

New Communication Models and Decoding of Maximum Rank Distance Codes

Wrya K. Kadir

Department of Informatics
University of Bergen, Norway
Email: wrya.kadir@uib.no

Abstract—In this paper an interpolation-based decoding algorithm to decode Gabidulin codes transmitted through a new communication model is proposed. The algorithm is able to decode rank errors beyond half the minimum distance by one unit. Also the existing decoding algorithms for generalized twisted Gabidulin codes and additive generalized twisted Gabidulin codes are improved.

I. INTRODUCTION

Delsarte [1], Gabidulin [2] and Roth [3] independently introduced *rank metric codes*. Those rank metric codes that achieve Singleton-like bound are called *maximum rank distance (MRD) codes*. Gabidulin codes are the most well known family of MRD codes. Later this family was generalized by Kshevetskiy and Gabidulin [4] to *generalized Gabidulin (GG) codes*. These codes are linear over \mathbb{F}_{q^n} . Sheekey in [5] defined *twisted Gabidulin (TG) codes* and established a way to generalize GG codes to linear MRD codes over a base fields and then he was followed by Lunardon *et al.* [6], Otal and Özbudak [7], Trombetti and Zhou [8] and Sheekey [9] to define *generalized twisted Gabidulin (GTG) codes*, *additive generalized twisted (AGTG) codes*, *Trombetti-Zhou (TZ) codes* and *new MRD codes by Sheekey*, respectively. For more constructions of MRD codes, please refer to [10].

Efficient decoding is required for the wide range of applications of MRD codes in storage system [3], network coding [11] and cryptography [12]. There are plenty of algorithms that decode Gabidulin codes up to half the minimum distance [2], [13]–[15] and some which decode Gabidulin codes beyond half the minimum distance by considering restricted communication models [16]–[20]. The previously proposed restricted models, can generate error vectors that hold some structure and they do not look random.

Randrianarisoa in [15] gave an interpolation-based decoding algorithm for Gabidulin codes and also for GTG codes. This idea is used later in [21], [22], [23] and [24] to decode AGTG [7], Non-additive partition MRD codes [25], TZ codes [8] and Hermitian Rank metric codes [26], respectively.

In this paper we decode Gabidulin codes beyond half the minimum distance and also improve the decoding algorithms for GTG in [15] and AGTG codes in [21], [27] by making some delicate restrictions on the communication model. In the previously defined restricted models, the error vectors hold some specific structures, for instance symmetric error vectors [16], space-symmetric error vectors [20], but the channels in

our model generate error vectors without any specific structure. Moreover, we use low rate GTG and AGTG codes at the end of this paper to decode error vectors with rank $\leq k$ where k is the dimension of the code.

II. PRELIMINARIES

Definition 1. Let q be a power of prime p and \mathbb{F}_{q^m} be an extension of the finite field \mathbb{F}_q . A q -polynomial is a polynomial of the form $L(x) = a_0x + a_1x^q + \dots + a_{k-1}x^{q^{k-1}}$ over \mathbb{F}_{q^m} . If $a_{k-1} \neq 0$, then we say that $L(x)$ has q -degree $k - 1$. The set of all linearized polynomials of the form $L(x)$ is denoted by $\mathcal{L}_k(\mathbb{F}_{q^m})$.

When q is fixed or the context is clear, it is also customary to speak of a *linearized polynomial* as it satisfies the linearity property: $L(c_1x + c_2y) = c_1L(x) + c_2L(y)$ for any $c_1, c_2 \in \mathbb{F}_q$ and any x, y in an arbitrary extension of \mathbb{F}_{q^m} . Hence a linearized polynomial $L(x) \in \mathcal{L}_k(\mathbb{F}_{q^m})$ defines an \mathbb{F}_q -linear transformation L from \mathbb{F}_{q^m} to itself. The rank of a nonzero linearized polynomial $L(x) = \sum_{i=0}^n a_i x^{q^i}$ over \mathbb{F}_{q^m} is given by $\text{Rank}(L) = n - \dim_{\mathbb{F}_q}(\text{Ker}(L))$, where $\text{Ker}(L)$ is the kernel of $L(x)$.

Proposition 1. Let $L(x) = \sum_{i=0}^{n-1} a_i x^{q^i}$ over \mathbb{F}_{q^m} be a linearized polynomial with rank t . Then its associated Dickson matrix

$$D = \left(a_{i-j \pmod{n}}^{q^i} \right)_{n \times n} = \begin{pmatrix} a_0 & a_{n-1}^q & \dots & a_1^{q^{n-1}} \\ a_1 & a_0^q & \dots & a_2^{q^{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2}^q & \dots & a_0^{q^{n-1}} \end{pmatrix}, \quad (1)$$

has rank t over \mathbb{F}_{q^m} [15]. Moreover, any $t \times t$ submatrix formed by t consecutive rows and t consecutive columns in D is non-singular [28], [29].

