# Simple Worst-Case Optimal Adaptive Prefix-Free Coding

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#### Abstract

Gagie and Nekrich (2009) gave an algorithm for adaptive prefix-free coding that, given a string S[1..n] over an alphabet of size  $\sigma = o(n/\log^{5/2} n)$ , encodes S in at most n(H+1)+o(n) bits, where H is the empirical entropy of S, such that encoding and decoding S take O(n) time. They also proved their bound on the encoding length is optimal, even when the empirical entropy is high. Their algorithm is impractical, however, because it uses complicated data structures. In this paper we give an algorithm with the same bounds (except that we require  $\sigma = o(n^{1/2}/\log n)$ ) that uses no data structures more complicated than a lookup table. Moreover, when Gagie and Nekrich's algorithm is used for optimal adaptive alphabetic coding it takes  $O(n \log \log n)$  time for decoding, but ours still takes O(n) time.

#### 1 Introduction

Adaptive prefix-free coding has a history stretching back nearly fifty years, with the best-known solutions being the Faller-Gallager-Knuth (FGK) algorithm [2, 6, 11] and Vitter's algorithm [15]. Those use at most 2 more bits [12] and 1 more bit [15] per character than non-adaptive Huffman coding, however, and Gagie [3] (see also [4]) gave an algorithm that uses at most only 1 more bit per character than the empirical entropy of the input. All three algorithms are based on what is called the sibling property of the code-trees, which means encoding and decoding takes time proportional to the number of bits in the encoding, but FGK and Vitter use Huffman coding [9] and Gagie uses Shannon coding [14].

It might seem Gagie and Nekrich's paper is the last word on adaptive prefix-free coding, but there are two drawbacks to it: first, their algorithm uses complicated and impractical data structures; second, although their algorithm can be used for adaptive alphabetic prefix-free coding — it then uses at most 2 more bits per character than the empirical entropy, which they showed is optimal — decoding slows down slightly. (With alphabetic coding, the lexicographic order of the possible encodings is the same as that of the encoded strings.) We note that Golin et al. [8] recently gave an algorithm for adaptive alphabetic prefix-free coding with encoding taking constant worst-case time per character, but they said only that it uses at most a constant number more bits per character than the empirical entropy and did not discuss decoding time.

Table 1: The per-character bounds for the algorithms discussed, ignoring lower-order terms and omitting asymptotic notation. Results above the double line are for adaptive non-alphabetic prefix-free coding, and those below it are for adaptive alphabetic prefix-free coding. GN's bounds hold in the non-alphabetic case when  $\sigma = o(n/\log^{5/2} n)$  and in the alphabetic case when  $\sigma = o((n/\log n)^{1/2})$ ; our bounds hold in both cases when  $\sigma = o(n^{1/2}/\log n)$ .

	encoding	encoding	decoding
authors	length	time	$_{ m time}$
FGK [2, 6, 11]	$H+2+\delta$	H+1	H+1
Vitter [15]	$H+1+\delta$	H+1	H+1
Gagie [3]	H+1	H+1	H+1
KN [10]	H+1	1	$\lg H + 1$
GN[5]	H+1	1	1
Thm 1	H+1	1	1
Gagie [3]	H+2	H+1	H+1
GN[5]	H+2	1	$\log \log n$
GILMN [8]	H + O(1)	1	?
Thm 2	H+2	1	1

In this paper we give an algorithm for adaptive Shannon coding that, like Gagie and Nekrich's, uses at most 1 more bit per character than the empirical entropy with encoding and decoding taking time proportional to the number of characters in the input; unlike Gagie and Nekrich's, however, we use no data structures more complicated than a lookup table. (We can make encoding and decoding times worst-case constant per character, like Gagie and Nekrich's, by staggering the construction of our lookup tables and building them in the background, but we leave the details for a later version of this paper.) Because our algorithm is not based on canonical coding, it can easily be used for adaptive alphabetic prefix-free coding with no asymptotic slowdown.

Suppose we want to encode a string S[1..n] over the alphabet  $\{1,\ldots,\sigma\}$ , with empirical entropy

$$H = \sum_{i=1}^{\sigma} \frac{\operatorname{occ}_{i}(S)}{n} \lg \frac{n}{\operatorname{occ}_{i}(S)},$$

where  $\operatorname{occ}_i(S)$  is the number of occurrences of i in S. Let  $\delta \geq 0$  be the amount by which the expected codeword length in a Huffman code for S exceeds H (see, e.g., [1]). Table 1 shows the per-character bounds of the algorithms we have discussed, ignoring lower-order terms and omitting asymptotic notation for the time bounds. Gagie and Nekrich's bounds hold in the non-alphabetic case when  $\sigma = o(n/\log^{5/2} n)$  and in the alphabetic case when  $\sigma = o(n/\log n)^{1/2}$ ; our bounds hold in both cases when  $\sigma = o(n^{1/2}/\log n)$ . Our model is the word RAM with  $\Omega(\log n)$ -bit words, but we do not use any sophisticated word-RAM techniques.

