Kähler toric manifolds from dually flat spaces

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Abstract

We present a correspondence between real analytic Kähler toric manifolds and dually flat spaces, similar to Delzant correspondence in symplectic geometry. This correspondence gives rise to a lifting procedure: if $f : M \to M'$ is an affine isometric map between dually flat spaces and if N and N' are Kähler toric manifolds associated to M and M', respectively, then there is an equivariant Kähler immersion $N \to N'$. For example, we show that the Veronese and Segre embeddings are lifts of inclusion maps between appropriate statistical manifolds. We also discuss applications to Quantum Mechanics.

1 Introduction and summary

1.1 From Delzant to Dually flat spaces. In 1988, Delzant showed the following result:

Theorem 1.1 ([Del88]). Given $i = 1, 2$, let (M_i, ω_i) be a connected compact symplectic manifold of dimension $2n$ equipped with an effective Hamiltonian torus action $\Phi_i : \mathbb{T}^n \times M_i \to M_i$, with momentum map $\mathbf{J}_i : M_i \to \mathbb{R}^n$. If $\mathbf{J}_1(M_1) = \mathbf{J}_2(M_2)$, then there is an equivariant symplectomorphism $\varphi : M_1 \to M_2$ satisfying $\mathbf{J}_1 = \mathbf{J}_2 \circ \varphi$.

A quadruplet $(M, \omega, \Phi, \mathbf{J})$ as in Theorem 1.1 is called a *symplectic toric manifold*. Two symplectic toric manifolds $(M_i, \omega_i, \Phi_i, \mathbf{J}_i)$, $i = 1, 2$, are said to be *equivalent* if there is an equivariant symplectomorphism $\varphi : M_1 \to M_2$ such that $\mathbf{J}_1 = \mathbf{J}_2 \circ \varphi$. Call any subset Δ of $\mathbb{R}^n$ a *Delzant polytope* if it is the image under the momentum map $\mathbf{J}$ of a symplectic toric manifold. Then Theorem 1.1 yields a bijection between the set of Delzant polytopes $\Delta \subset \mathbb{R}^n$ and the set of equivalence classes $[(M, \omega, \Phi, \mathbf{J})]$ of symplectic toric manifolds:

$$\begin{array}{ccc} \left\{ \begin{array}{c} \text{Equivalence classes of} \\ \text{symplectic toric manifolds} \end{array} \right\} & \to & \text{Delzant polytopes in } \mathbb{R}^n \\ [(M, \omega, \Phi, \mathbf{J})] & \mapsto & \mathbf{J}(M) \end{array} \tag{1.1}$$

This bijection is called *Delzant correspondence*. Delzant actually characterized Delzant polytopes: they are exactly the convex polytopes in $\mathbb{R}^n$ that are simple, rational and smooth (see, e.g., [CdS03]).

From a Kähler geometrical perspective, the constructions employed by Delzant show that the manifold M_Δ associated to a Delzant polytope Δ is canonically a Kähler manifold and that the torus acts by holomorphic and isometric transformations. Moreover, on the set M_Δ° of points

$p \in M_\Delta$ where the torus action is free, the momentum map $\mathbf{J} : M_\Delta^\circ \to \Delta^\circ$ is a principal $\mathbb{T}^n$-bundle (Δ° = topological interior of Δ) and hence the Kähler metric on M_Δ° descends to a Riemannian metric on Δ°. This metric is Hessian; a potential $\phi : \Delta^\circ \to \mathbb{R}$ was explicitly computed by Guillemin [Gui94a] (see also [CDG03]). Abreu generalized the results of Guillemin by characterizing all potentials ϕ on Δ° that are induced from a compatible Kähler metric g on M_Δ [Abr03].

Delzant correspondence and the results obtained by Guillemin and Abreu are very rigid in the sense that they depend in a crucial way on the geometry of the momentum polytope Δ (vertices, faces, etc.). The objective of this paper is to "reshape" Delzant's correspondence in a more flexible way, in the Kähler setting, by replacing Delzant polytopes by dually flat manifolds (see below). Our motivation comes from the geometrization program of Quantum Mechanics that we pursued in previous works [Mol12, Mol13, Mol14, Mol15][1]. Let us describe the objects involved.

- We call *Kähler toric manifold* a connected Kähler manifold N of complex dimension n equipped with an effective isometric and holomorphic action $\Phi : \mathbb{T}^n \times N \to N$ of the n-dimension real torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ that can be globally extended to a holomorphic action $\Phi^{\mathbb{C}} : (\mathbb{C}^*)^n \times N \to N$ of the algebraic torus $(\mathbb{C}^*)^n$. We say that a Kähler manifold is *regular* if it is complete, connected, simply connected and if the Kähler metric is real analytic.
- A *dually flat manifold* is a triple (M, h, ∇), where (M, h) is a Riemannian manifold and ∇ is a flat linear connection whose dual ∇^* with respect to h is also flat. (The dual connection ∇^* is the unique linear connection on M satisfying $Xh(Y,Z) = h(\nabla_X Y, Z) + h(Y, \nabla^*_X Z)$ for all vector fields X, Y, Z on M).

Two Kähler toric manifolds N_1 and N_2 with torus actions Φ_1 and Φ_2, respectively, are *equivalent* if there is a Kähler isomorphism $G : N_1 \to N_2$ and a Lie group isomorphism $\rho : \mathbb{T}^n \to \mathbb{T}^n$ such that

$$G \circ (\Phi_1)_a = (\Phi_2)_{\rho(a)} \circ G \tag{1.2}$$

for all $a \in \mathbb{T}^n$, where $(\Phi_i)_x : N_i \to N_i$, $y \mapsto \Phi_i(x, y)$. Two dually flat manifolds (M_i, h_i, ∇_i), $i = 1, 2$, are *equivalent* if there exists an afine isometric diffeomorphism between them.

With this notation, we show that there is a one-to-one correspondence, which we will regard as a duality, of the form (see Theorem 12.2) :

$$\left\{ \begin{array}{l} \text{Equivalence classes of} \\ \text{regular Kähler toric manifolds} \end{array} \right\} \xrightarrow{\text{1-to-1}} \left\{ \begin{array}{l} \text{Equivalence classes of} \\ \text{dually flat manifolds} \\ \text{(+ conditions)} \end{array} \right\} \tag{1.3}$$

The conditions that appear on the right hand side of the duality are discussed below, after we have defined the concept of torification.

There are naturally two equivalent ways to define the duality map (1.3), that we now describe.

[1] We learned after publication that many results in [Mol15] were already obtained independently by John David Lafferty in [Laf88] and Max von Renesse in [vR12].

The symplectic point of view. Since a regular toric Kähler manifold N is simply connected (by definition), its torus action is Hamiltonian and hence there is a momentum map $\mathbf{J} : N \to \mathbb{R}^n$. Let N° be the set of points $p \in N$ where the torus action is free. Since $\mathbf{J} : N^\circ \to \mathbf{J}(N^\circ)$ is a principal $\mathbb{T}^n$-bundle, the Kähler metric on N° descends to a Riemannian metric, say k, on $\mathbf{J}(N^\circ)$. Let ∇^k be the dual connection of the flat connection on $\mathbf{J}(N^\circ) \subset \mathbb{R}^n$. Then $[\mathbf{J}(N^\circ), k, \nabla^k]$ is the image under the duality map (1.3) of $[N]$ (here $[\,.\,]$ denotes the equivalence class).
The complex point of view. Let $\Phi^{\mathbb{C}} : (\mathbb{C}^*)^n \times N \to N$ be the holomorphic extension of the torus action. Choose any point $p \in N^\circ$ and consider the map $\Phi^{\mathbb{C}}_p : (\mathbb{C}^*)^n \to N^\circ$, $z \mapsto \Phi^{\mathbb{C}}(z, p)$. Then $\Phi^{\mathbb{C}}_p$ is a holomorphic diffeomorphism. Let $\sigma : (\mathbb{C}^*)^n \to \mathbb{R}^n$, $(z_1, ..., z_n) \mapsto (\frac{\ln(|z_1|)}{2\pi}, ..., \frac{\ln(|z_n|)}{2\pi})$. The map $\sigma \circ (\Phi^{\mathbb{C}}_p)^{-1} : N^\circ \to \mathbb{R}^n$ is a principal $\mathbb{T}^n$-bundle. Therefore the Kähler metric on N° descends to a Riemannian metric h on $\mathbb{R}^n$. Then $[\mathbb{R}^n, h, \nabla^{\text{flat}}]$ is the image under the duality map (1.3) of $[N]$, where ∇^{flat} is the flat connection on $\mathbb{R}^n$.

The equivalence between the two points of view was essentially proven by Guillemin [Gui94b], who noted that the metrics h on $\mathbb{R}^n$ and k on $\mathbf{J}(N^\circ)$ are related by a Legendre transform, as follows (see Appendix 12). Both metrics are Hessian, that is, there are smooth functions $\phi : \mathbb{R}^n \to \mathbb{R}$ and $\phi^* : \mathbf{J}(N^\circ) \to \mathbb{R}$ such that $h = \text{Hess}(\phi)$ and $k = \text{Hess}(\phi^*)$. With the right convention, ϕ^* is the Legendre transform of ϕ. Moreover, (minus) the gradient map $-\nabla\phi : \mathbb{R}^n \to \mathbf{J}(N^\circ)$ is an isometric diffeomorphism with inverse $-\nabla\phi^*$:

$$\begin{array}{ccc} & N^\circ & \\ {}^{\sigma \circ (\Phi^{\mathbb{C}}_p)^{-1}} \swarrow & & \searrow {}^{\mathbf{J}} \\ \mathbb{R}^n & \xrightarrow[-\nabla\phi = (-\nabla\phi^*)^{-1}]{} & \mathbf{J}(N^\circ) \end{array}$$

It is not difficult to see that the map $-\nabla\phi : \mathbb{R}^n \to \mathbf{J}(N^\circ)$ preserves the affine structures of $\mathbb{R}^n$ and $\mathbf{J}(N^\circ)$ described above and hence the dually flat spaces $(\mathbb{R}^n, h, \nabla^{\text{flat}})$ and $(\mathbf{J}(N^\circ), k, \nabla^k)$ are equivalent (see Proposition 7.12). This shows that both points of view can indeed be used to define the duality map (1.3), and illustrates the flexibility of using dually flat spaces.

The symplectic and complex points of view are well-known in the literature [Abr03, Gui94b] and they allow to define the duality map (1.3) easily. However, in this paper we focus on the inverse problem: given a dually flat manifold (M, h, ∇), how can we associate a Kähler toric manifold? Delzant used symplectic reduction techniques to construct a symplectic toric manifold from a Delzant polytope. Our approach is different (non constructive) and is based on what we call "torification" of a dually flat space.

1.2 Torification. Let (M, h, ∇) be a connected dually flat space. Suppose that there is a global frame $(X_1, ..., X_n)$ of vector fields on M. Then there is a natural action of $\mathbb{R}^n$ on the tangent bundle TM, defined by $x \cdot u := u + x_1X_1 + ... + x_nX_n$, where $x \in \mathbb{R}^n$ and $u \in TM$. It is a general fact that if a commutative group G acts on a set S and if K is any subgroup of G, then G/K acts on S/K. With $G = \mathbb{R}^n$ and $K = \mathbb{Z}^n$, we get an effective action of the real torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ on the quotient space $TM/\mathbb{Z}^n$:

$$\mathbb{T}^n \times TM/\mathbb{Z}^n \to TM/\mathbb{Z}^n. \tag{1.4}$$

From an analytical point of view, it follows from Dombrowski's construction that the tangent bundle TM is naturally a Kähler manifold (see Proposition 2.16). Moreover, if the vector fields X_j's are parallel with respect to ∇, then the $\mathbb{R}^n$-action on TM is holomorphic and isometric (Lemma 4.1). Thus the quotient $TM/\mathbb{Z}^n$ is naturally a Kähler manifold and the torus action (1.4) is isometric and holomorphic (Lemma 4.3).

Let N be a connected Kähler manifold equipped with an effective holomorphic and isometric torus action $\mathbb{T}^n \times N \to N$. We say that N is a *torification* of the dually flat space (M, h, ∇) if there is a global frame $(X_1, ..., X_n)$ of ∇-parallel vector fields on M and an equivariant Kähler isomorphism $F : TM/\mathbb{Z}^n \to N^\circ$ (see Definition 6.3).

Comments are in order. To say that N is a torification of M simply means that N° can be parametrized by points in $TM/\mathbb{Z}^n$ in a consistant way with the geometric structures involved, but the definition does not say anything about the set $N \backslash N^\circ$ of points where the torus action is not free. As a result, if a torification exists, then it is not unique in general (if N is a torification, then so does N°). In that sense, torification is a weak concept. However, the situation changes if the following conditions are imposed on the Kähler manifold N: completeness, simply connectedness and real analyticity of the Kähler metric. When these conditions are satisfied, we say that N is *regular*. If M has a regular torification, we say that it is *toric*.

One of the main results of this paper is the following: all regular torifications of a given dually flat manifold are equivalent (Theorem 9.11). Therefore one can unambiguously associate a Kähler manifold (equipped with a torus action) to a toric dually flat manifold, up to an equivariant Kähler isomorphism. This yields a map:

$$\begin{array}{ccc} \left\{ \text{ Toric dually flat manifolds } \right\} & \overset{\text{torification}}{\longrightarrow} & \left\{ \begin{array}{l} \text{Equivalence classes of} \\ \text{regular torifications} \end{array} \right\} \\ (M, h, \nabla) & \longmapsto & \big[\mathbb{T}^n \times N \to N\big]. \end{array} \tag{1.5}$$

The torification map (1.5) is analogous to the map $\Delta \mapsto [M_\Delta]$ given by Delzant correspondence, but there are two notable differences: N is not necessarily compact (see examples below) and the equivalence relation between torifications is weaker than the equivalence relation used in Delzant correspondence (the torus action can be reparametrized by a Lie group isomorphism $\rho : \mathbb{T}^n \to \mathbb{T}^n$, see (1.2)).