III. MAXIMUM RANK DISTANCE (MRD) CODES

The rank of a vector $a = (a_1, \dots, a_n)$ in $\mathbb{F}_{q^m}^n$, denoted as $\text{Rank}(a)$, is the number of its linearly independent components, that is the dimension of the vector space spanned by a_i 's over \mathbb{F}_q . The rank distance between two vectors $a, b \in \mathbb{F}_{q^m}^n$ is defined as $d_R(a, b) = \text{Rank}(a - b)$.

Definition 2. A subset $\mathcal{C} \subseteq \mathbb{F}_{q^m}^n$ with respect to the rank distance is called a *rank metric code*. When \mathcal{C} contains at least two elements, the minimum rank distance of \mathcal{C} is given by

$d(\mathcal{C}) = \min_{A, B \in \mathcal{C}, A \neq B} \{d_R(A, B)\}$. Furthermore, it is called a *maximum rank distance (MRD) code* if it attains the Singleton-like bound $|\mathcal{C}| \leq q^{\min\{m(n-d+1), n(m-d+1)\}}$.

The most famous MRD codes are Gabidulin codes [2] which were further generalized in [4], [30]. The generalized Gabidulin (GG) codes $\mathcal{GG}_{n,k}$ with length $n \leq m$ and dimension k over \mathbb{F}_{q^m} is defined by the evaluation of

$$\left\{ \sum_{i=0}^{k-1} f_i x^{q^{si}} \mid f_i \in \mathbb{F}_{q^m} \right\}, \quad (2)$$

where $(s, m) = 1$, on linearly independent points $\alpha_0, \alpha_1, \dots, \alpha_{n-1}$ in \mathbb{F}_{q^m} . The choice of α_i 's does not affect the rank property and it is customary to exhibit Gabidulin codes and its generalized families without the evaluation points as in (2). For consistency with the parameters of MRD codes in [5], [7], [8], through what follows we always assume $n = m$.

For a linearized polynomial $L(x) = \sum_{i=0}^k l_i x^{q^i}$ over \mathbb{F}_{q^n} , it is clear that $\text{Rank}(L) \geq n - k$ if $l_k \neq 0$. Gow and Quinlan in [31, Theorem 10] (see also [5]) characterize a necessary condition for $L(x)$ to have rank $n - k$ as below, see [32], [33] for other necessary conditions.

Lemma 1. [31] *Suppose a linearized polynomial $L(x) = l_0 x + l_1 x^q + \dots + l_k x^{q^k}$, $l_k \neq 0$, in $\mathcal{L}_n(\mathbb{F}_{q^n})$ has q^k roots in \mathbb{F}_{q^n} . Then $\text{Norm}_{q^n/q}(l_k) = (-1)^{nk} \text{Norm}_{q^n/q}(l_0)$, where $\text{Norm}_{q^n/q}(x) = x^{1+q+\dots+q^{n-1}}$ is the norm function from \mathbb{F}_{q^n} to \mathbb{F}_q .*

According to Lemma 1, a linearized polynomial $L(x)$ of q -degree k has rank at least $n - k + 1$ if the condition in Lemma 1 is not met. Sheekey [5] applied Lemma 1 and constructed a new family of \mathbb{F}_q -linear MRD codes, known as *twisted Gabidulin (TG) codes*, and the generalized TG codes are investigated in [6] as follows:

$$\mathcal{H}_{k,s}(\epsilon, h) = \left\{ \sum_{i=0}^{k-1} f_i x^{q^{si}} + \epsilon f_0^{q^h} x^{q^{sk}} \mid f_i \in \mathbb{F}_{q^n} \right\}, \quad (3)$$

where n, k, s, h are positive integers such that $k < n$ and $(s, n) = 1$. Here ϵ is a nonzero element in \mathbb{F}_{q^n} satisfying $\text{Norm}_{q^{sn}/q^s}(\epsilon) \neq (-1)^{nk}$. Later Otal and Özbudak [7] further generalized this family by manipulating some terms of linearized polynomials and constructed the following \mathbb{F}_{q_0} -linear MRD codes, known as *additive generalized twisted Gabidulin (AGTG) codes*

$$\mathcal{A}_{k,s,q_0}(\epsilon, h) = \left\{ \sum_{i=0}^{k-1} a_i x^{q^{si}} + \epsilon a_0^{q_0^h} x^{q^{sk}} \mid a_i \in \mathbb{F}_{q^n} \right\}, \quad (4)$$

where $q = q_0^u$ and nonzero ϵ in \mathbb{F}_{q^n} satisfies $\text{Norm}_{q_0^{sn}/q_0^s}(\epsilon) \neq (-1)^{nku}$.

For the rest of this paper, we use the notation $[i] := q^{si}$ for $i = 0, \dots, n-1$, where $\text{gcd}(s, n) = 1$, for simplicity.

IV. NEW COMMUNICATION MODELS

In this section we define two new communication models. The models contain two authorized parties as sender and receiver. The sender encodes his/her message and then an error vector with rank t is added to the encoded message. The receiver will be able to decode the error vector and recover the message. Each models uses a different form of interpolation polynomial to generate its corresponding error vector.

