism $\Sigma[\Omega_H^{\leq 2}] : \mathfrak{G}^{\text{pr}} \times \mathfrak{A}t^{\text{pr}}[\Omega_H^1] \rightarrow \mathcal{G}[\Omega_H^{\leq 2}] \times \mathcal{A}[\Omega_H^{\leq 2}]$, and the left inverse $A[\Omega_H^{\leq 2}]$ of $\Sigma[\Omega_H^{\leq 2}]$ satisfy the hypotheses of Lemma A.1. Thus, the equivalence kernel \sim of the map

$$\left(\nabla \mapsto \left[\pi[\Omega_H^{\leq 2}] \circ \Sigma[\Omega_H^{\leq 2}](\text{id}_P, \nabla) \right]_{\ker \mu[\Omega_H^{\leq 2}]} \right) : \mathfrak{A}t^{\text{pr}}[\Omega_H^1] \rightarrow \text{Ob}(\mathcal{G}[\Omega_H^{\leq 2}] / \ker \mu[\Omega_H^{\leq 2}])$$

is a \mathfrak{G}^{pr} -invariant equivalence relation, the subgroupoid $\ker \mu[\Omega_H^{\leq 2}]$ is wide and has trivial isotropy groups, the quotient groupoid $\mathcal{G}[\Omega_H^{\leq 2}] / \ker \mu[\Omega_H^{\leq 2}]$ is well-defined, and there exists a unique isomorphism $\tilde{\Sigma}^{\text{pr}}[\Omega_H^{\leq 2}] : \mathfrak{G}^{\text{pr}} \times \mathfrak{A}t^{\text{pr}}[\Omega_H^1] / \sim \xrightarrow{\sim} \mathcal{G}[\Omega_H^{\leq 2}] / \ker \mu[\Omega_H^{\leq 2}]$, such that

$$\forall(\phi, \nabla) \in \mathfrak{G}^{\text{pr}} \times \mathfrak{A}t^{\text{pr}}[\Omega_H^1], \quad \tilde{\Sigma}^{\text{pr}}[\Omega_H^{\leq 2}](\phi, [\nabla]_{\sim}) = \left[\pi[\Omega_H^{\leq 2}] \circ \Sigma[\Omega_H^{\leq 2}](\phi, \nabla) \right]_{\ker \mu[\Omega_H^{\leq 2}]}$$

Lemma 3.51 now implies that \sim is the orbit equivalence relation with respect to the translation action of $\text{at}_{\text{can}}^{\text{pr}}[\Omega_H^1] \leq \text{at}^{\text{pr}}$ on $\mathfrak{A}t^{\text{pr}}[\Omega_H^1] \subset \mathfrak{A}t^{\text{pr}}$. \square

Finally, observe that the subspace $\text{Inn}(\text{at}^{\text{pr}})$ of inner prolongable gauge potentials acts by translation on the affine space $\mathfrak{A}t^{\text{pr}}$. Given a bicovariant FODC (Ω_H^1, d_H) on H that is locally freeing for P , we can now characterise the stabiliser subgroup in $\text{Inn}(\text{at}^{\text{pr}})$ of the affine quadric subset $\mathfrak{A}t^{\text{pr}}[\Omega_H^1]$ of $\mathfrak{A}t^{\text{pr}}$. This will turn out to be the \mathfrak{G}^{pr} -invariant \mathbf{R} -linear subspace of all inner prolongable relative gauge potentials of the following form.

Definition 3.52. Let (Ω_H^1, d_H) be a bicovariant FODC on H that is locally freeing for P . Let \mathbf{A} be an inner prolongable relative gauge potential on P with respect to the second-order horizontal calculus $(\Omega_B, d_B; \Omega_{P,\text{hor}})$. We say that \mathbf{A} is (Ω_H^1, d_H) -*semi-adapted* whenever

$$F_{\text{rel}}[\mathbf{A}] = F_{\text{rel}}[\mathbf{A}] \circ d_{P,\text{ver}},$$

for some (necessarily unique) left H -covariant morphism $F_{\text{rel}}[\mathbf{A}] : \Omega_{P,\text{ver}}^1 \rightarrow \Omega_{P,\text{hor}}^2$ of P -*-bimodules, in which case, we call $F_{\text{rel}}[\mathbf{A}]$ the *relative curvature 2-form* of \mathbf{A} . We denote by $\text{Inn}(\text{at}^{\text{pr}}; \Omega_H^1)$ the subspace of all (Ω_H^1, d_H) -semi-adapted inner prolongable gauge potentials on P with respect to $(\Omega_B, d_B; \Omega_{P,\text{hor}})$.

Example 3.53. By Example 3.33, for every $\phi \in \text{Inn}(\mathfrak{G})$ and $\nabla \in \mathfrak{A}t^{\text{pr}}$, we have

$$\phi \triangleright \nabla - \nabla \in \text{Inn}(\text{at}^{\text{pr}}; \Omega_H^1), \quad F_{\text{rel}}[\phi \triangleright \nabla - \nabla] = 0.$$

Proposition 3.54. Let (Ω_H^1, d_H) be a bicovariant FODC on H that is locally freeing for P . Suppose that the (Ω_H^1, d_H) -adapted Atiyah space $\mathfrak{A}t^{\text{pr}}[\Omega_H^1]$ is non-empty. Let \mathbf{A} be an inner prolongable relative gauge potential on P with respect to $(\Omega_B, d_B; \Omega_{P,\text{hor}})$. The following are equivalent:

- (1) the inner prolongable relative gauge potential \mathbf{A} is (Ω_H^1, d_H) -semi-adapted;

- (2) for all $\nabla \in \mathfrak{A}t^{\text{Pr}}[\Omega_H^1]$, the prolongable gauge potential $\nabla + \mathbf{A}$ is also (Ω_H^1, d_H) -adapted;
 (3) there exists $\nabla \in \mathfrak{A}t^{\text{Pr}}[\Omega_H^1]$, such that $\nabla + \mathbf{A} \in \mathfrak{A}t^{\text{Pr}}$ is also (Ω_H^1, d_H) -adapted.

Thus, in particular, the affine quadric subset $\mathfrak{A}t^{\text{Pr}}[\Omega_H^1]$ of the affine space $\mathfrak{A}t^{\text{Pr}}$ is invariant under translation by $\text{Inn}(\mathfrak{a}t^{\text{Pr}}; \Omega_H^1)$.

Proof. First, suppose that $\mathbf{A} \in \text{Inn}(\mathfrak{a}t^{\text{Pr}}; \Omega_H^1)$. Let $\nabla \in \mathfrak{A}t^{\text{Pr}}[\Omega_H^1]$. Then $\nabla + \mathbf{A} \in \mathfrak{A}t^{\text{Pr}}$ with

$$F[\nabla + \mathbf{A}] = F[\nabla] + F_{\text{rel}}[\mathbf{A}] = F[\nabla] \circ d_{P,\text{ver}} + F_{\text{rel}}[\mathbf{A}] \circ d_{P,\text{ver}} = (F[\nabla] + F_{\text{rel}}[\mathbf{A}]) \circ d_{P,\text{ver}},$$

so that $\nabla + \mathbf{A} \in \mathfrak{A}t^{\text{Pr}}[\Omega_H^1]$ with $F[\nabla + \mathbf{A}] = F[\nabla] + F_{\text{rel}}[\mathbf{A}]$. Now, suppose that there exists $\nabla \in \mathfrak{A}t^{\text{Pr}}[\Omega_H^1]$, such that $\nabla + \mathbf{A} \in \mathfrak{A}t^{\text{Pr}}[\Omega_H^1]$. Then

$$F_{\text{rel}}[\mathbf{A}] = F[\nabla + \mathbf{A}] - F[\nabla] = F[\nabla + \mathbf{A}] \circ d_{P,\text{ver}} - F[\nabla] \circ d_{P,\text{ver}} = (F[\nabla + \mathbf{A}] - F[\nabla]) \circ d_{P,\text{ver}},$$

so that $\mathbf{A} \in \text{Inn}(\mathfrak{a}t^{\text{Pr}}; \Omega_H^1)$ with $F_{\text{rel}}[\mathbf{A}] = F[\nabla + \mathbf{A}] - F[\nabla]$. \square

We now summarise the basic properties of the subspace $\text{Inn}(\mathfrak{a}t^{\text{Pr}}; \Omega_H^1)$ and observe that the action of \mathfrak{G}^{Pr} on $\mathfrak{A}t^{\text{Pr}}[\Omega_H^1]$ descends to a second-order analogue of the action of $\text{Out}(\mathfrak{G})$ on $\text{Out}(\mathfrak{A}t)$; this all follows, *mutatis mutandis*, from the proof of Proposition 3.30.

Proposition 3.55. *Let (Ω_H^1, d_H) be a bicovariant FODC on H that is locally freeing for P . Suppose that $\mathfrak{A}t^{\text{Pr}}[\Omega_H^1]$ is non-empty. The subspace $\text{Inn}(\mathfrak{a}t^{\text{Pr}}; \Omega_H^1)$ of $\text{Inn}(\mathfrak{a}t^{\text{Pr}})$ consists of \mathfrak{G}^{Pr} -invariant vectors, so that the affine action of \mathfrak{G}^{Pr} on the affine space $\mathfrak{A}t^{\text{Pr}}$ descends to an affine action of \mathfrak{G}^{Pr} on the quotient affine space $\mathfrak{A}t^{\text{Pr}}/\text{Inn}(\mathfrak{a}t^{\text{Pr}}; \Omega_H^1)$. Furthermore, the inner gauge group $\text{Inn}(\mathfrak{G})$ acts trivially on $\mathfrak{A}t^{\text{Pr}}/\text{Inn}(\mathfrak{a}t^{\text{Pr}}; \Omega_H^1)$, so that the action of \mathfrak{G}^{Pr} on $\mathfrak{A}t^{\text{Pr}}/\text{Inn}(\mathfrak{a}t^{\text{Pr}}; \Omega_H^1)$ further descends to an affine action of the outer prolongable gauge group $\text{Out}(\mathfrak{G}^{\text{Pr}})$ on $\mathfrak{A}t^{\text{Pr}}/\text{Inn}(\mathfrak{a}t^{\text{Pr}}; \Omega_H^1)$ that restricts, in turn, to an action on the quadric subset*

$$\text{Out}(\mathfrak{A}t^{\text{Pr}}[\Omega_H^1]) := \mathfrak{A}t^{\text{Pr}}[\Omega_H^1]/\text{Inn}(\mathfrak{a}t^{\text{Pr}}; \Omega_H^1).$$

Finally, we record the basic properties of the relative curvature 2-form.

Corollary 3.56. *Let (Ω_H^1, d_H) be a bicovariant FODC on H that is locally freeing for P . The map $F_{\text{rel}} := (\mathbf{A} \mapsto F_{\text{rel}}[\mathbf{A}]) : \text{Inn}(\mathfrak{a}t^{\text{Pr}}; \Omega_H^1) \rightarrow \text{Hom}_P^H(\Omega_{P,\text{ver}}^1, \Omega_{P,\text{hor}}^2)$ is linear with range contained in $\text{Hom}_P^H(\Omega_{P,\text{ver}}^1, \Omega_{P,\text{hor}}^2)^{\mathfrak{G}^{\text{Pr}}}$. In particular, it satisfies*

$$\forall \nabla \in \mathfrak{A}t^{\text{Pr}}[\Omega_H^1], \forall \mathbf{A} \in \text{Inn}(\mathfrak{a}t^{\text{Pr}}; \Omega_H^1), \quad F[\nabla + \mathbf{A}] = F[\nabla] + F_{\text{rel}}[\mathbf{A}].$$

Proof. The map F_{rel} is \mathbf{R} -linear and \mathfrak{G}^{Pr} -equivariant by Corollary 3.32 and \mathfrak{G}^{Pr} -equivariance of $d_{P,\text{ver}}$. Since \mathfrak{G}^{Pr} acts trivially on $\text{Inn}(\mathfrak{a}t^{\text{Pr}}; \Omega_H^1)$, it follows that the range of F_{rel} is contained in $\text{Hom}_P^H(\Omega_{P,\text{ver}}^1, \Omega_{P,\text{hor}}^2)^{\mathfrak{G}^{\text{Pr}}}$. The relation between the maps F on $\mathfrak{A}t^{\text{Pr}}[\Omega_H^1]$ and F_{rel} on $\text{Inn}(\mathfrak{a}t^{\text{Pr}}; \Omega_H^1)$ now follows by the proof of Proposition 3.54. \square

4. GAUGE THEORY ON CROSSED PRODUCTS AS LAZY COHOMOLOGY

Unlike in the commutative case, trivial quantum principal bundles can encode non-trivial dynamical information. As Čačić–Mesland have already observed in the context of spectral triples [13, §3.4], the noncommutative \mathbf{T}^m -gauge theory of crossed products by \mathbf{Z}^m can be related to the degree 1 group cohomology of \mathbf{Z}^m with certain geometrically meaningful coefficients. In light of the groupoid equivalences of Proposition 2.40 and 3.43, we shall now similarly relate the gauge theory of (non-twisted) crossed product algebras to certain generalisations of group cohomology in degree 1.

4.1. Cohomological preliminaries. In effect, Čačić–Mesland computed the gauge group \mathfrak{G} and Atiyah space $\mathfrak{A}t$ of a crossed product by \mathbf{Z}^m in terms of the degree 1 group cohomology of \mathbf{Z}^m with coefficients in a certain group of unitaries and a certain $\mathbf{R}[\mathbf{Z}^m]$ -module of noncommutative 1-forms, respectively [13, Thm. 3.36]. In this section, we construct suitable generalisations of these two distinct but interrelated cases of degree 1 group cohomology to arbitrary Hopf $*$ -algebras; in the process, we provide an *ad hoc* generalisation of degree 1 Sweedler cohomology to not-necessarily-cocommutative Hopf $*$ -algebras and non-trivial coefficients. From now on, let H be a Hopf $*$ -algebra.

We begin by recalling more-or-less standard constructions of convolution algebras and bimodules on H with suitable coefficients.

Definition 4.1. Let B be a [graded] $*$ -algebra. The *B -valued convolution algebra on H* is the [graded] unital $*$ -algebra $C(H; B)$ defined by endowing $\text{Hom}_{\mathbf{C}}(H; B)$ with the product and $*$ -structure defined by

$$(4.1) \quad \forall f, g \in C(H; B), \forall h \in H, \quad (f \star g)(h) := f(h_{(1)})g(h_{(2)}),$$

$$(4.2) \quad \forall f \in C(H; B), \forall h \in H, \quad f^*(h) := f(S(h))^*,$$

respectively, and the unit $1_{C(H; B)} := \epsilon(\cdot)1_B$.

Definition 4.2. Let B be a $*$ -algebra, and let M be a B - $*$ -bimodule. The *M -valued convolution bimodule on H* is the $C(H, B)$ - $*$ -bimodule $C(H; M)$ defined by endowing the \mathbf{C} -vector space $\text{Hom}_{\mathbf{C}}(H, M)$ with the left $C(H, B)$ -module structure, right $C(H, B)$ -module structure, and $*$ -structure defined, respectively, by

$$(4.3) \quad \forall f \in C(H, B), \forall \mu \in C(H, M), \forall h \in H, \quad (f \star \mu)(h) := f(h_{(1)}) \cdot \mu(h_{(2)}),$$

$$(4.4) \quad \forall f \in C(H, B), \forall \mu \in C(H, M), \forall h \in H, \quad (\mu \star f)(h) := \mu(h_{(1)}) \cdot f(h_{(2)}),$$

$$(4.5) \quad \forall \mu \in C(H, M), \forall h \in H, \quad \mu^*(h) := \mu(S(h))^*.$$

In the case of H -module $*$ -algebras and H -equivariant $*$ -bimodules, one can canonically embed coefficient algebras and bimodules into the resulting convolution algebras and bimodules, respectively, in a manner respecting $*$ -bimodule structures. When H is no longer a group algebra, this will yield subtler notions of commutation than pointwise commutation.

Proposition 4.3. Let B be a [graded] H -module $*$ -algebra. Then $\rho_B : B \rightarrow C(H; B)$ defined by

$$\forall b \in B, \forall h \in H, \quad \rho_B(b)(h) := b \triangleleft h$$

is an injective [graded] $*$ -homomorphism.

Proposition 4.4. Let B be a right H -module $*$ -algebra, and let M be a right H -equivariant B - $*$ -bimodule. Then the map $\rho_M : M \rightarrow C(H; M)$ defined by

$$\forall m \in M, \forall h \in H, \quad \rho_M(m)(h) := m \triangleleft h$$

is an injective $*$ -preserving \mathbf{C} -linear map satisfying

$$\forall a, b \in B, \forall m \in M, \quad \rho_B(a) \star \rho_M(m) \star \rho_B(b) = \rho_M(a \cdot m \cdot b).$$

In particular, $C(H; M)$ defines a B - $*$ -bimodule with respect to ρ_B .

We can now provide a suitable generalisation of the degree 1 group cohomology of a group Γ with coefficients in the unitary group of a commutative Γ - $*$ -algebra. It can be viewed as an *ad hoc* generalisation of degree 1 Sweedler cohomology [39] in the spirit of Bichon–Carnovale’s ‘lazy’ cohomology [6] to the case of non-cocommutative Hopf

algebras and non-commutative coefficient algebras. We shall use it to compute the gauge group of a crossed product by H .

Proposition-Definition 4.5 (cf. Sweedler [39]). Let B be a right H -module $*$ -algebra, and let M be a right H -equivariant B - $*$ -bimodule.

- (1) A *lazy (B, M) -valued Sweedler 0-cochain on H* is an element of

$$\text{CS}_\ell^0(H; B, M) := \text{U}(C_B(B \otimes M)) \leq \text{U}(B).$$

- (2) A *lazy (B, M) -valued Sweedler 1-cocycle on H* is $\sigma \in \text{U}(C_{C(H, B)}(\rho_B(B) \otimes \rho_M(M)))$, such that $\sigma(1) = 1$ and

$$(4.6) \quad \forall h, k \in H, \quad \sigma(hk) = (\sigma(h) \triangleleft k_{(1)}) \sigma(k_{(2)});$$

we denote by $\text{ZS}_\ell^1(H; B, M)$ the set of all lazy (B, M) -valued Sweedler 1-cocycles on H , which defines a subgroup of $\text{U}(C(H, B))$.

- (3) The *coboundary map* is the homomorphism $D : \text{ZS}_\ell^0(H; B, M) \rightarrow \text{ZS}_\ell^1(H; B, M)$ defined by

$$(4.7) \quad \forall v \in \text{CS}_\ell^0(H; B, M), \forall h \in H, \quad Dv(h) := (v \triangleleft h) \cdot v^*;$$

thus, a *lazy (B, M) -valued Sweedler 1-coboundary on H* is an element of

$$\text{BS}_\ell^1(H; B, M) := D(\text{CS}_\ell^0(H; B, M)) \leq \text{Z}(\text{ZS}_\ell^1(H; B, M)).$$

- (4) The *lazy degree 1 Sweedler cohomology of H with coefficients in (B, M)* is the group

$$\text{HS}_\ell^1(H; B, M) := \text{ZS}_\ell^1(H; B, M) / \text{BS}_\ell^1(H; B, M).$$

The proof of this result—and several others besides—will require the following straightforward technical lemma.

Lemma 4.6. *Let B be a right H -module $*$ -algebra, and let M be a right H -equivariant B - $*$ -bimodule.*

- (1) *Let $b \in B$ and let $\mu \in C(H, M)$. Then $[\rho_B(b), \mu] = 0$ if and only if*

$$\forall h \in H, \quad (b \triangleleft h_{(1)}) \cdot \mu(h_{(2)}) = \mu(h_{(1)}) \cdot (b \triangleleft h_{(2)}).$$

- (2) *Let $\beta \in C(H, B)$ and let $m \in M$. Then $[\beta, \rho_M(m)] = 0$ if and only if*

$$\forall h \in H, \quad \beta(h_{(1)}) \cdot (m \triangleleft h_{(2)}) = (m \triangleleft h_{(1)}) \cdot \beta(h_{(2)}).$$

Proof. On the one hand, for all $b \in B$, $\mu \in C(H, M)$, and $h \in H$,

$$\begin{aligned} [\rho_B(b), \mu](h) &= \rho_B(b)(h_{(1)}) \cdot \mu(h_{(2)}) - \mu(h_{(1)}) \cdot \rho_B(b)(h_{(2)}) \\ &= (b \triangleleft h_{(1)}) \cdot \mu(h_{(2)}) - \mu(h_{(1)}) \cdot (b \triangleleft h_{(2)}). \end{aligned}$$

On the other hand, for all $\beta \in C(H, B)$, $m \in M$, and $h \in H$,

$$\begin{aligned} [\beta, \rho_M(m)](h) &= \beta(h_{(1)}) \cdot \rho_M(m)(h_{(2)}) - \rho_M(m)(h_{(1)}) \cdot \beta(h_{(2)}) \\ &= \beta(h_{(1)}) \cdot (m \triangleleft h_{(2)}) - (m \triangleleft h_{(1)}) \cdot \beta(h_{(2)}). \end{aligned} \quad \square$$

Proof of Proposition 4.5. Before continuing, note that by Lemma 4.6, $f \in C(H, B)$ centralises the subset $\rho_B(B) \otimes \rho_M(B)$ of $C(H; B \otimes M)$ if and only if

$$\begin{aligned} \forall b \in B, \forall h \in H, \quad f(h_{(1)}) \cdot (b \triangleleft h_{(2)}) &= (b \triangleleft h_{(1)}) \cdot f(h_{(2)}), \\ \forall m \in M, \forall h \in H, \quad f(h_{(1)}) \cdot (m \triangleleft h_{(2)}) &= (m \triangleleft h_{(1)}) \cdot f(h_{(2)}). \end{aligned}$$

We first show that $ZS_\ell^1(H; B, M)$ is a subgroup of $U(C_{C(H,B)}(\rho_B(B) \oplus \rho_M(M)))$. First, observe that $ZS_\ell^1(H; B, M) \ni 1_{C(H,B)}$. Next, let $\sigma_1, \sigma_2 \in ZS_\ell^1(H; B, M)$. Then for all $h, k \in H$,

$$\begin{aligned} \sigma_1 \star \sigma_2(hk) &= \sigma_1(h_{(1)}k_{(1)}) \cdot \sigma_2(h_{(2)}k_{(2)}) \\ &= (\sigma_1(h_{(1)}) \triangleleft k_{(1)}) \cdot \sigma_1(k_{(2)}) \cdot (\sigma_2(h_{(2)}) \triangleleft k_{(3)}) \cdot \sigma_2(k_{(4)}) \\ &= (\sigma_1(h_{(1)}) \triangleleft k_{(1)}) \cdot (\sigma_2(h_{(2)}) \triangleleft k_{(2)}) \cdot \sigma_1(k_{(3)}) \cdot \sigma_2(k_{(4)}) \\ &= (\sigma_1 \star \sigma_2(h) \triangleleft k_{(1)}) \cdot \sigma_1 \star \sigma_2(k_{(2)}), \end{aligned}$$

while $\sigma_1 \star \sigma_2(1) = \sigma_1(1) \cdot \sigma_2(1) = 1$, so that, indeed, $\sigma_1 \star \sigma_2 \in ZS_\ell^1(H; B, M)$. Finally, let $\sigma \in ZS_\ell^1(H; B, M)$; note that $\sigma^{-1} \in U(C_{C(H,B)}(\rho_B(B) \oplus \rho_M(M)))$. Then, for all $h, k \in H$,

$$\begin{aligned} \sigma^{-1}(hk) &= \sigma(S(h)^* S(k)^*)^* \\ &= ((\sigma(S(h)^*) \triangleleft (S(k)^*)_{(1)}) \cdot \sigma((S(k)^*)_{(2)}))^* \\ &= (\sigma((S(k)^*)_{(1)}) \cdot (\sigma(S(h)^*) \triangleleft (S(k)^*)_{(2)}))^* \\ &= (\sigma^*(h) \triangleleft k_{(1)}) \cdot \sigma^*(k_{(2)}), \end{aligned}$$

while $\sigma^*(1) = \sigma(1)^* = 1$, so that $\sigma^{-1} \in ZS_\ell^1(H; B, M)$.

Next, we show that $D : ZS_\ell^0(H; B, M) \rightarrow ZS_\ell^1(H; B, M)$ is well-defined as a function. Let $v \in CS_\ell^0(H; B, M)$, and set $\sigma := Dv$. First, σ is unitary, since for all $h \in H$,

$$\begin{aligned} \sigma \star \sigma^*(h) &= (v \triangleleft h_{(1)}) \cdot v^* \cdot ((v \triangleleft S(h_{(2)}))^* \cdot v^*)^* = (v \triangleleft h_{(1)}) \cdot v^* \cdot v \cdot (v^* \triangleleft h_{(2)}) = \epsilon(h)1_B, \\ \sigma^* \star \sigma(h) &= ((v \triangleleft S(h_{(1)}))^* \cdot v^*)^* \cdot (v \triangleleft h_{(2)}) \cdot v^* = v \cdot (v^* \triangleleft h_{(1)}) \cdot (v \triangleleft h_{(2)}) \cdot v^* = \epsilon(h)1_B; \end{aligned}$$

note, moreover, that $\sigma(1) = (v \triangleleft 1) \cdot v^* = v \cdot v^* = 1$. Next, $\sigma \in C_{C(H,B)}(\rho_B(B) \oplus \rho_M(M))$ since for all $b \in B$, $m \in M$, and $h \in H$,

$$\begin{aligned} \sigma(h_{(1)}) \cdot (b \triangleleft h_{(2)}) - (b \triangleleft h_{(1)}) \cdot \sigma(h_{(2)}) &= (v \triangleleft h_{(1)}) \cdot v^* \cdot (b \triangleleft h_{(2)}) - (b \triangleleft h_{(1)}) \cdot (v \triangleleft h_{(2)}) \cdot v^* \\ &= ((v \cdot b - b \cdot v) \triangleleft h) \cdot v^* \\ &= 0, \end{aligned}$$

and hence, *mutatis mutandis*, $\sigma(h_{(1)}) \cdot (m \triangleleft h_{(2)}) - (m \triangleleft h_{(1)}) \cdot \sigma(h_{(2)})$. Finally, for all $h, k \in H$,

$$\sigma(hk) = (v \triangleleft hk) \cdot v^* = ((v \triangleleft h) \triangleleft k_{(1)}) \cdot (v^* \triangleleft k_{(2)}) \cdot (v \triangleleft k_{(3)}) \cdot v^* = (\sigma(h) \triangleleft k_{(1)}) \cdot \sigma(k_{(2)}).$$

Hence, $Dv =: \sigma \in ZS_\ell^1(H; B, M)$.

Next, we show that D is a group homomorphism. First, note that

$$D1_B = (h \mapsto (1_B \triangleleft h) \cdot 1_B) = \epsilon(\cdot)1_B = 1_{C(H,B)}.$$

Now, let $v_1, v_2 \in CS_\ell^0(H; B, M)$. Then for all $h \in H$,

$$\begin{aligned} D(v_1 \cdot v_2)(h) &= ((v_1 \cdot v_2) \triangleleft h) \cdot (v_1 \cdot v_2)^* \\ &= (v_1 \triangleleft h_{(1)}) \cdot (v_2 \triangleleft h_{(2)}) \cdot v_1^* v_2^* \\ &= Dv_1(h_{(1)}) \cdot Dv_2(h_{(2)}), \end{aligned}$$

so that, indeed, $D(v_1 \cdot v_2) = Dv_1 \star Dv_2$.

Finally, we show that $BS_\ell^1(H; B, M)$ is central in $ZS_\ell^1(H; B, M)$. Let $v \in CS_\ell^0(H; B, M)$ and $\sigma \in ZS_\ell^1(H; B, M)$. Then, for all $h \in H$,

$$\sigma \star Dv \star \sigma^{-1}(h) = \sigma(h_{(1)}) \cdot (v \triangleleft h_{(2)}) \cdot v^* \cdot \sigma^{-1}(h_{(3)}) = (v \triangleleft h_{(1)}) \cdot v^* \cdot \sigma(h_{(2)}) \cdot \sigma^{-1}(h_{(3)}) = Dv(h),$$

so that, indeed, $\sigma \star Dv \star \sigma^{-1} = Dv$. \square

We now provide a suitable generalisation of the degree 1 group cohomology of a group Γ with coefficients in an \mathbf{R} -linear representation of Γ . That degree 1 group cohomology can be generalised using the appropriate Hochschild cohomology was first observed by Schürmann [37, §3]; this observation has been used, for instance, to characterise the Haagerup property on locally compact quantum groups [19]. We shall use the following variation on this theme to compute the Atiyah space of a crossed product by H .