Regarding the duality (1.3), we show that the torification map (1.5) depends only on the equivalence class of (M, h, ∇) and hence it descends to a quotient map between quotient spaces. When certain conditions are imposed on (M, h, ∇), this quotient map becomes the inverse of the duality map (1.3). These conditions are (see Theorem 12.2): (1) (M, h, ∇) is toric, (2) the torus action associated to the regular torification of (M, h, ∇) is holomorphically extendable, and (3) (M, h, ∇) admits a global pair of dual coordinate systems (see Definition 2.11). These three conditions are the conditions that appear on the right hand side of the duality (1.3).

1.3 Lifting property. One of the main results of this paper is that the torification map (1.5) gives rise to a lifting procedure. Suppose N and N' are regular torifications of the dually flat manifolds (M, h, ∇) and (M', h', ∇'), respectively. By definition of a torification, there are equivariant Kähler isomorphisms $N^\circ \cong TM/\mathbb{Z}^n$ and $(N')^\circ \cong TM'/\mathbb{Z}^d$. In this situation, we show that if $f : M \to M'$ is an affine isometric map, then its derivative $f_* : TM \to TM'$ is automatically equivariant with respect to the actions of $\mathbb{Z}^n$ and $\mathbb{Z}^d$ and hence it descends to a map $m : N^\circ \to (N')^\circ$ (Proposition 9.17). This map extends uniquely to an equivariant Kähler immersion $m : N \to N'$. We call it a *lift* of f. Lifts are not unique (they depend on the choice of the parametrizations $N^\circ \cong TM/\mathbb{Z}^n$), but they are conjugate (Proposition 9.16). Under suitable parametrizations, compositions of lifts are lifts (Proposition 9.9).

1.4 Fundamental lattice. If N is a torification of (M, h, ∇), then there is a frame $X = (X_1, ..., X_n)$ of ∇-parallel vector fields X_j's on M that induces a $\mathbb{Z}^n$-action on TM and there is an equivariant Kähler isomorphism $TM/\mathbb{Z}^n \to N^\circ$. The pair (X, F) is not unique in general. However, we show that if N is regular, then the set

$$\mathscr{L} = \{k_1 X_1(p) + ... + k_n X_n(p) \mid p \in M,\ k_1, ..., k_n \in \mathbb{Z}\} \subset TM$$

is independent of the the frame $(X_1, ..., X_n)$. We call $\mathscr{L} \subset TM$ the *fundamental lattice* of (M, h, ∇).

The fundamental lattice is closely related to the space of Kähler functions on TM. Recall that a Kähler function on a Kähler manifold is a smooth function whose Hamiltonian vector field is Killing. We show that when the space of Kähler functions separates the points of N, then $\mathscr{L}$ coincides with the set of diffeomorphisms $\varphi : TM \to TM$ satisfying $f \circ \varphi = f$ for all Kähler functions $f : TM \to \mathbb{R}$, where $\mathscr{L}$ is identified with a group of translations (see Proposition 10.4). Using this, we deduce a criteria for a dually flat manifold (M, h, ∇) not to be toric (Corollary 10.5).

1.5 Examples from Information Geometry. Information geometry is the branch of mathematics that studies statistical manifolds, that is, manifolds whose points are probability density functions defined over a fixed measure space [AJLS17, AN00, MR93]. For example, the set of all normal distributions $(\sqrt{2\pi}\sigma)^{-1}\exp(-(x-\mu)^2/(2\sigma^2))$ is a statistical manifold parametrized by the mean $\mu \in \mathbb{R}$ and $\sigma > 0$. Another important example is the set of all probabilities $p : \Omega \to \mathbb{R}$ $p > 0$, $\sum_{k=1}^{n} p(x_k) = 1$, defined over a finite set $\Omega = \{x_1, ..., x_n\}$. These probabilities are called *categorical distributions*. Together they form an $(n-1)$-dimensional statistical manifold.

In general, a statistical manifold M is equipped with a Riemannian metric h_F, namely the *Fisher metric*, and a linear connection $\nabla^{(e)}$, called *exponential connection*. In many cases, $(M, h_F, \nabla^{(e)})$ is a dually flat manifold. For example, this is true for the most common families of statistical distributions: Binomial, Poisson, Normal, Categorical, just to name a few. Throughout the rest of the introduction, we will use these names to denote the corresponding families of probability distributions. Some of these families are characterized by some parameters, like the number of trials for Binomial. For simplicity, we will omit these parameters from our notation. Thus, for instance, we will denote by $\mathsf{Binomial}$ the set of all Binomial distributions $p(k) = \binom{n}{k} q^k (1-q)^{n-k}$, $q \in (0,1)$, defined over the set $\{0, 1, ..., n\}$, instead of using an explicit notation like $\mathsf{Binomial}(n)$.

Let $\mathbb{P}_n(c)$ denote the complex projective space of complex dimension n and holomorphic sectional curvature $c > 0$. Let $\mathbb{D}(c)$ denote the unit disk in $\mathbb{C}$ endowed with the Hyperbolic metric of constant holomorphic sectional curvature $c < 0$. Below we give a list of statistical manifolds that are toric together with their regular torifications (see Proposition 8.11).

Kähler toric manifold	Dually flat space
$S^2 = \mathbb{P}_1(\frac{1}{n})$	Binomial
$\mathbb{C}$	Poisson
$\mathbb{P}_n(1)$	Categorical
$\mathbb{P}_n(\frac{1}{m})$	Multinomial
$\mathbb{D}(-\frac{1}{r})$	Negative Binomial

The numbers n, m, r are all positive integers. For the corresponding torus actions, see Proposition 8.11. A simple counterexample is the set of normal distributions with known variance $\sigma = 1$, which is not toric (see Example 10.6). Note that $\mathbb{C}$ and $\mathbb{D}$ are non-compact.

Given two dually flat manifolds M_1 and M_2 with torifications N_1 and N_2, respectively, we show that $N_1 \times N_2$ is a torification of $M_1 \times M_2$ (Proposition 11.5). Therefore one can construct new torifications from old ones. For example, $\mathbb{C}^n$ is the torification of a finite product of Poisson families.

An interesting connection with algebraic geometry and projective varieties arises when the measure space Ω of a dually flat statistical manifold M is finite. In this case, we prove that the natural inclusion $M \subset$ Categorical is always affine and isometric (Lemma 11.1). Thus if M has a regular torification N, then there is a lift $f : N \to \mathbb{P}_r(1)$ which is an equivariant Kähler immersion. For example (see Propositions 11.3 and 11.8),

- the Veronese embedding $\mathbb{P}_1(\frac{1}{n}) \to \mathbb{P}_n(1)$, $[z_1, z_2] \mapsto \left[z_1^n, ..., \binom{n}{k}^{1/2} z_1^{n-k} z_2^k, ..., z_2^n\right]$ (homogeneous coordinates) and
- the Segre embedding $\mathbb{P}_n(1) \times \mathbb{P}_m(1) \to \mathbb{P}_{(n+1)(m+1)-1}(1)$, $([z_i], [w_j]) \mapsto [z_i w_j]$, $i = 1, ..., n$, $j = 1, ..., m$ (lexicographic ordering),

are the lifts of the inclusion maps Binomial $\subset$ Categorical and Categorical$\times$Categorical $\subset$ Categorical, respectively. Both maps are well-known in algebraic geometry (see, e.g., [Har95]).

1.6 Applications to Geometric Quantum Mechanics. Geometric Quantum Mechanics (GQM) is a geometric reformulation of Quantum Mechanics (QM) based on the Kähler structure of the complex projective space $\mathbb{P}_n$ [AS99]. In GQM, the Schrödinger evolution is replaced by the Hamiltonian flows of Kähler functions $f : \mathbb{P}_n \to \mathbb{R}$, and the probabilistic features (quantum uncertainties and state vector reduction in a measurement process) are described by the Riemannian metric on $\mathbb{P}_n$, namely the Fubini-Study metric. This is an elegant formalism that enlightens many aspects of QM, such as Berry's phase, entanglement and the measurement problem [Spe12].

GQM naturally leads to Kähler manifolds other than just the complex projective space $\mathbb{P}_n$. Let us give two examples.

(1) The *Segre variety* characterizes the set of completely disentangled states (see [BS06]).

(2) The *Veronese variety* characterizes the so-called *spin coherent states* of $\mathrm{SU}(n)$ (see [BG10, Mol13]).

We stress the fact that these examples, and their connections with QM, are not the result of a quantization scheme (like geometric quantization [Bla07, Nai16, Woo92]); they arise in a purely quantum context, where no classical counterpart is required[2].

It is natural to ask whether or not GQM could be extended to a "natural" class of Kähler manifolds that would include the examples above (projective space, Segre and Veronese varieties), and if a simple mechanism could explain their connections with QM. It is our opinion that the torification approach provides an appropriate mathematical framework to tackle these questions. Let us elaborate this further by examining the examples above in more details.

The projective space $\mathbb{P}_n$, the Veronese and Segre varieties are all torifications of statistical manifolds (see examples above). Because of this, each one of them is canonically associated to a probability space, and thus a probabilistic interpretation is available. This is important because QM is probabilistic in nature. More concretely, if N is the torification of a dually flat manifold M, then there is a canonical projection $\kappa : N^\circ \to M$ (it is unique up to an equivariant Kähler isomorphism of N, see Lemma 9.15). If M is a statistical manifold, then $\kappa(p)$ is a probability for every $p \in N^\circ$. For example, let $M = \mathsf{Categorical}$ be the set of positive probability density functions defined on a finite set $\Omega = \{x_0, ..., x_n\}$. Then $N = \mathbb{P}_n$ is the regular torification of M and the corresponding projection $\kappa : \mathbb{P}_n^\circ \to M$ is defined by $\kappa([z])(x_k) = \frac{|z_k|^2}{|z_0|^2+...+|z_n|^2}$, where $[z] = [z_0, ..., z_n] \in \mathbb{P}_n^\circ$ (homogeneous coordinates) and $x_k \in \Omega$. (For more examples, see Proposition 8.12.) In [Mol13], we explain how to recover the probabilistic interpretation of GQM from the projection κ instead of the geodesic distance on $\mathbb{P}_n$. We then applied this approach to the torification of $\mathsf{Binomial}$, which is the sphere S^2. The result is essentially the mathematical description of the spin of a non-relativistic particle. This example shows that GQM can be generalized by considering torifications of appropriate statistical manifolds.

Regarding the Veronese variety, there is a simple way to understand its connection to QM, that we now describe. By definition, the Veronese variety is the image of the sphere S^2 under the Veronese embedding $S^2 \to \mathbb{P}_n$ and hence it is a complex submanifold of $\mathbb{P}_n$. Thus, one may consider the following problem: given a Kähler function f on S^2, is there a Kähler function $\hat{f}$ on $\mathbb{P}_n$ that extends f? In [Mol13], we proved that every Kähler function f on S^2 extends uniquely to a Kähler function $\hat{f}$ on $\mathbb{P}_n$ and that the map $\mathscr{K}(S^2) \to \mathscr{K}(\mathbb{P}_n)$, $f \mapsto \hat{f}$ is a Lie algebra homomorphism (here $\mathscr{K}(N)$ denotes the space of Kähler functions on N). Since $\mathscr{K}(S^2) \cong \mathfrak{u}(2)$ and $\mathscr{K}(\mathbb{P}_n) \cong \mathfrak{u}(n+1)$, one obtains a unitary representation of $\mathfrak{u}(2)$. From this one can extract all irreducible unitary representations of $\mathfrak{su}(2)$, which is what physicists use to describe the spin of a particle. The Veronese embedding itselft is interesting from the physical point of view, because it characterizes the so-called *spin coherent states* [BH01, Mol13]. In summary, the spin representations and spin coherent states are manifestations of the exterior geometry of S^2, that is, the way S^2 is embedded into $\mathbb{P}_n$. From a torification point of view, this embedding is just the lift of the inclusion map $\mathsf{Binomial} \subset \mathsf{Categorical}$.

Now we turn our attention to the Segre embedding. In QM, the Hilbert space for a combined system of two particles is the tensor product $H_1 \otimes H_2$, where each Hilbert space H_i is the state space of a single particle. If the state $\Psi \in H_1 \otimes H_2$ of the particles can be written as a single term $u \otimes v$ for some $u \in H_1$ and $v \in H_2$, then we say that the particles are *disentangled*. For simplicity, suppose that $H_1 = H_2 = \mathbb{C}^2$. In this case, it can be proven that the two particles ("qubits")

[2]GQM is not concerned with quantization.

are disentangled if and only if their state vector $\Psi \in \mathbb{P}_3 = \mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ is a point in the Segre variety $X \subset \mathbb{P}_3$. More generally, the generalized Segre Variety $X \subset \mathbb{P}_{2^n-1}$ coincides with the set of disentangled states of the n-qubit space [BS06]. Mathematically, the Segre variety $X \subset \mathbb{P}_3$ is the image of $\mathbb{P}_1 \times \mathbb{P}_1$ under the Segre embedding $f([z_1, z_2], [w_1, w_2]) = [z_1 w_1, z_1 w_2, z_2 w_1, z_2 w_2]$. From a torification point of view, the Segre embedding is just the lift of the inclusion map $\mathsf{Categorical} \times \mathsf{Categorical} \subset \mathsf{Categorical}$.