A. First Model

In this modes, a linearized polynomial of the form

$$e_{\theta_1, \theta_2}(x) = \sum_{i=0}^{n-1} z_i x^{[i]}, \quad z_i \in \mathbb{F}_{q^n}, \quad (5)$$

$$z_0^{[n/2]} - z_0 = \alpha_{\theta_1}, \quad (6)$$

$$z_{k-1}^{[n/2]} - z_{k-1} = \alpha_{\theta_2}, \quad (7)$$

is used as the error interpolation polynomial where $\theta_1, \theta_2 \in [0, n-1]$ are the models' public parameters. We denote this model by $\mathcal{Q}_{\theta_1, \theta_2}$.

B. Second Model

In this model we have two cases:

- **case 1.** Suppose n is an odd integer, then

$$b(x) = b_0 x^{[0]} + \sum_{i=1}^{\frac{n-1}{2}} (b_i x^{[i]} + (b_i x)^{[n-i]}), \quad (8)$$

is the error interpolation polynomial where $\tilde{b} = (b_0, \dots, b_{n-1})$, $b_i \in \mathbb{F}_{q^n}$ and

$$b_{n-i} = b_i^{[n-i]} \text{ for } i = 1, \dots, \frac{n-1}{2}. \quad (9)$$

- **case 2.** Suppose n is an even integer, then

$$h(x) = h_0 x^{[0]} + \sum_{i=1}^{\frac{n}{2}-1} (h_i x^{[i]} + (h_i x)^{[n-i-1]}) + h_{n-1} x^{[n-1]}, \quad (10)$$

is the error interpolation polynomial where $\tilde{h} = (h_0, \dots, h_{n-1})$, $h_i \in \mathbb{F}_{q^n}$, and

$$h_{n-i-1} = h_i^{[n-i-1]} \text{ for } i = 1, \dots, \frac{n}{2} - 1. \quad (11)$$

Suppose $s(x)$ be one of the polynomials $e_{\theta_1, \theta_2}, b(x)$ or $h(x)$. We use $s(x)$ such that

$$s(\alpha_i) = e_i, \quad i = 0, \dots, n-1, \quad (12)$$

where $e = (e_0, \dots, e_{n-1})$ is the error vector and $\alpha_0, \dots, \alpha_{n-1}$ are ordered linearly independent points in \mathbb{F}_{q^n} over \mathbb{F}_q .

V. DECODING GABIDULIN CODES BEYOND HALF THE MINIMUM DISTANCE

A. Encoding

Let $\mathcal{GG}_{n,k}$, where n is even and k is odd, be a Gabidulin code with ordered \mathbb{F}_q -linearly independent evaluation points $\alpha_0, \alpha_1, \dots, \alpha_{n-1}$. The encoding of a message $m = (m_0, \dots, m_{k-1})$ is the evaluation of the following linearized polynomial at points $\alpha_0, \alpha_1, \dots, \alpha_{n-1}$:

$$f(x) = \sum_{i=0}^{k-1} m_i x^{[i]}, \quad (13)$$

Let $\tilde{m} = (m_0, m_1, \dots, m_{k-1}, 0, \dots, 0)$ be a vector of length n over \mathbb{F}_{q^n} and $M = \left(\alpha_i^{[j]} \right)_{n \times n}$ be the *Moore matrix* generated by α_i 's, where $1 \leq i, j \leq n-1$. Then the encoding of the message m can be expressed as

$$(m_0, m_1, \dots, m_{k-1}) \mapsto c = (f(\alpha_0), \dots, f(\alpha_{n-1})) = \tilde{m} \cdot M^T, \quad (14)$$

where M^T is the transpose of matrix M . In this process since only the first k components of \tilde{m} are nonzero, so only the first k rows of M are involved.

B. Decoding errors with rank $t \leq \frac{n-k+1}{2}$

Let the error vector $e = (e_0, \dots, e_{n-1})$ of rank t be added to the codeword $c = (c_0, \dots, c_{n-1})$ during transmission and let $r = (r_0, \dots, r_{n-1}) = c + e$ be the received vector.

Suppose we use the communication model $\mathcal{Q}_{\theta_1, \theta_2}$ and let e_{θ_1, θ_2} in (5) be the error interpolation polynomial such that

$$e_{\theta_1, \theta_2}(\alpha_i) = e_i = r_i - c_i, \quad i = 0, \dots, n-1, \quad (15)$$

where $\alpha_0, \dots, \alpha_{n-1}$ are ordered linearly independent points over \mathbb{F}_q in \mathbb{F}_{q^n} . One can see that the error vector e is uniquely determined by the polynomial $e_{\theta_1, \theta_2}(x)$ and denote $z = (z_0, \dots, z_{n-1})$. From (14) and (15) it follows that

$$r = c + e = (\tilde{m} + z) \cdot M^T.$$

Since M is nonsingular, this can be rewritten as

$$r \cdot (M^T)^{-1} = (c_0, c_1, \dots, c_{k-1}, 0, \dots, 0) + (z_0, z_1, \dots, z_{k-1}, z_k, \dots, z_{n-1}).$$

Let $\tilde{r} = (\eta_0, \dots, \eta_{n-1}) = r \cdot (M^T)^{-1}$, then the known coefficients z_i 's are

$$(z_k, \dots, z_{n-1}) = (\eta_k, \dots, \eta_{n-1}), \quad (16)$$

and we also have the auxiliary equations (6) and (7) which we will use later.