Proposition-Definition 4.7. Let B be a right H -module $*$ -algebra, and let M be a right H -equivariant B - $*$ -bimodule.

(1) A lazy M -valued Hochschild 0-cochain on H is an element of

$$\mathrm{CH}_\ell^0(H; M) := Z_B(M)_{\mathrm{sa}}.$$

(2) A lazy M -valued Hochschild 1-cocycle on H is $\mu \in Z_B(C(H, M))_{\mathrm{sa}}$, such that

$$(4.8) \quad \forall h, k \in H, \quad \mu(hk) = \mu(h) \triangleleft k + \epsilon(h)\mu(k)$$

and $\mu(1) = 0$; we denote by $\mathrm{ZH}_\ell^1(H; M)$ the set of all lazy M -valued Hochschild 1-cocycles on H , which defines a \mathbf{R} -subspace of $C(H, M)$.

(3) The coboundary map is the homomorphism $D : \mathrm{CH}_\ell^0(H; M) \rightarrow \mathrm{ZH}_\ell^1(H; M)$ with

$$(4.9) \quad \forall m \in \mathrm{CH}_\ell^0(H; M), \forall h \in H, \quad D(m)(h) := m \triangleleft h - m\epsilon(h);$$

thus, a lazy M -valued Hochschild 1-coboundary on H is an element of

$$\mathrm{BH}_\ell^1(H; M) := D(\mathrm{CH}_\ell^0(H; M)) \leq \mathrm{ZH}_\ell^1(H; M).$$

(4) The lazy degree 1 Hochschild cohomology of H with coefficients in M is the group

$$\mathrm{HH}_\ell^1(H; M) := \mathrm{ZH}_\ell^1(H; M) / \mathrm{BH}_\ell^1(H; M).$$

Proof. Note that the right H -module M becomes an H -bimodule with respect to the trivial left H -action. From this perspective, the set $\mathrm{ZH}_\ell^1(H; B, M)$ is an \mathbf{R} -subspace of the \mathbf{C} -vector space of M -valued Hochschild 1-cocycles on H , the set $\mathrm{BH}_\ell^1(H; B, M)$ is an \mathbf{R} -subspace of the \mathbf{C} -vector space of M -valued Hochschild 1-coboundaries on H , and the map D is the restriction of the usual Hochschild coboundary operator. Thus, it suffices to check the inclusion $\mathrm{BH}_\ell^1(H; B, M) \subseteq Z_B(C(H, M))_{\mathrm{sa}}$. Let $m \in \mathrm{CH}_\ell^0(H; M)$, and set $\mu := Dm$. Then, for all $b \in B$ and $h \in H$,

$$\begin{aligned} [\mu, \rho_B(b)](h) &= (m \triangleleft h_{(1)} - m\epsilon(h_{(1)})) \cdot (b \triangleleft h_{(2)}) - (b \triangleleft h_{(1)}) \cdot (m \triangleleft h_{(2)} - m\epsilon(h_{(2)})) \\ &= [m, b] \triangleleft h - [m, b \triangleleft h] = 0. \end{aligned}$$

Hence, indeed, $[\mu, \rho_B(b)] = 0$. \square

We now show that the lazy Sweedler cohomology of H admits a canonical \mathbf{R} -linear action on the lazy Hochschild cohomology of H with relevant coefficients, with respect to which H -equivariant $*$ -derivations canonically yield 1-cocycles in the sense of ordinary group cohomology. We shall compute the gauge action of gauge transformations on gauge potentials for a crossed product by H in terms of the resulting \mathbf{R} -affine actions of lazy Sweedler cohomology on lazy Hochschild cohomology.

Theorem 4.8. Let B be a right H -module $*$ -algebra, and let M be a right H -equivariant B - $*$ -bimodule.

(1) The group $\mathrm{ZS}_\ell^1(H; B, M)$ acts \mathbf{R} -linearly on $\mathrm{ZH}_\ell^1(H; M)$ via conjugation, i.e.,

$$(4.10) \quad \forall \sigma \in \mathrm{ZS}_\ell^1(H; B, M), \forall \mu \in \mathrm{ZH}_\ell^1(H; M), \quad \sigma \triangleright \mu := \sigma \star \mu \star \sigma^{-1}.$$

- (2) The subgroup $BS_\ell^1(H; B, M)$ acts trivially on $ZH_\ell^1(H; M)$ and the subspace $BH_\ell^1(H; M)$ consists of $ZS_\ell^1(H; B, M)$ -invariant vectors, so that the linear action of $ZS_\ell^1(H; B, M)$ on $ZH_\ell^1(H; M)$ descends to a linear action of $HS_\ell^1(H; B, M)$ on $HH_\ell^1(H; M)$.
- (3) Let $\partial : B \rightarrow M$ be an H -equivariant $*$ -derivation. The action of $ZS_\ell^1(H; B, M)$ on $ZH_\ell^1(H; M)$ admits the 1-cocycle $MC[\partial] : ZS_\ell^1(H; B, M) \rightarrow ZH_\ell^1(H; M)$ given by

$$(4.11) \quad \forall \sigma \in ZS_\ell^1(H; B, M), \forall h \in H, \quad MC[\partial](\sigma)(h) := -\partial\sigma(h_{(1)}) \cdot \sigma^*(h_{(2)}).$$

Furthermore, the 1-cocycle $MC[\partial]$ satisfies

$$(4.12) \quad \forall v \in CS_\ell^0(H; B, M), \quad MC[\partial](Dv) = D(-\partial(v)v^*),$$

and hence descends to a 1-cocycle $\tilde{MC}[\partial] : HS_\ell^1(H; B, M) \rightarrow HH_\ell^1(H; M)$ for the induced action of $HS_\ell^1(H; B, M)$ on $HH_\ell^1(H; M)$.

Proof. Let us first show that $ZH_\ell^1(H; M)$ is invariant under conjugation by $ZS_\ell^1(H; B, M)$; note that $\{\sigma \triangleright \mu \mid \sigma \in ZS_\ell^1(H; B, M), \mu \in ZH_\ell^1(H; M)\} \subset Z_B(M)$ since $ZS_\ell^1(H; B, M)$ centralises $\rho_B(B)$. Let $\sigma \in ZS_\ell^1(H; B, M)$ and let $\mu \in ZH_\ell^1(H; M)$. First,

$$(\sigma \triangleright \mu)(1) = \sigma(1) \cdot \mu(1) \cdot \sigma^{-1}(1) = 0.$$

Next, since σ is unitary in the $*$ -algebra $C(H; B)$ and μ is self-adjoint in the $C(H; B)$ - $*$ -bimodule $C(H; M)$, it follows that $\sigma \triangleright \mu = \sigma \star \mu \star \sigma^*$ is self-adjoint. Finally, for all $h, k \in H$,

$$\begin{aligned} (\sigma \triangleright \mu)(hk) &= \sigma(h_{(1)}k_{(1)}) \cdot \mu(h_{(2)}k_{(2)}) \cdot \sigma^{-1}(h_{(3)}k_{(3)}) \\ &= (\sigma(h_{(1)}) \triangleleft k_{(1)}) \cdot \sigma(k_{(2)}) \cdot (\mu(h_{(2)}) \triangleleft k_{(3)} - \epsilon(h_{(2)})\mu(k_{(3)})) \cdot (\sigma^{-1}(h_{(3)}) \triangleleft k_{(4)}) \cdot \sigma^{-1}(k_{(5)}) \\ &= (\sigma(h_{(1)}) \triangleleft k_{(1)}) \cdot (\mu(h_{(2)}) \triangleleft k_{(2)}) \cdot \sigma(k_{(3)}) \cdot (\sigma^{-1}(h_{(3)}) \triangleleft k_{(4)}) \cdot \sigma^{-1}(k_{(5)}) \\ &\quad - (\sigma(h_{(1)}) \triangleleft k_{(1)}) \cdot \sigma(k_{(2)}) \cdot (\sigma^{-1}(h_{(2)}) \triangleleft k_{(3)}) \cdot \mu(k_{(4)}) \cdot \sigma^{-1}(k_{(5)}) \\ &= (\sigma(h_{(1)}) \triangleleft k_{(1)}) \cdot (\mu(h_{(2)}) \triangleleft k_{(2)}) \cdot (\sigma^{-1}(h_{(3)}) \triangleleft k_{(3)}) \cdot \sigma(k_{(4)}) \cdot \sigma^{-1}(k_{(5)}) \\ &\quad - (\sigma(h_{(1)}) \triangleleft k_{(1)}) \cdot (\sigma^{-1}(h_{(2)}) \triangleleft k_{(2)}) \cdot \sigma(k_{(3)}) \cdot \mu(k_{(4)}) \cdot \sigma^{-1}(k_{(5)}) \\ &= (\sigma \triangleright \mu)(h) \triangleleft k - \epsilon(h)(\sigma \triangleright \mu)(k) \end{aligned}$$

by repeated application of Lemma 4.6. Hence, $\sigma \triangleright \mu \in ZH_\ell^1(H; M)$.

Next, let us show that $BS_\ell^1(H; B, M)$ acts trivially by conjugation on $ZH_\ell^1(H; M)$ and that $BH_\ell^1(H; M)$ consists of $ZS_\ell^1(H; B, M)$ -invariant vectors. First, let $v \in CS_\ell^0(H; B, M)$ and let $\mu \in ZH_\ell^1(H; M)$. For all $h \in H$, by Lemma 4.6,

$$\begin{aligned} Dv \triangleright \mu(h) &= (v \triangleleft h_{(1)})v^* \cdot \mu(h_{(2)}) \cdot (v^* \triangleleft h_{(3)})v = (v \triangleleft h_{(1)})v^*(v^* \triangleleft h_{(2)}) \cdot \mu(h_{(3)}) \cdot v \\ &= (v \triangleleft h_{(1)})(v^* \triangleleft h_{(2)}) \cdot \mu(h_{(3)}) \cdot v^*v = \epsilon(h_{(1)})\mu(h_{(2)}) = \mu(h). \end{aligned}$$

so that, indeed, $Dv \triangleright \mu = \mu$. Now, let $m \in CH_\ell^0(H; M)$ and $\sigma \in ZS_\ell^1(H; B, M)$. Then for all $h \in H$, by Lemma 4.6,

$$\begin{aligned} (\sigma \triangleright Dm)(h) &= \sigma(h_{(1)}) \cdot (m \triangleleft h_{(2)} - \epsilon(h_{(2)})m) \cdot \sigma^{-1}(h_{(3)}) \\ &= (m \triangleleft h_{(1)}) \cdot \sigma(h_{(2)})\sigma^{-1}(h_{(3)}) - \sigma(h_{(1)})\sigma^{-1}(h_{(2)}) \cdot m = m \triangleleft h - \epsilon(h)m = Dm(h). \end{aligned}$$

From now on, let $\partial : B \rightarrow M$ be an H -equivariant $*$ -derivation. Let us now show that $MC[\partial]$ is a well-defined function. Let $\sigma \in ZS_\ell^1(H; B, M)$, and let $\mu := MC[\partial](\sigma) \in C(H, M)$. First, $\mu(1) = -\partial\sigma(1) \cdot \sigma^*(1) = 0$. Next, for all $b \in B$ and $h \in H$, by Lemma 4.6,

$$(b \triangleleft h_{(1)}) \cdot \mu(h_{(2)})$$

$$\begin{aligned}
 &= -(b \triangleleft h_{(1)}) \cdot \partial\sigma(h_{(2)}) \cdot \sigma^*(h_{(3)}) \\
 &= -\partial((b \triangleleft h_{(1)}) \cdot \sigma(h_{(2)})) \cdot \sigma^*(h_{(3)}) + \partial(b \triangleleft h_{(1)}) \cdot \sigma(h_{(2)}) \sigma^*(h_{(3)}) \\
 &= -\partial(\sigma(h_{(1)}) \cdot (b \triangleleft h_{(2)})) \cdot \sigma^*(h_{(3)}) + \partial(b) \triangleleft h \\
 &= -\partial\sigma(h_{(1)}) \cdot (b \triangleleft h_{(2)}) \sigma^*(h_{(3)}) - \sigma(h_{(1)}) \cdot (\partial(b) \triangleleft h_{(2)}) \cdot \sigma^*(h_{(3)}) + \partial(b) \triangleleft h \\
 &= -\partial\sigma(h_{(1)}) \cdot \sigma^*(h_{(2)})(b \triangleleft h_{(3)}) - \sigma(h_{(1)}) \sigma^*(h_{(2)}) \cdot (\partial(b) \triangleleft h) + \partial(b) \triangleleft h \\
 &= \mu(h_{(1)}) \cdot (b \triangleleft h_{(2)}),
 \end{aligned}$$

so that $\mu \in Z_B(C(H, M))$. Next, for all $h \in H$, since $\sigma(h_{(1)})\sigma^*(h_{(2)}) = \epsilon(h)1_B$,

$$\mu^*(h) = \left(-\partial\sigma(S(h)_{(1)}^*) \cdot \sigma(S(h)_{(2)}^*) \right)^* = \sigma(h_{(1)}) \cdot \partial\sigma^*(h_{(2)}) = -\partial\sigma(h_{(1)}) \cdot \sigma^*(h_{(2)}) = \mu(h),$$

so that $\mu^* = \mu$. Finally, for all $h, k \in H$, by Lemma 4.6 and the fact that $\sigma \in U(C(H, B))$,

$$\begin{aligned}
 \mu(hk) &= -\partial\sigma(h_{(1)}k_{(1)}) \cdot \sigma^*(h_{(2)}k_{(2)}) \\
 &= -\partial((\sigma(h_{(1)}) \triangleleft k_{(1)}) \cdot \sigma(k_{(2)})) \cdot (\sigma^*(h_{(2)}) \triangleleft k_{(3)}) \sigma^*(k_{(4)}) \\
 &= -\partial((\sigma(h_{(1)}) \triangleleft k_{(1)}) \cdot \sigma(k_{(2)})) \cdot \sigma^*(k_{(3)})(\sigma^*(h_{(2)}) \triangleleft k_{(4)}) \\
 &= -(\partial\sigma(h_{(1)}) \triangleleft k_{(1)}) \cdot \sigma(k_{(2)}) \sigma^*(k_{(3)})(\sigma^*(h_{(2)}) \triangleleft k_{(4)}) \\
 &\quad - (\sigma(h_{(1)}) \triangleleft k_{(1)}) \cdot \partial\sigma(k_{(2)}) \cdot \sigma^*(k_{(3)})(\sigma^*(h_{(2)}) \triangleleft k_{(4)}) \\
 &= -(\partial\sigma(h_{(1)}) \triangleleft k_{(1)}) \cdot \epsilon(k_{(2)})(\sigma^*(h_{(2)}) \triangleleft k_{(3)}) - (\sigma(h_{(1)}) \triangleleft k_{(1)}) \cdot \mu(k_{(2)}) \cdot (\sigma^*(h_{(2)}) \triangleleft k_{(3)}) \\
 &= -(\partial\sigma(h_{(1)}) \triangleleft k_{(1)})(\sigma^*(h_{(2)}) \triangleleft k_{(2)}) - (\sigma(h_{(1)}) \triangleleft k_{(1)})(\sigma^*(h_{(2)}) \triangleleft k_{(2)}) \cdot \mu(k_{(3)}) \\
 &= \mu(h) \triangleleft k + \epsilon(h)\mu(k).
 \end{aligned}$$

Hence, $\text{MC}[\partial](\sigma) =: \mu \in \text{ZH}_\ell^1(H; M)$.

Next, let us show that $\text{MC}[\partial]$ is a 1-cocycle for the conjugation action of $\text{ZS}_\ell^1(H; B, M)$ on $\text{ZH}_\ell^1(H; M)$. Let $\sigma, \tau \in \text{ZS}_\ell^1(H; B, M)$. Then, for all $h \in H$,

$$\begin{aligned}
 \text{MC}[\partial](\sigma \star \tau)(h) &= -\partial(\sigma(h_{(1)})\tau(h_{(2)})) \cdot \tau^*(h_{(3)})\sigma^*(h_{(4)}) \\
 &= -\partial\sigma(h_{(1)}) \cdot \tau(h_{(2)})\tau^*(h_{(3)})\sigma^*(h_{(4)}) - \sigma(h_{(1)}) \cdot \partial\tau(h_{(2)}) \cdot \tau^*(h_{(3)})\sigma^*(h_{(4)}) \\
 &= \text{MC}[\partial](\sigma)(h) + \sigma \triangleright \text{MC}[\partial](\tau)(h),
 \end{aligned}$$

so that, indeed, $\text{MC}[\partial](\sigma \star \tau) = \text{MC}[\partial](\sigma) + \sigma \triangleright \text{MC}[\partial](\tau)$.

Finally, let us show that $\text{MC}[\partial]$ satisfies (4.12). First, observe that $\partial(C_B(M)) \subset Z_B(M)$, since for all $b \in C_B(M)$ and $c \in B$,

$$\partial(b) \cdot c = \partial(bc) - b \cdot \partial(c) = \partial(cb) - b \cdot \partial(c) = \partial(c) \cdot b + c \cdot \partial(b) - b \cdot \partial(c) = c \cdot \partial(b).$$

Now, let $v \in \text{CS}_\ell^0(H; B, M)$. Then, for all $h \in H$, by the above observation,

$$\begin{aligned}
 \text{MC}[Dv](h) &= -\partial((v \triangleleft h_{(1)}) \cdot v^*) \cdot ((v^* \triangleleft h_{(2)}) \cdot v) \\
 &= -(\partial(v) \triangleleft h_{(1)}) \cdot v^*(v^* \triangleleft h_{(2)}) \cdot v - (v \triangleleft h_{(1)}) \cdot \partial(v^*) \cdot (v^* \triangleleft h_{(2)}) \cdot v \\
 &= -(\partial(v) \cdot v^*) \triangleleft h - \epsilon(h)v \cdot \partial(v^*) \\
 &= -(\partial(v) \cdot v^*) \triangleleft h + \epsilon(h)\partial(v) \cdot v^* \\
 &= D(-\partial(v) \cdot v^*)(h),
 \end{aligned}$$

as was claimed. \square

Definition 4.9. Let B be a right H -module $*$ -algebra, let M be a right H -equivariant B - $*$ -bimodule, and let $\partial : B \rightarrow M$ be a right H -equivariant $*$ -derivation. We define the *Maurer–Cartan 1-cocycle* of ∂ to be the 1-cocycle $\text{MC}[\partial] : \text{ZS}^1(H; B, M) \rightarrow \text{ZH}^1(H; M)$ of Theorem 4.8 induced by ∂ .

When H is a group algebra, our constructions reduce to degree 1 group cohomology in a manner compatible, e.g., with the results of Čačić–Mesland [13, Thm. 3.11].

Proposition 4.10 (cf. Sweedler [39, Thm. 3.1]). *Suppose that $H = \mathbb{C}[\Gamma]$ for Γ a group. Let B be a right Γ - $*$ -algebra, and let M be a right Γ -equivariant B - $*$ -bimodule, so that $\text{U}(\text{C}_B(B \oplus M))$ defines a multiplicative Γ -module and $Z_B(M)_{\text{sa}}$ defines a \mathbf{R} -linear representation of Γ .*

- (1) *The map $r_B := (f \mapsto f|_\Gamma) : \text{ZS}_\ell^1(H; B, M) \rightarrow Z^1(\Gamma, \text{U}(\text{C}_B(B \oplus M)))$ is a group isomorphism, such that $r_B \circ D = d_\Gamma$ and $r_B(\text{BS}_\ell^1(H; B, M)) = \text{B}^1(\Gamma, \text{U}(\text{C}_B(B \oplus M)))$; hence, in particular, it descends to an isomorphism of Abelian groups*

$$\text{HS}_\ell^1(H; M) \xrightarrow{\sim} \text{H}^1(\Gamma, \text{U}(\text{C}_B(B \oplus M))).$$

- (2) *The map $r_M := (\mu \mapsto \mu|_\Gamma) : \text{ZH}_\ell^1(H; M) \rightarrow Z^1(\Gamma, Z_B(M)_{\text{sa}})$ defines an isomorphism of \mathbf{R} -vector spaces, such that $r_M \circ D = d_\Gamma$ and $r_M(\text{BH}_\ell^1(H; M)) = \text{B}^1(\Gamma, Z_B(M)_{\text{sa}})$; hence, in particular, it descends to an isomorphism of Abelian groups*

$$\text{HH}_\ell^1(H; M) \xrightarrow{\sim} \text{H}^1(\Gamma, Z_B(M)_{\text{sa}}).$$

- (3) *The conjugation action of $\text{ZS}_\ell^1(H; B, M)$ on $\text{ZH}_\ell^1(H; M)$ is trivial. Thus, for every Γ -equivariant $*$ -derivation $\partial : B \rightarrow M$, the induced map*

$$\text{MC}_\Gamma := r_M \circ \text{MC}[\partial] \circ r_B^{-1} : Z^1(\Gamma, \text{U}(\text{C}_B(B \oplus M))) \rightarrow Z^1(\Gamma, Z_B(M)_{\text{sa}})$$

is a group homomorphism satisfying

$$\begin{aligned} \forall \sigma \in Z^1(\Gamma, \text{U}(\text{C}_B(B \oplus M))), \forall \gamma \in \Gamma, \quad \text{MC}_\Gamma(\sigma)(\gamma) &= -\partial\sigma(\gamma) \cdot \sigma(\gamma)^*, \\ \forall v \in \text{U}(\text{C}_B(B \oplus M)), \quad \text{MC}_\Gamma(d_\Gamma v) &= d_\Gamma(-\partial(v) \cdot v^*), \end{aligned}$$

so that, in particular, it descends to a group homomorphism

$$\text{H}^1(\Gamma, \text{U}(\text{C}_B(B \oplus M))) \rightarrow \text{H}^1(\Gamma, Z_B(M)_{\text{sa}}).$$

Proof. Given sets X and Y , let $\mathcal{F}(X, Y)$ be the set of all functions from X to Y . It follows that $\mathcal{F}(\Gamma, B)$ is a unital $*$ -algebra with respect to the pointwise operations, that any unital $*$ -subalgebra A of B yields a unital $*$ -subalgebra $\mathcal{F}(\Gamma, A)$ of $\mathcal{F}(\Gamma, B)$, that $\mathcal{F}(\Gamma, M)$ defines a $\mathcal{F}(\Gamma, B)$ - $*$ -bimodule with respect to pointwise operations, and that any B - $*$ -sub-bimodule L of M yields a $\mathcal{F}(\Gamma, B)$ - $*$ -sub-bimodule of $\mathcal{F}(\Gamma, M)$. By abuse of notation, let

$$r_B := (f \mapsto f|_\Gamma) : \mathcal{C}(H, B) \rightarrow \mathcal{F}(\Gamma, B), \quad r_M := (\mu \mapsto \mu|_\Gamma) : \mathcal{C}(H, M) \rightarrow \mathcal{F}(\Gamma, M).$$

Since Γ is a basis for H , it follows that r_B and r_M are bijections, and since Γ consists of group-like elements and satisfies $S \circ *|_\Gamma = \text{id}_\Gamma$, it follows that r_B is a $*$ -isomorphism and that r_M is a $*$ -preserving \mathbb{C} -linear map satisfying

$$\forall \sigma, \tau \in \mathcal{C}(H, B), \forall \mu \in \mathcal{C}(H, M), \quad r_M(\sigma \star \mu \star \tau) = r_B(\sigma) \cdot r_M(\mu) \cdot r_B(\tau).$$

The fact that Γ consists of group-like elements now implies that

$$\begin{aligned} r_B(\text{U}(\text{C}_{\mathcal{C}(H, B)}(\rho_B(B) \oplus \rho_M(M)))) &= \mathcal{F}(\Gamma, \text{U}(\text{C}_B(B \oplus M))), \\ r_M(Z_B(\mathcal{C}(H, M))_{\text{sa}}) &= \mathcal{F}(\Gamma, Z_B(M)_{\text{sa}}), \end{aligned}$$

from which our claims now follow by routine calculations. \square

Similarly, when H is the universal enveloping algebra of a real Lie algebra, our constructions reduce to standard degree 1 Lie cohomology (with a small caveat related to lazy Sweedler cohomology) when the coefficient algebra satisfies the following condition.

Definition 4.11. Let \mathfrak{g} be a real Lie algebra, and let A be a commutative right \mathfrak{g} -module $*$ -algebra. We say that A has \mathfrak{g} -equivariant exponentials if for every $a \in A_{\text{sa}}$ there exists a unitary $v_a \in U(A)$, such that

$$\forall X \in \mathfrak{g}, \quad (v_a) \triangleleft X = i(a \triangleleft X) \cdot v_a,$$

in which case we call v_a a \mathfrak{g} -equivariant exponential of a .