It is interesting to note that a multiparticle formalism is readily implemented in the torification approach. Suppose M_1 and M_2 are dually flat statistical manifolds defined over the finite sets $\Omega_1 = \{x_0, ..., x_r\}$ and $\Omega_2 = \{y_0, ..., y_s\}$, respectively. Then the product $M_1 \times M_2$ is naturally a dually flat statistical manifold defined over $\Omega_1 \times \Omega_2$ (see Definition 11.6). If N_1 and N_2 are regular torifications of M_1 and M_2, respectively, then the product $N_1 \times N_2$ is the regular torification of $M_1 \times M_2$ (Proposition 11.5). Moreover, since the natural inclusion $M_1 \times M_2 \subset \mathsf{Categorical}$ is affine and isometric, there is a lift $N_1 \times N_2 \to \mathbb{P}_{(r+1)(s+1)-1} = \mathbb{P}(\mathbb{C}^{r+1} \otimes \mathbb{C}^{s+1})$ which generalizes the Segre embedding.

The discussion above highlights some of the advantages of the torification approach, like its simplicity, minimality and naturalness, and paves the way for further developments of GQM. Perhaps more importantly, the torification approach places information geometry at the very heart of QM in finite dimension. This naturally leads to the idea that QM might be derived from information-theoretical principles, like, for instance, the constancy of speed of light in special relativity. Many authors already reached similar conclusions using axiomatic approaches [DB11, CBH03, Gri04, CDP11, Goy08, Goy10a, Goy10b, MM11, Rov96]. These approaches have their own merits and respective successes, but to our knowledge, no consensus has emerged yet. The torification approach, in this regard, might offer some guidance or insight.

1.7 Organization of the paper. For the convenience of the reader, the paper starts with a discussion on the relation between Kähler geometry and statistics (see Section 2). The material presented is mostly taken from [Mol14], except for Proposition 2.19 which seems to be new. We shall present the subject in a uniform way by using the concept of dually flat structure. In Section 3, we introduce the concept of *parallel lattice* on a dually flat manifold and show how to construct a torus action from it. The analytical and geometrical properties of this torus action are discussed in Sections 4 and 5. The concept of torification is defined in Section 6. In Section 7, we re-examine carefully some already known results in toric geometry, adopting systematically the torification point of view. In particular we show that every Kähler toric manifold is a torification of an appropriate dualistic structure on $\mathbb{R}^n$ (complex point of view, Theorem 7.1) but also the torification of its momentum polytope (symplectic point of view, Corollary 7.8). Examples from information geometry are presented in Section 8. The lifting property that comes with the concept of torification is discussed in Section 9. The fundamental lattice of a toric dually flat manifold is defined in Section 10. In Section 11, we describe a close connection between algebraic geometry and the torification of statistical manifolds. Finally, in Section 12 we discuss the duality between dually flat spaces and Kähler toric manifolds mentioned at the beginning of this paper.

Notation. Let M be a manifold. The map $\pi : TM \to M$ denotes the canonical projection. The set Diff(M) is the group of diffeomorphisms of M. The space of vector fields on M is denoted by $\mathfrak{X}(M)$. Let $\Phi : G\times M \to M$ be a Lie group action of a Lie group G on M. Given $g \in G$ and $p \in M$, we will denote by $\Phi_p : G \to M$ and $\Phi_g : M \to M$ the maps defined by $\Phi_p(g) = \Phi_g(p) = \Phi(g, p)$. If N is a Kähler manifold, we usually denote its metric by g, its complex structure by J and the corresponding symplectic form by ω. The set $\mathrm{GL}(n, \mathbb{K})$ is the group of $n \times n$ invertible matrices with entries in $\mathbb{K} \in \{\mathbb{R}, \mathbb{Z}\}$.

2 Preliminaries: dually flat spaces and Kähler geometry

2.1 Connections and connectors. This section follows closely [Dom62]. Let M be a manifold.

Definition 2.1. A *linear connection* ∇ on M is a map $\mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, $(X, Y) \mapsto \nabla_X Y$, satisfying the following properties:

(1) $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$,

(2) $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$,

(3) $\nabla_X(fY) = X(f)Y + f\nabla_X Y$,

for all vector fields $X, Y, Z \in \mathfrak{X}(M)$ and for all functions $f, g \in C^\infty(M)$.

In local coordinates $(x_1, ..., x_n)$ on $U \subseteq M$, if $X = \sum_{k=1}^n X^i \frac{\partial}{\partial x_k}$ and $Y = \sum_{k=1}^n Y^k \frac{\partial}{\partial x_k}$, then, by standard computations,

$$\nabla_X Y = \sum_{k=1}^n \left(X(Y^k) + \sum_{i,j=1}^n X^i Y^j \Gamma_{ij}^k \right) \frac{\partial}{\partial x_k}, \tag{2.1}$$

where $\Gamma_{ij}^k : U \to \mathbb{R}$ are the Christoffel symbols, defined by the formula

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{k=1}^n \Gamma_{ij}^k \frac{\partial}{\partial x_k} \qquad \text{for } i, j = 1, ..., n.$$

Let $\pi : TM \to M$ be the canonical projection and let (U, φ) be a chart for M with local coordinates $(x_1, ..., x_n)$. Define $\widetilde{\varphi} : \pi^{-1}(U) \to \mathbb{R}^{2n}$ by

$$\widetilde{\varphi}\left(\sum_{i=1}^n u_i \frac{\partial}{\partial x_i}\bigg|_p \right) = (x_1(p), ..., x_n(p), u_1, ..., u_n).$$

Then $(\pi^{-1}(U), \widetilde{\varphi})$ is a chart for TM. We will usually denote the corresponding local coordinates on $\pi^{-1}(U) \subseteq TM$ by $(q, r) = (q_1, ..., q_n, r_1, ..., r_n)$ (in particular, $q_i = x_i \circ \pi$ for every $i = 1, ..., n$). Informally, we sometimes write $(x, \dot{x}) = (x_1, ..., x_n, \dot{x}_1, ..., \dot{x}_n)$ instead of $(q, r) = (q_1, ..., q_n, r_1, ..., r_n)$.

Let $u = \sum_{k=1}^n u_k \frac{\partial}{\partial x_k}\big|_p \in \pi^{-1}(U)$ be arbitrary. Define a linear map $K_u : T_u(TM) \to T_pM$ by

$$K_u\left(\frac{\partial}{\partial q_a}\bigg|_u\right) := \sum_{k,j=1}^n \Gamma^k_{aj}(p) u_j \frac{\partial}{\partial x_k}\bigg|_p \qquad \text{for } a = 1, ..., n,$$

$$K_u\left(\frac{\partial}{\partial r_a}\bigg|_u\right) := \frac{\partial}{\partial x_a}\bigg|_p \qquad \text{for } a = 1, ..., n.$$

Lemma 2.2. Let X and Y be vector fields on M. Suppose $Y(p) = u$. Then $K_u(Y_{*_p}X_p) = (\nabla_X Y)(p)$.

Proof. By standard computations,

$$Y_{*_p}X_p = \sum_{a=1}^n \left(X^a(p)\frac{\partial}{\partial q_a}\bigg|_u + X_p(Y^a)\frac{\partial}{\partial r_a}\bigg|_u\right)$$

so

$$K_u(Y_{*_p}X_p) = \sum_{a=1}^n \left(X^a(p)\sum_{k,j=1}^n \Gamma^k_{aj}(p)Y^j(p)\frac{\partial}{\partial x_k}\bigg|_p + X_p(Y^a)\frac{\partial}{\partial x_a}\bigg|_p\right),$$

where we have used $u_j = Y^j(p)$. Comparing the above formula with the local expression for $\nabla_X Y$ in coordinates (see (2.1)), one obtains the desired formula. □

Clearly, vectors of the form $Y_{*_p}X_p$, with $Y_p = u$, generate $T_u(TM)$, and so the above lemma implies that the definition of K_u is independant of the choice of the chart (U, φ).

The map

$$K : TTM \to TM,$$

defined for $A \in T_u(TM)$ by $K(A) := K_u(A)$, is called *connector*, or *connection map*, associated to ∇.

The following result is an immediate consequence of the definition of K.

Proposition 2.3. Let K be the connector associated to a connection ∇ on M. The following holds.

(1) For every pair X, Y of vector fields on M, $\nabla_X Y = KY_*X$, where Y_*X denotes the derivative of Y in the direction of X.

(2) For every $u \in T_pM$, the restriction of K to $T_u(TM)$ is a linear map $T_u(TM) \to T_pM$.

If $A \in T_u(TM)$ is such that $\pi_{*_u}A = 0$ and $K(A) = 0$, then a simple calculation using local coordinates shows that $A = 0$. Therefore,

Proposition 2.4. Let K be the connector associated to a connection ∇ on M. Given $u \in T_pM$, the map $T_u(TM) \to T_pM \oplus T_pM$, defined by

$$A \mapsto (\pi_{*_u}A, KA), \tag{2.2}$$

is a linear bijection.

Thus, given a linear connection ∇, we can identify at any point $u \in T_pM$ the vector spaces $T_u(TM)$ and $T_pM \oplus T_pM$ via the map (2.2).

2.2 Dually flat spaces.

Definition 2.5. Let (M, h) be a Riemannian manifold endowed with a linear connection ∇. The *dual connection* of ∇ with respect to h is the unique linear connection, denoted by ∇^*, satisfying

$$Xh(Y, Z) = h(\nabla_X Y, Z) + h(Y, \nabla^*_X Z)$$

for all vector fields X, Y, Z in M. The triple (h, ∇, ∇^*) is called a *dualistic structure* on M.

Example 2.6. If ∇ is the Levi-Civita connection of h, then $\nabla^* = \nabla$.

As the literature is not uniform, let us agree that the torsion T and the curvature tensor R of a connection ∇ are defined as

$$T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y],$$
$$R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

where X, Y, Z are vector fields on M. By definition, a linear connection is *flat* if the torsion and curvature tensor are identically zero on M. A manifold endowed with a flat linear connection is called an *affine manifold*.

Definition 2.7. A dualistic structure (h, ∇, ∇^*) is *dually flat* if both ∇ and ∇^* are flat. A manifold endowed with a dually flat structure is called a *dually flat manifold*, or *dually flat space*.

Since the dual connection ∇^* is completely determined by h and ∇, we will often regard a dually flat manifold as a triple (M, h, ∇).

Proposition 2.8. Let (h, ∇, ∇^*) be a dualistic structure on a manfold M. Let R and R^* denote the curvature tensores of ∇ and ∇^*, respectively. Then

$$h(R(X, Y)Z, W) = -h(R^*(X, Y)W, Z)$$

for all vector fields X, Y, Z, W on M. In particular, $R \equiv 0$ if and only if $R^* \equiv 0$.

Proof. See [AN00], Theorem 3.3. □

Therefore an affine manifold (M, ∇) endowed with a Riemannian metric h is a dually flat manifold if and only if the torsion of the dual connection ∇^* is identically zero.

Let ∇^{flat} denote the canonical flat connection on $\mathbb{R}^n$ (or any open subset of it). Thus, if $x = (x_1, ..., x_n)$ are standard coordinates on $\mathbb{R}^n$, then

$$(\nabla^{\text{flat}})_X Y = \sum_{i=1}^{n} X(Y^i) \frac{\partial}{\partial x_i}, \tag{2.3}$$

where X, Y are vector fields on $\mathbb{R}^n$, $Y = \sum_{i=1}^n Y^i \frac{\partial}{\partial x_i}$.

The next result describes a simple yet important class of dually flat manifolds.

Lemma 2.9. Let $\psi : U \to \mathbb{R}$ be a smooth function defined on an open set $U \subseteq \mathbb{R}^n$. Suppose that the Hessian matrix of ψ, denoted by $\mathrm{Hess}(\psi)$, is positive definite at each point of U (thus $\mathrm{Hess}(\psi)$ can be regarded as a Riemannian metric on $\mathbb{R}^n$). Then $(U, \mathrm{Hess}(\psi), \nabla^{\mathrm{flat}})$ is a dually flat manifold.

Proof. Let ∇ be the dual connection of ∇^{flat} with respect to $h = \mathrm{Hess}(\psi)$. In view of Proposition 2.8, it suffices to show that the torsion T of ∇ is identically zero. Let $(x_1, ..., x_n)$ denote the usual coordinates on $\mathbb{R}^n$. Given $i, j, k = 1, ..., n$, we compute:

$$\begin{aligned} h\big(T(\tfrac{\partial}{\partial x_i}, \tfrac{\partial}{\partial x_j}), \tfrac{\partial}{\partial x_k}\big) &= h\Big(\nabla_{\frac{\partial}{\partial x_i}} \tfrac{\partial}{\partial x_j} - \nabla_{\frac{\partial}{\partial x_j}} \tfrac{\partial}{\partial x_i}, \tfrac{\partial}{\partial x_k}\Big) \\ &= \tfrac{\partial}{\partial x_i} h(\tfrac{\partial}{\partial x_j}, \tfrac{\partial}{\partial x_k}) - h\Big(\tfrac{\partial}{\partial x_j}, \nabla^{\mathrm{flat}}_{\frac{\partial}{\partial x_i}} \tfrac{\partial}{\partial x_k}\Big) - \tfrac{\partial}{\partial x_j} h(\tfrac{\partial}{\partial x_i}, \tfrac{\partial}{\partial x_k}) + h\Big(\tfrac{\partial}{\partial x_i}, \nabla^{\mathrm{flat}}_{\frac{\partial}{\partial x_j}} \tfrac{\partial}{\partial x_k}\Big) \\ &= \tfrac{\partial}{\partial x_i} \tfrac{\partial^2 \psi}{\partial x_j \partial x_k} - \tfrac{\partial}{\partial x_j} \tfrac{\partial^2 \psi}{\partial x_i \partial x_k} = 0. \end{aligned}$$

The result follows. □

Definition 2.10. Suppose (M, ∇) is an affine manifold. An *affine coordinate system* is a coordinate system $(x_1, ..., x_n)$ defined on some open set $U \subseteq M$ such that $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = 0$ for all $i, j = 1, ..., n$.