C. Reconstructing the interpolation polynomial $e_{\theta_1, \theta_2}(x)$

Let

$$E = \left(z_{i-j \pmod n}^{[j]} \right)_{n \times n} = (E_0 \ E_1 \ \dots \ E_{n-1}), \quad (17)$$

be the Dickson matrix associated with the linearized polynomial $e_{\theta_1, \theta_2}(x)$, where the indices i, j run through $\{0, 1, \dots, n-1\}$ and E_j is the j -th column of E .

According to Proposition 1, since $e_{\theta_1, \theta_2}(x)$ has rank t , so E has rank t and any $t \times t$ sub-matrix of E which contains t consecutive rows and columns is nonsingular. Hence the first column E_0 can be written as the linear combination of columns E_1, \dots, E_t as $E_0 = \gamma_1 E_1 + \gamma_2 E_2 + \dots + \gamma_t E_t$, where $\gamma_1, \dots, \gamma_t$ are elements in \mathbb{F}_{q^n} . Then we can obtain the following recursive equations

$$z_i = \gamma_1 z_{i-1}^{[1]} + \gamma_2 z_{i-2}^{[2]} + \dots + \gamma_t z_{i-t}^{[t]}, \quad 0 \leq i < n. \quad (18)$$

Due to the relation in (16), we already know z_k, \dots, z_{n-1} . These known coefficients leads us to the following linear recursive equation

$$z_i = \gamma_1 z_{i-1}^{[1]} + \gamma_2 z_{i-2}^{[2]} + \dots + \gamma_t z_{i-t}^{[t]}, \quad k+t \leq i < n, \quad (19)$$

where $\gamma_0, \dots, \gamma_t$ are unknowns. In [34], the q -linearized shift register is given and the above recursive relation (19) can be seen as its generalized version. Here $(\gamma_1, \dots, \gamma_t)$ is the connection vector of the shift register. We call the equation (19) as the *key equation* for the decoding algorithm in this paper and due to the properties of shift register, finding $\gamma_1, \dots, \gamma_t$ leads us to find the unknown coefficients z_0, \dots, z_{k-1} , recursively. The most complex task in our decoding algorithm is finding $\gamma_1, \dots, \gamma_t$ and then the remaining task (calculating unknown z_i 's) will be a recursive process. We consider $\text{Rank}(e) = t \leq \frac{n-k+1}{2}$, i.e., $2t+k \leq n+1$, and the task of finding $\gamma_1, \dots, \gamma_t$ via (19) is divided into two cases:

Case 1: If $2t+k < n+1$. In this case, (19) contains $n-k-t \geq t$ affine equations and t variables $\gamma_1, \dots, \gamma_t$, which has rank t . Hence the variables $\gamma_1, \dots, \gamma_t$ can be uniquely determined. Here any Gabidulin decoder can be applied, but here we assume the code has high code rate, for which the Berlekamp-Massey algorithm is more efficient and it has polynomial time complexity.

Case 2: If $2t+k = n+1$. In this case (19) is an under-determined system of $n-k-t = t-1$ equations with t variables $\gamma_1, \dots, \gamma_t$. A set of solutions $(\gamma_1, \dots, \gamma_t)$ with dimension one can be expressed of the form

$$\gamma + X\gamma' = (\gamma_1 + X\gamma'_1, \dots, \gamma_t + X\gamma'_t), \quad (20)$$

where γ, γ' are fixed elements in $\mathbb{F}_{q^n}^t$ and X runs through \mathbb{F}_{q^n} . The modified BM algorithm in [34, Th. 10] can give the solution with a free variable X .

If we take $i = 0$ and $i = k+t-1$ in (19) and substitute the solution (20), then we get

$$z_0 = \delta_0 + \delta_1 X, \quad (21)$$

and

$$z_{k+t-1} = \delta_2 + \delta_3 X + (\gamma_t + \gamma'_t X) z_{k-1}^{[t]}, \quad (22)$$

where in (21) and (22), z_0, z_{k-1} and X are the only unknowns and $\delta_0, \delta_1, \delta_2, \delta_3$ are derived from γ, γ' and known coefficients z_k, \dots, z_{n-1} . $X = -\gamma_t/\gamma'_t$ if $\gamma_t + \gamma'_t X = 0$ and this solution

can be verified by δ_2, δ_3 and a known coefficient z_i in (22). Substituting (21) in (6) gives

$$\tau_0 X^{[n/2]} + \tau_1 X + \tau_2 = 0. \quad (23)$$

As the next step, we rise both sides of (22) to the $[-t]$ -th power and obtain

$$z_{k-1} = \frac{a_1 + a_2 X^{[-t]}}{a_3 + a_4 X^{[-t]}}. \quad (24)$$

We also substitute (24) in (7) and rise both sides to the $[t]$ -th power to get

$$u_1 X^{[n/2]+1} + u_2 X^{[n/2]} + u_3 X + u_4 = 0. \quad (25)$$

Finally, one can substitute (23) into (25) and obtain the following quadratic polynomial equation over \mathbb{F}_{q^n}

$$\mu_1 X^2 + \mu_2 X + \mu_3 = 0. \quad (26)$$

If $\mu_1 = 0$, then $X = -\mu_3/\mu_2$ and if $\mu_1 \neq 0$, equation (26) can be reduced to

$$X^2 + rX + s = 0, \quad (27)$$

where $r = \mu_2/\mu_1$ and $s = \mu_3/\mu_1$. When the characteristic of \mathbb{F}_q is odd, equation (27) can be solved explicitly as follows:

- a) if $r^2 - 4s$ is a quadratic residue in \mathbb{F}_{q^n} , then it has two solutions $X = \frac{-r \pm \sqrt{r^2 - 4s}}{2}$;
- b) if $r^2 = 4s$, then it has a single solution $X = -r/2$;
- c) it has no solution in \mathbb{F}_{q^n} otherwise.

When the characteristic of \mathbb{F}_q is two, we have the following cases:

- 1) if $r = 0$, it has a single solution $X = s^{2^{n-1}}$, where $q = 2^l$;
- 2) if $r \neq 0$, the equation (27) can be reduced to $y^2 + y = \beta$, where $X = ry$ and $\beta = s/r^2$. Then $y^2 + y = \beta$ has
 - no zero if $\sum_{i=0}^{n-1} \beta^{2^i} = 1$;
 - two zeros of the form $W = \sum_{j=1}^{n-1} \beta^{2^j} (\sum_{k=0}^{j-1} c^{2^k})$ and $W + 1$ where $\sum_{i=0}^{n-1} \beta^{2^i} = 0$ and c is any fixed element such that $\sum_{i=0}^{n-1} c^{2^i} = 1$.

We expect our quadratic equation to have roots X in \mathbb{F}_{q^n} that lead to solutions $\gamma + X\gamma'$ in (19) and z_0 in (21). With the coefficients $\gamma_1, \dots, \gamma_t$ and also the initial state z_{n-1}, \dots, z_{n-t} , one can recursively compute z_1, \dots, z_{k-1} according to (18). Note that even if the equation (26) has two different solutions, they don't necessarily lead to correct coefficients of the error interpolation polynomial. In fact, by the expression of the Dickson matrix of $e_{\theta_1, \theta_2}(x)$, the correct $e_{\theta_1, \theta_2}(x)$ should have the sequence $(z_{n-1}, \dots, z_{n-t}, \dots)$ with period n . In other words, if the output sequence has period n , we know that the corresponding polynomial $e_{\theta_1, \theta_2}(x)$ is the desired error interpolation polynomial.

VI. AN IMPROVEMENT OF THE DECODING OF GTG AND AGTG CODES

In the interpolation-based decodings of GTG and AGTG codes in [15], [27], [35] and [21], when the rank of the error vector e is $t < \frac{n-k}{2}$, one can use any decoder of a Gaidulin code $\mathcal{GG}_{n, k+1}$ to recover the message. But when $t = \frac{n-k}{2}$, the problem of decoding the error vector is transformed to the problem of solving the projective polynomial $P(x) = x^{q^w+1} + u_1x + u_2 = 0$ over \mathbb{F}_{q^n} . In the following, we show that how one can decode GTG and AGTG codes more efficiently if he/she communicates via the communication model $\mathcal{Q}_{\theta_1, \theta_2}$. Moreover, we show that one will be able to decode any error vector with rank $t \leq k$ added to a low rate GTG and AGTG code if one uses the second communication model. In this paper by a low rate code we mean a code with $k \leq \lceil \frac{n-1}{2} \rceil$.

A. Decoding GTG and AGTG codes

Here we explain an improvement of the decoding algorithm for GTG codes and the same procedure can be applied to AGTG codes with some minor differences. In this subsection we assume n as an even positive integer. To be self-contained, we recall the decoding algorithm from [21] where the general communication model is replaced by the communication model $\mathcal{Q}_{\theta_1, \theta_2}$.

1) *Encoding*: The encoding of a message $m = (m_0, \dots, m_{k-1})$ is the evaluation of the following linearized polynomial at ordered points $\alpha_0, \alpha_1, \dots, \alpha_{n-1}$:

$$f(x) = \sum_{i=0}^{k-1} m_i x^{[i]} + \epsilon m_0^q x^{[k]}. \quad (28)$$

Then the encoding of GTG codes can be expressed as

$$(m_0, m_1, \dots, m_{k-1}) \mapsto c = (f(\alpha_0), \dots, f(\alpha_{n-1})) = \tilde{m} \cdot M^T, \quad (29)$$

where $\tilde{m} = (m_0, \dots, m_{k-1}, \epsilon m_0^q, 0, \dots, 0)$.