Proposition 4.12 (cf. Sweedler [39, Prop. 4.2]). *Suppose that $H = \mathcal{U}(\mathfrak{g})$ for \mathfrak{g} a real Lie algebra. Let B be a right \mathfrak{g} -module $*$ -algebra, and let M be a right \mathfrak{g} -equivariant B - $*$ -bimodule, so that $C_B(B \oplus M)_{\text{sa}}$ and $Z_B(M)_{\text{sa}}$ both define right \mathfrak{g} -modules over \mathbf{R} .*

- (1) *The map $r_B := (f \mapsto -i f|_{\mathfrak{g}}) : ZS_{\ell}^1(H; B, M) \rightarrow Z^1(\mathfrak{g}, C_B(B \oplus M)_{\text{sa}})$ defines an isomorphism of Abelian groups, such that*

$$r_B(\text{BS}_{\ell}^1(H; B, M)) = \{(X \mapsto -i(v \triangleleft X)v^*) \mid v \in U(C_B(B \oplus M))\}.$$

Moreover, if $C_B(B \oplus M)$ has \mathfrak{g} -equivariant exponentials, then for every $b \in C_B(B \oplus M)_{\text{sa}}$ and every \mathfrak{g} -equivariant exponential $v_b \in U(C_B(B \oplus M))$ of b ,

$$r_B \circ D(v_b) = d_{\mathfrak{g}}(b),$$

so that $B^1(\mathfrak{g}, C_B(B \oplus M)_{\text{sa}}) \leq r_B(\text{BS}_{\ell}^1(H; B, M))$, and hence, r_B induces a short exact sequence of Abelian groups

$$0 \rightarrow \frac{r_B(\text{BS}_{\ell}^1(H; B, M))}{B^1(\mathfrak{g}, C_B(B \oplus M)_{\text{sa}})} \rightarrow H^1(\mathfrak{g}, C_B(B \oplus M)_{\text{sa}}) \rightarrow \text{HS}_{\ell}^1(H; B, M) \rightarrow 0.$$

- (2) *The map $r_M := (\mu \mapsto \mu|_{\mathfrak{g}}) : ZH_{\ell}^1(H; M) \rightarrow Z^1(\mathfrak{g}, Z_B(M)_{\text{sa}})$ is an isomorphism of \mathbf{R} -vector spaces, such that $r_M \circ D = d_{\mathfrak{g}}$ and $r_M(\text{BH}_{\ell}^1(H; M)) = B^1(\mathfrak{g}, Z_B(M)_{\text{sa}})$; hence, in particular, r_M descends to an isomorphism of Abelian groups*

$$\text{HH}_{\ell}^1(H; M) \xrightarrow{\sim} H^1(\mathfrak{g}, Z_B(M)_{\text{sa}}).$$

- (3) *The conjugation action of $ZS_{\ell}^1(H; B, M)$ on $ZH_{\ell}^1(H; M)$ is trivial. Thus, for every \mathfrak{g} -equivariant $*$ -derivation $\partial : B \rightarrow M$, the induced map*

$$\text{MC}_{\mathfrak{g}} := r_M \circ \text{MC}[\partial] \circ r_B^{-1} : Z^1(\mathfrak{g}, C_B(B \oplus M)_{\text{sa}}) \rightarrow Z^1(\mathfrak{g}, Z_B(M)_{\text{sa}})$$

is a group homomorphism satisfying

$$\begin{aligned} \forall c \in Z^1(\mathfrak{g}, C_B(B \oplus M)_{\text{sa}}), \quad \forall X \in \mathfrak{g}, \quad \text{MC}_{\mathfrak{g}}(\sigma)(X) &= -i\partial c(X), \\ \forall b \in C_B(B \oplus M)_{\text{sa}}, \quad \text{MC}_{\mathfrak{g}}(d_{\mathfrak{g}} b) &= d_{\mathfrak{g}}(-i\partial b), \end{aligned}$$

so that, in particular, it descends to a group homomorphism

$$H^1(\mathfrak{g}, C_B(B \oplus M)_{\text{sa}}) \rightarrow H^1(\mathfrak{g}, Z_B(M)_{\text{sa}}).$$

Proof. Recall that $H = \mathcal{U}(\mathfrak{g})$ inherits a filtration from the complexified tensor algebra of \mathfrak{g} . Let $r_B := (\sigma \mapsto -i\sigma|_{\mathfrak{g}}) : ZS_{\ell}^1(H; B, M) \rightarrow \text{Hom}(\mathfrak{g}, B)$. First, given $\sigma \in ZS_{\ell}^1(H; B, M)$, for all $k \in \mathbf{N} \cup \{0\}$, $h \in H_k$, and $X \in \mathfrak{g}$,

$$\sigma(h \cdot X) = (\sigma(h) \triangleleft X_{(1)}) \sigma(X_{(2)}) = (\sigma(h) \triangleleft X) \sigma(1) + (\sigma(h) \triangleleft 1) \sigma(X) = \sigma(h) \triangleleft X + \sigma(h) \sigma(X),$$

so that by induction on $k \in \mathbf{N} \cup \{0\}$, the map σ is uniquely determined on $H = \bigcup_{k \geq 0} H_k$ by $r_B(\sigma)$; hence, r_B is injective. Next, note that for all $\sigma, \tau \in ZS_{\ell}^1(H; B, M)$ and $X \in \mathfrak{g}$,

$$(\sigma \star \tau)(X) = \sigma(X_{(1)})\tau(X_{(2)}) = \sigma(X)\tau(1) + \sigma(1)\tau(X) = \sigma(X) + \tau(X),$$

and that $1_{C(H,B)}|_{\mathfrak{g}} = -i\epsilon(\cdot)1_B|_{\mathfrak{g}} = 0$; hence, r_B is an injective group homomorphism from $ZS_\ell^1(H; B, M)$ to the additive group $\text{Hom}(\mathfrak{g}, B)$. Next, for $\sigma \in ZS_\ell^1(H; B, M)$ and $X \in \mathfrak{g}$,

$$0 = \epsilon(X)1_B = \sigma(X_{(1)})\sigma^*(X_{(2)}) = \sigma(X)\sigma(S(1)^*) + \sigma(1)\sigma(S(X)^*) = \sigma(X) + \sigma(X)^*;$$

hence, r_B maps into $\text{Hom}(\mathfrak{g}, B_{\text{sa}})$. Next, for $\sigma \in ZS_\ell^1(H; B, M)$, $b \in B$, and $X \in \mathfrak{g}$,

$$\begin{aligned} 0 &= [\sigma, \rho_B(b)](X) \\ &= \sigma(X_{(1)})(b \triangleleft X_{(2)}) - (b \triangleleft X_{(1)})\sigma(X_{(2)}) \\ &= \sigma(X)(b \triangleleft 1) - (b \triangleleft X)\sigma(1) + \sigma(1)(b \triangleleft X) - (b \triangleleft 1)\sigma(X) \\ &= [\sigma(X), b], \end{aligned}$$

so that r_B maps into $\text{Hom}(\mathfrak{g}, Z(B)_{\text{sa}})$; in fact, *mutatis mutandis*, this shows that r_B actually maps into $\text{Hom}(\mathfrak{g}, C_B(B \oplus M)_{\text{sa}})$. Finally, for all $\sigma \in ZS_\ell^1(H; B, M)$ and $X, Y \in \mathfrak{g}$,

$$\begin{aligned} 0 &= \sigma(XY - YX) - \sigma([X, Y]) \\ &= (\sigma(X) \triangleleft Y_{(1)}) \sigma(Y_{(2)}) - (\sigma(Y) \triangleleft X_{(1)}) \sigma(X_{(2)}) - \sigma([X, Y]) \\ &= (\sigma(X) \triangleleft Y) \sigma(1) + (\sigma(X) \triangleleft 1) \sigma(Y) - (\sigma(Y) \triangleleft X) \sigma(1) - (\sigma(Y) \triangleleft 1) \sigma(X) - \sigma([X, Y]) \\ &= \sigma(X) \triangleleft Y - \sigma(Y) \triangleleft X - \sigma([X, Y]) \\ &= \text{id}_{\mathfrak{g}}(r_B(\sigma)) \end{aligned}$$

so that r_B maps into $Z^1(\mathfrak{g}, C_B(B \oplus M)_{\text{sa}})$. Conversely, given $c \in Z^1(\mathfrak{g}, C_B(B \oplus M)_{\text{sa}})$, one can construct $r_B^{-1}(c) \in ZS_\ell^1(H; B, M)$ inductively by setting $r_B^{-1}(c)(1) := 1$ and setting

$$\forall k \in \mathbf{N} \cup \{0\}, \forall h \in H_k, \forall X \in \mathfrak{g}, \quad r_B^{-1}(c)(h \cdot X) := r_B^{-1}(c)(h) \triangleleft X + i r_B^{-1}(c)(h) \cdot c(X);$$

in particular, that $r_B^{-1}(c)$ is well-defined on $H = \mathcal{U}(\mathfrak{g})$ follows from the 1-cocycle identity

$$\forall X, Y \in \mathfrak{g}, \quad c(X) \triangleleft Y - c(Y) \triangleleft X - c([X, Y]) = 0$$

together with the universal property of $\mathcal{U}(\mathfrak{g})$. Thus, $r_B : ZS_\ell^1(H; B, M) \rightarrow Z^1(\mathfrak{g}, C_B(B \oplus M)_{\text{sa}})$ is a group isomorphism.

Next, by definition of $D : CS_\ell^0(H; B, M) \rightarrow ZS_\ell^1(H; B, M)$ and of r_B , it follows that

$$r_B(\text{BS}_\ell^1(H; B, M)) = \{(X \mapsto -i(v \triangleleft X)v^* \mid v \in U(C_B(B \oplus M)))\} \leq Z(\mathfrak{g}, C_B(B \oplus M)_{\text{sa}}).$$

Suppose, now, that B has \mathfrak{g} -equivariant exponentials. Given $b \in C_B(B \oplus M)_{\text{sa}}$, for every \mathfrak{g} -equivariant exponential $v_b \in U(C_B(B \oplus M))$ of b and every $X \in \mathfrak{g}$,

$$Dv_b(X) = (v_b \triangleleft X)v_b^* = i(b \triangleleft X)v_b v_b^* = i(b \triangleleft X) = \text{id}_{\mathfrak{g}}(b)(X),$$

so that $d_{\mathfrak{g}}(b) = r_B \circ D(v_b) \in r_B(ZS_\ell^1(H; B, M))$; hence, as additive groups,

$$B^1(\mathfrak{g}; C_B(B \oplus M)_{\text{sa}}) \leq r_B(\text{BS}_\ell^1(H; B, M)).$$

Now, the above proof that $r_B : ZS_\ell^1(H; B, M) \rightarrow Z^1(\mathfrak{g}, C_B(B \oplus M)_{\text{sa}})$ is a group isomorphism implies, *mutatis mutandis*, that

$$r_M := (\mu \mapsto -i\mu|_{\mathfrak{g}}) : ZH_\ell^1(H; M) \rightarrow Z^1(\mathfrak{g}, Z_B(M)_{\text{sa}})$$

is an isomorphism of \mathbf{R} -vector spaces; indeed, for all $m \in Z_B(M)_{\text{sa}}$ and $X \in \mathfrak{g}$,

$$Dm(X) = m \triangleleft X - \epsilon(X)m = m \triangleleft X = d_{\mathfrak{g}}(m)(X),$$

so that $r_M \circ D = d_{\mathfrak{g}}$, and hence $B^1(\mathfrak{g}, Z_B(M)_{\text{sa}}) = r_M(\text{BH}_\ell^1(H; M))$. Moreover, $ZS_\ell^1(H; B, M)$ acts trivially on $ZH_\ell^1(H; M)$, since for all $\sigma \in ZS_\ell^1(H; B, M)$, $\mu \in ZH_\ell^1(H; M)$, and $X \in \mathfrak{g}$,

$$\sigma \star \mu \star \sigma^{-1}(X) = \sigma(X_{(1)})\mu(X_{(2)})\sigma^{-1}(X_{(3)})$$

$$\begin{aligned}
 &= \sigma(X)\mu(1)\sigma^{-1}(1) + \sigma(1)\mu(X)\sigma^{-1}(1) + \sigma(1)\mu(1)\sigma^{-1}(X) \\
 &= \mu(X).
 \end{aligned}$$

Finally, let $\partial : B \rightarrow M$ be a \mathfrak{g} -equivariant $*$ -derivation. For all $c \in Z^1(\mathfrak{g}, Z_B(M)_{\text{sa}})$ and $X \in \mathfrak{g}$, we find that

$$\begin{aligned}
 \text{MC}_{\mathfrak{g}}[\partial](c)(X) &= -\partial(r_B^{-1}(c)(X_{(1)})) \cdot r_B^{-1}(c)^*(X_{(2)}) \\
 &= -\partial(r_B^{-1}(c)(X)) \cdot r_B^{-1}(c)^*(1) - \partial(r_B^{-1}(c)(1)) \cdot r_B^{-1}(c)^*(X) \\
 &= -i\partial c(X).
 \end{aligned}$$

Hence, in particular, for all $b \in B$ and $X \in \mathfrak{g}$,

$$\text{MC}_{\mathfrak{g}}[\partial](d_{\mathfrak{g}}(b))(X) = -i\partial(b \triangleleft X) = \partial(-ib) \triangleleft X = d_{\mathfrak{g}}(-i\partial b). \quad \square$$

We now refine our construction of lazy Hochschild cohomology in the manner that, as we shall see, will encode prolongability of (relative) gauge potentials.

Proposition-Definition 4.13. Let B be a right H -module $*$ -algebra, let M be a right H -equivariant B - $*$ -module, and let Ω be a right H -module graded $*$ -algebra Ω with $\Omega^0 = B$ and $\Omega^1 = M$ that is generated over B by M . Then

$$\begin{aligned}
 \text{ZH}_{\ell}^1(H; M, \Omega) &:= \text{ZH}_{\ell}^1(H; M) \cap C_{C(H; \Omega)}(\rho_{\Omega}(\Omega)), \\
 \text{BH}_{\ell}^1(H; M, \Omega) &:= D(Z_B(M)_{\text{sa}} \cap Z(\Omega)) \subset \text{BH}_{\ell}^1(H; M) \cap C_{C(H; \Omega)}(\rho_{\Omega}(\Omega)), \\
 \text{HH}_{\ell}^1(H; M, \Omega) &:= \text{ZH}_{\ell}^1(H; M, \Omega) / \text{BH}_{\ell}^1(H; M, \Omega).
 \end{aligned}$$

Moreover, $\text{ZH}_{\ell}^1(H; M, \Omega)$ is a $\text{ZS}_{\ell}^1(H; B, M)$ -invariant subspace of $\text{ZH}_{\ell}^1(H; M)$, and hence the conjugation action of $\text{ZS}_{\ell}^1(H; B, M)$ on $\text{ZH}_{\ell}^1(H; M, \Omega)$ descends to a \mathbf{R} -linear action on the quotient space $\text{HH}_{\ell}^1(H; M, \Omega)$.

Proof. The first non-trivial claim to check is the inclusion $\text{BH}_{\ell}^1(H; M, \Omega) \subset C_{C(H; \Omega)}(\rho_{\Omega}(\Omega))$. Let $m \in C_{\Omega}(M)_{\text{sa}}$. Then for all $m' \in M$ and $h \in H$,

$$\begin{aligned}
 [Dm, \rho_M(m')](h) &= (m \triangleleft h_{(1)} - \epsilon(h_{(1)})m) \wedge (m' \triangleleft h_{(2)}) + (m' \triangleleft h_{(1)}) \wedge (m \triangleleft h_{(2)} - \epsilon(h_{(2)})m) \\
 &= [m, m'] \triangleleft h - [m, m' \triangleleft h] = 0,
 \end{aligned}$$

so that, indeed, $Dm \in C_{C(H; \Omega)}(\rho_{\Omega}(\Omega))$.

The other non-trivial claim to check is $\text{ZS}_{\ell}^1(H; B, M)$ -invariance of $\text{ZH}_{\ell}^1(H; M, \Omega)$. Let $\sigma \in \text{ZS}_{\ell}^1(H; B, M)$ and let $\mu \in \text{ZH}_{\ell}^1(H; M, \Omega)$. Then, for all $m \in M$ and $h \in H$,

$$\begin{aligned}
 [\sigma \triangleright \mu, \rho_M(m)](h) &= \sigma(h_{(1)}) \cdot \mu(h_{(2)}) \cdot \sigma^{-1}(h_{(3)}) \wedge m \triangleleft h_{(4)} \\
 &\quad - m \triangleleft h_{(1)} \wedge \sigma(h_{(2)}) \cdot \mu(h_{(3)}) \cdot \sigma^{-1}(h_{(4)}) \\
 &= \sigma(h_{(1)}) \cdot \mu(h_{(2)}) \wedge m \triangleleft h_{(3)} \cdot \sigma^{-1}(h_{(4)}) \\
 &\quad - \sigma(h_{(1)}) \cdot m \triangleleft h_{(2)} \wedge \mu(h_{(3)}) \cdot \sigma^{-1}(h_{(4)}) \\
 &= \sigma \star [\mu, \rho_M(m)] \star \sigma^{-1}(h) = 0,
 \end{aligned}$$

so that, indeed, $\sigma \triangleright \mu \in C_{C(H; \Omega)}(\rho_{\Omega}(\Omega))$. \square

We conclude by observing that Maurer–Cartan 1-cocycles are automatically compatible with the above refinement.

Corollary 4.14. *Let B be a right H -module $*$ -algebra, and let (Ω_B, d_B) be an H -equivariant SODC over B . Then*

$$\text{MC}[d_B](ZS_\ell^1(H; B, \Omega_B^1)) \subset ZH_\ell^1(H; \Omega_B^1, \Omega_B), \quad \text{MC}[d_B](BS_\ell^1(H; B, \Omega_B^1)) \subset BH_\ell^1(H; \Omega_B^1, \Omega_B),$$

so that $\text{MC}[d_B]$ is a 1-cocycle for the restricted action of $ZS_\ell^1(H; B, \Omega_B^1)$ on $ZH_\ell^1(H; \Omega_B^1, \Omega_B)$, and hence descends to a 1-cocycle $\widetilde{\text{MC}}[d_B, \Omega_B] : \text{HS}_\ell^1(H; B, \Omega_B^1) \rightarrow \text{HH}_\ell^1(H; \Omega_B^1, \Omega_B)$ for the action of $\text{HS}_\ell^1(H; B, \Omega_B^1)$ on $\text{HH}_\ell^1(H; \Omega_B^1, \Omega_B)$.

Proof. First, let $\sigma \in ZS_\ell^1(H; B, \Omega_B^1)$. Then, for all $b \in B$ and $h \in H$,

$$\begin{aligned} & [\text{MC}[d_B](\sigma), \rho_M(d_B b)](h) \\ &= -d_B \sigma(h_{(1)}) \cdot \sigma^{-1}(h_{(2)}) \wedge (d_B(b) \triangleleft h_{(3)}) - (d_B(b) \triangleleft h_{(1)}) \wedge d_B \sigma(h_{(2)}) \cdot \sigma^{-1}(h_{(3)}) \\ &= -d_B \sigma(h_{(1)}) \wedge d_B(b \triangleleft h_{(2)}) \cdot \sigma^{-1}(h_{(3)}) - d_B(b \triangleleft h_{(1)}) \wedge d_B \sigma(h_{(2)}) \cdot \sigma^{-1}(h_{(3)}) \\ &= -d_B(\sigma(h_{(1)}) \cdot (d_B(b) \triangleleft h_{(2)}) - (d_B(b) \triangleleft h_{(1)}) \cdot \sigma(h_{(2)})) \cdot \sigma^{-1}(h_{(3)}) \\ &= 0, \end{aligned}$$

so that, indeed, $\text{MC}[d_B](\sigma) \in C_{C(H; \Omega_B)}(\rho_{\Omega_B}(\Omega_B))$.

Now, let $v \in \text{CS}_\ell^0(H, B, \Omega_B^1) = \text{U}(C_B(B \otimes \Omega_B^1))$, so that $\text{MC}[d_B](Dv) = D(-d_B(v)v^*)$ by Theorem 4.8. Then, for all $b \in B$,

$$\begin{aligned} [-d_B(v)v^*, d_B(b)] &= -d_B(v)v^* \wedge d_B(b) - d_B(b) \wedge d_B(v)v^* \\ &= -(d_B(v) \wedge d_B(b) + d_B(b) \wedge d_B(v))v^* \\ &= -d_B(v \cdot d_B(b) - d_B(b) \cdot v)v^* = 0, \end{aligned}$$

so that, indeed, $-d_B(v)v^* \in Z(\Omega)$. \square

4.2. Gauge transformations and (relative) gauge potentials. In this section, we use lazy Sweedler cohomology, lazy Hochschild cohomology, and Maurer–Cartan 1-cocycles to compute the gauge group \mathfrak{G} , the Atiyah space $\mathfrak{A}t$, and the affine action of \mathfrak{G} on $\mathfrak{A}t$ in the case of a crossed product by the Hopf $*$ -algebra H . This yields a far-reaching generalisation of the computation by Čačić–Mesland [13, §3.4] of the noncommutative T^m -gauge theory in this sense of a crossed product by Z^m .

Let B be a right H -module $*$ -algebra with fixed right H -invariant SODC (Ω_B, d_B) . Let

$$P := B \rtimes H,$$

so that $P = H \otimes_C B$ together with the multiplication, $*$ -structure, and left H -coaction defined, respectively, by

$$\begin{aligned} \forall h, h' \in H, \forall b, b' \in B, \quad (h \otimes b)(h' \otimes b') &:= hh'_{(1)} \otimes (b \triangleleft h'_{(2)})b', \\ \forall h \in H, \forall b \in B, \quad (h \otimes b)^* &:= h_{(1)}^* \otimes b^* \triangleleft h_{(2)}^*, \\ \forall h \in H, \forall b \in B, \quad \delta_P(h \otimes b) &:= h_{(1)} \otimes (h_{(2)} \otimes b); \end{aligned}$$

it follows that P defines a principal left H -comodule $*$ -algebra, such that ${}^{\text{co}H}P = B$; in fact, P can be viewed as a globally trivial (cleft) principal H -comodule algebra with trivialisation given by the injective left H -covariant $*$ -homomorphism $(h \mapsto h \otimes 1_B) : H \hookrightarrow P$. There is a canonical second-order horizontal calculus on P , which we shall use exclusively from now on; it can be straightforwardly constructed as follows.

Proposition 4.15. *Let $\Omega_{P, \text{hor}} := \Omega_B \rtimes H$, so that $\Omega_{P, \text{hor}} = H \otimes_C \Omega_B$ with the grading*

$$\forall k \in \{0, 1, 2\}, \quad \Omega_{P, \text{hor}}^k = H \otimes_C \Omega_B^k$$

and the multiplication, $*$ -structure, and left H -coaction defined, respectively, by

$$\begin{aligned} \forall h, h' \in H, \forall \beta, \beta' \in \Omega_B, \quad (h \otimes \beta) \wedge (h' \otimes \beta') &:= hh'_{(1)} \otimes (\beta \triangleleft h'_{(2)})\beta', \\ \forall h \in H, \forall \beta \in \Omega_B, \quad (h \otimes \beta)^* &:= h_{(1)}^* \otimes \beta^* \triangleleft h_{(2)}^*, \\ \forall h \in H, \forall \beta \in B, \quad \delta_{\Omega_{P,\text{hor}}}(h \otimes \beta) &:= h_{(1)} \otimes (h_{(2)} \otimes \beta). \end{aligned}$$

Then $(\Omega_B, d_B; \Omega_{P,\text{hor}}, \beta \mapsto 1 \otimes \beta)$ defines a second-order horizontal calculus on $P := B \rtimes H$.

In what follows, we will find it notationally convenient to view $\Omega_{P,\text{hor}}$ as the graded left H -comodule $*$ -algebra generated by the graded $*$ -subalgebra Ω_B of H -coinvariants together with the left H -subcomodule $*$ -subalgebra H in degree 0, subject to the relation $1_H = 1_{\Omega_B}$ and the braided supercommutation relation

$$\forall h \in H, \forall \beta \in \Omega_B, \quad \beta h = h_{(1)}(\beta \triangleleft h_{(2)}).$$

We begin by expressing the gauge group \mathfrak{G} of P with respect to the canonical first-order horizontal calculus $(\Omega_B^1, d_B; \Omega_{P,\text{hor}}^1)$ in terms of lazy Sweedler cohomology; note that inner gauge transformations will correspond precisely to lazy Sweedler 1-coboundaries.

Proposition 4.16 (cf. Brzeziński [8, Thm. 5.4], Ćaćić–Mesland [13, Thm. 3.36.(2)]). *The function $\text{Op} : \text{ZS}_\ell^1(H; B, \Omega_B^1) \rightarrow \mathfrak{G}$ given by*

$$(4.13) \quad \forall \sigma \in \text{ZS}_\ell^1(H; B, \Omega_B^1), \forall h \in H, \forall b \in B, \quad \text{Op}(\sigma)(hb) := h_{(1)}\sigma(h_{(2)})b$$

is a group isomorphism. Moreover,

$$(4.14) \quad \forall v \in \text{CS}_\ell^0(H; B, \Omega_B^1), \quad \text{Op}(Dv) = \text{Ad}_v,$$

so that $\text{Op}(\text{BS}_\ell^1(H; B, \Omega_B^1)) = \text{Inn}(\mathfrak{G})$ is the central subgroup of inner gauge transformations on P with respect to $(\Omega_B^1, d_B; \Omega_{P,\text{hor}}^1)$, and hence Op descends to a group isomorphism

$$\widetilde{\text{Op}} : \text{HS}_\ell^1(H; B, \Omega_B^1) \xrightarrow{\sim} \mathfrak{G}/\text{Inn}(\mathfrak{G}) =: \text{Out}(\mathfrak{G}).$$

Proof. Let $\text{Aut}_B(P)$ denote the group of all automorphisms $f : P \rightarrow P$ of P as a left H -comodule right B -module, such that $f|_B = \text{id}_B$, and let $\mathcal{A}(P)$ denote the group of all invertible elements $\sigma \in C(H, B)$, such that $\sigma(1) = 1$; observe that

$$\text{Inn}(\mathfrak{G}) \leq \mathfrak{G} \leq \text{Aut}_B(P), \quad \text{BS}_\ell^1(H; B, \Omega_B^1) \leq \text{ZS}_\ell^1(H; B, \Omega_B^1) \leq \mathcal{A}(P).$$

By [8, Thm. 5.4], *mutatis mutandis*, the map $\text{Op} : \mathcal{A}(P) \rightarrow \text{Aut}_B(P)$ defined by

$$\forall \sigma \in \mathcal{A}(P), \forall h \in H, \forall b \in B, \quad \text{Op}(\sigma)(hb) := h_{(1)}\sigma(h_{(2)})b$$

is a well-defined group isomorphism with inverse given by

$$\forall f \in \text{Aut}_B(P), \forall h \in H, \quad \text{Op}^{-1}(f)(h) := S(h_{(1)})f(h_{(2)}).$$

Note, moreover, that for all $\sigma \in \mathcal{A}(P)$, the inverse σ^{-1} is given by

$$\forall h \in H, \quad \sigma^{-1}(h) := \sigma(S(h_{(1)})) \triangleleft h_{(2)},$$

since, for all $h \in H$,

$$\epsilon(h)1_B = \sigma(S(h_{(1)})h_{(2)}) = (\sigma(S(h_{(1)}) \triangleleft h_{(2)}) \cdot \sigma(h_{(3)}).$$

Let us first show that $\text{Op}^{-1}(\mathfrak{G}) = \text{ZS}_\ell^1(H; B, \Omega_B^1)$. Let $\sigma \in \mathcal{A}(P)$ and set $f := \text{Op}(\sigma)$. First, for all $h, k \in H$,

$$\begin{aligned} f(hk) - f(h)f(k) &= h_{(1)}k_{(1)}\sigma(h_{(2)}k_{(2)}) - h_{(1)}\sigma(h_{(2)})k_{(1)}\sigma(k_{(2)}) \\ &= h_{(1)}k_{(1)}(\sigma(h_{(2)}k_{(2)})\epsilon(k_{(3)}) - (\sigma(h_{(2)}) \triangleleft k_{(2)})\sigma(k_{(3)})), \end{aligned}$$

while for all $h \in H$ and $b \in B$, since $bh = h_{(1)}(b \triangleleft h_{(2)})$,

$$\begin{aligned} f(bh) - f(b)f(h) &= h_{(1)}\sigma(h_{(2)})(b \triangleleft h_{(3)}) - bh_{(1)}\sigma(h_{(2)}) \\ &= h_{(1)}(\sigma(h_{(2)})(b \triangleleft h_{(3)}) - (b \triangleleft h_{(2)})\sigma(h_{(3)})), \end{aligned}$$

so that $f \in \text{Aut}_B(P)$ is an algebra automorphism if and only if $\sigma \in C_{C(H,B)}(\rho_B(B))$ and

$$\forall h, k \in H, \quad \sigma(hk) = (\sigma(h) \triangleleft k_{(1)})\sigma(k_{(2)}).$$

In particular, if f is an algebra automorphism, then σ^{-1} is given by

$$\forall h \in H, \quad \sigma^{-1}(h) := \sigma(S(h_{(1)})) \triangleleft h_{(2)},$$

since, for all $h \in H$,

$$\epsilon(h)1_B = \sigma(S(h_{(1)})h_{(2)}) = (\sigma(S(h_{(1)}) \triangleleft h_{(2)}) \cdot \sigma(h_{(3)}).$$

Now, suppose that f is an algebra automorphism. Then, for all $h \in H$,

$$\begin{aligned} f(S(h)^*)^* - f(S(h)) &= (S(h_{(2)})^* \sigma(S(h_{(1)}))^*)^* - S(h_{(2)})\sigma(S(h_{(1)})) \\ &= \sigma(S(h_{(1)}))^* S(h_{(2)})^* - (\sigma(S(h_{(1)}))^* S(h_{(2)})^*)^* \\ &= \sigma(S(h_{(1)}))^* S(h_{(2)})^* - (S(h_{(3)})^* (\sigma(S(h_{(1)}))^* \triangleleft S(h_{(2)}))^*)^* \\ &= (\sigma(S(h_{(1)}))^* \epsilon(h_{(2)}) - \sigma(S(h_{(1)})) \triangleleft h_{(2)}) S(h_{(2)})^* \\ &= (\sigma^*(h_{(1)}) - \sigma^{-1}(h_{(1)})) S(h_{(2)})^*, \end{aligned}$$

so that f is a $*$ -automorphism if and only if σ is unitary. Finally, suppose that f is a $*$ -automorphism. Then, for all $h \in H$ and $\beta \in B$, since $\beta h = h_{(1)} \cdot (\beta \triangleleft h_{(2)})$,

$$\begin{aligned} f(h_{(1)}) \cdot (\beta \triangleleft h_{(2)}) - \beta \cdot f(h) &= h_{(1)}\sigma(h_{(2)}) \cdot (\beta \triangleleft h_{(2)}) - \beta \cdot h_{(1)}\sigma(h_{(2)}) \\ &= h_{(1)} \cdot (\sigma(h_{(2)}) \cdot (\beta \triangleleft h_{(2)}) - (\beta \triangleleft h_{(2)}) \cdot \sigma(h_{(3)})), \end{aligned}$$

so that $f \in \mathfrak{G}$ if and only if $\sigma \in C_{C(H,B)}(\rho_{\Omega_B^1}(\Omega_B^1))$.