It can be shown that for every point p in an affine manifold M, there is an affine coordinate system $(x_1, ..., x_n)$ defined on some neighborhood $U \subseteq M$ of p (see [Shi07], Proposition 1.1).

Definition 2.11. Let (M, h, ∇) be a dually flat manifold, with dual connection ∇^*, and let $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ be coordinate systems on the same open set $U \subseteq M$. The pair (x, y) is said to be a *pair of dual coordinate systems* if the following conditions are satisfied.

(1) x is affine with respect to ∇.

(2) y is affine with respect to ∇^*.

(3) $h\left(\dfrac{\partial}{\partial x_i}, \dfrac{\partial}{\partial y_j}\right) = \delta_{ij}$ (= Kronecker delta) for all $i, j = 1, ..., n$.

When $U = M$ in the above definition, the pair (x, y) is said to be a *global pair of dual coordinate systems*. It can be shown that there exists a pair of dual coordinate systems around each point $p \in M$ (see [AN00], Section 3.3).

Proposition 2.12. Let (M, h, ∇) be a dually flat manifold and let $(x, y) = \big((x_1, ..., x_n), (y_1, ..., y_n)\big)$ be a pair of dual coordinate systems on $U \subset M$. Given $i, j = 1, ..., n$, let $h_{ij} : U \to \mathbb{R}$ and $h^{ij} : U \to \mathbb{R}$ be defined by $h_{ij}(x) = h_x(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ and $h^{ij}(x) = h_x(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j})$.

(1) For every $x \in U$, the matrices $[h_{ij}(x)]$ and $[h^{ij}(x)]$ are inverses of each other.

(2) $\dfrac{\partial x_i}{\partial y_j} = h^{ij}$ and $\dfrac{\partial y_i}{\partial x_j} = h_{ij}$.

(3) $\dfrac{\partial}{\partial x_i} = \displaystyle\sum_{j=1}^{n} h_{ij} \dfrac{\partial}{\partial y_j}$ and $\dfrac{\partial}{\partial y_i} = \displaystyle\sum_{j=1}^{n} h^{ij} \dfrac{\partial}{\partial x_j}$.

(4) If U is connected and simply connected, then there are smooth functions $\psi, \phi : U \to \mathbb{R}$ such that $\dfrac{\partial \psi}{\partial x_i} = y_i$ and $\dfrac{\partial \phi}{\partial y_i} = x_i$ for all $i = 1, ..., n$.

(5) Suppose U connected and simply connected. Let $\psi, \phi : U \to \mathbb{R}$ be as in (4). Then,

(a) $\dfrac{\partial^2 \psi}{\partial x_i \partial x_j} = h_{ij}$ and $\dfrac{\partial^2 \phi}{\partial y_i \partial y_j} = h^{ij}$ for all $i, j = 1, ..., n$.

(b) $\psi + \phi - \displaystyle\sum_{k=1}^{n} x_k y_k$ is constant on U.

Proof. See [AN00], Section 3.3. □

Lemma 2.13. Let (M, h, ∇) be a dually flat manifold and let $x = (x_1, ..., x_n)$ be a ∇-affine coordinate system on $U \subseteq M$ such that $x(U) \subseteq \mathbb{R}^n$ is convex. Suppose there exists a smooth function $\psi : U \to \mathbb{R}$ such that $\frac{\partial^2 \psi}{\partial x_i \partial x_j} = h_{ij} = h(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ for all $i, j = 1, ..., n$ on U. Let $y = (y_1, ..., y_n) : U \to \mathbb{R}^n$ be defined by $y = \text{grad}(\psi) = (\frac{\partial \psi}{\partial x_1} ..., \frac{\partial \psi}{\partial x_n})$. Then (x, y) is a pair of dual coordinate systems on U.

Proof. In order to simplify notation, we identify U and $x(U)$. Thus ψ is a smooth function defined on the convex set $U \subseteq \mathbb{R}^n$, whose Hessian is the matrix representation of h. In particular, ψ is strictly convex. Because $\frac{\partial y_i}{\partial x_j} = \frac{\partial^2 \psi}{\partial x_j \partial x_i} = h_{ji}$, y is a local diffeomorphism and $y(U)$ is an open subset of $\mathbb{R}^n$. To see that y is injective, let x_1, x_2 be a pair of points in U such that $x_1 \neq x_2$. Since ψ is strictly convex, $\psi(x_2) > \psi(x_1) + \langle \text{grad}(\psi)(x_1), x_2 - x_1 \rangle$, where $\langle\, ,\, \rangle$ is the Euclidean pairing on $\mathbb{R}^n$. Reversing the roles of x_1 and x_2, and adding the two inequalities we get

$$\langle y(x_1) - y(x_2), x_1 - x_2 \rangle > 0.$$

This forces $y(x_1) \neq y(x_2)$ and shows that y is injective. It follows that $y : U \to y(U)$ is a diffeomorphism. In particular, $(y_1, ..., y_n)$ is a coordinate system on U. Given $a = 1, ..., n$, we compute:

$$\frac{\partial}{\partial x_a} = \sum_{k=1}^{n} \frac{\partial y_k}{\partial x_a} \frac{\partial}{\partial y_k} = \sum_{k=1}^{n} \frac{\partial}{\partial x_a} \left(\frac{\partial \psi}{\partial x_k} \right) \frac{\partial}{\partial y_k} = \sum_{k=1}^{n} h_{ak} \frac{\partial}{\partial y_k},$$

where we have used $y_a = (\text{grad}(\psi))_a = \frac{\partial \psi}{\partial x_a}$ and the fact that h is the Hessian of ψ. Inverting these equations, we get

$$\frac{\partial}{\partial y_a} = \sum_{k=1}^{n} h^{ak} \frac{\partial}{\partial x_k},$$

where h^{ij} is the (i,j)-entry of the inverse of the matrix $[h_{ij}]$. It follows that

$$h\left(\frac{\partial}{\partial x_a}, \frac{\partial}{\partial y_b}\right) = \sum_{k=1}^{n} h^{bk} h\left(\frac{\partial}{\partial x_a}, \frac{\partial}{\partial x_k}\right) = \sum_{k=1}^{n} h^{bk} h_{ak} = \delta_{ab}.$$

It remains to show that y is affine with respect to the dual connection ∇^* of ∇. Given $i,j,k = 1,...,n$, we have

$$h\left(\nabla^*_{\frac{\partial}{\partial y_i}} \frac{\partial}{\partial y_j}, \frac{\partial}{\partial x_k}\right) = \frac{\partial}{\partial y_i} h\left(\frac{\partial}{\partial y_j}, \frac{\partial}{\partial x_k}\right) - h\left(\frac{\partial}{\partial y_j}, \nabla_{\frac{\partial}{\partial y_i}} \frac{\partial}{\partial x_k}\right) = 0,$$

where we have used the following facts: (1) $h(\frac{\partial}{\partial y_j}, \frac{\partial}{\partial x_k}) = \delta_{jk}$ is constant and (2) $\frac{\partial}{\partial x_k}$ is parallel with respect to ∇ (since $x = (x_1,...,x_n)$ is ∇-affine), which implies $\nabla_{\frac{\partial}{\partial y_i}} \frac{\partial}{\partial x_k} = 0$. Therefore $\nabla^*_{\frac{\partial}{\partial y_i}} \frac{\partial}{\partial y_j} = 0$ for all i,j, which shows that y is ∇^*-affine. □

Let $\varphi : M \to M'$ be a diffeomorphism between manifolds and ∇ a linear connection on M'. We will use the following notation:

- Given a vector field X on M, $\varphi_* X$ is the vector field on M' defined by $(\varphi_* X)(p) = \varphi_{*_{\varphi^{-1}(p)}} X(\varphi^{-1}(p))$.
- $\varphi^*\nabla$ is the unique connection on M satisfying $(\varphi^*\nabla)_X Y = (\varphi^{-1})_*\big(\nabla_{\varphi_* X} \varphi_* Y\big)$ for all vector fields X, Y on M.

Definition 2.14. Let (M, h, ∇) and (M', h', ∇') be dually flat spaces. Let ∇^* and $(\nabla')^*$ denote the dual connections of ∇ and ∇' with respect to h and h', respectively. A diffeomorphism $\varphi : M \to M'$ is said to be an *isomorphism of dually flat spaces* if $\varphi^* h' = h$, $\varphi^*\nabla' = \nabla$ and $\varphi^*(\nabla')^* = \nabla^*$.

In practice, it suffices to prove that $\varphi^* h' = h$ and $\varphi^*\nabla' = \nabla$, as the following lemma shows.

Lemma 2.15. Let (M, h) and (M', h') be Riemannian manifolds and let ∇ and ∇' be affine connections on M and M', respectively. We denote the dual connections by ∇^* and $(\nabla')^*$, respectively. Let $\varphi : M \to M'$ be a diffeomorphism. If $\varphi^* h' = h$ and $\varphi^*\nabla' = \nabla$, then $\varphi^*(\nabla')^* = \nabla^*$.

Proof. Let X, Y, Z be arbitrary vector fields on M. Given $x \in M$, we compute

$$\begin{aligned}
& h_x\big((\varphi^*(\nabla')^*)_X Y, Z\big) = (\varphi^* h')_x\big((\varphi^*(\nabla')^*)_X Y, Z\big) \\
= \;& h'_{\varphi(x)}\big(\varphi_{*_x}(\varphi^*(\nabla')^*)_X Y, \varphi_{*_x} Z\big) = h'_{\varphi(x)}\big((\nabla')^*_{\varphi_* X} \varphi_* Y, \varphi_* Z\big) \\
= \;& (\varphi_* X)\big(h'(\varphi_* Y, \varphi_* Z)\big) - h'_{\varphi(x)}\big(\varphi_* Y, \nabla'_{\varphi_* X} \varphi_* Z\big) \\
= \;& X\big(h'(\varphi_* Y, \varphi_* Z) \circ \varphi\big) - h'_{\varphi(x)}\big(\varphi_* Y, \varphi_{*_x} (\varphi^{-1})_{*_{\varphi(x)}} \nabla'_{\varphi_* X} \varphi_* Z\big) \\
= \;& X\big(h(Y,Z)\big) - h_x\big(Y, (\varphi^*\nabla')_X Z\big) = X\big(h(Y,Z)\big) - h_x\big(Y, \nabla_X Z\big) \\
= \;& h_x\big(\nabla_X Y, Z\big)
\end{aligned}$$

and hence $h_x\big((\varphi^*(\nabla')^*)_X Y, Z\big) = h_x\big(\nabla_X Y, Z\big)$. It follows that $\varphi^*(\nabla')^* = \nabla$. □

2.3 Dombrowski's construction. Let M be a smooth manifold endowed with a connection ∇. We will denote by $\pi : TM \to M$ the canonical projection.

Given $p \in M$ and $u \in T_pM$, the connection ∇ induces an identification of vector spaces $T_u(TM) \cong T_pM \oplus T_pM$ given by $A \in T_u(TM) \mapsto (\pi_{*_u}A, KA) \in T_pM \oplus T_pM$, where π_{*_u} is the derivative of π at u and K is the connector associated to ∇ (see Section 2.1). If there is no danger of confusion, we will therefore regard an element of $T_u(TM)$ as a pair (v, w), where $v, w \in T_pM$.

Let h be a Riemannian metric on M. The pair (h, ∇) determines an almost Hermitian structure on TM via the following formulas:

$$\begin{aligned} g_u\big((v,w),(\overline{v},\overline{w})\big) &:= h_p(v,\overline{v}) + h_p(w,\overline{w}), && \text{(metric)} \\ \omega_u\big((v,w),(\overline{v},\overline{w})\big) &:= h_p(v,\overline{w}) - h_p(w,\overline{v}), && \text{(2-form)} \\ J_u\big((v,w)\big) &:= (-w,v), && \text{(almost complex structure)} \end{aligned}$$

where $u, v, w, \overline{v}, \overline{w} \in T_pM$.

The tensors g, J, ω are smooth (this will follow from their coordinate representation, see Proposition 2.17 below) and clearly, $J^2 = -Id$, $g(Ju, Jv) = g(u, v)$ and $\omega(u, v) = g(Ju, v)$ for all $u, v \in TM$ such that $\pi(u) = \pi(v)$. Thus, (TM, g, J) is an almost Hermitian manifold with fundamental form ω. This is *Dombrowski's construction* [Dom62].

Note that $\pi : (TM, g) \to (M, h)$ is a Riemannian submersion.

We now review the analytical properties of Dombrowski's construction. Recall that an almost Hermitian structure (g, J, ω) is Kähler when the following two analytical conditions are met: (1) J is integrable; (2) $d\omega = 0$.

Proposition 2.16. Let (h, ∇, ∇^*) be a dualistic structure on M and let (g, J, ω) be the almost Hermitian structure on TM associated to (h, ∇) via Dombrowski's construction. The following are equivalent.

(1) (TM, g, J, ω) is a Kähler manifold.