2) *Decoding*: Let the error vector $e = (e_0, \dots, e_{n-1})$ of rank t be added to the codeword $c = (c_0, \dots, c_{n-1})$ during transmission and let $r = (r_0, \dots, r_{n-1}) = c + e$ be the received vector. Take $e(x)$ be the error interpolation polynomial of the form given in (5) where instead of (7) we have

$$z_k^{[n/2]} - z_k = \alpha_{\theta_2}. \quad (30)$$

Then

$$e(\alpha_i) = e_i = r_i - c_i, \quad i = 0, \dots, n-1. \quad (31)$$

As we mentioned before, e is uniquely determined by the polynomial $e(x)$ and denote $z = (z_0, \dots, z_{n-1})$. From (14) and (15) it follows that

$$r = c + e = (\tilde{m} + z) \cdot M^T.$$

This is equivalent to

$$r \cdot (M^T)^{-1} = (m_0, m_1, \dots, m_{k-1}, \epsilon m_0^q, 0, \dots, 0) + (z_0, z_1, \dots, z_{k-1}, z_k, z_{k+1}, \dots, z_{n-1}).$$

Letting $\tilde{r} = (\eta_0, \dots, \eta_{n-1}) = r \cdot (M^T)^{-1}$, we obtain

$$(z_{k+1}, \dots, z_{n-1}) = (\eta_{k+1}, \dots, \eta_{n-1}), \quad (32)$$

and we also have the relations (6) and (30). In (32) we have $n - k - 1$ known coefficients z_i 's, while in (16) we had $n - k$ known coefficients γ_i 's.

3) *Reconstructing the interpolation polynomial $e(x)$* : If we write the 0th column E_0 of the Dickson matrix associated to $e(x)$ as the linear combination of E_1, \dots, E_t we will get the recursive equation

$$z_i = \gamma_1 z_{i-1}^{[1]} + \gamma_2 z_{i-2}^{[2]} + \dots + \gamma_t z_{i-t}^{[t]}, \quad 0 \leq i < n, \quad (33)$$

same as (18), where the subscripts in z_i 's are taken modulo n . Recall that the elements z_{k+1}, \dots, z_{n-1} are known from (32). Hence we obtain the following linear equations to replace the key equation in (19), with known coefficients z_i and variables $\gamma_1, \dots, \gamma_t$:

$$z_i = \gamma_1 z_{i-1}^{[1]} + \gamma_2 z_{i-2}^{[2]} + \dots + \gamma_t z_{i-t}^{[t]}, \quad k+t+1 \leq i < n. \quad (34)$$

For an error vector with $\text{Rank}(e) = t \leq \frac{n-k}{2}$, i.e., $2t+k \leq n$, we can divide the discussion into two cases.

Case 1: $2t+k < n$. In this case, (34) contains $n-k-t-1 \geq t$ affine equations in variables $\gamma_1, \dots, \gamma_t$, which has rank t . Hence the variables $\gamma_1, \dots, \gamma_t$ can be uniquely determined. Any Gabidulin $\mathcal{GG}_{n,k+1}$ decoder can be applied. Here we assume the code has high code rate, for which the Berlekamp-Massey algorithm gives a better complexity. Although the recurrence equation (34) is a generalized version of the ones in [13] and [34], the modified Berlekamp-Massey algorithm can be applied here to recover the coefficients $\gamma_1, \dots, \gamma_t$.

Case 2: $2t+k = n$. In this case (34) gives $n-k-t-1 = t-1$ independent affine equations in variables $\gamma_1, \dots, \gamma_t$. For such an under-determined system of linear equations, we will have a set of solutions $(\gamma_1, \dots, \gamma_t)$ that has dimension 1 over \mathbb{F}_{q^n} . Namely, the solutions will be of the form

$$\gamma + X\gamma' = (\gamma_1 + X\gamma'_1, \dots, \gamma_t + X\gamma'_t),$$

where γ, γ' are fixed elements in $\mathbb{F}_{q^n}^t$ and X runs through \mathbb{F}_{q^n} . As shown in [34, Th. 10], the solution can be derived from the modified BM algorithm with a free variable X .

Observe that in (33), by taking $i = 0$ and $i = k+t$ and substituting the solution $\gamma + X\gamma'$, one gets the following two equations

$$z_0 = \delta'_0 + \delta'_1 X, \quad (35)$$

and

$$z_{k+t} = \delta_2 + \delta_3 X + (\gamma_t + \gamma'_t X) z_k^{[t]}, \quad (36)$$

where in (35) and (36), z_0, z_k and X are unknowns. Using equations (6),(30), (35) and (36) instead of (6),(7), (21) and (22) and going through the same procedure in Subsection V-C, we can get a quadratic equation of the form

$$\mu_1 X^2 + \mu_2 X + \mu_3 = 0. \quad (37)$$

which can be solved in polynomial time as discussed in Subsection V-C. Hence, if the communication parties use the model $\mathcal{Q}_{\theta_1, \theta_2}$ to transfer their messages, then GTG and AGTG codes can be decoded with less time complexity.

VII. DECODING ERROR RANK VECTORS WITH ANY RANK $t \leq k$

In this subsection we consider the second communication model described in IV-B, but the generated error vectors are still look random and they can have any rank up to n .