## 2 Algorithm

Let  $b = \lceil \sigma \lg n \rceil$  and suppose we encode S[1..b] using a code that assigns every character a codeword of length  $\lceil \lg \sigma \rceil$ ; then, after encoding S[1..kb] for  $k \geq 1$ , we build a Shannon code for the distribution in which character i is assigned probability

$$\frac{\lg n - 1}{\lg n} \cdot \frac{\operatorname{occ}_i(S[1..kb])}{kb} + \frac{1}{\sigma \lg n},$$

and use that code to encode S[kb+1..(k+1)b]. That is, we encode each block of b characters, except the first such block, using a Shannon code for a distribution that is a weighted average of the distribution we saw in earlier blocks and of the uniform distribution.

To see how we can encode and decode quickly, notice we can amortize the cost of building each Shannon code over the b characters we encode with it. We can keep the characters sorted by their frequencies so far, but sorting them before building a code takes  $O(\sigma \log \sigma) = o(b)$  time anyway, assuming  $\sigma = o(n^{1/2}/\log n)$ , and then building the code takes  $O(\sigma) = o(b)$  time as well. We can also build an  $O(\sigma)$ -space lookup table (that is,  $O(\sigma)$  words or  $O(\sigma \log n)$  bits) mapping characters to their codewords, allowing us to encode characters in constant amortized time.

More interestingly, because a Shannon code assigns any character with probability p a codeword of length  $\lceil \lg(1/p) \rceil$ , each the code-tree has height at most  $\lceil \lg \sigma + \lg \lg n \rceil$ , meaning that in

$$O(2^{\lceil \lg \sigma + \lg \lg n \rceil}) = O(\sigma \log n) = O(b)$$

time and space we can build a lookup table storing, for every binary string of length  $\lceil \lg \sigma + \lg \lg n \rceil$ , the character whose codeword is a prefix of that string and the length of that codeword (if there is such a codeword), allowing us to decode characters in constant amortized time.

### 3 Analysis

By the nature of a Shannon code, we encode each S[j] with fewer than

$$\left[\lg \frac{1}{\frac{\lg n-1}{\lg n} \cdot \frac{\max(\operatorname{occ}_{S[j]}(S[1..j])-b,0)}{j} + \frac{1}{\sigma \lg n}}\right]$$

bits. Therefore, we encode the first b copies of each distinct character with fewer than  $\lg \sigma + \lg \lg n + 1$  bits each, and we encode each other copy S[j] with fewer than

$$\lg \frac{j}{\operatorname{occ}_{S[j]}(S[1..j]) - b} + \lg \frac{\lg n}{\lg n - 1}$$

$$< \lg j - \lg(\operatorname{occ}_{S[j]}(S[1..j]) - b) + \frac{\lg e}{\lg n - 1}$$

bits.

Since

$$\sum \left\{ \lg(\operatorname{occ}_{S[j]}(S[1..j]) - b) : S[j] = i, \operatorname{occ}_{S[j]}(S[1..j]) > b \right\}$$

$$= \lg \max(\operatorname{occ}_{i}(S) - b, 0)!$$

$$\geq \lg \operatorname{occ}_{i}(S)! - b \lg \operatorname{occ}_{i}(S)$$

$$\geq \lg \operatorname{occ}_{i}(S)! - b \lg n,$$

we use a total of fewer than

$$\sum_{j=1}^{n} \lg j - \sum_{i=1}^{\sigma} \lg \operatorname{occ}_{i}(S)! + \sigma b \lg n + \sigma b (\lg \sigma + \lg \lg n + 1) + \frac{n \lg e}{\lg n - 1}$$

$$= \lg n! - \sum_{i=1}^{\sigma} \lg \operatorname{occ}_{i}(S)! + O(\sigma^{2} \log^{2} n)$$

$$= \lg \frac{n!}{\prod_{i=1}^{\sigma} \operatorname{occ}_{i}(S)!} + O(\sigma^{2} \log^{2} n)$$

bits.