(2) (h, ∇, ∇^*) is dually flat.

Proof. See [Dom62, Mol13]. □

We now direct our attention to the coordinate expressions for g, J and ω.

Proposition 2.17. Let (M, h, ∇) be a dually flat manifold and let (g, J, ω) be the Kähler structure on TM associated to (h, ∇) via Dombrowski's construction. Let $x = (x_1, ..., x_n)$ be an affine coordinate system with respect to ∇ on $U \subseteq M$, and let $(q, r) = (q_1, ..., q_n, r_1, ..., r_n)$ denote the corresponding coordinates on $\pi^{-1}(U)$, as described before Lemma 2.2. Then, in the coordinates (q, r),

$$g = \begin{bmatrix} h_{ij} & 0 \\ 0 & h_{ij} \end{bmatrix}, \quad J = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}, \quad \omega = \begin{bmatrix} 0 & h_{ij} \\ -h_{ij} & 0 \end{bmatrix}, \tag{2.4}$$

where $h_{ij} = h\big(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\big)$, $i, j = 1, ..., n$.

Proof. See [Mol14]. ◻

In what follows, we will identify $T\mathbb{R}^n$ with $\mathbb{C}^n$ via the correspondence $T\mathbb{R}^n \to \mathbb{C}^n$ defined by

$$\sum_{k=1}^{n} b_k \frac{\partial}{\partial x_k}\bigg|_a \mapsto a + ib,$$

where $a, b \in \mathbb{R}^n$, $b = (b_1, ..., b_n)$ and where $(x_1, ..., x_n)$ are standard coordinates on $\mathbb{R}^n$.

An immediate consequence of Proposition 2.17 is the following result.

Corollary 2.18. Let h be a Riemannian metric on $\mathbb{R}^n$ and let ∇^{flat} be the canonical flat connection on $\mathbb{R}^n$. Let (g, J) be the almost Hermitian structure on $T\mathbb{R}^n = \mathbb{C}^n$ associated to $(h, \nabla^{\text{flat}})$ via Dombrowski's construction. Then,

(1) J is the canonical complex structure of $\mathbb{C}^n$ (= multiplication by i).

(2) $g_{x+iy}(a + ib, a' + ib') = h_x(a, a') + h_x(b, b')$ for all $x, y, a, a', b, b' \in \mathbb{R}^n$.

Now we turn our attention to affine maps between dually flat manifolds. Let (M, h, ∇) and (M', h', ∇') be connected dually flat manifolds of dimension n and d, respectively. We endow TM and TM' with their natural Kähler structures (coming from Dombrowski's construction).

Recall that a smooth function $f : M \to M'$ is *affine* if for every $p \in M$, there is a ∇-affine coordinate system $x : U \subseteq M \to \mathbb{R}^n$ and a ∇'-affine coordinate system $x' : U' \subseteq M' \to \mathbb{R}^d$ such that $p \in U$, $f(U) \subseteq U'$ and $x' \circ f \circ x^{-1} : x(U) \to \mathbb{R}^d$ is the restriction of an affine map (thus $(x' \circ f \circ x^{-1})(y) = Ay + B$ for all $y \in x(U)$, where A is a $d \times n$ real matrix and $B \in \mathbb{R}^d$).

When f is a diffeomorphism, an easy verification shows that f is affine if and only if $f^*\nabla' = \nabla$.

Proposition 2.19. Let $f : M \to M'$ be a smooth map.

(1) The derivative $f_* : TM \to TM'$ is holomorphic if and only if f is affine.

(2) Suppose f affine. Then f_* is isometric if and only if f is isometric.

Consequently, f_* is a Kähler immersion if and only if f is an isometric affine immersion.

Proof. Let $x = (x_1, ..., x_n)$ be a ∇-affine coordinate system on $U \subseteq M$ and let $x' = (x'_1, ..., x'_d)$ be a ∇'-affine coordinate system on $U' \subseteq M'$. Suppose U connected and $f(U) \subseteq U'$. We denote by $(q, r) = (q_1, ..., q_n, r_1, ..., r_n)$ and $(q', r') = (q'_1, ..., q'_d, r'_1, ..., r'_d)$ the corresponding coordinates on TM and TM', respectively (see before Lemma 2.2). For simplicity, we will use the same symbols "f" and "f_*" for the local expressions for f and f_*, respectively. Thus we write $f(x) = (f^1(x), ..., f^d(x))$ and $f_*(q, r) = (f(q), f_{*_q} r)$.

The derivative of f_* at (q, r) in the direction $(u, v) \in \mathbb{R}^n \times \mathbb{R}^n$ is given by

$$(f_*)_{*_{(q,r)}}(u, v) = \big(f_{*_q}(u), A(q, u)r + f_{*_q} v\big),$$

where $A(q, u)$ is the $d \times n$ matrix whose (i, j)-entry is $\big(\frac{\partial f^i}{\partial x_j}\big)_{*_q} u = \sum_{k=1}^{n} u_k \frac{\partial^2 f^i(q)}{\partial x_k \partial x_j}$. Let J and J' be the complex structures of M and M', respectively. Locally, $J_{(q,r)}(u, v) = (-v, u)$ and

$J'_{(q',r')}(u',v') = (-v',u')$ (see Proposition 2.17). Therefore $(f_*)_* \circ J = J' \circ (f_*)_*$ on $\pi^{-1}(U)$ if and only if

$$\begin{aligned} & \big(-f_{*_q}(v), A(q,-v)r + f_{*_q}u\big) = \big(-A(q,u)r - f_{*_q}v, f_{*_q}(u)\big) \qquad \forall\, q \in U,\ \forall\, u,v,r \in \mathbb{R}^n \\ \Leftrightarrow\quad & A(q,u)r = A(q,v)r = 0 \qquad \forall\, q \in U,\ \forall\, u,v,r \in \mathbb{R}^n \\ \Leftrightarrow\quad & \tfrac{\partial^2 f^i}{\partial x_k \partial x_j} = 0 \ \text{ for all } k,i,j \text{ on } U. \end{aligned}$$

Therefore f_* is holomorphic on $\pi^{-1}(U)$ if and only if all partial derivatives $\frac{\partial^2 f^i}{\partial x_k \partial x_j}$ vanish on U. Since U is connected, this is equivalent to the existence of a $d \times n$ real matrix B and $C \in \mathbb{R}^n$ such that $f(x) = Bx + C$ for all $x \in U$. It follows that f_* is holomorphic on TM if and only if f is affine. This shows (1).

Assume now that f is affine. Let g and g' be the Riemannian metrics on TM and TM', respectively. With the same notation as above, the coordinate expression for g and g' are given by $g(q,r) = \begin{pmatrix} h_{ij} & 0 \\ 0 & h_{ij} \end{pmatrix}$ and $g'(q',r') = \begin{pmatrix} h'_{ij} & 0 \\ 0 & h'_{ij} \end{pmatrix}$, where $h_{ij} = h(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ and $h'_{ij} = h'(\frac{\partial}{\partial x'_i}, \frac{\partial}{\partial x'_j})$ (see Proposition 2.17). Given $(q,r) \in U \times \mathbb{R}^n$ and $(u,v),(u',v') \in \mathbb{R}^n \times \mathbb{R}^n$, we compute:

$$\begin{aligned} & \big((f_*)^* g'\big)_{(q,r)}\big((u,v),(u',v')\big) = g'_{(f_*)(q,r)}\big((f_*)_*(u,v),(f_*)_*(u',v')\big) \\ = \ & g'_{(f_*)(q,r)}\big[\big(f_{*_q}(u), A(q,u)r + f_{*_q}v\big), \big(f_{*_q}(u'), A(q,u')r + f_{*_q}v'\big)\big] \\ = \ & g'_{(f_*)(q,r)}\big[\big(f_{*_q}(u), f_{*_q}v\big), \big(f_{*_q}(u'), f_{*_q}v'\big)\big] \\ = \ & h'_{f(q)}\big(f_{*_q}(u), f_{*_q}(u')\big) + h'_{f(q)}\big(f_{*_q}(v), f_{*_q}(v')\big) \\ = \ & (f^*h')_q(u,u') + (f^*h')_q(v,v'), \end{aligned}$$

where we have used the fact that $A(q,u) = 0$, since f is affine. It follows that $(f_*)^* g' = g$ if and only if $f^*h' = h$. This shows (2) and concludes the proof of the proposition. □

3 Parallel lattices and torus actions

Throughout this section (M, ∇) is a connected affine manifold of dimension n.

Definition 3.1. A subset $L \subset TM$ is said to be a *parallel lattice* with respect to ∇ if there are n parallel vector fields $X_1, ..., X_n$ on M such that:

(1) $\{X_1(p), ..., X_n(p)\}$ is a basis for T_pM for every $p \in M$,

(2) $L = \big\{k_1 X_1(p) + ... + k_n X_n(p) \ \big|\ p \in M,\ k_1, ..., k_n \in \mathbb{Z}\big\}$.

In that case we shall write $L = L_X$, where $X = (X_1, ..., X_n) \in \mathfrak{X}(M)^n$, and call X a *generator* for L.

As a matter of notation, we will denote the set of all generators of L by $\text{gen}(L)$. Given $X \in \text{gen}(L)$, we say that L is *generated by* X.

Remark 3.2. Let $L \subset TM$ be a parallel lattice and $X = (X_1, ..., X_n) \in \text{gen}(L)$.

(1) For every $p \in M$, the intersection $L \cap T_pM$ is a full rank lattice.

(2) $X = (X_1, ..., X_n)$ is a global frame for M. Therefore M is parallelizable.

(3) $[X_i, X_j] = 0$ for all $i, j = 1, ..., n$. This comes from the fact that parallel vector fields on an affine manifold commute.

In what follows, let $L \subset TM$ be a fixed parallel lattice. If $X = (X_1, ..., X_n) \in \mathrm{gen}(L)$, then, for any $p \in M$ and any $n \times n$ real matrix $A = (a_{ij})$, we let

$$X(p) := (X_1(p), ..., X_n(p)) \in (T_pM)^n,$$
$$AX := \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} := \left(\sum_{i=1}^{n} a_{1i} X_i, ..., \sum_{i=1}^{n} a_{ni} X_i \right) \in \mathfrak{X}(M)^n.$$

Let $\mathrm{GL}(n, \mathbb{Z})$ denote the group of invertible $n \times n$ matrices with integer entries.

Lemma 3.3. Let $X, X' \in \mathfrak{X}(M)^n$ be frames.

(1) If $X \in \mathrm{gen}(L)$, then a vector field Y on M is parallel if and only if there are real numbers $\lambda_1, ..., \lambda_n$ such that $Y = \lambda_1 X_1 + ... + \lambda_n X_n$.

(2) If X and X' are both generators for L, then there exists $A \in \mathrm{GL}(n, \mathbb{Z})$ such that $X = AX'$.

Proof. (1) Let Y be a parallel vector field on M. Fix $p_0 \in M$. Since $\{X_1(p_0), ..., X_n(p_0)\}$ is a basis for $T_{p_0}M$, there are real numbers $\lambda_1, ..., \lambda_n$ such that

$$Y(p_0) = \lambda_1 X_1(p_0) + ... + \lambda_n X_n(p_0).$$

Let $p \in M$ be arbitary. Since M is connected, there exists a piecewise smooth curve $c : [0, 1] \to M$ such that $c(0) = p_0$ and $c(1) = p$. Because $(X_1, ..., X_n)$ is a global frame, there are functions $f_1, ..., f_n : [0, 1] \to \mathbb{R}$ such that

$$Y(c(t)) = \sum_{k=1}^{n} f_k(t) X_k(c(t))$$

for every $t \in [0, 1]$. Note that $f_k(0) = \lambda_k$ for every $k = 1, .., n$. Let $(X_1^*, ..., X_n^*)$ denote the dual coframe of $(X_1, ..., X_n)$. Thus, by definition, $\langle X_i^*, X_j \rangle = \delta_{ij}$ for every i, j, where $\langle \, , \, \rangle$ is the natural pairing between TM and T^*M. It is immediate that

$$f_i(t) = \langle X_i^*, Y \rangle (c(t))$$

for every $t \in [0, 1]$ and every $i = 1, ..., n$, where $\langle X_i^*, Y \rangle : M \to \mathbb{R}$, $q \mapsto \langle X_i^*(q), Y(q) \rangle$. Since the latter function is smooth and since c is piecewise smooth, each f_i is continuous on $[0, 1]$ and smooth wherever c is.

Let $\frac{D}{\partial t}$ denote the covariant derivative operator of vector fields along c induced by ∇. Because Y and X_k are parallel, we have for all $t \in [0, 1]$ where c is smooth:

$$0 = \frac{D}{\partial t} Y(c(t)) = \sum_{k=1}^{n} \left[f_k'(t) X_k(c(t)) + f_k(t) \frac{D}{\partial t} X_k(c(t)) \right] = \sum_{k=1}^{n} f_k'(t) X_k(c(t))$$

and hence $f_k'(t) = 0$ for all k and all t, except for finitely many points. It follows from this and the continuity of f_k that f_k is constant. Thus

$$Y(p) = \sum_{k=1}^{n} f_k(1) X_k(p) = \sum_{k=1}^{n} f_k(0) X_k(p) = \sum_{k=1}^{n} \lambda_k X_k(p).$$

Since $p \in M$ is arbitrary, $Y = \lambda_1 X_1 + ... + \lambda_n X_n$.