In the decoding of GTG codes in Subsection VI-A, let $\tilde{r} = (\eta_0, \dots, \eta_{n-1}) = r \cdot (M^T)^{-1}$, then we obtain

$$(z_{k+1}, \dots, z_{n-1}) = (\eta_{k+1}, \dots, \eta_{n-1}), \quad (38)$$

and also based on the definition of GTG codes we have an auxiliary equation

$$-\epsilon z_0^{q^h} + z_k = \eta_k - \epsilon \eta_0^{q^h}, \quad (39)$$

since $\epsilon m_0^{q^h} + z_k = \eta_k$, and $m_0 + z_0 = \eta_0$. Let $k \leq \lceil \frac{n-1}{2} \rceil$. If we use (8) ((10)) as the error interpolation polynomial, one can employ (9) ((11)) and directly obtain z_1, \dots, z_k from the known coefficients in (38). The only remaining unknown coefficient z_0 can be calculated using the auxiliary equation (39) since z_k is already calculated.

Hence, by restricting the error interpolation polynomial we can decode any rank error vector with rank $t \leq k$ added to a low rate GTG (AGTG) code.

Remark 1. In [20], an application of space-symmetric rank errors in code-based cryptography is proposed. But space-symmetric rank errors similar to symmetric rank errors [16], contain some structures and this may lead to a new structural attack. If we use rank error vectors defined in Subsection VII instead of space-symmetric rank errors and use GTG codes instead of Gabidulin codes in GPT variants [36] and [37], we can avoid potential structural attacks and possibly get the same key size found in [20, Section VI.]. This will be investigated in future works.

Remark 2. The advantage of the model $\mathcal{Q}_{\theta_1, \theta_2}$ or even the second model IV-B is that it can generate error vectors that do not carry a specific structure since the structured coefficients' vector of the error interpolation polynomial goes through an interpolation process on linearly independent points. Even in subsection VI. the error space has dimension $n/2$ but it contains error with high or low ranks with no specific structure. So based on this observation, to find more suitable rank-based scheme, besides looking for new MRD codes and find the most efficient one, one can also look for new communication models with higher error correctness.

VIII. CONCLUSION

In this paper we made some delicate restrictions on the communication model and decode Gabidulin codes beyond half the minimum distance by one unit in polynomial time. The error vectors which are added to the codewords in our model, do not carry a specific structure. Moreover, we improved the decoding algorithms for GTG and AGTG codes proposed in [15] and [21], if two parties communicate through the first defined models. We are also able to decode any error vector with any rank $t \leq k$ added to low rate ($k \leq \lceil \frac{n-1}{2} \rceil$) GTG and AGTG codes if we employ the second communication model.

ACKNOWLEDGMENT

The author would like to thank Dr. Chunlei Li and Dr. Ferdinando Zullo for their helpful advice and also the anonymous reviewers for their valuable suggestions and comments.