Let us now show that $\text{Op} \circ D = (v \mapsto \text{Ad}_v)$ on $\text{CS}_\ell^0(H; B, \Omega_B^1) = \text{U}(C_B(B \oplus \Omega_B^1))$, which will imply the rest of the claim. Let $v \in \text{U}(C_B(B \oplus \Omega_B^1))$. Then, for all $h \in H$ and $b \in B$,

$$\text{Op}(Dv)(hb) = h_{(1)}(v \triangleleft h_{(2)})v^*b = vhbv^* = \text{Ad}_v(hb). \quad \square$$

Next, we express the Atiyah space \mathfrak{At} of P with respect to the canonical first-order horizontal calculus $(\Omega_B^1, d_B; \Omega_{P, \text{hor}}^1)$ and its space of translations at in terms of lazy Hochschild cohomology; note that inner relative gauge potentials will correspond precisely to lazy Hochschild 1-coboundaries.

Proposition 4.17 (cf. Ćačić–Mesland [13, Thm. 3.36.(1)]). *We have an isomorphism of \mathbb{R} -affine spaces $\text{Op} : \text{ZH}_\ell^1(H; \Omega_B^1) \rightarrow \mathfrak{At}$ given by*

$$(4.15) \quad \forall \mu \in \text{ZH}_\ell^1(H; \Omega_B^1) \forall h \in H, \forall b \in B, \quad \text{Op}(\mu)(hb) := h \cdot d_B(b) + h_{(1)} \cdot \mu(h_{(2)}) \cdot b,$$

whose linear part $\text{Op}_0 : \text{ZH}_\ell^1(H; \Omega_B^1) \xrightarrow{\sim} \text{at}$ is given by

$$(4.16) \quad \forall \mu \in \text{ZH}_\ell^1(H; \Omega_B^1), \forall h \in H, \forall b \in B, \quad \text{Op}_0(\mu)(hb) := h_{(1)} \cdot \mu(h_{(2)}) \cdot b.$$

Moreover,

$$(4.17) \quad \forall \alpha \in \text{CH}_\ell^0(H; \Omega_B^1), \quad \text{Op}_0(D\alpha) = \text{ad}_\alpha,$$

so that $\text{Op}_0(\text{BH}_\ell^1(H; \Omega_B^1) = \text{Inn}(\mathfrak{at})$ is the subspace of inner relative gauge potentials on P with respect to $(\Omega_B^1, d_B; \Omega_{P, \text{hor}}^1)$, and hence Op descends to an isomorphism of \mathbf{R} -affine spaces

$$\overline{\text{Op}} : \text{HH}_\ell^1(H; \Omega_B^1) \xrightarrow{\sim} \mathfrak{at}/\text{Inn}(\mathfrak{at}) =: \text{Out}(\mathfrak{at})$$

with linear part $\overline{\text{Op}}_0 : \text{HH}_\ell^1(H; \Omega_B^1) \xrightarrow{\sim} \mathfrak{at}/\text{Inn}(\mathfrak{at}) =: \text{Out}(\mathfrak{at})$ descending from Op_0 .

Proof. Let $\text{Hom}_B(P, \Omega_{P, \text{hor}}^1)_0$ denote the \mathbf{R} -vector space of all morphisms $f : P \rightarrow \Omega_{P, \text{hor}}^1$ of left H -comodule right B -modules, such that $f|_B = 0$, and let $C(H, \Omega_B^1)_0$ denote the \mathbf{R} -vector space of all $\mu \in C(H, \Omega_B^1)$, such that $\mu(1) = 0$; observe that

$$\text{Inn}(\mathfrak{at}) \leq \mathfrak{at} \leq \text{Hom}_B(P, \Omega_{P, \text{hor}}^1)_0, \quad \text{BH}_\ell^1(H; \Omega_B^1) \leq \text{ZH}_\ell^1(H; \Omega_B^1) \leq C(H, \Omega_B^1)_0.$$

By [8, Thm. 5.4], *mutatis mutandis*, the map $\text{Op}_0 : C(H, \Omega_B^1)_0 \rightarrow \text{Hom}_B(P, \Omega_{P, \text{hor}}^1)_0$ defined by

$$\forall \mu \in C(H, \Omega_B^1)_0, \forall h \in H, \forall b \in B, \quad \text{Op}_0(\mu)(hb) := h_{(1)}\mu(h_{(2)}) \cdot b$$

is a well-defined \mathbf{R} -linear isomorphism with inverse given by

$$\forall A \in \text{Hom}_B(P, \Omega_{P, \text{hor}}^1)_0, \forall h \in H, \quad \text{Op}_0^{-1}(A)(h) := S(h_{(1)})A(h_{(2)}).$$

Let us first show that $\text{Op}_0^{-1}(\mathfrak{at}) = \text{ZH}_\ell^1(H; \Omega_B^1)$ and that $\text{Op}_0 \circ D = (\alpha \mapsto \text{ad}_\alpha)$ on the domain $\text{CH}_\ell^0(H; \Omega_B^1) = Z_B(\Omega_B^1)_{\text{sa}}$. Let $\mu \in C(H, \Omega_B^1)_0$ and set $A := \text{Op}(\mu)$. By the proof of Proposition 4.16, *mutatis mutandis*, it follows that $A \in \text{Hom}_B(P, \Omega_{P, \text{hor}}^1)_0$ is a P -bimodule derivation if and only if $\mu \in Z_B(C(H, \Omega_B^1))$ and

$$\forall h, k \in H, \quad \mu(hk) = \mu(h) \triangleleft k + \epsilon(h)\mu(k),$$

in which case, for all $h \in H$,

$$0 = \mu(S(h_{(1)})h_{(2)}) = \mu(S(h_{(1)})) \triangleleft h_{(2)} + \epsilon(S(h_{(1)}))\mu(h_{(2)}) = \mu(S(h_{(1)})) \triangleleft h_{(2)} + \mu(h).$$

Thus, if A is a P -bimodule derivation, then, by the proof of Proposition 4.16,

$$\forall h \in H, \quad A(S(h)^*)^* + A(S(h)) = (\mu^*(h_{(1)}) - \mu(h_{(1)})) S(h_{(2)})^*,$$

so that the A is a $*$ -derivation if and only if $\mu = \mu^*$. The rest now follows from the proof of Proposition 4.16, *mutatis mutandis*.

Let us now show that $\text{Op} : \text{ZH}_\ell^1(H; \Omega_B^1) \rightarrow \mathfrak{at}$ is a well-defined isomorphism of \mathbf{R} -affine spaces with linear part Op_0 ; it suffices to show that the map $\nabla_0 : P \rightarrow \Omega_{P, \text{hor}}^1$ defined by

$$\forall h \in H, \forall b \in B, \quad \nabla_0(hb) := h \cdot d_B(b)$$

is a gauge potential, so that $\text{Op}(0) = \nabla_0$ is well-defined. First, for all $h, k \in H$ and $b, c \in B$, since $hbkc = hk_{(1)}(b \triangleleft h_{(2)})c$,

$$\begin{aligned} \nabla_0(hbkc) - \nabla_0(hb)kc - hb\nabla_0(kc) &= hk_{(1)} d_B((b \triangleleft k_{(2)})c) - h d_B(b)kc - hbk d_B(c) \\ &= hk_{(1)} \cdot (d_B((b \triangleleft k_{(2)})c) - d_B(b \triangleleft k_{(1)}) \cdot (b \triangleleft k_{(1)}) \cdot d_B(c)) \\ &= 0. \end{aligned}$$

Next, for all $h \in H$ and $b \in B$, since $(hb)^* = h_{(1)}^*(b^* \triangleleft h_{(2)}^*)$ and $(h d_B(b))^* = h_{(1)}^* \cdot d_B(b)^* \triangleleft h_{(2)}^*$,

$$\nabla_0((hb)^*) + \nabla_0(hb)^* = h_{(1)}^* d_B(b^* \triangleleft h_{(2)}^*) + (h d_B(b))^* = h_{(1)}^* \cdot (d_B(b^*) + d_B(b)^*) \triangleleft h_{(2)}^* = 0.$$

Finally, by construction, $\nabla_0|_B = d_B$. \square

We now relate the affine action of the gauge group \mathfrak{G} on the Atiyah space $\mathfrak{A}t$ to the affine action of lazy Sweedler cohomology with coefficients in (B, Ω_B^1) on lazy Hochschild cohomology with coefficients in Ω_B^1 induced by the Maurer–Cartan 1-cocycle $MC[d_B]$ of the derivation $d_B : B \rightarrow \Omega_B^1$.

Proposition 4.18 (cf. Brzeziński–Majid [11, Prop. 4.8], Čačić–Mesland [13, Thm. 3.36.(3)]). *For every $\sigma \in ZS_\ell^1(H; B, \Omega_B^1)$ and every $\mu \in ZH_\ell^1(H; \Omega_B^1)$, we have*

$$(4.18) \quad \text{Op}(\sigma) \triangleright \text{Op}(\mu) = \text{Op}(\sigma \triangleright \mu + MC[d_B](\sigma)),$$

$$(4.19) \quad \text{Op}(\sigma) \triangleright \text{Op}_0(\mu) = \text{Op}_0(\sigma \triangleright \mu).$$

and hence, at the level of cohomology,

$$(4.20) \quad \widetilde{\text{Op}}([\sigma]) \triangleright \widetilde{\text{Op}}([\mu]) = \widetilde{\text{Op}}([\sigma] \triangleright [\mu] + \widetilde{MC}[d_B]([\sigma])),$$

$$(4.21) \quad \widetilde{\text{Op}}([\sigma]) \triangleright \widetilde{\text{Op}}_0([\mu]) = \widetilde{\text{Op}}_0([\sigma] \triangleright [\mu]).$$

Thus, the maps $\text{Op} \times \text{Op}$ and $\widetilde{\text{Op}} \times \widetilde{\text{Op}}$ define groupoid isomorphisms

$$\begin{aligned} ZS_\ell^1(H; B, \Omega_B^1) \times ZH_\ell^1(H; \Omega_B^1) &\rightarrow \mathfrak{G} \times \mathfrak{A}t, \\ HS_\ell^1(H; B, \Omega_B^1) \times HH_\ell^1(H; \Omega_B^1) &\rightarrow \text{Out}(\mathfrak{G}) \times \text{Out}(\mathfrak{A}t), \end{aligned}$$

respectively, where $ZS_\ell^1(H; B, \Omega_B^1)$ acts affine-linearly on $ZH_\ell^1(H; \Omega_B^1)$ with 1-cocycle $MC[d_B]$ and $HS_\ell^1(H; B, \Omega_B^1)$ acts affine-linearly on $HH_\ell^1(H; \Omega_B^1)$ with 1-cocycle $\widetilde{MC}[d_B]$.

Proof. Let $\sigma \in ZS_\ell^1(H; B, \Omega_B^1)$ and $\mu \in ZH_\ell^1(H; \Omega_B^1)$; note that

$$\text{Op}(\sigma) \triangleright \text{Op}(\mu) = \text{Op}(\sigma) \triangleright (\text{Op}(0) + \text{Op}_0(\mu)) = \text{Op}(\sigma) \triangleright \text{Op}(0) + \text{Op}(\sigma) \triangleright \text{Op}_0(\mu).$$

On the one hand, for all $h \in H$ and $b \in B$, since $\sigma \star \sigma^* = \epsilon(\cdot)1_B$,

$$\begin{aligned} \text{Op}(\sigma) \triangleright \text{Op}(0)(hb) &= \text{Op}(\sigma) \circ \text{Op}(0) \circ \text{Op}(\sigma^*)(hb) \\ &= h_{(1)}\sigma(h_{(2)}) \cdot (d_B \sigma^*(h_{(3)}) \cdot b + \sigma^*(h_{(3)}) d_B(b)) \\ &= -h_{(1)} \cdot d_B \sigma(h_{(2)}) \cdot \sigma^*(h_{(3)})b + h_{(1)}\sigma(h_{(2)})\sigma^*(h_{(3)}) \cdot d_B(b) \\ &= h_{(1)} \cdot MC[d_B](\sigma)(h_{(2)}) \cdot b + h \cdot d_B(b) \\ &= \text{Op}(MC[d_B](\sigma))(hb), \end{aligned}$$

so that $\text{Op}(\sigma) \triangleright \text{Op}(0) = \text{Op}(MC[d_B](\sigma))$. On the other hand, for all $h \in H$ and $b \in B$,

$$\begin{aligned} \text{Op}(\sigma) \triangleright \text{Op}_0(\mu)(hb) &= \text{Op}(\sigma) \circ \text{Op}_0(\mu) \circ \text{Op}(\sigma^*)(hb) \\ &= h_{(1)}\sigma(h_{(2)}) \cdot \mu(h_{(3)}) \cdot \sigma^*(h_{(4)})b \\ &= h_{(1)} \cdot (\sigma \triangleright \mu)(h_{(2)}) \cdot b \\ &= \text{Op}_0(\sigma \triangleright \mu)(hb), \end{aligned}$$

so that $\text{Op}(\sigma) \triangleright \text{Op}_0(\mu) = \text{Op}_0(\sigma \triangleright \mu)$. \square

All of the above results yield straightforward characterisations of prolongable gauge transformations and prolongable (relative) gauge potentials in terms of lazy Sweedler cohomology and lazy Hochschild cohomology, respectively; in particular, we shall find that every gauge transformation is automatically prolongable.

Corollary 4.19. *Let \mathfrak{G}^{Pr} be the prolongable gauge group of P with respect to the canonical second-order horizontal calculus $(\Omega_B, d_B; \Omega_{P, \text{hor}})$. Let $\mathfrak{A}t^{\text{Pr}}$ be the prolongable Atiyah space*

of P with respect to $(\Omega_B, d_B; \Omega_{P, \text{hor}})$, let at^{Pr} be its translation space, and let $\text{Inn}(\text{at}^{\text{Pr}}) \subset \text{at}^{\text{Pr}}$ be the subspace of all inner prolongable gauge potentials. Then $\mathfrak{G}^{\text{Pr}} = \mathfrak{G}$ and

$$\text{Op}^{-1}(\mathfrak{A}t^{\text{Pr}}) = \text{Op}_0^{-1}(\text{at}^{\text{Pr}}) = \text{ZH}_\ell^1(H; \Omega_B^1, \Omega_B),$$

$$\text{Op}_0^{-1}(\text{Inn}(\text{at}^{\text{Pr}})) = \text{BH}_\ell^1(H; \Omega_B^1, \Omega_B),$$

hence Op induces an isomorphism $\widetilde{\text{Op}}^{\text{Pr}} : \text{HH}_\ell^1(H; \Omega_B^1, \Omega_B) \xrightarrow{\sim} \mathfrak{A}t^{\text{Pr}} / \text{Inn}(\text{at}^{\text{Pr}}) =: \text{Out}(\mathfrak{A}t^{\text{Pr}})$ of \mathbf{R} -affine spaces with linear part $\widetilde{\text{Op}}_0^{\text{Pr}} : \text{HH}_\ell^1(H; \Omega_B^1, \Omega_B) \xrightarrow{\sim} \text{at}^{\text{Pr}} / \text{Inn}(\text{at}^{\text{Pr}}) =: \text{Out}(\text{at}^{\text{Pr}})$ induced by Op_0 . Moreover, for all $\sigma \in \text{ZS}_\ell^1(H; B, \Omega_B^1, \Omega_B)$ and $\mu \in \text{ZH}_\ell^1(H; \Omega_B^1, \Omega_B)$,

$$(4.22) \quad \widetilde{\text{Op}}([\sigma]) \triangleright \widetilde{\text{Op}}^{\text{Pr}}([\mu]) = \widetilde{\text{Op}}^{\text{Pr}}([\sigma] \triangleright [\mu] + \widetilde{\text{MC}}[d_B, \Omega_B]([\sigma])),$$

$$(4.23) \quad \widetilde{\text{Op}}([\sigma]) \triangleright \widetilde{\text{Op}}_0^{\text{Pr}}([\mu]) = \widetilde{\text{Op}}_0^{\text{Pr}}([\sigma] \triangleright [\mu]).$$

Thus, the maps $\text{Op} \times \text{Op}|_{\text{ZH}_\ell^1(H; \Omega_B^1, \Omega_B)}$ and $\widetilde{\text{Op}} \times \widetilde{\text{Op}}^{\text{Pr}}$ define groupoid isomorphisms

$$\text{ZS}_\ell^1(H; B, \Omega_B^1) \ltimes \text{ZH}_\ell^1(H; \Omega_B^1, \Omega_B) \rightarrow \mathfrak{G}^{\text{Pr}} \ltimes \mathfrak{A}t^{\text{Pr}},$$

$$\text{HS}_\ell^1(H; B, \Omega_B^1) \ltimes \text{HH}_\ell^1(H; \Omega_B^1, \Omega_B) \rightarrow \text{Out}(\mathfrak{G}^{\text{Pr}}) \ltimes \text{Out}(\mathfrak{A}t^{\text{Pr}}),$$

respectively, where $\text{ZS}_\ell^1(H; B, \Omega_B^1)$ acts affinely on $\text{ZH}_\ell^1(H; \Omega_B^1, \Omega_B)$ with 1-cocycle $\text{MC}[d_B]$ and $\text{HS}_\ell^1(H; B, \Omega_B^1)$ acts affinely on $\text{HH}_\ell^1(H; \Omega_B^1, \Omega_B)$ with 1-cocycle $\widetilde{\text{MC}}[d_B]$.

Proof. Before continuing, note that for all $h, k \in H$ and $b, b', c \in B$,

$$hb \cdot d_B(b') \cdot kc = hk_{(1)}(b \triangleleft k_{(2)}) \cdot d_B(b' \triangleleft k_{(3)}) \cdot c.$$

Let us first show that $\mathfrak{G} = \mathfrak{G}^{\text{Pr}}$. Let $\sigma \in \text{ZS}_\ell^1(H; B, \Omega_B^1)$ be given. Then, for all $h, k \in H$, $b, c \in B$, and $\alpha, \beta \in \Omega_B^1$,

$$\begin{aligned} & \text{Op}(\sigma)(hb) \cdot \alpha \wedge \beta \cdot \text{Op}(\sigma)(kc) \\ &= h_{(1)}\sigma(h_{(2)})b \cdot \alpha \wedge \beta \cdot k_{(1)}\sigma(k_{(2)})c \\ &= h_{(1)}k_{(1)}(\sigma(h_{(2)}) \triangleleft k_{(2)})(b \triangleleft k_{(3)}) \cdot (\alpha \triangleleft k_{(4)}) \wedge (\beta \triangleleft k_{(5)}) \cdot \sigma(k_{(6)})c \\ &= h_{(1)}k_{(1)}(\sigma(h_{(2)}) \triangleleft k_{(2)})\sigma(k_{(3)})(b \triangleleft k_{(4)}) \cdot (\alpha \triangleleft k_{(5)}) \wedge (\beta \triangleleft k_{(6)}) \cdot c \\ &= h_{(1)}k_{(1)}\sigma(h_{(2)}k_{(2)})(b \triangleleft k_{(3)}) \cdot (\alpha \triangleleft k_{(4)}) \wedge (\beta \triangleleft k_{(5)}) \cdot c \\ &= \text{Op}(\sigma)(hk_{(1)}) \cdot ((b \triangleleft k_{(2)}) \cdot (\alpha \triangleleft k_{(3)}) \wedge (\beta \triangleleft k_{(4)}) \cdot c), \end{aligned}$$

where $hb \cdot \alpha \wedge \beta \cdot kc = hk_{(1)} \cdot ((b \triangleleft k_{(2)}) \cdot (\alpha \triangleleft k_{(3)}) \wedge (\beta \triangleleft k_{(4)}) \cdot c)$, so that $\text{Op}(\sigma) \in \mathfrak{G}^{\text{Pr}}$ with

$$\forall h \in H, \forall \eta \in \Omega_B^2, \quad \text{Op}(\sigma)_*(h\eta) = h_{(1)}\sigma(h_{(2)}) \cdot \eta.$$

Next, observe that $\text{Op}(0) \in \mathfrak{A}t^{\text{Pr}}$. Indeed, for all $h, k \in H$ and $b, b', c \in B$,

$$\begin{aligned} & \text{Op}(0)(hb) \wedge d_B(b') \cdot kc - hb \cdot d_B(b') \wedge \text{Op}(0)(kc) \\ &= h \cdot d_B(b) \wedge d_B(b') \cdot kc - hb \cdot d_B(b') \wedge k \cdot d_B(c) \\ &= hk_{(1)} \cdot d_B((b \triangleleft k_{(2)}) \cdot d_B(b' \triangleleft k_{(3)}) \cdot c), \end{aligned}$$

where $hb \cdot d_B(b') \cdot kc = hk_{(1)}(b \triangleleft k_{(2)}) \cdot d_B(b' \triangleleft k_{(3)}) \cdot c$, so that $\text{Op}(0) \in \mathfrak{A}t^{\text{Pr}}$ with

$$\forall h \in H, \forall \beta \in \Omega_B^1, \quad \text{Op}(0)^{\text{Pr}}(h \cdot \beta) = h \cdot d_B(\beta).$$

Next, let us show that $\text{Op}_0^{-1}(\text{at}^{\text{Pr}}) = \text{ZH}_\ell^1(H; \Omega_B^1, \Omega_B)$; since $\text{Op}(0) \in \text{at}^{\text{Pr}}$, this will also imply that $\text{Op}^{-1}(\mathfrak{A}t^{\text{Pr}}) = \text{ZH}_\ell^1(H; \Omega_B^1, \Omega_B)$. Let $\mu \in \text{ZH}_\ell^1(H; \Omega_B^1)$ be given. Then, for all $h \in H$ and $\beta \in \Omega_B^1$, so that $\beta \cdot h = h_{(1)} \cdot \beta \triangleleft h_{(2)}$,

$$\text{Op}_0(\mu)(h_{(1)}) \wedge \beta \triangleleft h_{(2)} + \beta \wedge \text{Op}_0(\mu)(h) = h_{(1)} \cdot \mu(h_{(2)}) \wedge \beta \triangleleft h_{(3)} + \beta \wedge h_{(1)} \cdot \mu(h_{(2)})$$

$$\begin{aligned}
&= h_{(1)} \cdot (\mu(h_{(2)}) \wedge \beta \triangleleft h_{(3)} + \beta \triangleleft h_{(2)} \wedge \mu(h_{(3)})) \\
&= h_{(1)} \cdot [\mu, \rho_{\Omega_B^1}(\beta)](h_{(2)}).
\end{aligned}$$

Thus, $\text{Op}_0(\mu) \in \mathfrak{at}^{\text{PF}}$ if and only if $\mu \in \text{ZH}_\ell^1(H; \Omega_B^1, \Omega_B)$, in which case

$$\forall h \in H, \forall \beta \in \Omega_B^1, \quad \text{Op}_0(\mu)^{\text{PF}}(h \cdot \beta) := h_{(1)} \cdot \mu(h_{(2)}) \wedge \beta.$$

Finally, let us show that $\text{Op}_0^{-1}(\text{Inn}(\mathfrak{at}^{\text{PF}})) = \text{BH}_\ell^1(H; \Omega_B^1, \Omega_B)$. Let $\alpha \in Z_B(\Omega_B^1)_{\text{sa}}$. Then, for all $h \in H$ and $\beta \in \Omega_B^1$,

$$\begin{aligned}
\text{Op}_0(D\alpha)^{\text{PF}}(h \cdot \beta) - [\alpha, h \cdot \beta] &= h_{(1)} \cdot (\alpha \triangleleft h_{(2)} - \epsilon(h_{(2)})\alpha) \wedge \beta - \alpha \wedge h \cdot \beta - h \cdot \beta \wedge \alpha \\
&= -h \cdot [\alpha, \beta]_{C(H, \Omega_B)},
\end{aligned}$$

so that $\text{Op}_0(D\alpha) \in \text{Inn}(\mathfrak{at}^{\text{PF}})$ if and only if $\alpha \in (\Omega_B^1)_{\text{sa}} \cap Z(\Omega_B) = \text{CH}_\ell^0(H; \Omega_B^1, \Omega_B)$. \square

We conclude by observing that field strength as a map on the prolongable Atiyah space \mathfrak{A}^{PF} admits a straightforward reinterpretation as an \mathbf{R} -affine quadratic map between spaces of lazy Hochschild 1-cocycles that, in turn, descends to a map between lazy Hochschild cohomology groups. Given a choice of bicovariant FODC (Ω_H^1, d_H) on H , this reinterpretation will prove crucial to characterising (Ω_H^1, d_H) -adapted prolongable gauge potentials in terms of lazy Hochschild cohomology.

Proposition 4.20. *The map $\mathcal{F} : \text{ZH}_\ell^1(H; \Omega_B^1, \Omega_B) \rightarrow \text{ZH}_\ell^1(H; \Omega_B^2)$ defined by*

$$\forall h \in H, \forall \mu \in \text{ZH}_\ell^1(H; \Omega_B^1, \Omega_B), \quad \mathcal{F}[\mu](h) := S(h_{(1)}) \cdot \mathbf{F}[\text{Op}(\mu)](h)$$

is an \mathbf{R} -affine quadratic map, satisfying

$$\begin{aligned}
\forall \mu \in \text{ZH}_\ell^1(H; \Omega_B^1, \Omega_B), \quad \mathcal{F}[\mu] &= -i(d_B \mu(\cdot) + \tfrac{1}{2}[\mu, \mu]_{C(H, \Omega_B)}), \\
\forall \mu, \nu \in \text{ZH}_\ell^1(H; \Omega_B^1, \Omega_B), \quad \mathcal{F}[\mu + \nu] - \mathcal{F}[\mu] - \mathcal{F}[\nu] &= -i[\mu, \nu]_{C(H, \Omega_B)}, \\
\forall \alpha \in (\Omega_B^1)_{\text{sa}} \cap Z(\Omega_B), \quad \mathcal{F}[D\alpha] &= D(-i d_B \alpha),
\end{aligned}$$

$$\forall \sigma \in \text{ZS}_\ell^1(H; B, \Omega_B^1), \forall \mu \in \text{ZH}_\ell^1(H; \Omega_B^1, \Omega_B), \quad \mathcal{F}[\sigma \triangleright \mu + \text{MC}[d_B](\sigma)] = \sigma \triangleright \mathcal{F}[\mu].$$

Hence, \mathcal{F} descends to an \mathbf{R} -affine quadratic map $\widetilde{\mathcal{F}} : \text{HH}_\ell^1(H; \Omega_B^1, \Omega_B) \rightarrow \text{HH}_\ell^1(H; \Omega_B^2)$, satisfying

$$\begin{aligned}
\forall \sigma \in \text{ZS}_\ell^1(H; B, \Omega_B^1), \forall \mu \in \text{ZH}_\ell^1(H; \Omega_B^1, \Omega_B), \\
\widetilde{\mathcal{F}}[[\sigma] \triangleright [\mu] + \widetilde{\text{MC}}[d_B, \Omega_B]([\sigma)]] = [\sigma] \triangleright \widetilde{\mathcal{F}}[[\mu]].
\end{aligned}$$

Proof. First, note that $\mathcal{F} : \text{ZH}_\ell^1(H; \Omega_B^1, \Omega_B) \rightarrow \text{ZH}_\ell^1(H; \Omega_B^2)$ is well-defined by the proof of Proposition 4.17, *mutatis mutandis*, hence \mathbf{R} -affine quadratic by Proposition 3.27.