(2) By the first item there exists an invertible $n \times n$ real matrix A such that $X = AX'$, Thus, at a particular point $p \in M$, we have the formula $X(p) = AX'(p)$, which can be interpreted as a change of basis for the full rank lattice $L \cap T_pM$. By the general theory of lattices, $A \in \mathrm{GL}(n, \mathbb{Z})$. □

Given $X = (X_1, ..., X_n) \in \mathrm{gen}(L)$, we will denote by $\Gamma(L, X)$ the group of transformations of TM of the form

$$TM \to TM, \;\; u \mapsto u + k_1 X_1 + ... + k_n X_n. \quad (k_1, ..., k_n \in \mathbb{Z})$$

The group $\Gamma(L, X)$ is obviously isomorphic to $\mathbb{Z}^n$.

Lemma 3.4. Suppose $X, X' \in \mathrm{gen}(L)$. Then $\Gamma(L, X) = \Gamma(L, X')$.

Proof. This follows immediately from the preceding lemma. □

It follows that $\Gamma(L, X)$ is independent of the choice of $X \in \mathrm{gen}(L)$. We shall thus write $\Gamma(L)$ instead of $\Gamma(L, X)$. Note that:

- $\Gamma(L)$ is a subgroup of $\mathrm{Diff}(TM)$, the group of diffeomorphisms of TM.
- For every $\gamma \in \Gamma(L)$, $\pi \circ \gamma = \pi$, where $\pi : TM \to M$ is the canonical projection.
- $\Gamma(L)$ characterizes L, for $L = \{\gamma(0_p) \mid p \in M, \; \gamma \in \Gamma(L)\}$, where 0_p is the zero vector in T_pM.

Because the action of $\Gamma(L)$ on TM is free and proper, the orbit space

$$M_L := TM/\Gamma(L)$$

is a smooth manifold and the quotient map

$$q_L : TM \to M_L$$

is a covering map whose Deck transformation group is $\Gamma(L)$. Moreover, the fact that $\pi \circ \gamma = \pi$ for every $\gamma \in \Gamma(L)$ implies that there exists a surjective submersion $\pi_L : M_L \to M$ such that the following diagram commutes:

$$\begin{array}{ccc} TM & \xrightarrow{q_L} & M_L \\ \pi \downarrow & & \downarrow \pi_L \\ M & \xrightarrow[\mathrm{Id}]{} & M \end{array} \tag{3.1}$$

Let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ denote the n-dimensional torus. Given $t = (t_1, ..., t_n) \in \mathbb{R}^n$, we will denote by $[t] = [t_1, ..., t_n]$ the corresponding equivalence class in $\mathbb{R}^n/\mathbb{Z}^n$.

Given $X \in \mathrm{gen}(L)$, we will denote by

$$\Phi_X : \mathbb{T}^n \times M_L \to M_L$$

the torus action defined by

$$\Phi_X\big([t], q_L(u)\big) := q_L(u + t_1X_1 + ... + t_nX_n), \tag{3.2}$$

where $t = (t_1, ..., t_n) \in \mathbb{R}^n$ and $u \in TM$.

Remark 3.5. A simple verification shows that:

(1) Φ is effective, that is, the map $\mathbb{T}^n \to \mathrm{Diff}(M_L)$, $a \mapsto (\Phi_X)_a$ is injective, and

(2) $\pi_L \circ (\Phi_X)_a = \pi_L$ for every $a \in \mathbb{T}^n$.

Lemma 3.6. The map $f : M_L \to \mathbb{T}^n \times M$ given by

$$f\big(q_L(u_1X_1(p) + ... + u_nX_n(p))\big) := \big([u_1, ..., u_n], p\big)$$

is a $\mathbb{T}^n$-equivariant diffeomorphism ($\mathbb{T}^n$ acts on $\mathbb{T}^n \times M$ via translations on the first factor).

Proof. By a direct verification. □

Lemma 3.7. Suppose $X, X' \in \mathrm{gen}(L)$. Let $A \in \mathrm{GL}(n, \mathbb{Z})$ be the unique matrix satisfying $X = AX'$. Then for every $[t] \in \mathbb{T}^n$ and every $q_L(u) \in M_L$,

$$\Phi_X\big([t], q_L(u)\big) = \Phi_{X'}\big(\rho_{A^T}([t]), q_L(u)\big),$$

where $\rho_{A^T} : \mathbb{T}^n \to \mathbb{T}^n$, $[t] \mapsto [A^Tt]$ (A^T is the transpose of A).

Proof. By a direct verification. □

To summarize, a parallel lattice L on a connected affine manifold (M, ∇) induces an effective torus action on the quotient space $M_L = TM/\Gamma(L)$, which is unique modulo $\mathrm{Aut}(\mathbb{T}^n) \cong \mathrm{GL}(n, \mathbb{Z})$.

4 Analytic properties

Throughout this section (M, h, ∇) is a dually flat connected manifold and (g, J, ω) is the Kähler structure on TM associated to (h, ∇) via Dombrowski's construction.

Let $L \subset TM$ be a fixed parallel lattice with respect to ∇ and let $\Gamma(L)$ be the corresponding subgroup of $\mathrm{Diff}(TM)$.

Lemma 4.1. Suppose $X = (X_1, ..., X_n) \in \mathrm{gen}(L)$. For any $t = (t_1, ..., t_n) \in \mathbb{R}^n$, the map $T_t : TM \to TM$ defined by

$$T_t(u) := u + t_1X_1 + ... + t_nX_n,$$

is a holomorphic isometry.

Proof. Let (U, φ) be an affine chart for M with respect to ∇, with local coordinates $(x_1, ..., x_n)$. We denote by $(q_1, ..., q_n, r_1, ..., r_n)$ the corresponding coordinates on $\pi^{-1}(U) \subseteq TM$ as described before Lemma 2.2, where $\pi : TM \to M$ is the canonical projection.

Since each X_i is parallel and since parallel vector fields are constant in affine coordinates, there exists an invertible real matrix $A = (a_{ij}) \in \mathrm{GL}(n, \mathbb{R})$ such that for every $p \in U$ and every $i = 1, ..., n$,

$$X_i(p) = \sum_{j=1}^{n} a_{ij} \frac{\partial}{\partial x_j}\bigg|_p,$$

so if $u \in \pi^{-1}(U)$, then

$$T_t(u) = u + \sum_{j=1}^{n} b_j \frac{\partial}{\partial x_j}\bigg|_p,$$

where $b_j = \sum_{i=1}^{n} t_i a_{ij}$. It follows that the local expression for T_t in the coordinates $(q_1, ..., q_n, r_1, ..., r_n)$ is given by

$$T_t(q, r) = (q, r + b),$$

where $b = (b_1, ..., b_n) \in \mathbb{R}^n$. Now, a simple calculation using the local expressions for g and J in the coordinates $(q_1, ..., q_n, r_1, ..., r_n)$ shows that $T_t^* g = g$ and $(T_t)_* \circ J = J \circ (T_t)_*$ on $\pi^{-1}(U)$. It follows that T_t is isometric and holomorphic. □

Corollary 4.2. Each $\gamma \in \Gamma(L)$ is a holomorphic and isometric map.

It follows from the corollary above that $M_L = TM/\Gamma(L)$ is a Kähler manifold for which the quotient map

$$q_L : TM \to M_L$$

is a holomorphic and locally isometric covering map with Deck transformation group $\Gamma(L)$. Moreover, it follows from the formula $\pi = \pi_L \circ q_L$ (see (3.1)) and the fact that the canonical projection $\pi : TM \to M$ is a Riemannian submersion that $\pi_L : M_L \to M$ is a Riemannian submersion.

Let $X \in \mathrm{gen}(L)$ be arbitrary and let $\Phi_X : \mathbb{T}^n \times M_L \to M_L$ be the corresponding torus action as defined in (3.2).

Lemma 4.3. For every $[t] \in \mathbb{T}^n$, the map $M_L \to M_L$, $p \mapsto \Phi_X([t], p)$, is holomorphic and isometric.

Proof. Let $\Phi : M_L \to M_L$, $p \mapsto \Phi_X([t], p)$. By definition of Φ_X (see (3.2)), we have

$$\Phi \circ q_L = q_L \circ T_t,$$

and since q_L is a holomorphic and locally isometric covering map, we see that Φ is holomorphic and isometric if and only if T_t is, which is the case by Lemma 4.1. □

Therefore, a parallel lattice $L \subset TM$ on a dually flat manifold M induces a holomorphic and isometric torus action $\mathbb{T}^n \times M_L \to M_L$.

5 Momentum map

We start with some definitions. Let G be a Lie group with Lie algebra $\mathrm{Lie}(G) = \mathfrak{g}$. Given $g \in G$, we denote by $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ and $\mathrm{Ad}_g^* : \mathfrak{g}^* \to \mathfrak{g}^*$ the adjoint and coadjoint representations of G, respectively; they are related as follows:

$$\langle \mathrm{Ad}_g^* \alpha, \xi \rangle = \langle \alpha, \mathrm{Ad}_{g^{-1}} \xi \rangle,$$

where $\xi \in \mathfrak{g}$, $\alpha \in \mathfrak{g}^*$ (the dual of $\mathfrak{g}$) and $\langle \, , \, \rangle$ is the natural pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$.

Let $\Phi : G \times M \to M$ be a Lie group action of G on a manifold M. The *fundamental vector field* associated to $\xi \in \mathfrak{g}$ is the vector field on M, denoted by ξ_M, defined by

$$(\xi_M)(p) := \left.\frac{d}{dt}\right|_0 \Phi(c(t), p),$$

where $p \in M$ and $c(t)$ is a smooth curve in G satisfying $c(0) = e$ (neutral element) and $\dot{c}(0) = \xi$.

Given a map $\mathbf{J} : M \to \mathfrak{g}^*$ and a vector $\xi \in \mathfrak{g}$, we will denote by $\mathbf{J}^\xi : M \to \mathbb{R}$ the function given by

$$\mathbf{J}^\xi(p) := \langle \mathbf{J}(p), \xi \rangle.$$

We shall say that $\mathbf{J}$ is *G-equivariant* if for every $g \in G$,

$$\mathbf{J} \circ \Phi_g = \mathrm{Ad}_g^* \circ \mathbf{J},$$

where $\Phi_g : M \to M$, $p \mapsto \Phi(g, p)$.

Finally, given a symplectic form ω on M, the say that Φ is *symplectic* if $(\Phi_g)^*\omega = \omega$ for all $g \in G$.

Definition 5.1. Let (M, ω) be a symplectic manifold. A symplectic action $\Phi : G \times M \to M$ is said to be *Hamiltonian* if there exists a G-equivariant map $\mathbf{J} : M \to \mathfrak{g}^*$, called *momentum map*, such that

$$\omega(\xi_M, \, . \,) = d\mathbf{J}^\xi(.)$$

(equality of 1-forms) for all $\xi \in \mathfrak{g}$.

When $G = \mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$ is a torus, it is convenient to identify

- the Lie algebra of the torus $\mathbb{T}^n$ with $\mathbb{R}^n$ via the derivative at $0 \in \mathbb{R}^n$ of the quotient map $\mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$,
- $\mathbb{R}^n$ and its dual $(\mathbb{R}^n)^*$ via the Euclidean metric.

Upon these identifications, a momentum map for a Hamiltonian torus action $\mathbb{T}^n \times M \to M$ can be regarded as a map $\mathbf{J} : M \to \mathbb{R}^n$. Moreover, since the coadjoint action of a commutative group is trivial, the equivariance condition reduces to $\mathbf{J} \circ \Phi_g = \mathbf{J}$ for all $g \in \mathbb{T}^n$.

If the symplectic manifold M is connected, then it is easy to see that two momentum maps $\mathbf{J}_1$ and $\mathbf{J}_2$ for the same Hamiltonian torus action $\Phi : \mathbb{T}^n \times M \to M$ differ by a constant, that is $\mathbf{J}_1 = \mathbf{J}_2 + c$, $c \in \mathbb{R}^n$. Reciprocally, if $\mathbf{J} : M \to \mathbb{R}^n$ is a moment map for a Hamiltonian torus action, then so is $\mathbf{J} + c$, where $c \in \mathbb{R}^n$ is any constant.