REFERENCES

- [1] P. Delsarte, "Bilinear forms over a finite field, with applications to coding theory," *Journal of Combinatorial Theory, Series A*, vol. 25, no. 3, pp. 226–241, 1978.
- [2] E. M. Gabidulin, "Theory of codes with maximum rank distance," *Problemy Peredachi Informatsii*, vol. 21, no. 1, pp. 3–16, 1985.
- [3] R. M. Roth, "Maximum-rank array codes and their application to crisscross error correction," *IEEE Transactions on Information Theory*, vol. 37, no. 2, pp. 328–336, 1991.
- [4] A. Kshevetskiy and E. Gabidulin, "The new construction of rank codes," in *International Symposium on Information Theory (ISIT)*. IEEE, 2005, pp. 2105–2108.
- [5] J. Sheekey, "A new family of linear maximum rank distance codes," *Advances in Mathematics of Communications*, vol. 10, p. 475, 2016.
- [6] G. Lunardon, R. Trombetti, and Y. Zhou, "Generalized twisted gabidulin codes," *Journal of Combinatorial Theory, Series A*, vol. 159, pp. 79–106, 2018.
- [7] K. Otal and F. Özbudak, "Additive rank metric codes," *IEEE Transactions on Information Theory*, vol. 63, no. 1, pp. 164–168, 2017.
- [8] R. Trombetti and Y. Zhou, "A new family of MRD codes in $\mathbb{F}_q^{2n \times 2n}$ with right and middle nuclei \mathbb{F}_{q^n} ," *IEEE Transactions on Information Theory*, vol. 65, no. 2, pp. 1054–1062, 2019.
- [9] J. Sheekey, "New semifields and new MRD codes from skew polynomial rings," *Journal of the London Mathematical Society*, vol. 101, no. 1, pp. 432–456, 2020.
- [10] —, "MRD codes: Constructions and connections," *arXiv.org.*, vol. abs/1904.05813, 2019.
- [11] D. Silva, F. R. Kschischang, and R. Koetter, "A rank-metric approach to error control in random network coding," *IEEE Transactions on Information Theory*, vol. 54, no. 9, pp. 3951–3967, Sept 2008.
- [12] E. M. Gabidulin, A. V. Paramonov, and O. V. Tretjakov, "Ideals over a non-commutative ring and their application in cryptology," in *Advances in Cryptology – EUROCRYPT'91*, D. W. Davies, Ed. Springer, 1991, pp. 482–489.
- [13] G. Richter and S. Plass, "Fast decoding of rank-codes with rank errors and column erasures," in *International Symposium on Information Theory (ISIT)*, June 2004, pp. 398–398.
- [14] P. Loidreau, "A Welch–Berlekamp like algorithm for decoding Gabidulin codes," in *International Workshop on Coding and Cryptography (WCC)*, Ø. Ytrehus, Ed. Berlin, Heidelberg: Springer, 2006, pp. 36–45.
- [15] T. H. Randriantarisoa, "A decoding algorithm for rank metric codes," *arXiv.org.*, vol. abs/1712.07060, 2017.
- [16] E. M. Gabidulin and N. I. Pilipchuk, "Symmetric rank codes," *Problems of Information Transmission*, vol. 40, p. 103–117, 2004.
- [17] N. I. Pilipchuk and E. M. Gabidulin, "On codes correcting symmetric rank errors," in *Coding and Cryptography*, Ø. Ytrehus, Ed. Berlin, Heidelberg: Springer Berlin Heidelberg, 2006, pp. 14–21.
- [23] W. K. Kadir, C. Li, and F. Zullo, "On interpolation-based decoding of a class of maximum rank distance codes," in *International Symposium on Information Theory (ISIT)*, 2021.
- [18] E. M. Gabidulin and N. I. Pilipchuk, "Symmetric matrices and codes correcting rank errors beyond the $\lfloor (d-1)/2 \rfloor$ bound," *Discrete Applied Mathematics*, vol. 154, no. 2, pp. 305–312, 2006, coding and Cryptography.
- [19] J. Renner, T. Jerkovits, H. Bartz, S. Puchinger, P. Loidreau, and A. Wachter-Zeh, "Randomized decoding of gabidulin codes beyond the unique decoding radius," in *Post-Quantum Cryptography*, J. Ding and J.-P. Tillich, Eds. Cham: Springer International Publishing, 2020, pp. 3–19.
- [20] T. Jerkovits, V. Sidorenko, and A. Wachter-Zeh, "Decoding of space-symmetric rank errors," 2021.
- [21] W. K. Kadir and C. Li, "On decoding additive generalized twisted Gabidulin codes," *Cryptography and Communications*, vol. 12, pp. 987–1009, 2020.
- [22] C. Li, "Interpolation-based decoding of nonlinear maximum rank distance codes," in *International Symposium on Information Theory (ISIT)*, 2019.
- [24] —, "Decoding a class of maximum hermitian rank metric codes," *Submitted to The 6th International Workshop on Boolean Functions and their Applications (BFA)*, 2021.
- [25] K. Otal and F. Özbudak, "Some new non-additive maximum rank distance codes," *Finite Fields and Their Applications*, vol. 50, pp. 293–303, 2018.
- [26] K.-U. Schmidt, "Hermitian rank distance codes," *Designs, Codes and Cryptography*, vol. 86, no. 7, pp. 1469–1481, 2018.
- [27] C. Li and W. K. Kadir, "On decoding additive generalized twisted Gabidulin codes," *presented at the International Workshop on Coding and Cryptography (WCC)*, 2019.
- [28] G. Menichetti, "Roots of affine polynomials," in *Combinatorics '84*, ser. North-Holland Mathematics Studies, A. Barlotti, M. Biliotti, A. Cossu, G. Korchmaros, and G. Tallini, Eds. North-Holland, 1986, vol. 123, pp. 303–310.
- [29] L. Dickson, *Linear Groups, with an Exposition of the Galois Field Theory - Scholar's Choice Edition*. Creative Media Partners, LLC, 2015.
- [30] R. M. Roth, "Tensor codes for the rank metric," *IEEE Transactions on Information Theory*, vol. 42, no. 6, pp. 2146–2157, 1996.
- [31] R. Gow and R. Quinlan, "Galois theory and linear algebra," *Linear Algebra and its Applications*, vol. 430, no. 7, pp. 1778–1789, 2009, special Issue in Honor of Thomas J. Laffey.
- [32] B. Csajbók, G. Marino, O. Polverino, and F. Zullo, "A characterization of linearized polynomials with maximum kernel," *Finite Fields and Their Applications*, vol. 56, pp. 109–130, 2019.
- [33] G. McGuire and J. Sheekey, "A characterization of the number of roots of linearized and projective polynomials in the field of coefficients," *Finite Fields and Their Applications*, vol. 57, pp. 68–91, 2019.
- [34] V. Sidorenko, G. Richter, and M. Bossert, "Linearized shift-register synthesis," *IEEE Transactions on Information Theory*, vol. 57, no. 9, pp. 6025–6032, Sep. 2011.
- [35] J. Rosenthal and T. H. Randriantarisoa, "A decoding algorithm for twisted Gabidulin codes," in *International Symposium on Information Theory (ISIT)*. IEEE, 2017, pp. 2771–2774.
- [36] P. Loidreau, "An evolution of gpt cryptosystem," in *Int. Workshop Alg. Combin. Coding Theory (ACCT)*, 2016.
- [37] —, "A new rank metric codes based encryption scheme," in *International Workshop on Post-Quantum Cryptography*. Springer, 2017, pp. 3–17.