Notice

$$\frac{n!}{\prod_{i=1}^{\sigma} \operatorname{occ}_i(S)!}$$

is the number of distinct strings consisting of the characters in S, so intuitively it is approximately nH, and we can prove that using Stirling's approximation:

$$\lg n! - \sum_{i=1}^{\sigma} \lg \operatorname{occ}_{i}(S)!$$

$$= n \lg n - n \lg e + O(\log n) - \sum_{i=1}^{\sigma} \left(\operatorname{occ}_{i}(S) \lg \operatorname{occ}_{i}(S) - \operatorname{occ}_{i}(S) \lg e + O(\log \operatorname{occ}_{i}(S))\right)$$

$$= \sum_{i=1}^{\sigma} \operatorname{occ}_{i}(S) \lg \frac{n}{\operatorname{occ}_{i}(S)} + O(\log n) - O(\sigma \log(n/\sigma))$$

$$= nH \pm o(n).$$

As long as  $\sigma = o(n^{1/2}/\log n)$ , we use nH + o(n) bits in total.

**Theorem 1** There is an algorithm for adaptive prefix-free coding that uses no complicated data structures and, given a string S[1..n] over the alphabet  $\{1, ..., \sigma\}$  with  $\sigma = o(n^{1/2}/\log n)$ , encodes S in at most n(H+1) + o(n) bits, such that encoding and decoding S take O(n) time.

## 4 Alphabetic

Gilbert and Moore [7] modified Shannon's construction to produce an alphabetic code that assigns any character with probability p a codeword of length  $\lceil \lg(1/p) \rceil + 1$ , with no asymptotic increase in the time required. If we use their construction instead of Shannon's, then by the same arguments we just gave but with an extra 1 inserted where appropriate, we obtain the following theorem:

**Theorem 2** There is an algorithm for adaptive alphabetic prefix-free coding that uses no complicated data structures and, given a string S[1..n] over the alphabet  $\{1, ..., \sigma\}$  with  $\sigma = o(n^{1/2}/\log n)$ , encodes S in at most n(H+2) + o(n) bits, such that encoding and decoding S take O(n) time.

To see why we need n(H+2) bit in general, suppose  $\sigma = 2^k + 1$ . Any algorithm for adaptive alphabetic prefix-free coding uses a code-tree for S[j] has at least one even leaf at depth k+1. Suppose the adversary always choose S[j] to be the character assigned to that deep leaf. Then the algorithm uses n(k+1) bits but, because only the even characters occur in S,

$$H \le \lg 2^{k-1} = k - 1 \, .$$

- [1] I. Blanes, M. Hernández-Cabronero, J. Serra-Sagristà, and M. W. Marcellin. Lower bounds on the redundancy of Huffman codes with known and unknown probabilities. *IEEE Access*, 7:115857–115870, 2019.
- [2] N. Faller. An adaptive system for data compression. In *Record of the 7th Asilomar Conference on Circuits, Systems and Computers*, pages 593–597, 1973.
- [3] T. Gagie. Dynamic Shannon coding. In *Proceedings of the 12th European Symposium on Algorithms*, pages 359–370, 2004.
- [4] T. Gagie. Dynamic Shannon coding. *Information Processing Letters*, 102(2–3):113–117, 2007.
- [5] T. Gagie and Y. Nekrich. Worst-case optimal adaptive prefix coding. in *Proceedings* of the 11th Symposium on Algorithms and Data Structures, pages 315–326, 2009.
- [6] R. G. Gallager. Variations on a theme by Huffman. IEEE Transactions on Information Theory, 24(6):668-674, 1978.
- [7] E. N. Gilbert and E. F. Moore. Variable-length binary encodings. Bell System Technical Journal, 38:933–967, 1959.
- [8] M. J. Golin, J. Iacono, S. Langerman, J. I. Munro, and Y. Nekrich. Dynamic trees with almost-optimal access cost. In *Proceedings of the 26th Annual European Symposium on Algorithms*, pages 38:1–38:14, 2018.
- [9] D. A. Huffman. A method for the construction of minimum-redundancy codes. *Proceedings of the IRE*, 40(9):1098–1101, 1952.
- [10] M. Karpinski and Y. Nekrich. A fast algorithm for adaptive prefix coding. *Algorithmica*, 55(1): 29–41, 2009.
- [11] D. E. Knuth. Dynamic Huffman coding. Journal of Algorithms, 6(2):163–180, 1985.
- [12] R. L. Milidiú, E. S. Laber, and A. A. Pessoa. Bounding the compression loss of the FGK algorithm. *Journal of Algorithms*, 32(2):195–211, 1999.
- [13] E. S. Schwartz and B. Kallick. Generating a canonical prefix encoding. *Communications of the ACM*, 7(3):166–169, 1964.
- [14] C. E. Shannon. A mathematical theory of communication. *Bell System Technical Journal*, 27:379–423, 623–656, 1948.
- [15] J. S. Vitter. Design and analysis of dynamic Huffman codes. *Journal of the ACM*, 1987(4):825–845, 1987.