Proposition 5.2. Let (M, h, ∇) be a connected dually flat manifold endowed with a parallel lattice $L = L_X \subset TM$ with respect to ∇, where $X = (X_1, ..., X_n) \in \mathrm{gen}(L)$, and let $\Phi_X : \mathbb{T}^n \times M_L \to M_L$ be the corresponding torus action. If (x, y) is a global pair of dual coordinate systems on M, then Φ_X is Hamiltonian with momentum map $\mathbf{J}_X : M_L \to \mathbb{R}^n$ given by

$$\mathbf{J}_X = -A \circ y \circ \pi_L,$$

where $A = (a_{ij}) \in \mathrm{GL}(n, \mathbb{R})$ is the matrix defined via the formula

$$X_i = \sum_{j=1}^{n} a_{ij} \frac{\partial}{\partial x_j}, \qquad i = 1, ..., n.$$

Proof. Let ω^{TM} and ω^{M_L} be the symplectic forms on TM and M_L, respectively. Let $T : \mathbb{R}^n \times TM \to TM$ be the Lie group action of $\mathbb{R}^n$ on TM given by

$$T(t, u) := u + t_1 X_1 + ... + t_n X_n,$$

where $t = (t_1, ..., t_n) \in \mathbb{R}^n$ and $u \in TM$. We claim that T is Hamiltonian with momentum map $\mathbf{J} : TM \to \mathbb{R}^n \cong (\mathbb{R}^n)^*$ given by

$$\mathbf{J} = -A \circ y \circ \pi,$$

where $\pi : TM \to M$ is the canonical projection. To see this, let $(x_1, ..., x_n)$ be an affine coordinate system on M with respect to ∇, and let $(q, r) = (q_1, ..., q_n, r_1, ..., r_n)$ be the corresponding coordinates on $\pi^{-1}(U) \subseteq TM$ as described before Lemma 2.2. Since $X_i = \sum_{j=1}^{n} a_{ij} \frac{\partial}{\partial x_j}$ by hypothesis, the local expression for T is the coordinates (q, r) is given by

$$T(t, (q, r)) = (q, r + A^T t),$$

and so, the fundamental vector field of $\xi \in \mathbb{R}^n \cong T_0\mathbb{R}^n$ is given by

$$(\xi_{TM})(q, r) = \frac{d}{dt}\bigg|_0 T(t\xi, (q, r)) = \frac{d}{dt}\bigg|_0 (q, r + tA^T\xi) = (0, A^T\xi).$$

It follows from this together with the coordinate expression for ω^{TM} (see Proposition 2.17) that

$$\omega^{TM}(\xi_{TM}, \,.\,) = (0, A^T\xi) \begin{bmatrix} 0 & h \\ -h & 0 \end{bmatrix} = (-\xi^T A h, 0), \tag{5.1}$$

where $h = (h_{ij})$ is the coordinate expression for the metric h. On the other hand, the derivative of the map $\mathbf{J}^\xi : TM \to \mathbb{R}$, $u \mapsto -\langle \xi, (A \circ y \circ \pi)(u) \rangle$ is given in matrix notation by

$$d\mathbf{J}^\xi = \left(\frac{\partial \mathbf{J}^\xi}{\partial q_i}, \frac{\partial \mathbf{J}^\xi}{\partial r_i} \right) = \left(-\,\xi^T A \left(\frac{\partial y_i}{\partial x_j} \right)_{ij}, 0 \right) = (-\xi^T A h, 0), \tag{5.2}$$

where we have used $\pi(q, r) = q$ and $\frac{\partial y_i}{\partial x_j} = h_{ij}$ (see Proposition 2.12). Comparing (5.1) and (5.2) we find that $\omega^{TM}(\xi_{TM}, \,.\,) = d\mathbf{J}^\xi$ for all $\xi \in \mathbb{R}^n$. Obviously $\mathbf{J}$ is equivariant. Therefore $\mathbf{J}$ is a momentum map for the action T. This concludes the proof of the claim.

Next we show that $\mathbf{J}_X : M_L \to \mathbb{R}^n$ is a momentum map for the action Φ_X. Let $\xi \in \mathbb{R}^n \cong \mathrm{Lie}(\mathbb{R}^n) \cong \mathrm{Lie}(\mathbb{T}^n)$ be arbitrary. By inspection of the definition of Φ_X, we see that $q_L : TM \to M_L$ is equivariant, that is (with obvious notation):

$$(\Phi_X)_{[t]} \circ q_L = q_L \circ T_t$$

for every $t \in \mathbb{R}^n$. Taking the derivative along the curve $t\xi$ at $t = 0$ we find that

$$\xi_{M_L} \circ q_L = (q_L)_* \circ \xi_{TM}.$$

Because $q_L : TM \to M_L$ is holomorphic and locally isometric, we have that $q_L^* \omega^{M_L} = \omega^{TM}$. Thus for any $A \in TTM$,

$$\begin{aligned}
\omega^{M_L}(\xi_{M_L} \circ q_L, (q_L)_* A) &= \omega^{M_L}((q_L)_* \xi_{TM}, (q_L)_* A) \\
&= (q_L^* \omega^{M_L})(\xi_{TM}, A) \\
&= \omega^{TM}(\xi_{TM}, A) \\
&= d\mathbf{J}^\xi(A) \\
&= d\mathbf{J}_X^\xi\big((q_L)_* A\big),
\end{aligned}$$

where in the last equality we have used $\mathbf{J} = \mathbf{J}_X \circ q_L$. It follows that $\omega^{M_L}(\xi_{M_L}, \,.\,) = d\mathbf{J}_X^\xi$ for every $\xi \in \mathrm{Lie}(\mathbb{T}^n)$. Obviously $\mathbf{J}_X$ is equivariant. Therefore $\mathbf{J}_X$ is a momentum map for the action Φ_X. This concludes the proof of the proposition. □

Lemma 5.3. Let the hypotheses be as in Proposition 5.2. Suppose $X, X' \in \mathrm{gen}(L)$. Let $B \in \mathrm{GL}(n, \mathbb{Z})$ be the unique matrix such that $X = BX'$, as in Lemma 3.3. Then

$$\mathbf{J}_X = B \circ \mathbf{J}_{X'}.$$

Proof. By a direct verification. □

6 Torification

Throughout this section,

- (M, h, ∇, ∇^*) is a connected dually flat manifold of dimension n and
- (N, g, J, ω) is a connected Kähler manifold of real dimension $2n$, equipped with an effective holomorphic and isometric torus action $\Phi : \mathbb{T}^n \times N \to N$.

We will denote by N° the set of points $p \in N$ where the action Φ is free, that is,

$$N^\circ = \{p \in N \mid \Phi(a, p) = p \;\; \Rightarrow \;\; a = e\}.$$

Then N° is a $\mathbb{T}^n$-invariant connected open dense subset of N[3].

Recall the notation Φ_X defined in (3.2) for a generator X of a parallel lattice $L \subset TM$.

[3]This follows from the following result (see [GGK02], Corollary B.48.). Let $\Phi : G \times M \to M$ be a proper effective Lie group action of a commutative Lie group G on a connected manifold M. Then the set M° of points where the action is free is open and dense in M. If in addition M is orientable and G is connected, then M° is connected.

Lemma 6.1. Let $L \subset TM$ be a parallel lattice with respect to ∇, $U \subseteq N$ a $\mathbb{T}^n$-invariant set and $F : M_L \to U$ a map. The following are equivalent:

(a) There exists $X \in \text{gen}(L)$ such that $F \circ (\Phi_X)_{[t]} = \Phi_{[t]} \circ F$ for all $[t] \in \mathbb{T}^n$.

(b) For every $X \in \text{gen}(L)$, there exists $A \in \text{GL}(n, \mathbb{Z})$ such that $F \circ (\Phi_X)_{[t]} = \Phi_{[At]} \circ F$ for all $t \in \mathbb{R}^n$.

Proof. Use Lemma 3.7. □

Definition 6.2. Let $L \subset TM$ be a parallel lattice with respect to ∇, $U \subseteq N$ a $\mathbb{T}^n$-invariant set and $F : M_L \to U$ a map. We shall say that F is *equivariant with respect to* L, or simply L-*equivariant*, if any of the conditions in Lemma 6.1 holds.

Now we can define the main concept of this paper.

Definition 6.3. We shall say that N is a *torification* of M if there exist a parallel lattice $L \subset TM$ with respect to ∇ and a L-equivariant holomorphic and isometric diffeomorphism $F : M_L \to N^\circ$.

By abuse of language, we will often say that the torus action $\Phi : \mathbb{T}^n \times N \to N$ is a torification of M.

Remark 6.4.

(1) If N is a torification of M, then so does N°. Therefore torifications are not unique in general.

(2) Let $A \in \text{GL}(n, \mathbb{Z})$. If $\Phi : \mathbb{T}^n \times N \to N$ is a torification of M, then so does $\widetilde{\Phi} : \mathbb{T}^n \times N \to N$, $([t], p) \mapsto \Phi([At], p)$.

(3) If $L \subset TM$ is a parallel lattice with respect to ∇ with generator X, then $\Phi_X : \mathbb{T}^n \times M_L \to M_L$ is trivially a torification of M.

Below is an alternative definition, in terms of covering maps.

Proposition 6.5. Let M and N be as defined in the beginning of this section. The following are equivalent:

(1) N is a torification of M.

(2) There exist a holomorphic and isometric covering map $\tau : TM \to N^\circ$, a parallel lattice $L \subset TM$ with respect to ∇ and $X = (X_1, ..., X_n) \in \text{gen}(L)$ such that:

   (i) $\Gamma(L) = \text{Deck}(\tau)$ (= Deck transformation group of τ).

   (ii) $\tau \circ T_t = \Phi_{[t]} \circ \tau$ for every $t \in \mathbb{R}^n$, where $T : \mathbb{R}^n \times TM \to TM$ is the Lie group action of $\mathbb{R}^n$ on TM given by $T(t, u) = u + t_1 X_1 + ... + t_n X_n$, where $t = (t_1, ..., t_n) \in \mathbb{R}^n$ and $u \in TM$.

Sketch of proof. (1) ⇒ (2). If N is a torification of M, then there exist a parallel lattice $L \subset TM$ with respect to ∇, $X \in \mathrm{gen}(L)$ and a holomorphic and isometric diffeomorphism $F : M_L \to N^\circ$ such that $F \circ (\Phi_X)_a = \Phi_a \circ F$ for all $a \in \mathbb{T}^n$. Then it is easy to check that the map $\tau : TM \to N^\circ$ defined by $\tau(u) = (F \circ q_L)(u)$ has the required properties.
(2) ⇒ (1). Let $\tau : TM \to N^\circ$, $L \subset TM$ and $X \in \mathrm{gen}(L)$ be as in the second item of the proposition. Because τ is $\Gamma(L)$-invariant, it descends to a Kähler isomorphism $F : M_L \to N^\circ$, and a straightforward computation using $\tau \circ T_t = \Phi_{[t]} \circ \tau$ shows that F is equivariant in the sense that $F \circ (\Phi_X)_a = \Phi_a \circ F$ for all $a \in \mathbb{T}^n$. It follows that N is a torification of M. □

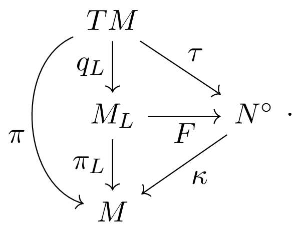
Before proceeding, we introduce some terminology.

Definition 6.6. Suppose $\Phi : \mathbb{T}^n \times N \to N$ is a torification of (M, h, ∇).

(1) A *toric parametrization* is a triple (L, X, F), where

- $L \subset TM$ is parallel lattice with respect to ∇, generated by X,
- $F : M_L \to N^\circ$ is a holomorphic and isometric diffeomorphism that is equivariant in the sense that $F \circ (\Phi_X)_a = \Phi_a \circ F$ for all $a \in \mathbb{T}^n$.

(2) Let $\tau : TM \to N^\circ$ and $\kappa : N^\circ \to M$ be smooth maps. We say that the pair (τ, κ) is a *toric factorization* if there exists a toric parametrization (L, X, F) that makes the following diagram commutative:



In this case, we say that (τ, κ) is *induced by the toric parametrization* (L, X, F).

(3) We say that $\kappa : N^\circ \to M$ is the *compatible projection induced by the toric parametrization* (L, X, F) if there exists a map $\tau : TM \to N^\circ$ such that (τ, κ) is the toric factorization induced by (L, X, F). When it is not necessary to mention (L, X, F) explicitly, we just say that κ is a *compatible projection.* Analogously, one defines a *compatible covering map* $\tau : TM \to N^\circ$.

By abuse of language, we will often say that the formula $\pi = \kappa \circ \tau$ is a toric factorization. If $\pi = \kappa \circ \tau$ is a toric factorization, then τ is a Kähler covering map whose Deck transformation group is $\Gamma(L)$, and κ is naturally a principal $\mathbb{T}^n$-bundle and a Riemannian submersion.

Proposition 6.7. Let (M, h, ∇) and (M', h', ∇') be connected dually flat spaces and $f : M \to M'$ an isomorphism of dually flat spaces. Suppose $\Phi : \mathbb{T}^n \times N \to N$ is a torification of M, with toric factorization $\pi = \kappa \circ \tau : TM \to M$. Then $\Phi : \mathbb{T}^n \times N \to N$ is a torification of M', with toric factorization $\pi' = \kappa' \circ \tau' : TM' \to M'$, where $\tau' = \tau \circ (f_*)^{-1}$ and $\kappa' = f \circ \kappa$.

Sketch of proof. By Proposition 2.19, $f_* : TM \to TM'$ is a Kähler isomorphism, and thus $\tau' = \tau \circ (f_*)^{-1} : TM' \to N^\circ$ is a Kähler covering map. Now apply Proposition 6.5. □

Now we focus our attention on torifications whose torus action is Hamiltonian.

Proposition 6.8. Let $\Phi : \mathbb{T}^n \times N \to N$ be a torification of (M, h, ∇). Suppose that Φ is Hamiltonian with momentum map $\mathbf{J} : N \to \mathbb{R}^n$ and that (x, y) are global pair of dual coordinate systems on M.

(1) Let $\kappa : N^\circ \to M$ be the compatible projection induced by a toric parametrization (L, X, F). Then there exists a constant $C \in \mathbb{R}^n$ such that on N°,

$$\mathbf{J} = -A \circ y \circ \kappa + C,$$

where $A = (a_{ij}) \in \mathrm{GL}(n, \mathbb{R})$ is the matrix defined via the formula $X_i = \sum_{j=1}^{n} a_{ij} \frac{\partial}{\partial x_j}$, $i = 1, ..., n$ (here $X = (X_1, ..., X_n)$ and $x = (x_1, ..., x_n)$). In particular, $\mathbf{J}(N^\circ)$ is an open subset of $\mathbb{R}^n$.

(2) If $x(M) = \mathbb{R}^n$, then $\mathbf{J}(N^\circ)$ is a convex subset of $\mathbb{R}^n$. If in addition $\mathbf{J}$ is proper, that is, if $\mathbf{J}^{-1}(K)$ is compact whenever $K \subseteq \mathbb{R}^n$ is compact, then $\mathbf{J}(N) \subseteq \mathbb{R}^n$ is convex.