Next, let $\mu \in \text{ZH}_\ell^1(H; \Omega_B^1, \Omega_B)$. Then, for all $h \in H$,

$$\begin{aligned}
i\mathcal{F}[\mu](h) &= S(h_{(1)}) \cdot \text{Op}(\mu)^{\text{PF}} \circ \text{Op}(\mu)(h_{(2)}) \\
&= S(h_{(1)}) \cdot \text{Op}(\mu)^{\text{PF}}(h_{(2)} \cdot \mu(h_{(3)})) \\
&= S(h_{(1)}) \cdot (h_{(2)} \cdot \mu(h_{(3)}) \wedge \mu(h_{(4)}) + h_{(2)} \cdot d_B \mu(h_{(3)})) \\
&= d_B \mu(h) + \mu(h_{(1)}) \wedge \mu(h_{(2)}),
\end{aligned}$$

so that $\mathcal{F}[\mu] = -i(d_B \mu(\cdot) + \tfrac{1}{2}[\mu, \mu]_{C(H, \Omega_B)})$. Thus, for all $\mu, \nu \in \text{ZH}_\ell^1(H; \Omega_B^1, \Omega_B)$,

$$\mathcal{F}[\mu + \nu] = d_B(\mu + \nu)(\cdot) + \tfrac{1}{2}[\mu + \nu, \mu + \nu]_{C(H, \Omega_B)} = \mathcal{F}[\mu] + \mathcal{F}[\nu] + [\mu, \nu]_{C(H, \Omega_B)}.$$

Now, let $\alpha \in (\Omega_B^1)_{\text{sa}} \cap Z(\Omega_B)$; note that $-i d_B \alpha \in Z_B(\Omega_B^2)_{\text{sa}}$, since for all $b \in B$,

$$b \cdot d_B \alpha - d_B \alpha \cdot b = d_B(b \cdot \alpha) - d_B b \wedge \alpha - d_B(\alpha \cdot b) - \alpha \wedge d_B b = 0.$$

Then, for all $h \in H$,

$$\begin{aligned} i\mathcal{F}[D\alpha](h) &= d_B(\alpha \triangleleft h + \epsilon(h)\alpha) + (\alpha \triangleleft h_{(1)} + \epsilon(h_{(1)})\alpha) \wedge (\alpha \triangleleft h_{(2)} + \epsilon(h_{(2)})\alpha) \\ &= (d_B \alpha + \frac{1}{2}[\alpha, \alpha]_{\Omega_B}) \triangleleft h + \epsilon(h) (d_B \alpha + \frac{1}{2}[\alpha, \alpha]_{\Omega_B}) + [\alpha, \alpha \triangleleft h]_{\Omega_B} \\ &= d_B(\alpha) \triangleleft h + \epsilon(h) d_B(\alpha), \end{aligned}$$

so that $\mathcal{F}[D\alpha] = D(-i d_B \alpha) \in \text{BH}_\ell^1(H; \Omega_B^2)$.

Finally, let $\sigma \in \text{ZS}_\ell^1(H; B, \Omega_B^1)$ and $\mu \in \text{ZH}_\ell^1(H; \Omega_B^1, \Omega_B)$. Then, for all $h \in H$,

$$\begin{aligned} \mathcal{F}[\sigma \triangleright \mu + \text{MC}[d_B](\sigma)](h) &= S(h_{(1)}) \cdot \mathbf{F}[\text{Op}(\sigma) \triangleright \text{Op}(\mu)](h_{(2)}) \\ &= S(h_{(1)}) \cdot \text{Op}(\sigma)_* \circ \mathbf{F}[\text{Op}(\mu)] \circ \text{Op}(\sigma^*)(h_{(2)}) \\ &= S(h_{(1)}) \cdot \text{Op}(\sigma)_* \circ \mathbf{F}[\text{Op}(\mu)](h_{(2)} \sigma^*(h_{(3)})) \\ &= S(h_{(1)}) \cdot \text{Op}(\sigma)_*(h_{(2)} \cdot \mathcal{F}[\mu](h_{(3)}) \cdot \sigma^*(h_{(4)})) \\ &= S(h_{(1)}) h_{(2)} \sigma(h_{(3)}) \cdot \mathcal{F}[\mu](h_{(4)}) \cdot \sigma^*(h_{(5)}) \\ &= \sigma(h_{(1)}) \cdot \mathcal{F}[\mu](h_{(2)}) \cdot \sigma^*(h_{(3)}) \\ &= \sigma \triangleright \mathcal{F}[\mu](h), \end{aligned}$$

so that, indeed, $\mathcal{F}[\sigma \triangleright \mu + \text{MC}[d_B](\sigma)](h) = \sigma \triangleright \mathcal{F}[\mu]$. \square

4.3. Reconstruction of total calculi. As we have just seen, all purely horizontal aspects of noncommutative gauge theory on the trivial quantum principal H -bundle $B \rtimes H$ can be computed solely in terms of lazy Sweedler cohomology and lazy Hochschild cohomology on H with coefficients arising from the basic calculus (Ω_B, d_B) . We now turn to those aspects that depend on a choice of bicovariant FODC (Ω_H^1, d_H) on H , e.g., the \mathfrak{G} -equivariant moduli space $\mathfrak{A}t/\text{at}[\Omega_H^1] \cong \text{Ob}(\mathcal{G}[\Omega_H^1]/\ker \mu[\Omega_H^1])$ of total FODC on P and the \mathfrak{G}^{Pr} -invariant quadratic subset $\mathfrak{A}t^{\text{Pr}}[\Omega_H^1]$ of Ω_H^1 -adapted prolongable gauge potentials.

Let us fix a bicovariant FODC (Ω_H^1, d_H) on H with left crossed H - $*$ -module Λ_H^1 of right H -covariant 1-forms and quantum Maurer–Cartan form $\omega : H \rightarrow \Lambda_H^1$. Let (Ω_H, d_H) denote its canonical prolongation to a bicovariant sodc on H , let $(\Omega_{P, \text{ver}}, d_{P, \text{ver}})$ denote the resulting second-order vertical calculus of P , and let $\Omega_{P, \otimes} := \Lambda_H \widehat{\otimes}^{\leq 2} \Omega_{P, \text{hor}}$, which therefore contains $\Omega_{P, \text{ver}}$ as a left H -subcomodule graded $*$ -algebra.

Using the multiplication map $(\omega \otimes h \mapsto \omega \cdot h) : \Lambda_H \otimes H \rightarrow \Omega_H$, we can identify Ω_H with the graded $*$ -subalgebra of $\Omega_{P, \otimes}$ generated by $H \subset P$ and $\Lambda_H^1 \widehat{\otimes} 1_P \subset \Omega_{P, \text{ver}}^1$. Hence, we can view $\Omega_{P, \otimes}$ as the graded left H -comodule $*$ -algebra, truncated at degree 2, generated by the graded left H -subcomodule $*$ -subalgebras Ω_H and Ω_B subject to the relation $1_{\Omega_H} = 1_{\Omega_B}$ and the braided graded commutation relations

$$(4.24) \quad \forall \omega \in \Omega_H, \forall \alpha \in \Omega_B, \quad \alpha \wedge \omega := (-1)^{|\alpha||\omega|} \omega_{[0]} \wedge \alpha \triangleleft \omega_{[1]};$$

indeed, $\Omega_{P, \otimes} = \Omega_H \cdot \Omega_B$ is freely generated as a graded right Ω_B -module by Ω_H . In particular, the graded left H -subcomodule $*$ -subalgebra $\Omega_{P, \text{ver}} = \Omega_H \cdot B$ is freely generated as a right B -module by Ω_H , and

$$(4.25) \quad \forall \omega \in \Omega_H, \forall b \in B, \quad d_{P, \text{ver}}(\omega \cdot b) = d_H(\omega) \cdot b,$$

so that (Ω_H^1, d_H) is necessarily locally free for P .

We can now characterise the space $\text{at}[\Omega_H^1]$ of (Ω_H^1, d_H) -adapted relative gauge potentials on P with respect to the canonical second-order horizontal calculus $(\Omega_B, d_B; \Omega_{P, \text{hor}})$, and hence relate the \mathfrak{G} -equivariant moduli space $\mathfrak{A}t/\text{at}[\Omega_H^1] \cong \text{Ob}(\mathcal{G}[\Omega_H^1]/\ker \mu[\Omega_H^1])$

of strongly $(H; \Omega_H^1, d_H)$ -principal sodc on P inducing $(\Omega_B, d_B; \Omega_{P, \text{hor}})$ to lazy Hochschild cohomology on H .

Proposition 4.21. *The groupoid isomorphism $ZS_\ell^1(H; B, \Omega_B^1) \ltimes ZH_\ell^1(H; \Omega_B^1) \xrightarrow{\sim} \mathfrak{G} \ltimes \mathfrak{At}$ of Proposition 4.18 descends to a groupoid isomorphism*

$$ZS_\ell^1(H; B, \Omega_B^1) \ltimes \left(\frac{ZH_\ell^1(H; \Omega_B^1)}{\text{Op}_0^{-1}(\text{at}[\Omega_H^1])} \right) \xrightarrow{\sim} \mathfrak{G} \ltimes \left(\frac{\mathfrak{At}}{\text{at}[\Omega_H^1]} \right),$$

where

$$\text{Op}_0^{-1}(\text{at}[\Omega_H^1]) = \{\mu \in ZH_\ell^1(H; \Omega_B^1) \mid \ker(\mu \circ S^{-1}) \supseteq \ker \omega\}.$$

Proof. Before continuing, recall that $\omega : H \rightarrow \Lambda_H^1$ satisfies

$$\forall h \in H, \quad \omega(h)_{[-1]} \otimes \omega(h)_{[0]} = h_{(1)} S(h_{(3)}) \otimes \omega(h_{(2)}).$$

so that that $\Delta(\ker \omega) \subseteq \ker(\text{id}_H \otimes \omega) = H \otimes \ker \omega$. Let $\mu \in ZH_\ell^1(H; \Omega_B^1)$ and set $\mathbf{A} := \text{Op}_0(\mu)$.

First, suppose that $\mathbf{A} \in \text{at}[\Omega_H^1]$. Then for all $h \in H$,

$$\begin{aligned} \mu \circ S^{-1}(h) &= S(S^{-1}(h)_{(1)}) \cdot \mathbf{A}(S^{-1}(h)_{(2)}) \\ &= h_{(2)} \cdot \mathbf{A}(S^{-1}(h_{(1)})) \\ &= \mathbf{A}(S^{-1}(h_{(1)} S(h_{(2)}))) - \mathbf{A}(h_{(2)}) \cdot S^{-1}(h_{(1)}) \\ &= -\omega[\mathbf{A}](\omega(h_{(2)}) \cdot h_{(3)}) \cdot S^{-1}(h_{(1)}) \\ &= -\omega[\mathbf{A}](\omega(h)_{[0]} \cdot S^{-1}(\omega(h)_{[-1]})), \end{aligned}$$

so that, indeed, $\ker(\mu \circ S^{-1}) \supseteq \ker \omega$.

Now, suppose that $\ker(\mu \circ S^{-1}) \supseteq \ker \omega$. Then, for all $h \in H$,

$$\begin{aligned} \mathbf{A}(h_{(1)}) \cdot S(h_{(2)}) &= -\mathbf{A}(h_{(1)}^*)^* \cdot S(h_{(2)}) = -\left(h_{(1)}^* \cdot \mu(h_{(2)}^*)\right)^* \cdot S(h_{(3)}) = -\mu(h_{(2)}^*)^* \cdot h_{(1)} S(h_{(3)}) \\ &= -\mu(S(h_{(2)}^*)^*) \cdot h_{(1)} S(h_{(3)}) = -\mu(S^{-1}(h_{(2)})) \cdot h_{(1)} S(h_{(3)}); \end{aligned}$$

hence, if $h \in \ker \omega$, then $h_{(1)} S(h_{(3)}) \otimes h_{(2)} \in H \otimes \ker \omega$, so that $\mathbf{A}(h_{(1)}) \cdot S(h_{(2)}) = 0$. Since $\omega : H \rightarrow \Lambda_H^1$ is surjective, $\mathbf{A} = N \circ d_{P, \text{ver}}$ for the morphism of left H -comodule right P -modules $N : \Omega_{P, \text{ver}}^1 \rightarrow \Omega_{P, \text{hor}}^1$ defined by

$$\forall h, k \in H, \forall b \in B, \quad N(\omega(k) \cdot hb) := \mathbf{A}(k_{(1)}) \cdot S(k_{(2)}) hb;$$

it therefore suffices to show that N is left P -linear and $*$ -preserving, for then $\mathbf{A} \in \text{at}[\Omega_H^1]$ with relative connection 1-form $\omega[\mathbf{A}] = N$. Let $h, k \in H$ and $b \in B$, so that

$$\begin{aligned} hb \cdot \omega(k) &= h \cdot \omega(k) \cdot (b \triangleleft 1) \\ &= (\omega(h_{(1)} k) - \epsilon(k) \omega(h_{(1)})) \cdot h_{(2)} b \\ &= \omega(h_{(1)} k) \cdot h_{(2)} b - \omega(h_{(1)}) \cdot h_{(2)} \epsilon(k) b. \end{aligned}$$

On the one hand,

$$\begin{aligned} &N(\omega(h_{(1)} k) \cdot h_{(2)} b - \omega(h_{(1)}) \cdot h_{(2)} \epsilon(k) b) \\ &= \mathbf{A}(h_{(1)} k_{(1)}) \cdot S(h_{(2)} k_{(2)}) \cdot h_{(3)} b - \mathbf{A}(h_{(1)}) \cdot S(h_{(2)}) h_{(3)} \epsilon(k) b \\ &= \mathbf{A}(h_{(1)} k_{(1)}) \cdot S(k_{(2)}) S(h_{(2)}) h_{(3)} b - \epsilon(k) \mathbf{A}(h) b \\ &= \mathbf{A}(h) k_{(1)} \cdot S(k_{(2)}) b + h \mathbf{A}(k_{(1)}) \cdot S(k_{(2)}) b - \epsilon(k) \mathbf{A}(h) b \\ &= h \mathbf{A}(k_{(1)}) \cdot S(k_{(2)}) b, \end{aligned}$$

while on the other,

$$\begin{aligned}
 hbN(\omega(k)) &= hb \cdot \mathbf{A}(k_{(1)})S(k_{(2)}) \\
 &= -hb k_{(1)}\mathbf{A}(S(k_{(2)})) \\
 &= -hk_{(1)}S(k_{(5)}) \left((b \triangleleft k_{(2)}) \triangleleft S(k_{(4)}) \right) \cdot \mu(S(k_{(3)})) \\
 &= -hk_{(1)}S(k_{(5)}) \cdot \mu(S(k_{(4)})) \cdot \left((b \triangleleft k_{(2)}) \triangleleft S(k_{(3)}) \right) \\
 &= -hk_{(1)}S(k_{(3)})\mu(S(k_{(2)}))b \\
 &= -hk_{(1)} \cdot \mathbf{A}(S(k_{(2)})) \cdot b \\
 &= h \cdot \mathbf{A}(k_{(1)}) \cdot S(k_{(2)})b,
 \end{aligned}$$

so that $hbN(\omega(k)) = N(\omega(h_{(1)}k) \cdot h_{(2)}b - \omega(h_{(1)}) \cdot h_{(2)}\epsilon(k)b)$. Hence, N is left P -linear. Similar calculations now show that N is also $*$ -preserving. \square

The techniques of this proof can also be used to characterise the quadric set $\mathfrak{A}^{\text{Pr}}[\Omega_H^1]$ of (Ω_H^1, d_H) -adapted prolongable gauge potentials on P with respect to $(\Omega_B, d_B; \Omega_{P,\text{hor}})$ —hence, also, the \mathfrak{G}^{Pr} -equivariant moduli space

$$\mathfrak{A}^{\text{Pr}}[\Omega_H^1]/\mathfrak{at}_{\text{can}}^{\text{Pr}}[\Omega_H^1] \cong \text{Ob}(\mathcal{C}[\Omega_H^{\leq 2}]/\ker \mu[\Omega_H^{\leq 2}])$$

of strongly $(H; \Omega_H, d_H)$ -principal sodc on P with respect to $(\Omega_B, d_B; \Omega_{P,\text{hor}})$ —in terms of lazy Hochschild cohomology. Recall that $\mathfrak{at}_{\text{can}}^{\text{Pr}}[\Omega_H^1]$ denotes the space of all (Ω_H, d_H) -adapted prolongable relative gauge potentials on P with respect to $(\Omega_B, d_B; \Omega_{P,\text{hor}})$ and that $\text{Inn}(\mathfrak{at}^{\text{Pr}}; \Omega_H^1)$ denotes the space of all (Ω_H^1, d_H) -semi-adapted inner prolongable relative gauge potentials on P with respect to $(\Omega_B, d_B; \Omega_{P,\text{hor}})$, so that

$$\text{Out}(\mathfrak{at}^{\text{Pr}}[\Omega_H^1]) := \mathfrak{A}^{\text{Pr}}[\Omega_H^1]/\text{Inn}(\mathfrak{at}^{\text{Pr}}; \Omega_H^1).$$

Theorem 4.22. *The groupoid isomorphism $ZS_\ell^1(H; B, \Omega_B^1) \times ZH_\ell^1(H; \Omega_B^1, \Omega_B) \xrightarrow{\sim} \mathfrak{G}^{\text{Pr}} \times \mathfrak{A}^{\text{Pr}}$ of Corollary 4.19 restricts to a groupoid isomorphism*

$$ZS_\ell^1(H; B, \Omega_B^1) \times \text{Op}^{-1}(\mathfrak{A}^{\text{Pr}}[\Omega_H^1]) \xrightarrow{\sim} \mathfrak{G}^{\text{Pr}} \times \mathfrak{A}^{\text{Pr}}[\Omega_H^1],$$

where

$$(4.26) \quad \text{Op}^{-1}(\mathfrak{A}^{\text{Pr}}[\Omega_H^1]) = \{\mu \in ZH_\ell^1(H; \Omega_B^1, \Omega_B) \mid \ker(\mathcal{F}[\mu] \circ S^{-1}) \supseteq \ker \omega\}.$$

In turn, this restricted groupoid isomorphism descends to respective groupoid isomorphisms

$$\begin{aligned}
 ZS_\ell^1(H; B, \Omega_B^1) \times \left(\frac{\text{Op}^{-1}(\mathfrak{A}^{\text{Pr}}[\Omega_H^1])}{\text{Op}_0^{-1}(\mathfrak{at}_{\text{can}}^{\text{Pr}}[\Omega_H^1])} \right) &\xrightarrow{\sim} \mathfrak{G}^{\text{Pr}} \times \left(\frac{\mathfrak{A}^{\text{Pr}}[\Omega_H^1]}{\mathfrak{at}_{\text{can}}^{\text{Pr}}[\Omega_H^1]} \right), \\
 \text{HS}_\ell^1(H; B, \Omega_B^1) \times \left(\frac{\text{Op}^{-1}(\mathfrak{A}^{\text{Pr}}[\Omega_H^1])}{\text{Op}_0^{-1}(\text{Inn}(\mathfrak{at}^{\text{Pr}}; \Omega_H^1))} \right) &\xrightarrow{\sim} \text{Out}(\mathfrak{G}^{\text{Pr}}) \times \text{Out}(\mathfrak{A}^{\text{Pr}}[\Omega_H^1]),
 \end{aligned}$$

where

$$(4.27) \quad \text{Op}_0^{-1}(\mathfrak{at}_{\text{can}}^{\text{Pr}}[\Omega_H^1]) = \{\mu \in ZH_\ell^1(H; \Omega_B^1, \Omega_B) \mid \ker(\mu \circ S^{-1}) \supseteq \ker \omega\},$$

(4.28)

$$\text{Op}_0^{-1}(\text{Inn}(\mathfrak{at}^{\text{Pr}}; \Omega_H^1)) = \{D\beta \mid \beta \in Z_B(\Omega_B^1)_{\text{sa}} \cap Z(\Omega_B), \ker(D(-i d_B \beta) \circ S^{-1}) \supseteq \ker \omega\}.$$

In particular, $\mathfrak{at}_{\text{can}}^{\text{Pr}}[\Omega_H^1] = \mathfrak{at}^{\text{Pr}} \cap \mathfrak{at}[\Omega_H^1]$.

Proof. Let us first prove (4.26). Let $\mu \in \text{ZH}_\ell^1(H; \Omega_B^1, \Omega_B)$, so that

$$\forall h \in H, \forall b \in B, \quad \mathbf{F}[\text{Op}(\mu)](hb) = h_{(1)} \cdot \mathcal{F}[\mu](h_{(2)}) \cdot b.$$

By applying the proof of Proposition 4.21, *mutatis mutandis*, to the left H -covariant $*$ -derivation $\mathbf{F}[\text{Op}(\mu)] : P \rightarrow \Omega_{P, \text{hor}}^2$, it follows that $\text{Op}(\mu) \in \mathfrak{A}^{\text{tot}}$ if and only if

$$\ker(\mathcal{F}[\mu] \circ S^{-1}) \supseteq \ker \omega.$$

This argument now also proves (4.22), since for all $\beta \in Z_B(\Omega_B^1)_{\text{sa}} \cap Z(\Omega_B)$ and $h \in H$

$$D(-i_{d_B} \beta)(h) = \mathcal{F}[D\alpha](h) = S(h_{(1)}) \cdot \mathbf{F}[\text{Op}(\alpha)](h_{(2)}) = S(h_{(1)}) \cdot \mathbf{F}_{\text{rel}}[\text{Op}_0(\alpha)](h_{(2)})$$

by (4.20) and the fact that $\mathbf{F}[\text{Op}(0)] = 0$. Hence, it remains to prove (4.22).

On the one hand, $\mathfrak{A}^{\text{tPr}}[\Omega_H] \subseteq \mathfrak{a}^{\text{tPr}} \cap \mathfrak{a}[\Omega_H^1]$ by Theorem 3.48; on the other hand, by Corollary 4.19 and Proposition 4.21,

$$\text{Op}_0^{-1}(\mathfrak{a}^{\text{tPr}} \cap \mathfrak{a}[\Omega_H^1]) = \{\mu \in \text{ZH}_\ell^1(H; \Omega_B^1, \Omega_B) \mid \ker(\mu \circ S^{-1}) \supseteq \ker \omega\} =: \text{ZH}_\ell^1(H; \Omega_B^1, \Omega_B)[\Omega_H^1].$$

Hence, it suffices to show that $\mathfrak{A}^{\text{tPr}}[\Omega_H] \supseteq \mathfrak{a}^{\text{tPr}} \cap \mathfrak{a}[\Omega_H^1]$.

Let $\mathbf{A} \in \mathfrak{a}^{\text{tPr}} \cap \mathfrak{a}[\Omega_H^1]$ be given; set $\mu := \text{Op}_0^{-1}(\mathbf{A})$ and $N := \omega[\mathbf{A}]$, so that for all $h \in H$,

$$N(\omega(h)) = \mathbf{A}(h_{(1)}) \cdot S(h_{(2)}) = -h_{(1)} \cdot \mathbf{A}(S(h_{(2)})) = -h_{(1)} S(h_{(3)}) \cdot \mu(S(h_{(2)}))$$

by the proof of Proposition 4.21. We will show that $\mathbf{A} \in \mathfrak{A}^{\text{tPr}}[\Omega_H]$ by showing that N satisfies both (3.33) and (3.34).

First, let $h \in H$ and $\beta \in \Omega_B^1$. Since $\mu \in \text{ZH}_\ell^1(H; \Omega_B^1, \Omega_B)$, it follows that

$$\begin{aligned} \beta \wedge N(\omega(h)) &= -\beta \wedge h_{(1)} S(h_{(3)}) \cdot \mu(S(h_{(2)})) \\ &= -h_{(1)} S(h_{(5)}) \left((\beta \triangleleft h_{(2)}) \triangleleft S(h_{(4)}) \right) \wedge \mu(S(h_{(3)})) \\ &= h_{(1)} S(h_{(5)}) \cdot \mu(S(h_{(4)})) \wedge \left((\beta \triangleleft h_{(2)}) \triangleleft S(h_{(3)}) \right) \\ &= h_{(1)} S(h_{(3)}) \cdot \mu(S(h_{(3)})) \cdot \beta \\ &= -N(\omega(h)) \wedge \beta \end{aligned}$$

by Lemma 4.6. Hence, N satisfies (3.33).

Let us now check that N satisfies (3.34). Let $h, k \in H$; note that

$$\begin{aligned} \omega(k)_{[-1]} \triangleright \omega(h) \otimes \omega(k)_{[0]} &= k_{(1)} S(h_{(k)}) \triangleright \omega(h) \otimes \omega(k_{(2)}) \\ &= \omega(h_{(1)} S(h_{(3)}) k) \otimes \omega(h_{(2)}) - \epsilon(k) \omega(h_{(1)} S(h_{(3)})) \otimes \omega(h_{(2)}), \end{aligned}$$

so that it suffices to show that

$$N(\omega(h)) \wedge N(\omega(k)) + N(\omega(h_{(1)} S(h_{(3)}) k)) \wedge N(\omega(h_{(2)})) - \epsilon(k) N(\omega(h_{(1)} S(h_{(3)}))) N(\omega(h_{(2)})) = 0.$$

First, by repeated applications of Lemma 4.6 together with the fact that ω is a 1-cocycle valued in the left crossed H - $*$ -module Λ_H^1 , we find that

$$\begin{aligned} N(\omega(h)) \wedge N(\omega(k)) &= \mathbf{A}(h_{(1)}) \cdot S(h_{(2)}) \wedge \mathbf{A}(k_{(1)}) \cdot S(k_{(2)}) \\ &= h_{(1)} \cdot \mathbf{A}(S(h_{(2)})) \wedge k_{(1)} \cdot \mathbf{A}(S(k_{(2)})) \\ &= h_{(1)} \cdot \mathbf{A}(S(h_{(2)}) k_{(1)}) \wedge \mathbf{A}(k_{(2)}) - h_{(1)} S(h_{(2)}) \cdot \mathbf{A}(k_{(1)}) \wedge \mathbf{A}(S(k_{(2)})) \\ &= h_{(1)} S(h_{(3)}) k_{(1)} \cdot \mu(S(h_{(2)}) k_{(2)}) \wedge S(k_{(4)}) \cdot \mu(S(k_{(3)})) \\ &\quad - \epsilon(h) k_{(1)} \cdot \mu(k_{(2)}) \wedge S(k_{(4)}) \cdot \mu(S(k_{(3)})) \\ &= h_{(1)} S(h_{(3)}) k_{(1)} S(k_{(5)}) \cdot \mu(S(h_{(2)}) k_{(2)}) \triangleleft S(k_{(4)}) \wedge \mu(S(k_{(3)})) \\ &\quad - \epsilon(h) k_{(1)} S(k_{(5)}) \cdot \mu(k_{(2)}) \triangleleft S(k_{(4)}) \wedge \mu(S(k_{(3)})) \\ &= -h_{(1)} S(h_{(3)}) k_{(1)} S(k_{(5)}) \cdot \mu(S(k_{(4)})) \wedge \mu(S(h_{(2)}) k_{(2)}) \triangleleft S(k_{(3)}) \end{aligned}$$

$$\begin{aligned}
 & + \epsilon(h)k_{(1)}S(k_{(5)}) \cdot \mu(S(k_{(4)})) \wedge \mu(k_{(2)}) \triangleleft S(k_{(3)}) \\
 = & -h_{(1)}S(h_{(3)})k_{(1)}S(k_{(3)}) \cdot \mu(S(k_{(2)})) \wedge \mu(S(h_{(2)})) \\
 & + \epsilon(h)k_{(1)}S(k_{(4)}) \cdot \mu(S(k_{(3)})) \wedge \mu(S(k_{(2)})) \\
 & - \epsilon(h)k_{(1)}S(k_{(4)}) \cdot \mu(S(k_{(3)})) \wedge \mu(S(k_{(2)})) \\
 = & -h_{(1)}S(h_{(3)})k_{(1)}S(k_{(3)}) \cdot \mu(S(k_{(2)})) \wedge \mu(S(h_{(2)})).
 \end{aligned}$$

Next, by applying the last calculation, *mutatis mutandis*, we find that

$$\begin{aligned}
 & N(\varpi(h_{(1)}S(h_{(3)})k)) \wedge N(\varpi(h_{(2)})) \\
 = & -h_{(1)}S(h_{(9)})k_{(1)}S(k_{(3)})S^2(h_{(7)})S(h_{(3)})h_{(4)}S(h_{(6)}) \cdot \mu(S(h_{(5)})) \wedge \mu(S(k_{(2)}))S^2(h_{(8)})S(h_{(2)}) \\
 = & -h_{(1)}S(h_{(5)})k_{(1)}S(k_{(3)}) \cdot \mu(S(h_{(3)})) \wedge \mu(S(k_{(2)}))S^2(h_{(4)})S(h_{(2)}) \\
 = & -h_{(1)}S(h_{(5)})k_{(1)}S(k_{(3)}) \cdot \mu(S(h_{(3)})) \wedge (\mu(S(k_{(2)}))S^2(h_{(4)})) \triangleleft S(h_{(2)}) + \epsilon(k_{(2)})\epsilon(h_{(4)})\mu(S(h_{(2)})) \\
 = & h_{(1)}S(h_{(5)})k_{(1)}S(k_{(3)}) \cdot \mu(S(k_{(2)}))S^2(h_{(4)}) \triangleleft S(h_{(3)}) \wedge \mu(S(h_{(2)})) \\
 & - \epsilon(k)h_{(1)}S(h_{(4)}) \cdot \mu(S(h_{(3)})) \wedge \mu(S(h_{(2)})) \\
 = & h_{(1)}S(h_{(3)})k_{(1)}S(k_{(3)}) \cdot \mu(S(k_{(2)})) \wedge \mu(S(h_{(2)})) - \epsilon(k)h_{(1)}S(h_{(4)}) \cdot \mu(S(h_{(3)})) \wedge \mu(S(h_{(2)})) \\
 & - \epsilon(k)h_{(1)}S(h_{(4)}) \cdot \mu(S(h_{(3)})) \wedge \mu(S(h_{(2)})) \\
 = & -N(\varpi(h)) \wedge N(\varpi(k)) - 2\epsilon(k)h_{(1)}S(h_{(4)}) \cdot \mu(S(h_{(3)})) \wedge \mu(S(h_{(2)})).
 \end{aligned}$$

Finally, by applying the last calculation, *mutatis mutandis*, we find that

$$N(\varpi(h_{(1)}S(h_{(3)}))) \wedge N(\varpi(h_{(2)})) = -2h_{(1)}S(h_{(4)}) \cdot \mu(S(h_{(3)})) \wedge \mu(S(h_{(2)})),$$

which we can substitute into the calculation of $N(\varpi(h_{(1)}S(h_{(3)})k) \wedge N(\varpi(h_{(2)}))$ to obtain

$$N(\varpi(h_{(1)}S(h_{(3)})k) \wedge N(\varpi(h_{(2)})) = -N(\varpi(h)) \wedge N(\varpi(k)) + 2\epsilon(k)N(\varpi(h_{(1)}S(h_{(3)}))) \wedge N(\varpi(h_{(2)})),$$

which, in turn, yields our claim. \square

Remark 4.23. Note, in particular, that $\mathfrak{A}^{\text{Pr}}[\Omega_H^1]$ necessarily contains $\text{Op}(0)$, which corresponds to the trivial flat connection.

The proof of the above result almost immediately yields the following expression for the curvature 2-form of an (Ω_H^1, d_H) -adapted prolongable gauge potential.

Corollary 4.24. *Let $\mu \in \text{Op}^{-1}(\mathfrak{A}^{\text{Pr}}[\Omega_H^1])$. Then*

$$\forall h \in H, \quad F[\text{Op}(\mu)](\varpi(h)) = h_{(1)}S(h_{(4)}) \cdot i(d_B \mu(S(h_{(3)}))\epsilon(h_{(2)}) + \mu(S(h_{(3)})) \wedge \mu(S(h_{(2)})))$$

5. q -MONOPOLES OVER REAL MULTIPLICATION NONCOMMUTATIVE 2-TORI

Let $\theta \in \mathbf{R} \setminus \mathbf{Q}$ be a quadratic irrationality, and let \mathcal{A}_θ be the corresponding smooth noncommutative 2-torus. In this section, we show how to subsume Connes's constant curvature connections [15] and Polishchuk–Schwarz's holomorphic structures [32] on the self-Morita equivalence bimodules amongst the basic Heisenberg modules over \mathcal{A}_θ into gauge theory on a certain canonical non-trivial principal $\mathcal{O}(U(1))$ -module algebra P over \mathcal{A}_θ implicit in Manin's 'Alterstraum' [28]. In the process, we will encounter striking formal similarities with the q -deformed complex Hopf fibration; in this case, however, there is a canonical value for the parameter q arising from the algebraic number theory of the real quadratic irrationality θ .

5.1. Number-theoretic preliminaries. We begin by recalling relevant folklore about real quadratic number fields; our primary reference is the monograph of Halter–Koch [25]. Let $\theta \in \mathbf{R} \setminus \mathbf{Q}$ be a quadratic irrationality. Recall that $\mathrm{SL}(2, \mathbf{Z})$ acts on $\mathbf{R} \setminus \mathbf{Q}$ by fractional linear transformations, i.e., by

$$\forall g \in \mathrm{SL}(2, \mathbf{Z}), \forall \xi \in \mathbf{R} \setminus \mathbf{Q}, \quad g \triangleright \xi := \frac{g_{11}\xi + g_{12}}{g_{21}\xi + g_{22}},$$

and observe that this action descends to an action of $\mathrm{PSL}(2, \mathbf{Z})$ on $\mathbf{R} \setminus \mathbf{Q}$; denote the stabilizers of θ in $\mathrm{SL}(2, \mathbf{Z})$ and $\mathrm{PSL}(2, \mathbf{Z})$ by $\mathrm{SL}(2, \mathbf{Z})_\theta$ and $\mathrm{PSL}(2, \mathbf{Z})_\theta$. Our goal is to compute $\mathrm{SL}(2, \mathbf{Z})_\theta$ (and hence $\mathrm{PSL}(2, \mathbf{Z})_\theta$) in terms of the solutions of the positive Pell’s equation associated with the real quadratic irrationality θ .

First, the *type* of θ is the unique coprime triple $(a, b, c) \in (\mathbf{Z} \setminus \{0\}) \times \mathbf{Z}^2$, such that

$$\theta = \frac{b + \sqrt{b^2 - 4ac}}{2a}$$

and $b^2 - 4ac$ is not a square, so that $\Delta := b^2 - 4ac$ is the *discriminant* of θ . On the one hand, then, the *norm* on the real quadratic number field $\mathbf{Q}[\theta] = \mathbf{Q}[\sqrt{\Delta}]$ is the multiplicative unit-preserving map $\mathcal{N} : \mathbf{Q}[\theta] \rightarrow \mathbf{Q}$ defined by

$$\forall r, s \in \mathbf{Q}, \quad \mathcal{N}(r + s\sqrt{\Delta}) := r^2 - \Delta s^2;$$

on the other hand, the *quadratic order* of discriminant Δ in $\mathbf{Q}[\theta] = \mathbf{Q}[\sqrt{\Delta}]$ is the subring

$$\mathcal{O}_\Delta = \left\{ \frac{u+v\sqrt{\Delta}}{2} \mid u, v \in \mathbf{Z}, u \equiv v\Delta \pmod{2} \right\}.$$

Thus, we can define the multiplicative group of *norm-positive* units in \mathcal{O}_Δ by

$$\mathcal{O}_\Delta^{\times,+} := \{ \epsilon \in \mathcal{O}_\Delta^\times \mid \mathcal{N}(\epsilon) > 0 \} = \left\{ \frac{u+v\sqrt{\Delta}}{2} \mid u, v \in \mathbf{Z}, u^2 - \Delta v^2 = 4 \right\},$$

which can therefore be viewed as the group of positive solutions of the Pell’s equation

$$x^2 - \Delta y^2 = 4.$$

By applying Dirichlet’s unit theorem to the quadratic order \mathcal{O}_Δ of the real quadratic field $\mathbf{Q}[\theta]$, one finds that $\mathcal{O}_\Delta^\times \cap \mathbf{R}_{>0}$ is infinite cyclic and generated by the *fundamental unit*

$$\epsilon_\Delta := \min\{ \epsilon \in \mathcal{O}_\Delta^\times \cap \mathbf{R}_{>0} \mid \epsilon > 1 \} = \min\{ \epsilon \in \mathcal{O}_\Delta^\times \mid \epsilon > 1 \},$$

so that, in turn, the subgroup $\mathcal{O}_\Delta^{\times,+} \cap \mathbf{R}_{>0}$ is also infinite cyclic and generated by the *norm-positive fundamental unit* (or *Pell’s unit*)

$$\epsilon_\Delta^+ := \begin{cases} \epsilon_\Delta & \text{if } \mathcal{N}(\epsilon_\Delta) = 1, \\ \epsilon_\Delta^2 & \text{if } \mathcal{N}(\epsilon_\Delta) = -1. \end{cases}$$

The following folkloric result now gives the desired number-theoretic characterization of the stabilizer groups $\mathrm{SL}(2, \mathbf{Z})_\theta$ and $\mathrm{PSL}(2, \mathbf{Z})_\theta$.

Proposition 5.1 (Folklore [25, Thm. 5.2.10]). *The map $\Phi : \mathcal{O}_\Delta^{\times,+} \rightarrow \mathrm{SL}(2, \mathbf{Z})_\theta$ given by*

$$(5.1) \quad \forall \frac{u+v\sqrt{\Delta}}{2} \in \mathcal{O}_\Delta^{\times,+}, \quad \Phi \left(\frac{u+v\sqrt{\Delta}}{2} \right) := \begin{pmatrix} \frac{u+bv}{2} & -cv \\ av & \frac{u-bv}{2} \end{pmatrix}$$

defines a group isomorphism satisfying $\Phi(-1) = -I$ and

$$(5.2) \quad \forall g \in \mathrm{SL}(2, \mathbf{Z})_\theta, \quad \Phi^{-1}(g) := g_{21}\theta + g_{22}.$$

Thus, in particular, $\mathrm{SL}(2, \mathbf{Z})_\theta$ decomposes as the internal direct product

$$\mathrm{SL}(2, \mathbf{Z})_\theta = \Phi(\mathcal{O}_\Delta^{\times,+} \cap \mathbf{R}_{>0}) \times \{\pm I\},$$

where $\Phi(\mathcal{O}_\Delta^{\times,+} \cap \mathbf{R}_{>0})$ is infinite cyclic with canonical generator $\Phi(\epsilon_\Delta^+)$, so that $\mathrm{PSL}(2, \mathbf{Z})_\theta$ is infinite cyclic with canonical generator $[\Phi(\epsilon_\Delta^+)]$.

5.2. Basic Heisenberg modules over irrational noncommutative 2-tori. In this section, we recall the construction of basic Heisenberg modules over an irrational noncommutative 2-torus $C^\infty(\mathbf{T}_\theta^2)$; when $\theta \in \mathbf{R} \setminus \mathbf{Q}$ is a quadratic irrationality, we assemble the self-Morita equivalence bimodules of positive rank among the basic Heisenberg modules into a canonical non-trivial principal $\mathcal{O}(\mathrm{U}(1))$ -comodule algebra P over $C^\infty(\mathbf{T}_\theta^2)$.

Let us first recall the relevant algebraic notions of (Morita) equivalence bimodule and self-Morita equivalence bimodule.

Definition 5.2. Let A and B be unital \mathbf{C} -algebras. An (A, B) -equivalence bimodule is an (A, B) -bimodule ${}_A E_B$ satisfying the following:

- (1) the right B -module E_B is finitely generated, projective, and full in the sense that

$$B = \mathrm{Span}_{\mathbf{C}}\{f(e) \mid e \in E, f \in \mathrm{Hom}_B(E_B, B_B)\};$$

- (2) the left A -module structure on ${}_A E_B$ is an algebra isomorphism $A \xrightarrow{\sim} \mathrm{End}_B(E_B)$.

In particular, a *self-Morita equivalence bimodule* over B is an (B, B) -equivalence bimodule.

Example 5.3. Let B be a unital \mathbf{C} -algebra. The *trivial* self-Morita equivalence bimodule over B is B itself equipped with the left and right B -module structures induced by multiplication in the algebra B .

Next, recall that the *smooth noncommutative 2-torus* with deformation parameter $\theta \in \mathbf{R}$ is the unital $*$ -algebra \mathcal{A}_θ of rapidly decaying Laurent series in two unitary generators U_θ and V_θ satisfying the commutation relation

$$(5.3) \quad V_\theta U_\theta = e^{2\pi i \theta} U_\theta V_\theta;$$

note that \mathcal{A}_θ can be canonically topologised as a Fréchet pre- C^* -algebra, whose C^* -algebraic completion can be identified with the rotation algebra $A_\theta := C(\mathrm{U}(1)) \rtimes_\theta \mathbf{Z}$. As Rieffel famously observed [34, Thm. 1.1], the action of $\mathrm{SL}(2, \mathbf{Z})$ on $\mathbf{R} \setminus \mathbf{Q}$ by fractional linear transformations manifests itself as distinguished family of equivalence bimodules, the *basic Heisenberg modules*, for smooth noncommutative 2-tori with irrational deformation parameter. These bimodules had already been constructed *qua* right modules by Connes [15] as the very first examples of noncommutative smooth vector bundles in noncommutative differential geometry.

Theorem-Definition 5.4 (Connes [15], Rieffel [33; 34, Thm. 1.1; 35, §§2–3, 5]). Let $\theta \in \mathbf{R} \setminus \mathbf{Q}$. For every $g \in \mathrm{SL}(2, \mathbf{Z})$, we can construct a $(\mathcal{A}_{g \triangleright \theta}, \mathcal{A}_\theta)$ -equivalence bimodule $\mathcal{E}(g, \theta)$ as follows:

- (1) if $g_{21} = 0$, so that $\mathcal{A}_{g \triangleright \theta} = \mathcal{A}_\theta$ and $g_{22} \neq 0$, let $\mathcal{E}(g, \theta) := \mathcal{A}_\theta$ be the trivial self-Morita equivalence bimodule over \mathcal{A}_θ ;
- (2) if $g_{22} \neq 0$, let $\mathcal{E}(g, \theta) := \mathcal{S}(\mathbf{R}) \otimes \mathbf{C}[\mathbf{Z}_{g_{21}}]$ with the unique right \mathcal{A}_θ -module structure satisfying

$$\forall f \in \mathcal{E}(g, \theta), \forall (x, k) \in \mathbf{R} \times \mathbf{Z}_{g_{21}}, \quad (f \cdot U_\theta)(x, k) = \exp\left(2\pi i \left(x - \frac{k g_{22}}{g_{21}}\right)\right) f(x, k),$$

$$\forall f \in \mathcal{E}(g, \theta), \forall (x, k) \in \mathbf{R} \times \mathbf{Z}_{g_{21}}, \quad (f \cdot V_\theta)(x, k) = f\left(x - \frac{g_{21} \theta + g_{22}}{g_{21}}, k - 1\right),$$

and the unique left $\mathcal{A}_{g \triangleright \theta}$ -module structure satisfying

$$\forall f \in \mathcal{E}(g, \theta), \forall (x, k) \in \mathbf{R} \times \mathbf{Z}_{g_{21}}, \quad (U_{g \triangleright \theta} \cdot f)(x, k) = \exp\left(2\pi i \left(\frac{x}{g_{21} \theta + g_{22}} - \frac{k}{g_{21}}\right)\right) f(x, k),$$

$$\forall f \in \mathcal{E}(g, \theta), \forall (x, k) \in \mathbf{R} \times \mathbf{Z}_{g_{21}}, \quad (V_{g \succ \theta} \cdot f)(x, k) = f\left(x - \frac{1}{g_{21}}, k - g_{11}\right).$$

We call $\mathcal{E}(g, \theta)$ the *basic Heisenberg module* over \mathcal{A}_θ with *rank* $g_{21}\theta + g_{22}$ and *degree* g_{21} .

Now, given $\theta \in \mathbf{R} \setminus \mathbf{Q}$ and $g \in \mathrm{SL}(2, \mathbf{Z})$, we see that $g \succ \theta = \theta$ if and only if θ is a quadratic irrationality and $g \in \mathrm{SL}(2, \mathbf{Z})_\theta$. Thus, in light of Proposition 5.1, if θ is a real quadratic irrationality, then the family of basic Heisenberg bimodules over \mathcal{A}_θ contains a canonical family of self-Morita equivalence bimodules, which we can now assemble into a principal $\mathcal{O}(\mathrm{U}(1))$ -module $*$ -algebra P with ${}^{\mathrm{co}}\mathcal{O}(\mathrm{U}(1))P = \mathcal{A}_\theta$.

Theorem 5.5 (Schwarz [38, §3], Dieng–Schwarz [20], Polishchuk–Schwarz [32, §1.3], Polishchuk [31, §2.2], Vlasenko [40, Thm. 6.1]). *Let $\theta \in \mathbf{R} \setminus \mathbf{Q}$ be a quadratic irrationality with norm-positive fundamental unit ϵ , and let $\Phi : \langle \epsilon \rangle \times \{\pm 1\} \rightarrow \mathrm{SL}(2, \mathbf{Z})_\theta$ be the isomorphism of Proposition 5.1; hence, given $m \in \mathbf{Z}$, let*

$$a_m := \Phi(\epsilon^m)_{11}, \quad b_m := \Phi(\epsilon^m)_{12}, \quad c_m := \Phi(\epsilon^m)_{21}, \quad d_m := \Phi(\epsilon^m)_{22}.$$

For each $n \in \mathbf{Z}$, let $P_n := \mathcal{E}(\Phi(\epsilon^n), \theta)$, and define a $\mathcal{O}(\mathrm{U}(1))$ -comodule \mathcal{A}_θ -bimodule

$$P := \bigoplus_{n \in \mathbf{Z}} P_n,$$

where the corepresentation of $\mathcal{O}(\mathrm{U}(1))$ is induced by the obvious \mathbf{Z} -grading. Then P defines a principal $\mathcal{O}(\mathrm{U}(1))$ -comodule $*$ -algebra over ${}^{\mathrm{co}}\mathcal{O}(\mathrm{U}(1))P = P_0 = \mathcal{A}_\theta$ when endowed with the $*$ -operation defined by

$$\forall m \in \mathbf{Z}, \forall f \in P_m, \forall (x, k) \in \mathbf{R} \times \mathbf{Z}_{c_m}, \quad f^*(x, k) := \overline{f(\epsilon^m x, -a_m k)}$$

and the multiplication $P \otimes_{\mathbf{C}} P \rightarrow P$ defined as follows:

- (1) the restrictions of the multiplication to $P_0 \otimes_{\mathbf{C}} P = \mathcal{A}_\theta \otimes_{\mathbf{C}} P$ and $P \otimes_{\mathbf{C}} P_0 = P \otimes_{\mathbf{C}} \mathcal{A}_\theta$ are given by the left and right \mathcal{A}_θ -module structures on P , respectively;
- (2) for all non-zero $m \in \mathbf{Z}$, for all $f \in P_{-m}$, and for all $g \in P_m$,

$$f \cdot g := \sum_{(n_1, n_2) \in \mathbf{Z}^2} U^{n_1} V^{n_2} \sum_{k \in \mathbf{Z}_{c_m}} \int_{\mathbf{R}} (V_\theta^{-n_2} U_\theta^{-n_1} \cdot f)\left(\frac{x}{\epsilon^m}, k\right) g(x, -a_m k) dx;$$

- (3) for all non-zero $m, n \in \mathbf{Z}$ with $m + n \neq 0$, for all $f \in P_m$, and for all $g \in P_n$,

$$\forall (x, k) \in \mathbf{R} \times \mathbf{Z}_{c_{m+n}},$$

$$(f \cdot g)(x, k) := \sum_{j \in \mathbf{Z}} f\left(\frac{x}{\epsilon^n} + \epsilon^m \left(\frac{d_{m+n}k}{c_{m+n}} - \frac{j}{c_m}\right), a_m d_{m+n}k - j\right) \cdot g\left(x - \left(\frac{d_{m+n}k}{c_{m+n}} - \frac{j}{c_n}\right), a_n j\right).$$

Proof. First, observe that the results of Schwarz [38, §3], Dieng–Schwarz [20], and Polishchuk–Schwarz [32, §1.3] specialise to yield the multiplication $P \otimes P \rightarrow P$ that makes P into a unital \mathbf{C} -algebra. Next, the results of Polishchuk [31, §2.2] specialise to show that $*$: $P \rightarrow P$ makes P into a B - $*$ -bimodule, while results of Vlasenko [40, Thm. 6.1], with our notation and conventions, specialise to show that

$$\forall m, n \in \mathbf{Z}, \forall p_1, q_1 \in P_m, \forall p_2, q_2 \in P_n, \quad (p_1 \cdot p_2) \cdot (q_1 \cdot q_2)^* = (p_1 \cdot (p_2 \cdot q_2^*)) \cdot q_1^*,$$

so that, by associativity of the multiplication on P ,

$$(5.4) \quad \forall m, n \in \mathbf{Z}, \forall p_1, q_1 \in P_m, \forall p_2, q_2 \in P_n, \quad (p_1 \cdot p_2) \cdot (q_2^* \cdot q_1).$$

Finally, by observations of Dieng–Schwarz [20, §2] and Polishchuk–Schwarz [32, Proof of Prop. 1.2], for all $m, n \in \mathbf{Z}$, multiplication in P yields an isomorphism $P_m \otimes_B P_n \xrightarrow{\sim} P_{m+n}$. On the one hand, for all $m, n \in \mathbf{Z}$, $q_1 \in P_m$, and $q_2 \in P_n$, since $P_{m+n} = P_m \cdot P_n$ and since

the multiplication on P restricts to an isomorphism of B -bimodules $P_{m+n} \otimes_B P_{-m-n} \xrightarrow{\sim} B$, Equation 5.4 implies that $(q_1 \cdot q_2)^* = q_2^* \cdot q_1$, so that $*$: $P \rightarrow P$ makes P into a unital $*$ -algebra. On the other hand, since $P_m \cdot P_n = P_{m+n}$ for all $m, n \in \mathbb{Z}$, it follows that the $*$ -algebra P defines a principal $\mathcal{O}(U(1))$ -comodule $*$ -algebra [1, Thm. 4.4]. \square

Now, if B is a $\mathcal{O}(U(1))$ -module $*$ -algebra, then the trivial principal $\mathcal{O}(U(1))$ -comodule $*$ -algebra $B \rtimes \mathcal{O}(U(1))$ admits the left $\mathcal{O}(U(1))$ -covariant $*$ -homomorphism

$$(h \mapsto h \otimes 1_P) : \mathcal{O}(U(1)) \rightarrow B \rtimes \mathcal{O}(U(1)),$$

which one views as a global trivialisation of $B \rtimes \mathcal{O}(U(1))$. Indeed, more generally, a principal $\mathcal{O}(U(1))$ -comodule $*$ -algebra P is left $\mathcal{O}(U(1))$ -covariant $*$ -isomorphic to such a trivial principal $\mathcal{O}(U(1))$ -comodule $*$ -algebra if and only if it admits a left $\mathcal{O}(U(1))$ -covariant $*$ -homomorphism $\mathcal{O}(U(1)) \rightarrow P$. We now show that Theorem 5.5 yields non-trivial principal $\mathcal{O}(U(1))$ -comodule $*$ -algebra in this technically precise sense.

Proposition 5.6. *Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$ be a quadratic irrationality, and let P be the resulting principal $\mathcal{O}(U(1))$ -comodule $*$ -algebra of Theorem 5.5. There does not exist a left H -covariant unital $*$ -homomorphism $\mathcal{O}(U(1)) \rightarrow P$.*

Proof. Let ϵ be the norm-positive fundamental unit of $\mathbb{Q}[\theta]$ induced by the real quadratic irrationality θ , and let $\Phi : \langle \epsilon \rangle \times \{\pm 1\} \rightarrow \mathrm{SL}(2, \mathbb{Z})_\theta$ is the isomorphism of Proposition 5.1; note, in particular, that $\Phi_{21}(\epsilon)\epsilon + \Phi_{22}(\epsilon) = \epsilon \in \mathbb{R} \setminus \mathbb{Q}$, where $\Phi_{21}(\epsilon), \Phi_{22}(\epsilon) \in \mathbb{Z}$, so that, necessarily, $\Phi_{21}(\epsilon) \neq 0$. Thus, by a result of Connes [15, Thm. 7], it follows $P_1 := \mathcal{E}(\Phi(\epsilon), \theta)$ is not free as a right \mathcal{A}_θ -module. But now, if $\psi : \mathcal{O}(U(1)) \rightarrow P$ were a left H -covariant $*$ -homomorphism, then P_1 would be freely generated as a right \mathcal{A}_θ -module by the unitary element $\psi((z \mapsto z)) \in P_1 \cap U(P)$. \square

Remark 5.7. In fact, this shows that P is not cleft, i.e., that there does not exist a unital convolution-invertible left $\mathcal{O}(U(1))$ -covariant map $\mathcal{O}(U(1)) \rightarrow P$. Indeed, P_1 is the quantum vector bundle (in the sense of Brzeziński–Majid [11, Def. A.3]) associated to P induced by the standard irreducible corepresentation of $\mathcal{O}(U(1))$, so that P_1 would be free as a right \mathcal{A}_θ -module if P were cleft [11, Appx. A].

From now on, we will be concerned with the $\mathcal{O}(U(1))$ -gauge theory of the canonical non-trivial principal left $\mathcal{O}(U(1))$ -comodule $*$ -algebra of Theorem 5.5 induced by a real quadratic irrationality.

5.3. Constant curvature connections as q -monopoles. From now on, let $\theta \in \mathbb{R} \setminus \mathbb{Q}$ be a real quadratic irrationality, let $\epsilon \in \mathbb{Q}[\theta]^\times \cap (1, \infty)$ be its norm-positive fundamental unit, let $B := \mathcal{A}_\theta$ be the smooth noncommutative 2-torus with deformation parameter θ , and let P be the non-trivial principal $\mathcal{O}(U(1))$ -comodule $*$ -algebra with ${}^{\mathrm{co}}\mathcal{O}(U(1))P = \mathcal{A}_\theta$ of Theorem 5.5. We shall now show that Connes’s constant curvature connections [15] on the isotypical subspaces of P *qua* basic Heisenberg modules on \mathcal{A}_θ combine to yield a ϵ^{-1} -monopole closely analogous to Brzeziński–Majid’s q -monopole on the q -deformed complex Hopf fibration [11, §5.2]. In fact, we shall see that Polishchuk–Schwarz’s holomorphic structures [32] exhaust the $\mathcal{O}(U(1))$ -gauge theory of P in a manner that demonstrates the necessity of considering *all* principal FODC on P compatible with given vertical and horizontal calculi.

Before continuing, let us fix notation. On the one hand, define a left $\mathcal{O}(U(1))$ -covariant unital \mathbb{C} -algebra automorphism $\sigma : P \rightarrow P$ by

$$(5.5) \quad \forall m \in \mathbb{Z}, \forall p \in P_m, \quad \sigma(p) := \epsilon^{-m} p.$$

Although neither σ nor σ^2 is a $*$ -automorphism, they do respectively satisfy

$$(\sigma \circ *)^2 = \text{id}_P, \quad (\sigma^2 \circ *)^2 = \text{id}_P.$$

On the other hand, given $m \in \mathbf{Z}$, define $a_m, b_m, c_m, d_m \in \mathbf{Z}$ by

$$(5.6) \quad \begin{pmatrix} a_m & b_m \\ c_m & d_m \end{pmatrix} := \Phi(\epsilon^m),$$

where $\Phi : \langle \epsilon \rangle \times \{\pm 1\} \xrightarrow{\sim} \text{SL}(2, \mathbf{Z})_\theta$ is the isomorphism of Proposition 5.1. Thus, by (5.2),

$$\forall m \in \mathbf{Z}, \quad c_m \epsilon + d_m = \epsilon^m,$$

so that $\{m \in \mathbf{Z} \mid c_m = 0\} = \{0\}$. Moreover, by an observation of Polishchuk–Schwarz [32, Eq. 1.2], the map $(m \mapsto c_m) : \mathbf{Z} \rightarrow \mathbf{Z}$ satisfies

$$(5.7) \quad \forall m, n \in \mathbf{Z}, \quad c_{m+n} = c_m \epsilon^{-n} + \epsilon^m c_n$$

Let us now recall the construction of the standard $*$ -differential calculus on the smooth noncommutative 2-torus $B := \mathcal{A}_\theta$, which consists of rapidly decaying Laurent series in unitary generators U_θ and V_θ satisfying $V_\theta U_\theta = \epsilon^{2\pi i \theta} U_\theta V_\theta$. Recall that B admits a commuting pair of $*$ -derivations $\delta_1, \delta_2 : B \rightarrow B$ uniquely determined by

$$\delta_1(U) := 2\pi U, \quad \delta_1(V) := 0, \quad \delta_2(U) := 0, \quad \delta_2(V) := 2\pi V.$$

Hence, the canonical $*$ -differential calculus (Ω_B, d_B) on B is given by

$$\forall k \in \mathbf{Z}_{\geq 0}, \quad \Omega_B^k := \begin{cases} B & \text{if } k = 0, 2, \\ B^{\otimes 2} & \text{if } k = 1, \\ 0 & \text{else,} \end{cases}$$

where B is viewed as the trivial B - $*$ -bimodule, the non-trivial product $\wedge : \Omega_B^1 \otimes_B \Omega_B^1 \rightarrow \Omega_B^2$ is given by

$$\forall (b_1, b_2), (c_1, c_2) \in \Omega_B^1, \quad (b_1, b_2) \wedge (c_1, c_2) := b_2 c_1 - b_1 c_2,$$

and the exterior derivative d_B given by

$$\begin{aligned} \forall b \in \Omega_B^0, \quad d_B(b) &:= (\delta_1(b), \delta_2(b)), \\ \forall (b_1, b_2) \in \Omega_B^1, \quad d_B(b_1, b_2) &:= \delta_2(b_1) - \delta_1(b_2). \end{aligned}$$

For notational convenience, let

$$d\tau^1 := \frac{1}{2\pi i} U_\theta^* d_B(U_\theta) = (-i, 0), \quad d\tau^2 := \frac{1}{2\pi i} V_\theta^* d_B(V_\theta) = (0, -i), \quad \text{vol}_B := d\tau^1 \wedge d\tau^2 = 1,$$

so that $d\tau^1$ and $d\tau^2$ are central in the graded $*$ -algebra Ω_B and skew-adjoint, vol_B is central in Ω_B and self-adjoint, $\{d\tau^1, d\tau^2\}$ is a basis for Ω_B^1 as both a left and right B -module, and $\{\text{vol}_B\}$ is a basis for Ω_B^2 as both a left and right B -module. In particular, can write

$$\begin{aligned} \forall b \in B, \quad d_B(b) &= i\partial_1(b)d\tau^1 + i\partial_2(b)d\tau^2, \\ \forall (b_1, b_2) \in B^{\otimes 2}, \quad d_B(b_1 d\tau^1 + b_2 d\tau^2) &= -i(\partial_2(b_1) - \partial_1(b_2)) \text{vol}_B \end{aligned}$$

Now, suppose that we have completed (Ω_B, d_B) to a second-order horizontal calculus $(\Omega_B, d_B; \Omega_{P, \text{hor}})$ on the principal left H -comodule $*$ -algebra P . Since P is principal and since $\Omega_{P, \text{hor}}^1$ and $\Omega_{P, \text{hor}}^2$ are projectable horizontal lifts for Ω_B^1 and Ω_B^2 , respectively, it follows by the generalised Hopf module lemma [5, Lemma 5.29] that $\Omega_{P, \text{hor}}^1$ and $\Omega_{P, \text{hor}}^2$ are free as left P -modules with respective bases $\{d\tau^1, d\tau^2\}$ and $\{\text{vol}_B\}$. Thus, for every prolongable gauge potential ∇ on P with respect to $(\Omega_B, d_B; \Omega_{P, \text{hor}})$, there necessarily exist

unique left H -covariant maps $\partial_1, \partial_2 : P \rightarrow P$ extending $\delta_1, \delta_2 : B \rightarrow B$, respectively, such that

$$(5.8) \quad \forall p \in P, \quad \nabla(p) = i\partial_1(p)d\tau^1 + i\partial_2(p)d\tau^2;$$

in this case, it follows that the canonical prolongation ∇^{Pf} of ∇ is given by

$$(5.9) \quad \forall (p_1, p_2) \in P^{\otimes 2}, \quad \nabla(p_1 d\tau^1 + p_2 d\tau^2) = -i(\partial_2(p_1) - \partial_1(p_2)) \text{vol}_B,$$

and hence that the field strength $F[\nabla]$ of ∇ is given by

$$(5.10) \quad \forall p \in P, \quad F[\nabla](p) = i[\partial_1, \partial_2](p) \text{vol}_B.$$

As it turns out, we have the following canonical candidate for $\{\partial_1, \partial_2\}$ due originally to Connes [15, Thm. 7]; our immediate goal, then, will be to reverse-engineer $\Omega_{P, \text{hor}}$ so that (5.8) defines a prolongable gauge potential ∇ on P with respect to $(\Omega_B, d_B; \Omega_{P, \text{hor}})$.

Theorem 5.8 (Connes [15, Thm. 7], Polishchuk–Schwarz [32, Propp. 2.1, 2.2]). *Define $\mathcal{O}(U(1))$ -covariant \mathbb{C} -linear extensions $\partial_1, \partial_2 : P \rightarrow P$ of δ_1 and δ_2 , respectively, by*

$$(5.11) \quad \forall m \in \mathbb{Z} \setminus \{0\}, \forall f \in P_m, \forall (x, k) \in \mathbb{R} \times \mathbb{Z}_{c_m}, \quad \partial_1 f(x, k) := -i \frac{\partial}{\partial t} f(x, k),$$

$$(5.12) \quad \forall m \in \mathbb{Z} \setminus \{0\}, \forall f \in P_m, \forall (x, k) \in \mathbb{R} \times \mathbb{Z}_{c_m}, \quad \partial_2 f(x, k) := 2\pi \epsilon^{-n} c_n x f(x, \alpha).$$

where $\frac{\partial}{\partial t}$ denotes differentiation on $S(\mathbb{R}) \otimes \mathbb{C}[Z_{c_m}]$ with respect to the continuous variable. Then the maps ∂_1 and ∂_2 satisfy the following relations:

$$(5.13) \quad \forall j \in \{1, 2\}, \forall p, p' \in P, \quad \partial_j(pp') = \partial_j(p) \cdot \sigma(p') + p \cdot \partial_j(p'),$$

$$(5.14) \quad \forall j \in \{1, 2\}, \forall p \in P, \quad \partial_j(p^*) = -\sigma(\partial_j(p)^*).$$

Moreover, the commutator $[\partial_1, \partial_2] := \partial_1 \circ \partial_2 - \partial_2 \circ \partial_1$ satisfies

$$(5.15) \quad \forall m \in \mathbb{Z}, \forall f \in P_m, \quad [\partial_1, \partial_2](f) = -2\pi i \epsilon^{-m} c_m f.$$

Remark 5.9 (Polishchuk [30]). Let $\tau \in \{z \in \mathbb{C} \mid \text{Im } z < 0\}$; let $g := \Phi(\epsilon)$. Then

$$B_g(\theta, \tau) := \ker(\partial_1 + \tau \partial_2)$$

is a unital (non- $*$ -closed) subalgebra of P , which can be interpreted as the homogeneous coordinate ring of \mathcal{A}_θ with the complex structure τ *qua* noncommutative projective variety.

We can now use (5.13) and (5.14) to reverse-engineer a canonical second-order horizontal calculus on P of the form $(\Omega_B, d_B; \Omega_{P, \text{hor}})$ encoding the canonical twisted derivations ∂_1 and ∂_2 on P as a prolongable gauge potential. This will require the following definition.

Definition 5.10. Let H be a Hopf $*$ -algebra, let A be a left H -comodule $*$ -algebra, and let E be a left H -covariant A - $*$ -bimodule. Let $\phi : A \rightarrow A$ be a left H -covariant unital \mathbb{C} -algebra automorphism, such that $(\phi \circ *)^2 = \text{id}_A$. We define the left H -covariant A - $*$ -bimodule E_ϕ to be the left H -covariant left A -module E together with right A -module structure $\cdot_\phi : E \otimes_{\mathbb{C}} A \rightarrow E$ and $*$ -structure $*_\phi : E \rightarrow E$ defined, respectively, by

$$\begin{aligned} \forall e \in E, \forall a \in A, \quad e \cdot_\phi a &:= e \cdot \phi(a), \\ \forall e \in E, \quad e^*_\phi &:= \phi(e^*). \end{aligned}$$

Proposition 5.11. *Define a $\mathbb{Z}_{\geq 0}$ -graded left $\mathcal{O}(U(1))$ -comodule P - $*$ -bimodule $\Omega_{P,\text{hor}}$ by*

$$\forall k \in \mathbb{Z}_{\geq 0}, \quad \Omega_{P,\text{hor}}^k := \begin{cases} P & \text{if } k = 0, \\ (P_\sigma)^{\otimes 2} & \text{if } k = 1, \\ P_{\sigma^2} & \text{if } k = 2, \\ 0 & \text{else;} \end{cases}$$

let $\iota : \Omega_B \hookrightarrow \Omega_{P,\text{hor}}$ be the map induced by the inclusion $B \hookrightarrow P$. Then $(\Omega_B, d_B; \Omega_{P,\text{hor}}, \iota)$ defines a second-order horizontal calculus on P when $\Omega_{P,\text{hor}}$ is endowed with the extension of the P -bimodule structure to a multiplication $\Omega_{P,\text{hor}} \otimes \Omega_{P,\text{hor}} \rightarrow \Omega_{P,\text{hor}}$ given by

$$\begin{aligned} \forall (p_1, p_2), (p'_1, p'_2) \in \Omega_{P,\text{hor}}^1, \\ (p_1 \cdot d\tau^1 + p_2 \cdot d\tau^2) \wedge (p'_1 \cdot d\tau^1 + p'_2 \cdot d\tau^2) := (p_1 \sigma(p'_2) - p_2 \sigma(p'_1)) \cdot \text{vol}_B. \end{aligned}$$

Moreover, the left $\mathcal{O}(U(1))$ -covariant map $\nabla_0 : P \rightarrow \Omega_{P,\text{hor}}^1$ given by

$$(5.16) \quad \forall p \in P, \quad \nabla_0(p) = i\partial_1(p) \cdot d\tau^1 + i\partial_2(p) \cdot d\tau^2,$$

where $\partial_1, \partial_2 : P \rightarrow P$ are the maps constructed in Theorem 5.8, defines a prolongable gauge potential ∇_0 on P with respect to $(\Omega_B, d_B; \Omega_{P,\text{hor}})$ with canonical prolongation ∇_0^{pr} given by

$$(5.17) \quad \forall (p_1, p_2) \in P^{\otimes 2}, \quad \nabla_0^{\text{pr}}(p_1 \cdot d\tau^1 + p_2 \cdot d\tau^2) = -i(\partial_2(p_1) - \partial_1(p_2)) \cdot \text{vol}_B,$$

and field strength $F[\nabla_0]$ given by

$$(5.18) \quad \forall p \in P, \quad F[\nabla_0](p) = \left[2\pi \frac{\epsilon c_1}{\epsilon^2 - 1} \text{vol}_B, p \right].$$

Proof. Let us first show that $(\Omega_B, d_B; \Omega_{P,\text{hor}}, \iota)$ correctly defines a second-order horizontal calculus on P ; since $\sigma|_B = \text{id}_B$, the only non-trivial point is that the $*$ -structure on $\Omega_{P,\text{hor}}$ is graded-antimultiplicative with respect to the multiplication $\Omega_{P,\text{hor}}^1 \times \Omega_{P,\text{hor}}^1 \rightarrow \Omega_{P,\text{hor}}^2$. Indeed, for all $(p_1, p_2), (p'_1, p'_2) \in P^{\otimes 2}$,

$$\begin{aligned} & ((p_1 \cdot d\tau^1 + p_2 \cdot d\tau^2) \wedge (p'_1 \cdot d\tau^1 + p'_2 \cdot d\tau^2))^* \\ &= ((p_2 \sigma(p'_1) - p_1 \sigma(p'_2)) \cdot \text{vol}_B)^* \\ &= \text{vol}_B \cdot (\sigma(p'_1)^* p_2^* - \sigma(p'_2)^* p_1^*) \\ &= (\sigma^2(\sigma(p'_1)^*) \sigma^2(p_2^*) - \sigma^2(\sigma(p'_2)^*) \sigma^2(p_1^*)) \cdot \text{vol}_B \\ &= -(\sigma((p'_1)^*) \cdot d\tau^1 + \sigma((p'_2)^*) \cdot d\tau^2) \wedge (\sigma(p_1^*) \cdot d\tau^1 + \sigma(p_2^*) \cdot d\tau^2) \\ &= -(d\tau^1 \cdot (p'_1)^* + d\tau^2 \cdot (p'_2)^*) \wedge (d\tau^1 \cdot p_1^* + d\tau^2 \cdot p_2^*) \\ &= -(p'_1 \cdot d\tau^1 + p'_2 \cdot d\tau^2)^* \wedge (p_1 \cdot d\tau^1 + p_2 \cdot d\tau^2)^*. \end{aligned}$$

Let us now show that the map ∇_0 defines a prolongable gauge potential on P with respect to $(\Omega_B, d_B; \Omega_{P,\text{hor}})$ with the correct canonical prolongation and field strength. Before continuing, note that δ_1, δ_2 , and σ are all block-diagonal with respect to the decomposition $P = \bigoplus_{m \in \mathbb{Z}} P_m$ and that σ acts as multiplication by a scalar on each isotypical subspace P_m , so that $\sigma \circ \delta_1 = \delta_1 \circ \sigma$ and $\sigma \circ \delta_2 = \delta_2 \circ \sigma$.

First, equations (5.13) and (5.14) together with the construction of $\Omega_{P,\text{hor}}$ imply that the left $\mathcal{O}(U(1))$ -covariant map $\nabla_0 : P \rightarrow \Omega_{P,\text{hor}}^1$ is a $*$ -derivation, while the fact that $\partial_j|_B = \delta_j$ for $j = 1, 2$ implies that $\nabla_0|_B = d_B$. Hence, ∇_0 defines a gauge potential with respect to the first-order horizontal calculus $(\Omega_B^1, d_B; \Omega_{P,\text{hor}}^1)$ induced by $(\Omega_B, d_B; \Omega_{P,\text{hor}})$.

Next, for all $p, q \in P$ and $b \in B$,

$$\begin{aligned}
 \nabla_0(p) \wedge d_B(b) \cdot q - p \cdot d_B(b) \wedge \nabla_0(q) &= i(\partial_1(p) \cdot d\tau^1 + \partial_2(p) \cdot d\tau^2) \wedge i(\delta_1(b) \cdot d\tau^1 + \delta_2(b) \cdot d\tau^2) \cdot q \\
 &\quad - p \cdot i(\delta_1(b) \cdot d\tau^1 + \delta_2(b) \cdot d\tau^2) \wedge i(\partial_1(q) \cdot d\tau^1 + \partial_2(q) \cdot d\tau^2) \\
 &= -(\partial_1(p)\delta_2(b) - \delta_2(p)\delta_1(b))\sigma^2(q) \cdot \text{vol}_B + p(\delta_1(b)\sigma(\partial_2(q)) - \delta_2(b)\sigma(\partial_1(q))) \cdot \text{vol}_B \\
 &= (\partial_2(p)\delta_1(b)\sigma(q) - \partial_1(p)\delta_2(b)\sigma(q)) \cdot \text{vol}_B,
 \end{aligned}$$

where

$$p \cdot d_B(b) \cdot q = p \cdot i(\delta_1(b) \cdot d\tau^1 + \delta_2(b) \cdot d\tau^2) \cdot q = ip\delta_1(b)\sigma(q) \cdot d\tau^1 + ip\delta_2(b)\sigma(q) \cdot d\tau^2.$$

Hence, the gauge potential ∇_0 is prolongable with canonical prolongation ∇_0^{pr} given by

$$\forall (p_1, p_2) \in P^{\oplus 2}, \quad \nabla_0^{\text{pr}}(p_1 \cdot d\tau^1 + p_2 \cdot d\tau^2) = -i(\partial_2(p_1) - \partial_1(p_2)) \cdot \text{vol}_B.$$

Finally, by Equation 5.15, we see that for all $m \in \mathbf{Z}$ and $p \in P_m$,

$$\begin{aligned}
 F[\nabla_0](p) &= -i\nabla_0(i\partial_1(p) \cdot d\tau^1 + i\partial_2(p) \cdot d\tau^2) \\
 &= -i(\partial_2(\partial_1(p)) - \partial_1(\partial_2(p))) \cdot \text{vol}_B = 2\pi\epsilon^{-m}c_m p \cdot \text{vol}_B.
 \end{aligned}$$

But now, since

$$\forall m, n \in \mathbf{Z}, \quad \epsilon^{-(m+n)}c_{m+n} = \epsilon^{-m-n}(q^n c_m + q^{-m}c_n) = (\epsilon^{-m}c_m) + (\epsilon^{-2})^m(\epsilon^{-n}c_n),$$

it follows that for all $m \in \mathbf{Z}$,

$$\epsilon^{-m}c_m = \frac{1 - \epsilon^{-2m}}{1 - \epsilon^{-2}}\epsilon^{-1}c_1 = -(1 - \epsilon^{-2m})\frac{\epsilon c_1}{\epsilon^2 - 1},$$

so that, in turn, for all $m \in \mathbf{Z}$ and $p \in P_m$,

$$F[\nabla_0](p) = 2\pi\epsilon^{-m}c_m p \cdot \text{vol}_B = 2\pi(1 - \epsilon^{-2m})\frac{\epsilon c_1}{\epsilon^2 - 1}p \cdot \text{vol}_B = \left[2\pi\frac{\epsilon c_1}{\epsilon^2 - 1}\text{vol}_B, p\right],$$

as was claimed. \square

We can now readily compute the Atiyah space \mathfrak{At} and gauge group \mathfrak{G} of P with respect to $(\Omega_B^1, d_B; \Omega_{P, \text{hor}}^1)$ as well as their prolongable analogues with respect to $(\Omega_B, d_B; \Omega_{P, \text{hor}})$. In fact, we shall see that every gauge potential and gauge transformation is automatically prolongable and that the group $\mathfrak{G} \cong \text{U}(1)$ acts trivially on the affine space $\mathfrak{At} \cong \mathbf{R}^2$.

Proposition 5.12. *Let $(\Omega_B, d_B; \Omega_{P, \text{hor}})$ be the second-order horizontal calculus on P and let ∇_0 be the prolongable gauge potential on P with respect to $(\Omega_B, d_B; \Omega_{P, \text{hor}})$ of Proposition 5.11.*

- (1) *Let \mathfrak{at} be the space of relative gauge potentials on P with respect to the first-order horizontal calculus $(\Omega_B^1, d_B; \Omega_{P, \text{hor}}^1)$. The map $\psi_{\mathfrak{at}} : \mathbf{R}^2 \rightarrow \mathfrak{at}$ defined by*

$$\forall (s_1, s_2) \in \mathbf{R}^2, \forall p \in P, \quad \psi_{\mathfrak{at}}(s_1, s_2)(p) := [i(s_1 d\tau^1 + s_2 d\tau^2), p]$$

is an isomorphism of \mathbf{R} -vector spaces; hence, in particular, the subspace $\text{Inn}(\mathfrak{at})$ of inner relative gauge potentials satisfies $\text{Inn}(\mathfrak{at}) = \mathfrak{at}$.

- (2) *Let $\mathfrak{at}^{\text{pr}}$ be the space of prolongable relative gauge potentials on P with respect to the second-order horizontal calculus $(\Omega_B, d_B; \Omega_{P, \text{hor}})$, and let $\text{Inn}(\mathfrak{at}^{\text{pr}})$ be the subspace of inner prolongable gauge potentials. Then $\mathfrak{at}^{\text{pr}} = \text{Inn}(\mathfrak{at}^{\text{pr}}) = \mathfrak{at}$, where*

$$\forall (s_1, s_2) \in \mathbf{R}^2, \quad F[\nabla_0 + \sigma(s_1, s_2)] = F[\nabla_0].$$

- (3) Let \mathfrak{G} be the gauge group of P with respect to $(\Omega_B^1, d_B; \Omega_{P, \text{hor}}^1)$. Then the function $\psi_{\mathfrak{G}} : U(1) \rightarrow \mathfrak{G}$ defined by

$$\forall \zeta \in U(1), \forall m \in \mathbb{Z}, \forall p \in P_m, \quad \psi_{\mathfrak{G}}(\zeta)(p) := \zeta^m p$$

is a group isomorphism; hence, in particular, the subgroup $\text{Inn}(\mathfrak{G})$ of inner gauge transformations and the subgroup \mathfrak{G}^{Pr} of prolongable gauge transformations with respect to $(\Omega_B, d_B; \Omega_{P, \text{hor}})$ satisfy $\text{Inn}(\mathfrak{G}) = \{\text{id}_P\}$ and $\mathfrak{G}^{\text{Pr}} = \mathfrak{G}$. Moreover, the group \mathfrak{G} acts trivially on the Atiyah space $\mathfrak{A}t$ of P with respect to $(\Omega_B^1, d_B; \Omega_{P, \text{hor}}^1)$.

Proof. Let us begin by proving part 1. Note that ψ_{at} is a well-defined \mathbf{R} -linear map with range contained in $\text{Inn}(\text{at})$. Note, moreover, that ψ_{at} is injective: indeed, if $(s_1, s_2) \in \ker \psi_{\text{at}}$, then for all $p \in P_1$,

$$0 = \psi_{\text{at}}(s_1, s_2)(p) = i(s_1 d\tau^1 + s_2 d\tau^2) \cdot p - p \cdot i(s_1 d\tau^1 + s_2 d\tau^2) = i(\epsilon^{-1} - 1)(s_1 p \cdot d\tau^1 + s_2 p \cdot d\tau^2),$$

so that $(s_1, s_2) = (0, 0)$ as was claimed. Thus, it suffices to show that ψ_{at} is surjective.

Let $\mathbf{A} \in \text{at}$ be given. Given $m \in \mathbb{Z}$, since \mathbf{A} is left $\mathcal{O}(U(1))$ -covariant and right B -linear, since the left P -module $\Omega_{P, \text{hor}}^1$ is free with basis $\{d\tau^1, d\tau^2\} \subset Z_B(\Omega_B^1)$, and since P_m is a self-Morita equivalence bimodule for B , there exist unique $\alpha_{m,1}, \alpha_{m,2} \in B$, such that

$$\forall p \in P_m, \quad \mathbf{A}(p) = \alpha_{m,1} p \cdot d\tau^1 + \alpha_{m,2} p \cdot d\tau^2;$$

note that $\alpha_{0,1} = \alpha_{0,2} = 0$ since $\mathbf{A}|_{P_0} = 0$. First, for all $m \in \mathbb{Z}$, we have $\alpha_{m,1}, \alpha_{m,2} \in \mathbf{C}$; indeed, given $b \in B$, for all $p \in P_m$,

$$\begin{aligned} 0 &= \mathbf{A}(bp) - b \cdot \mathbf{A}(p) \\ &= \alpha_{m,1} bp \cdot d\tau^1 + \alpha_{m,2} bp \cdot d\tau^2 - b\alpha_{m,1} p \cdot d\tau^1 - b\alpha_{m,2} p \cdot d\tau^2 \\ &= [\alpha_{m,1}, b]p \cdot d\tau^1 + [\alpha_{m,2}, b]p \cdot d\tau^2, \end{aligned}$$

so that $[\alpha_{m,1}, b] = [\alpha_{m,2}, b] = 0$ since P_m is a self-Morita equivalence bimodule for B , and hence $\alpha_{m,1}, \alpha_{m,2} \in \mathbf{C}$ since the algebra $B := \mathcal{A}_\theta$ is central. Next, given $m, n \in \mathbb{Z}$, we see that for all $p \in P_m$ and $q \in P_n$,

$$\begin{aligned} 0 &= \mathbf{A}(pq) - \mathbf{A}(p) \cdot q - p \cdot \mathbf{A}(q) \\ &= (\alpha_{m+n,1} pq \cdot d\tau^1 + \alpha_{m+n,2} pq \cdot d\tau^2) - (\alpha_{m,1} p \cdot d\tau^1 + \alpha_{m,2} p \cdot d\tau^2) \cdot q \\ &\quad - p \cdot (\alpha_{n,1} q \cdot d\tau^1 + \alpha_{n,2} q \cdot d\tau^2) \\ &= (\alpha_{m+n,1} - \epsilon^{-n} \alpha_{m,1} - \alpha_{n,1})p \cdot d\tau^1 + (\alpha_{m+n,2} - \epsilon^{-n} \alpha_{m,2} - \alpha_{n,2})p \cdot d\tau^2; \end{aligned}$$

again, since $P_{m+n} = P_m \cdot P_n$ is a self-Morita equivalence bimodule, it follows that

$$(5.19) \quad \forall j \in \{1, 2\}, \quad \alpha_{m+n,j} = \epsilon^{-n} \alpha_{m,j} + \alpha_{n,j}$$

On the one hand, given $m \in \mathbb{Z}$, for all $p \in P_m$,

$$\begin{aligned} 0 &= \mathbf{A}(p) + \mathbf{A}(p^*)^* \\ &= (\alpha_{m,1} p \cdot d\tau^1 + \alpha_{m,2} p \cdot d\tau^2) + (\alpha_{-m,1} p^* \cdot d\tau^1 + \alpha_{-m,2} p^* \cdot d\tau^2)^* \\ &= (\alpha_{m,1} p \cdot d\tau^1 + \alpha_{m,2} p \cdot d\tau^2) + (-\epsilon^{-m} \alpha_{m,1} p^* \cdot d\tau^1 - \epsilon^{-m} \alpha_{m,2} p^* \cdot d\tau^2)^* \\ &= (\alpha_{m,1} p \cdot d\tau^1 + \alpha_{m,2} p \cdot d\tau^2) + (\alpha_{m,1} d\tau^1 \cdot p^* + \alpha_{m,2} d\tau^2 \cdot p^*)^* \\ &= (\alpha_{m,1} + \overline{\alpha_{m,1}})p \cdot d\tau^1 + (\alpha_{m,2} + \overline{\alpha_{m,2}})p \cdot d\tau^2, \end{aligned}$$

so that again, since P_m is a self-Morita equivalence bimodule, $\alpha_{m,1}, \alpha_{m,2} \in i\mathbb{R}$. On the other hand, by induction together with the 1-cocycle identity (5.19), for all $m \in \mathbb{Z}$ and $j \in \{1, 2\}$,

$$\alpha_{m,j} = \frac{1 - \epsilon^{-m}}{1 - \epsilon^{-1}} \alpha_{1,j} = (1 - \epsilon^{-m}) \frac{\alpha_{1,j}}{1 - \epsilon^{-1}}.$$

Hence, for all $m \in \mathbb{Z}$ and $p \in P_m$,

$$\mathbf{A}(p) = \sum_{j=1}^2 \alpha_{m,j} p \cdot d\tau^j = \sum_{j=1}^2 (1 - \epsilon^{-m}) \frac{\alpha_{1,j}}{1 - \epsilon^{-1}} p \cdot d\tau^j = \left[\sum_{j=1}^2 \frac{\alpha_{1,j}}{1 - \epsilon^{-1}} d\tau^j, p \right],$$

so that $\mathbf{A} = \sigma(s_1, s_2)$ for $s = (-i(1 - \epsilon^{-1})^{-1} \alpha_{1,1}, -i(1 - \epsilon^{-1})^{-1} \alpha_{1,2}) \in \mathbb{R}^2$.

Let us now turn to part 2. First, since $d\tau^1, d\tau^2 \in Z_B(\Omega_B^1)$, it follows that

$$\mathfrak{at} = \text{ran } \psi_{\mathfrak{at}} \subseteq \text{Inn}(\mathfrak{at}^{\text{Pr}}),$$

so that, indeed, $\mathfrak{at}^{\text{Pr}} = \text{Inn}(\mathfrak{at}^{\text{Pr}}) = \mathfrak{at}$. Now, given $(s_1, s_2) \in \mathbb{R}^2$, it follows by Corollary 3.32 that for all $p \in P$,

$$(\mathbf{F}[\nabla_0 + \sigma(s_1, s_2)] - \mathbf{F}[\nabla_0])(p) = [-\text{id}_B(i(s_1 d\tau^2 + s_2 d\tau^2)), p] = 0.$$

Finally, let us consider part 3. Before continuing, note that $\text{Inn}(\mathfrak{G}) = \{\text{id}_P\}$ since B is central. By construction, the map $\psi_{\mathfrak{G}} : U(1) \rightarrow \mathfrak{G}$ is a well-defined injective group homomorphism. Furthermore, for every $\zeta \in U(1)$, the gauge transformation $\psi_{\mathfrak{G}}(\zeta)$ acts diagonally on $P = \bigoplus_{m \in \mathbb{Z}} P_m$ by scalar multiplication; hence, the range of $\psi_{\mathfrak{G}}$ is contained in \mathfrak{G}^{Pr} and acts trivially on \mathfrak{At} . The proof that $\psi_{\mathfrak{at}} : \mathbb{R}^2 \rightarrow \mathfrak{at}$ is an \mathbb{R} -vector space isomorphism, *mutatis mutandis*, now shows that $\psi_{\mathfrak{G}}$ is surjective. \square

Now, recall that the usual de Rham calculus on $\mathcal{O}(U(1))$ fits into a canonical 1-parameter family of 1-dimensional bicovariant $*$ -differential calculi on $\mathcal{O}(U(1))$ defined as follows. Let $q \in \mathbb{R}^\times$ be given, and recall that the corresponding q -numbers are defined by

$$\forall n \in \mathbb{Z}, \quad [n]_q := \begin{cases} \frac{1-q^n}{1-q} & \text{if } q \neq 1, \\ n & \text{if } q = 1. \end{cases}$$

We define (Ω_q, d_q) to be the unique $*$ -differential calculus on $\mathcal{O}(U(1))$ with $*$ -closed left $\mathcal{O}(U(1))$ -comodule $\Lambda_q^1 =: (\Omega_q^1)^{\mathcal{O}(U(1))}$ of right $\mathcal{O}(U(1))$ -covariant 1-forms and quantum Maurer–Cartan form $\omega_q : \mathcal{O}(U(1)) \rightarrow \Lambda_q^1$ defined as follows:

- (1) the left crossed $\mathcal{O}(U(1))$ - $*$ -module Λ_q^1 is defined to be \mathbb{C} with the left $\mathcal{O}(U(1))$ -module structure given by

$$\forall m \in \mathbb{Z}, \forall \mu \in \Lambda_q^1, \quad (z \mapsto z^m) \triangleright \mu := q^m \mu$$

and the $*$ -structure given by complex conjugation;

- (2) the Λ_q^1 -valued Ad-invariant 1-cocycle $\omega_q : \mathcal{O}(U(1)) \rightarrow \Lambda_q^1$ is given by

$$\forall m \in \mathbb{Z}, \quad \omega_q((z \mapsto z^m)) := 2\pi[m]_q.$$

Since elements of Λ_q^1 are also bicovariant and since Λ_q^1 is spanned by the image under ω_q of the group-like unitary $(z \mapsto z) \in \mathcal{O}(U(1))$, it follows $\Omega_q^k = 0$ for $k \geq 2$. Note that Ω_q^1 is freely generated as both a left and right $\mathcal{O}(U(1))$ -module by the skew-adjoint bicovariant element $d_q t := (2\pi i)^{-1} \omega_q((z \mapsto z)) = -i$; note also that setting $q = 1$ recovers the de Rham calculus on $\mathcal{O}(U(1))$. We now show that there exists a unique value of q , such that P admits (Ω_q^1, d_q) -adapted prolongable gauge potentials with respect to the second-order horizontal calculus $(\Omega_B, d_B; \Omega_{P, \text{hor}})$.

Theorem 5.13. *Let $q \in \mathbb{R}^\times$.*

- (1) *The quadric subset $\mathfrak{At}^{\text{PR}}[\Omega_q^1]$ of all (Ω_q^1, d_q) -adapted prolongable gauge potentials on P with respect to $(\Omega_B, d_B; \Omega_{P, \text{hor}})$ satisfies*

$$\mathfrak{At}^{\text{PR}}[\Omega_q^1] = \begin{cases} \mathfrak{At}, & \text{if } q = \epsilon^2, \\ \emptyset, & \text{else.} \end{cases}$$

In particular, for every $\nabla \in \mathfrak{At} = \mathfrak{At}^{\text{PR}}[\Omega_{\epsilon^2}^1]$, the curvature 2-form $F[\nabla]$ is non-zero, independent of ∇ , and uniquely determined by

$$F[\nabla](d_{\epsilon^2} t) = -i\epsilon c_1 \text{vol}_B.$$

- (2) *The space $\text{at}[\Omega_q^1]$ of all (Ω_q^1, d_q) -adapted relative gauge potentials on P with respect to $(\Omega_B^1, d_B; \Omega_{P, \text{hor}}^1)$ and the subspace $\text{at}^{\text{PR}}[\Omega_q^{\leq 2}]$ of all (Ω_q, d_q) -adapted prolongable relative gauge potentials on P with respect to $(\Omega_B, d_B; \Omega_{P, \text{hor}})$ satisfy*

$$\text{at}[\Omega_q^1] = \text{at}^{\text{PR}}[\Omega_q^{\leq 2}] = \begin{cases} \text{at}, & \text{if } q = \epsilon, \\ 0 & \text{else;} \end{cases}$$

hence, in particular, it follows that

$$\mathfrak{At}/\text{at}^{\text{PR}}[\Omega_{\epsilon^2}^1] = \mathfrak{At}^{\text{PR}}[\Omega_{\epsilon^2}^1]/\text{at}^{\text{PR}}[\Omega_{\epsilon^2}^{\leq 2}] = \mathfrak{At}.$$

- (3) *The space $\text{Inn}(\text{at}^{\text{PR}}; \Omega_q^1)$ of all (Ω_q^1, d_q) -semi-adapted inner prolongable gauge potentials on P with respect to $(\Omega_B, d_B; \Omega_{P, \text{hor}})$ satisfies $\text{Inn}(\text{at}^{\text{PR}}; \Omega_q^1) = \text{at}$; hence, in particular,*

$$\text{Out}(\mathfrak{At}^{\text{PR}}[\Omega_{\epsilon^2}^1]) := \mathfrak{At}^{\text{PR}}[\Omega_{\epsilon^2}^1]/\text{Inn}(\text{at}^{\text{PR}}; \Omega_{\epsilon^2}^1) = \mathfrak{At}/\text{at} = \{\nabla_0 + \text{at}\}.$$

Proof. Let $(\Omega_{P, \text{ver}}, d_{P, \text{ver}})$ be the second-order vertical calculus on P induced by the unique bicovariant prolongation (Ω_q, d_q) of (Ω_q^1, d_q) . Since $\Omega_{P, \text{ver}}^1$ and $\Omega_{P, \text{hor}}^2$ are free as left P -modules with respective bases $\{d_q t\} \subset {}^{\text{co } \mathcal{O}(U(1))} \Omega_{P, \text{ver}}^1$ and $\{\text{vol}_B\} \subset {}^{\text{co } \mathcal{O}(U(1))} \Omega_{P, \text{hor}}^2 = \Omega_B^2$, it follows that a left $\mathcal{O}(U(1))$ -covariant left P -linear map $\phi : \Omega_{P, \text{ver}}^1 \rightarrow \Omega_{P, \text{hor}}^2$ is completely determined by the unique element $c \in {}^{\text{co } \mathcal{O}(U(1))} P = B$, such that $\phi(d_q t) = c \text{vol}_B$, and vice versa. Thus, given $c \in B$ with corresponding map ϕ , for all $b \in B$, $m \in \mathbb{Z}$, and $p \in P_m$,

$$\phi(d_q t \cdot b) - \phi(d_q t) \cdot b = \phi(b \cdot d_q t) - c \text{vol}_B \cdot b = (bc - cb) \text{vol}_B,$$

$$\phi(d_q t \cdot p) - \phi(d_q t) \cdot p = \phi(q^{-m} p \cdot d_q t) - c \text{vol}_B \cdot p = (q^{-m} p c - c \epsilon^{-2m} p) \text{vol}_B,$$

so that ϕ is a P -bimodule map if and only if $c = 0$ or $c \in \mathbb{C} \setminus \{0\}$ and $q = \epsilon^2$. We can now proceed with the proof of this corollary.

Let us first check part 1. On the one hand, suppose that there exists $\nabla \in \mathfrak{At}^{\text{PR}}[\Omega_q^1]$; recall that $F[\nabla]$ denotes its curvature 2-form. Since $\mathbf{F}[\nabla] = \mathbf{F}[\nabla_0] \neq 0$ by Propositions 5.11 and 5.12, it follows that $F[\nabla] : \Omega_{P, \text{ver}}^1 \rightarrow \Omega_{P, \text{hor}}^2$ is a non-zero left $\mathcal{O}(U(1))$ -covariant morphism of P -bimodules, so that $q = \epsilon^2$ by the above discussion. On the other hand, let $\nabla \in \mathfrak{At}$ be given. By the proof of Proposition 5.11 together with Proposition 5.12, for all $m \in \mathbb{Z}$ and $p \in P_m$,

$$\mathbf{F}[\nabla](p) = \mathbf{F}[\nabla_0](p) = 2\pi[m]_{\epsilon^{-2}} \epsilon^{-1} c_1 p \cdot \text{vol}_B,$$

$$d_{P, \text{ver}}(p) = 2\pi i[m]_q d_q t \cdot p = 2\pi i[m]_q q^{-m} p \cdot d_q t = 2\pi i q^{-1}[m]_{-q} p \cdot d_q t$$

so that $\nabla \in \mathfrak{At}^{\text{PR}}[\Omega_{\epsilon^2}^1]$ with $F[\nabla] : \Omega_{P, \text{ver}}^1 \rightarrow \Omega_{P, \text{hor}}^2$ uniquely defined by

$$F[\nabla](d_{\epsilon^2} t) := -i\epsilon c_1 \text{vol}_B.$$

Let us now turn to parts 2 and 3. Note that $\Omega_{P,\text{hor}}^1$ is free as a left P -module with basis given by $\{d\tau^1, d\tau^2\} \subset {}^{\text{co}}\mathcal{O}(\text{U}(1))\Omega_{P,\text{hor}}^1 = \Omega_B^1$. Hence, the above argument, *mutatis mutandis*, shows that $\text{at}[\Omega_q^1] = \text{at}$ when $q = \epsilon$ and $\text{at}[\Omega_q^1] = 0$ otherwise; indeed, for all $(s_1, s_2) \in \mathbb{R}^2$, the relative connection 1-form $\omega[\sigma(s_1, s_2)]$ of $\sigma(s_1, s_2) \in \text{at}[\Omega_\epsilon^1]$ is given by

$$\omega[\sigma(s_1, s_2)](d_\epsilon t) := -\frac{\epsilon - 1}{2\pi}(s_1 d\tau^1 + s_2 d\tau^2) \in \text{Span}_{\mathbb{R}}\{d\tau^1, d\tau^2\}.$$

Since $\text{at}^{\text{pr}} = \text{at}$, since $\Omega_q^2 = 0$, and since $d\tau^1 \wedge d\tau^1 = d\tau^1 \wedge d\tau^2 + d\tau^2 \wedge d\tau^1 = d\tau^2 \wedge d\tau^2 = 0$, it now follows that $\text{at}^{\text{pr}}[\Omega_q^{\leq 2}] = \text{at}[\Omega_q^1]$ in each case. Since $\mathbf{F}[\nabla] = \mathbf{F}[\nabla_0]$ for all $\nabla \in \mathfrak{A}t^{\text{pr}} = \mathfrak{A}t$, it now follows by Proposition 5.12 that $\text{Inn}(\text{at}^{\text{pr}}; \Omega_q^1) = \text{at}$ in general. \square

It therefore follows that the norm-positive fundamental unit ϵ of the real quadratic field $\mathbb{Q}[\theta]$ induced by θ is the unique value of $q \in \mathbb{R}^x$ for which P admits $(\Omega_{q^2}^1, d_{q^2})$ -adapted prolongable gauge potentials. In this case, every gauge potential is prolongable $(\Omega_{q^2}^1, d_{q^2})$ -adapted with the same non-zero constant curvature 2-form

$$d_{q^2} t \mapsto -i\epsilon c_1 \text{vol}_B$$

thereby defining a q -monopole analogous to the q -monopole of Brzeziński–Majid [11, §5.2] on the q -deformed complex Hopf fibration; mdistinct gauge potentials are gauge-inequivalent and yield non-isomorphic $(\mathcal{O}(\text{U}(1)); \Omega_{q^2}^1, d_{q^2})$ -principal FODC on P .

APPENDIX A. GROUPOIDS

In this appendix, we fix notation and terminology related to groupoids, and we state and prove a key technical lemma used in the proofs of Theorems 2.44 and 3.48.

Recall that a *groupoid* is a small category \mathcal{G} whose morphisms are all invertible; thus, in particular, a group is a groupoid with a single object. By abuse of notation, we conflate \mathcal{G} with the set of all morphisms in \mathcal{G} , and we further identify the set $\text{Ob}(\mathcal{G})$ of all objects in \mathcal{G} with a subset of \mathcal{G} via the injection $e \mapsto \text{id}_e$. If $f \in \mathcal{G}$ is an *arrow* (morphism) in \mathcal{G} , we denote its *source* (domain) by $s(f)$ and its *target* (codomain) by $t(f)$. Given $f, g \in \mathcal{G}$ satisfying $t(f) = s(g)$, we denote the composition $g \circ f$ by gf or $g \cdot f$. Given $e \in \text{Ob}(\mathcal{G})$, the *isotropy group* of e is the automorphism group $\mathcal{G}(e)$ of e in the category \mathcal{G} and the *star* of e is the set $\text{St}_{\mathcal{G}}(e)$ of all arrows in \mathcal{G} with source e ; given $e_1, e_2 \in \text{Ob}(\mathcal{G})$, we denote by $\mathcal{G}(e_1, e_2)$ the set of all arrows in \mathcal{G} with source e_1 and target e_2 .

A *subgroupoid* of a groupoid \mathcal{G} is a subcategory \mathcal{H} of \mathcal{G} whose morphisms are all invertible; we say that \mathcal{H} is *wide* whenever $\text{Ob}(\mathcal{G}) = \text{Ob}(\mathcal{H})$, and we say that \mathcal{H} has *trivial isotropy groups* whenever $\mathcal{G}(e) = \{\text{id}_e\}$ for all $e \in \text{Ob}(\mathcal{G})$. Given a groupoid \mathcal{G} and a wide subgroupoid \mathcal{H} of \mathcal{G} that has trivial isotropy groups, the *quotient* of \mathcal{G} by \mathcal{H} is the groupoid \mathcal{G}/\mathcal{H} constructed as follows. Define an equivalence relation

$$\sim_{\mathcal{H}} := \{(f, g) \in \mathcal{G}^2 \mid \exists (a, b) \in \mathcal{H}^2, g = afb\}$$

on \mathcal{G} , and denote:

$$\forall e \in \text{Ob}(\mathcal{G}), \quad [e]_{\mathcal{H}} := [\text{id}_e]_{\sim_{\mathcal{H}}}; \quad \forall f \in \mathcal{G}, \quad [f]_{\mathcal{H}} := [f]_{\sim_{\mathcal{H}}}.$$

Then $\mathcal{G}/\mathcal{H} := \mathcal{G}/\sim_{\mathcal{H}}$ as a set of arrows with set of objects $\text{Ob}(\mathcal{G}/\mathcal{H}) := \{[e]_{\mathcal{H}} \mid e \in \text{Ob}(\mathcal{G})\}$ and with the unique composition of arrows, such that:

$$\begin{aligned} \forall e \in \text{Ob}(\mathcal{G}), \quad \text{id}_{[e]_{\mathcal{H}}} &:= [\text{id}_e]_{\mathcal{H}} = [e]_{\mathcal{H}}, \\ \forall e_1, e_2, e_3 \in \text{Ob}(\mathcal{G}), \forall f \in \mathcal{G}(e_1, e_2), \forall g \in \mathcal{G}(e_2, e_3), \quad [g]_{\mathcal{H}} \cdot [f]_{\mathcal{H}} &:= [g \cdot f]_{\mathcal{H}}. \end{aligned}$$

A homomorphism from a groupoid \mathcal{G}_1 to a groupoid \mathcal{G}_2 is a functor $F : \mathcal{G}_1 \rightarrow \mathcal{G}_2$; the kernel of F is the subgroupoid

$$\ker F := \{f \in \mathcal{G}_1 \mid F(f) \in \text{Ob}(\mathcal{G}_2)\}$$

of \mathcal{G}_2 . A homomorphism $F : \mathcal{G}_1 \rightarrow \mathcal{G}_2$ is *star-injective* if $F|_{\text{St}_{\mathcal{G}_1}(e)} : \text{St}_{\mathcal{G}_1}(e) \rightarrow \text{St}_{\mathcal{G}_2}(F(e))$ is injective for all $e \in \text{Ob}(\mathcal{G}_1)$; it is *acovering* if, in addition, $F|_{\text{Ob}(\mathcal{G}_1)} : \text{Ob}(\mathcal{G}_1) \rightarrow \text{Ob}(\mathcal{G}_2)$ is surjective; it is an *isomorphism* if it is an isomorphism of categories; and it is an *equivalence* if it is an equivalence of categories, in which case a *homotopy inverse* of F is a weak inverse of F . A *homotopy* from a homomorphism $F_1 : \mathcal{G}_1 \rightarrow \mathcal{G}_2$ to a homomorphism $F_2 : \mathcal{G}_1 \rightarrow \mathcal{G}_2$ is a natural isomorphism $\eta : F_1 \Rightarrow F_2$.

Finally, given a groupoid \mathcal{G} , a set X , and a function $p : X \rightarrow \text{Ob}(\mathcal{G})$, an *action* of \mathcal{G} on X via p is a function

$$\{(g, x) \in \mathcal{G} \times X \mid p(x) = s(g)\} \rightarrow X, \quad (g, x) \mapsto g \triangleright x$$

satisfying the following conditions:

$$\begin{aligned} \forall e \in \text{Ob}(\mathcal{G}), \forall x \in p^{-1}(e), \quad \text{id}_e \triangleright x &= x, \\ \forall (e_1, e_2, e_3) \in \text{Ob}(\mathcal{G}), \forall f \in \mathcal{G}(e_1, e_2), \forall g \in \mathcal{G}(e_2, e_3), \forall x \in p^{-1}(e_1), \\ g \triangleright (f \triangleright x) &= (g \cdot f) \triangleright x. \end{aligned}$$

In this case, the *action groupoid* of the action of \mathcal{G} on X via p is the groupoid $\mathcal{G} \triangleright X$ with set of objects $\text{Ob}(\mathcal{G}) \times X$ and set of arrows $\{(g, x) \in \mathcal{G} \times X \mid p(x) = s(g)\}$, with the following definitions:

$$\begin{aligned} \forall (g, x) \in \mathcal{G} \times X, \quad s(g, x) &:= (s(g), x), \quad t(g, x) := (t(g), g \triangleright x); \\ \forall (g_1, x) \in \mathcal{G} \times X, \forall g_2 \in \text{St}_{\mathcal{G}}(t(g_1)), \quad (g_2, g_1 \triangleright x) \cdot (g_1, x) &:= (g_2 \cdot g_1, x); \\ \forall (e, x) \in \text{Ob}(\mathcal{G} \times X), \quad \text{id}_{(e, x)} &:= (\text{id}_e, x). \end{aligned}$$

When \mathcal{G} is a group, this recovers the usual notions of group action and action groupoid.

We can now state and prove our technical lemma, which we shall need for the proofs of Theorems 2.44 and 3.48.

Lemma A.1. *Let $\pi : \mathcal{G} \rightarrow \mathcal{H}$ be surjective covering of groupoids, and suppose that we have a star-injective homomorphism $\mu : \mathcal{H} \rightarrow H$ from \mathcal{H} to a group H . Suppose that we are given a homomorphism $S : G \times X \rightarrow \mathcal{G}$ and a left inverse $T : \mathcal{G} \rightarrow G \times X$ of S , such that:*

- (1) *there exists a natural isomorphism $\eta : \text{id}_{\mathcal{G}} \Rightarrow S \circ T$ satisfying*

$$\forall e \in \text{Ob}(\mathcal{G}), \quad \mu \circ \pi(\eta_e) = \text{id}_H;$$

- (2) *we have commutative diagrams*

$$\begin{array}{ccc} G \times X & \xrightarrow{S} & \mathcal{G} \\ \downarrow & & \downarrow \mu \circ \pi \\ G & \hookrightarrow & H \end{array} \quad \begin{array}{ccc} G \times X & \xleftarrow{T} & \mathcal{G} \\ \downarrow & & \downarrow \mu \circ \pi \\ G & \hookrightarrow & H \end{array}$$

where $G \hookrightarrow H$ is the inclusion map and $G \times X \twoheadrightarrow G$ is the surjective covering homomorphism given by $(g, x) \mapsto g$.

Then $\mu(\mathcal{H}) = G$, so that μ defines a surjective morphism $\mathcal{H} \twoheadrightarrow G$; the subgroupoid $\ker \mu$ of \mathcal{H} is wide and has trivial isotropy groups; the equivalence kernel \sim of the set function

$$(x \mapsto [\pi \circ \Sigma(1_G, x)]_{\ker \mu}) : X \rightarrow \text{Ob}(\mathcal{H} / \ker \mu)$$

is a G -invariant; and there exists a unique isomorphism $\tilde{S} : G \ltimes (X/\sim) \xrightarrow{\sim} \mathcal{H}/\ker \mu$ with

$$\forall (g, x) \in G \ltimes X, \quad \tilde{S}(g, [x]_{\sim}) = [\pi \circ S(g, x)]_{\ker \mu}.$$

Proof. Let us first show that $\mu(\mathcal{H}) = G$. Let $p := ((g, x) \mapsto g) : G \ltimes X \rightarrow G$, so that

$$p = \mu \circ \pi \circ S, \quad p \circ T = \mu \circ \pi.$$

Since $\pi : \mathcal{G} \rightarrow \mathcal{H}$ and $p : G \ltimes X \rightarrow G$ are surjections, it follows that

$$G = p(G \ltimes X) = (p \circ T)(\mathcal{G}) = (\mu \circ \pi)(\mathcal{G}) = \mu(\mathcal{H}).$$

Next, let us show that \sim is G -invariant. Let $x, y \in X$, and suppose that $x \sim y$, so that there exists $(f : \pi \circ S(x) \rightarrow \pi \circ S(y)) \in \ker \mu$. Given $g \in G$, the arrow

$$(\pi \circ S)(g, y) \cdot f \cdot (\pi \circ S)(g^{-1}, gx) : \pi \circ S(gx) \rightarrow \pi \circ S(gy)$$

in \mathcal{H} satisfies

$$\mu((\pi \circ S)(g, y) \cdot f \cdot (\pi \circ S)(g^{-1}, gx)) = g \cdot \text{id} \cdot g^{-1} = \text{id};$$

hence, $gx \sim gy$.

Next, let us check that $\ker \mu$ has trivial isotropy groups; it is wide since $\text{id}_e \in \ker \mu$ for all $e \in \text{Ob}(\mathcal{H})$. Let $e \in \text{Ob}(\mathcal{H})$ and let $h \in (\ker \mu)(e)$. Since π is a covering of groupoids, let $\tilde{h} \in \pi^{-1}(h)$. Then

$$\text{id} = \mu(h) = (\mu \circ \pi)(\tilde{h}) = (p \circ T)(\tilde{h}),$$

so that $T(\tilde{h}) = (\text{id}, T(s(\tilde{h})))$, and hence

$$\tilde{e} := s(\tilde{h}) = ST(s(\tilde{h})) = ST(t(\tilde{h})) = t(\tilde{h});$$

in other words, $\tilde{h} \in \mathcal{G}(\tilde{e})$ and $ST(\tilde{h}) = \text{id}_{\tilde{e}}$ for some $\tilde{e} \in \pi^{-1}(e)$. It now follows that

$$h = \pi(\tilde{h}) = \pi(\eta_{\tilde{e}}^{-1} \cdot \text{id}_{\tilde{e}} \cdot \eta_{\tilde{e}}) = \pi(\eta_{\tilde{e}})^{-1} \cdot \text{id}_e \cdot \pi(\eta_{\tilde{e}}) = \text{id}_e.$$

Since $\ker \mu$ is wide and has trivial isotropy groups, it now follows that $\mathcal{H}/\ker \mu$ is well-defined.

Next, let us show that \tilde{S} is well-defined. First, let $g \in G$, let $x, y \in X$, and suppose that $x \sim y$, so that there exists $(f : \pi \circ S(x) \rightarrow \pi \circ S(y)) \in \ker \mu$. Since

$$\mu((\pi \circ S)(g, y) \cdot f \cdot (\pi \circ S)(g^{-1}, gx)) = g \cdot \text{id} \cdot g^{-1} = \text{id},$$

it follows that $(\pi \circ S)(g, y) \circ f \circ (\pi \circ S)(g^{-1}, gx) \in \ker \mu$, so that

$$\begin{aligned} [(\pi \circ S)(g, y)]_{\ker \mu} &= [((\pi \circ S)(g, y) \cdot f \cdot (\pi \circ S)(g^{-1}, gx)) \cdot (\pi \circ S)(g, x) \cdot \text{id}_{\pi \circ S(x)}]_{\ker \mu} \\ &= [(\pi \circ S)(g, x)]_{\ker \mu}, \end{aligned}$$

so that $\tilde{S}(g, [x]_{\sim})$ is well-defined. A straightforward calculation now shows that \tilde{S} , which is well-defined as a map between sets of arrows, is a homomorphism.

Now, since \sim is G -invariant, the quotient map $X \rightarrow X/\sim$ is G -equivariant and hence extends to a covering of groupoids $\sigma := (g, x) \mapsto (g, [x]) : G \ltimes X \rightarrow G \ltimes (X/\sim)$. We wish to show that there exists a unique homomorphism $\tilde{T} : \mathcal{H}/\ker \mu \rightarrow G \ltimes (X/\sim)$ with

$$\forall f \in \mathcal{G}, \quad \tilde{T}([\pi(f)]_{\ker \mu}) = (\sigma \circ T)(f).$$

Let $f_1, f_2 \in \mathcal{G}$, and suppose that $[\pi(f_1)]_{\ker \mu} = [\pi(f_2)]_{\ker \mu}$, so that there exist $a, b \in \ker \mu$, such that $\pi(f_1) \cdot a = b \cdot \pi(f_2)$; we wish to show that $\sigma \circ T(f_1) = \sigma \circ T(f_2)$. On the one hand,

$$p \circ T(f_1) = \mu \circ \pi(f_1) = \mu(\pi(f_1) \cdot a) = \mu(b \cdot \pi(f_2)) = \mu \circ \pi(f_2) = p \circ T(f_2);$$

on the other hand,

$$((\pi \circ \eta_{s(f_2)}) \cdot a \cdot (\pi \circ \eta_{s(f_1)})^{-1} : (\pi \circ S)(T(s(f_1))) \rightarrow (\pi \circ S)(T(s(f_2)))) \in \ker \mu,$$

so that $T(s(f_1)) \sim T(s(f_2))$. Hence,

$$\sigma \circ T(f_1) = (p \circ T(f_1), [T(s(f_1))]) = (p \circ T(f_2), [T(s(f_2))]) = \sigma \circ T(f_2).$$

Again, a straightforward calculation now shows that \tilde{T} , which is well-defined as a map between sets of arrows, is a homomorphism.

Finally, let us show that \tilde{S} and \tilde{T} are mutually inverse homomorphisms. On the one hand, for every $(g, x) \in G \ltimes X$,

$$\tilde{T} \circ \tilde{S}(g, [x]) = \tilde{T}([S(g, x)]_{\ker \mu}) = (\sigma \circ T)(S(g, x)) = \sigma(g, x) = (g, [x]);$$

On the other hand, for every $f \in \mathcal{G}$,

$$\begin{aligned} \tilde{S} \circ \tilde{T}([\pi(f)]_{\ker \mu}) &= \tilde{S}(\sigma \circ T(f)) = [\pi \circ S \circ T(f)]_{\ker \mu} \\ &= [\pi(\eta_{t(f)}) \cdot \pi(f) \cdot \pi(\eta_{s(f)})^{-1}]_{\ker \mu} = [\pi(f)]_{\ker \mu}. \end{aligned}$$

Thus, the homomorphisms \tilde{S} and \tilde{T} are mutually inverse. \square

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