(3) $\mathbf{J} : N^\circ \to \mathbf{J}(N^\circ)$ is naturally a principal $\mathbb{T}^n$-bundle. In particular, there exists a unique Riemannian metric k on $\mathbf{J}(N^\circ)$ that makes $\mathbf{J} : N^\circ \to \mathbf{J}(N^\circ)$ a Riemannian submersion.

Proof. (1) By Proposition 5.2, $\Phi_X : \mathbb{T}^n \times M_L \to M_L$ is Hamiltonian with momentum map $\mathbf{J}' : M_L \to \mathbb{R}^n$ given by $\mathbf{J}' = -A \circ y \circ \pi_L$. Since $F : M_L \to N^\circ$ is an equivariant symplectomorphism, $\mathbf{J}' \circ F^{-1} : N^\circ \to \mathbb{R}^n$ is a momentum map with respect to the action $\Phi : \mathbb{T}^n \times N^\circ \to N^\circ$. But then $\mathbf{J}' \circ F^{-1}$ and $\mathbf{J}$ are two momentum maps for the same torus action on the connected set N°. Therefore there is a constant $C \in \mathbb{R}^n$ such that $\mathbf{J} = \mathbf{J}' \circ F^{-1} + C$ on N°. The lemma follows from this and the fact that $\mathbf{J}' \circ F^{-1} = -A \circ y \circ \pi_L \circ F^{-1} = -A \circ y \circ \kappa$.
(2) By (1), there are an invertible matrix $A = (a_{ij})$ and a constant $C \in \mathbb{R}^n$ such that $\mathbf{J}(N^\circ) = -A(y(M)) + C$. Therefore $\mathbf{J}(N^\circ)$ is convex if and only if $y(M)$ is convex. By Proposition 2.12, there is a smooth function $\psi : M \to \mathbb{R}$ such that $y = \mathrm{grad}(\psi) = (\frac{\partial \psi}{\partial x_1}, ..., \frac{\partial \psi}{\partial x_n})$ and $\frac{\partial^2 \psi}{\partial x_i \partial x_j} = h_{ij} = h(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ for all $i, j = 1, ..., n$ on M. Thus y can be regarded as a strictly convex function on the convex set $x(M) = \mathbb{R}^n$. By the general properties of the Legendre transform (see Theorem A.3), $y(\mathbb{R}^n)$ is convex. If $\mathbf{J}$ is proper, then a simple exercice in topology shows that $\mathbf{J}(N) = \overline{\mathbf{J}(A)}$ for any dense subset A of N, where $\overline{\mathbf{J}(A)}$ denotes the closure of $\mathbf{J}(A)$ in $\mathbb{R}^n$. Since N° is dense in N, $\mathbf{J}(N) = \overline{\mathbf{J}(N^\circ)}$. It follows from this and the fact that the closure of a convex set is convex that $\mathbf{J}(N)$ is convex.
(3) By (1), the following diagram is commutative:

$$\begin{array}{ccc} & N^\circ & \\ {}^{\kappa}\swarrow & & \searrow^{\mathbf{J}} \\ M & \xrightarrow[-A \circ y + C]{} & \mathbf{J}(N^\circ) \end{array}$$

Since κ is a principal $\mathbb{T}^n$-bundle and $-A \circ y + C$ is a diffeomorphism, $\mathbf{J} : N^\circ \to \mathbf{J}(N^\circ)$ is a principal $\mathbb{T}^n$-bundle. $\square$

Lemma 6.9. Let $\mathcal{O} \subset N$ be the $\mathbb{T}^n$-orbit of $p \in N^\circ$. Then $T_pN = T_p\mathcal{O} \oplus J\big(T_p\mathcal{O}\big)$, and the direct sum is orthogonal relative to the Kähler metric g of N.

Proof. Because Φ is free at p, $\dim(\mathcal{O}) = \dim(\mathbb{T}^n) = n$ and hence $\dim(T_p\mathcal{O}) = \dim(JT_p\mathcal{O}) = n$. Thus, to prove that $T_pN = T_p\mathcal{O} \oplus J\big(T_p\mathcal{O}\big)$, it suffices to show that $T_p\mathcal{O} \cap J\big(T_p\mathcal{O}\big)$ is trivial (recall that $\dim(N) = 2n$ by hypothesis). So let $u = Jv \in T_p\mathcal{O} \cap J\big(T_p\mathcal{O}\big)$ be arbitrary, where $u, v \in T_p\mathcal{O}$. Since $T_p\mathcal{O} = \{\xi_N(p) \mid \xi \in \mathrm{Lie}(\mathbb{T}^n)\}$, where ξ_N denotes the fundamental vector field on N associated to $\xi \in \mathrm{Lie}(\mathbb{T}^n)$, there are $\xi, \eta \in \mathbb{R}^n = \mathrm{Lie}(\mathbb{T}^n)$ such that $u = \xi_N(p)$ and $v = \eta_N(p)$. From this it follows that

$$g_p(u,u) = g_p(Jv,u) = \omega_p(v,u) = \omega_p(\eta_N, \xi_N) = (d\mathbf{J}^\eta)_p(\xi_N) = \frac{d}{dt}\bigg|_0 \mathbf{J}^\eta(\Phi(\exp(t\xi), p)) = 0,$$

where we have used the following facts: (1) $\mathbf{J}^\eta$ is constant along $\mathbb{T}^n$-orbits and (2) $(t,p) \mapsto \Phi(\exp(t\xi), p)$ is the flow of ξ_N. Thus $u = 0$. It follows that $T_pN = T_p\mathcal{O} \oplus J\big(T_p\mathcal{O}\big)$. The computation above also shows that this direct sum is orthogonal. □

Proposition 6.10. Let the hypotheses be as in Proposition 6.8. Let k be the Riemannian metric on $\mathbf{J}(N^\circ)$ described in Proposition 6.8. Given $p \in N^\circ$ and $\xi, \eta \in \mathrm{Lie}(\mathbb{T}^n)$, we have

$$k_{\mathbf{J}(p)}\big(\mathbf{J}_{*_p} J\xi_N, \mathbf{J}_{*_p} J\eta_N\big) = g_p(\xi_N, \eta_N). \tag{6.1}$$

Proof. This follows from the following facts: (1) $\mathbf{J} : N^\circ \to \mathbf{J}(N^\circ)$ is a Riemannian submersion, (2) the fibers of $\mathbf{J}|_{N^\circ}$ are $\mathbb{T}^n$-orbits and (3) the orthogonal complement of $T_p\mathcal{O}$ is $J(T_p\mathcal{O})$, by Lemma 6.9. □

Let ∇^{flat} denote the canonical flat connection on $\mathbb{R}^n$, or any open subset of it (see (2.3)).

Definition 6.11. Let the hypotheses be as in Proposition 6.8. Let ∇^k be the dual connection of ∇^{flat} with respect to the Riemannian metric k on $\mathbf{J}(N^\circ)$. We call $(k, \nabla^k, \nabla^{\text{flat}})$ the *canonical dualistic structure* of $\mathbf{J}(N^\circ)$ and $(\mathbf{J}(N^\circ), k, \nabla^k)$ the *canonical dually flat space* associated to $\Phi : \mathbb{T}^n \times N \to N$.

The terminology is justified by the following result.

Proposition 6.12. Let the hypotheses be as in Proposition 6.8. Let $(k, \nabla^k, \nabla^{\text{flat}})$ be the canonical dualistic structure of $\mathbf{J}(N^\circ)$. The following hold.

(1) $(k, \nabla^k, \nabla^{\text{flat}})$ is dually flat.

(2) $\Phi : \mathbb{T}^n \times N \to N$ is a torification of $(\mathbf{J}(N^\circ), k, \nabla^k)$.

(3) Given a compatible projection $\kappa : N^\circ \to M$, there exists an isomorphism of dually flat spaces f from (M, h, ∇) to $(\mathbf{J}(N^\circ), k, \nabla^k)$ such that $f \circ \kappa = \mathbf{J}$ on N°.

Proof. Let $\kappa : N^\circ \to M$ be a compatible projection. By Proposition 6.8(1), there exist a constant $C \in \mathbb{R}^n$ and an invertible matrix $A = (a_{ij}) \in \mathrm{GL}(n, \mathbb{R})$ such that $\mathbf{J} = -A \circ y \circ \kappa + C$ on N°. In particular, the map $f = -A \circ y + C$ is a diffeomorphism from M to $\mathbf{J}(N^\circ)$ satisfying $f \circ \kappa = \mathbf{J}$ on N°. We claim that f is an isometry. Indeed, since κ is a surjective submersion, we have

$$f^*k = h \qquad \Leftrightarrow \qquad \kappa^* f^* k = \kappa^* h \qquad \Leftrightarrow \qquad \mathbf{J}^* k = \kappa^* h \;\text{ on } \mathbf{J}(N^\circ).$$

Therefore it suffices to show that $\mathbf{J}^*k = \kappa^* h$ on N°. Let $p \in N^\circ$ and $u, v \in T_pN$ be arbitrary. Let $\mathcal{O} \subset N$ denotes the $\mathbb{T}^n$-orbit of p and let $T_p\mathcal{O}^\perp = J\big(T_p\mathcal{O}\big)$ denotes the orthogonal complement of $T_p\mathcal{O}$ in T_pN with respect to the Kähler metric g. Write $u = u_1 + u_2$ and $v = v_1 + v_2$, where $u_1, v_1 \in T_p\mathcal{O}$ and $u_2, v_2 \in T_p\mathcal{O}^\perp$. Because $\mathbf{J} : N^\circ \to \mathbf{J}(N^\circ)$ is a Riemannian submersion with fiber $\mathbf{J}^{-1}(\mathbf{J}(p)) = \mathcal{O}$, we have $\mathbf{J}_{*_p} u_1 = \mathbf{J}_{*_p} v_1 = 0$ and $k_{\mathbf{J}(p)}(\mathbf{J}_{*_p} u_2, \mathbf{J}_{*_p} v_2) = g_p(u_2, v_2)$. Thus

$$(\mathbf{J}^*k)_p(u, v) = k_{\mathbf{J}(p)}(\mathbf{J}_{*_p} u_2, \mathbf{J}_{*_p} v_2) = g_p(u_2, v_2).$$

The exact same argument applied to κ shows that $(\kappa^* h)_p(u, v) = g_p(u_2, v_2)$. It follows that $(\mathbf{J}^*k)_p(u, v) = (\kappa^* h)_p(u, v)$ and concludes the proof of the claim. Let ∇^* be the dual connection of ∇ with respect to ∇. The map f is trivially affine from (M, ∇^*) to $(\mathbf{J}(N^\circ), \nabla^{\text{flat}})$ (since f is an affine function of y and y is a ∇^*-affine coordinate system on M). By Lemma 2.15 and the claim, f is also affine from (M, ∇) to $(\mathbf{J}(N^\circ), \nabla^k)$. This forces ∇^k to be flat. It follows that $(k, \nabla^k, \nabla^{\text{flat}})$ is dually flat and that f is an isomorphism of dually flat spaces. This shows (1) and (3). (2) is a consequence of (3) and Proposition 6.7. □

Proposition 6.13. Given $i = 1, 2$, let (M_i, h_i, ∇_i) be a connected dually flat space that has a global pair of dual coordinate systems. Suppose that $\Phi_i : \mathbb{T}^n \times N_i \to N_i$ is a torification of (M_i, h_i, ∇_i), $i = 1, 2$, and that Φ_i is Hamiltonian. If there exists a Kähler isomorphism $G : N_1 \to N_2$ and a Lie group isomorphism $\rho : \mathbb{T}^n \to \mathbb{T}^n$ such that $G \circ (\Phi_1)_a = (\Phi_2)_{\rho(a)} \circ G$ for all $a \in \mathbb{T}^n$, then there exists an isomorphism of dually flat spaces between (M_1, h_1, ∇_1) and (M_2, h_2, ∇_2).

Proof. Let $\mathbf{J}_1 : N_1 \to \mathbb{R}^n$ be a momentum map and $L = \rho_{*_e} : \mathbb{R}^n \to \mathbb{R}^n = \text{Lie}(\mathbb{T}^n)$. Because G is equivariant relative to ρ, the fundamental vector fields of $\xi \in \text{Lie}(\mathbb{T}^n)$ and $L(\xi)$ are related by

$$G_{*_p} \xi_{N_1}(p) = (L(\xi))_{N_2}(G(p)) \tag{6.2}$$

for all $p \in N_1$. From this, a straightforward verification shows that $\mathbf{J}_2 = (L^{-1})^* \circ \mathbf{J}_1 \circ G^{-1}$ is a momentum map with respect to Φ_2, where $(L^{-1})^*$ denotes the adjoint of L^{-1} with respect to the Euclidean scalar product (thus $\langle L^{-1}(x), y\rangle = \langle x, (L^{-1})^*(y)\rangle$ for all $x, y \in \mathbb{R}^n$). Let $(\mathbf{J}_1(N_1^\circ), k_1, \nabla^{k_1})$ and $(\mathbf{J}_2(N_2^\circ), k_2, \nabla^{k_2})$ be the corresponding canonical dually flat manifolds (see Definition 6.11). Given $p \in N_i$, $i = 1, 2$, the Kähler metric g_i induces an orthogonal decomposition $T_pN_i = T_p\mathcal{O}_i \oplus (T_p\mathcal{O}_i)^\perp$, where $\mathcal{O}_i \subset N_i$ is the $\mathbb{T}^n$-orbit of p. We shall denote by $u^\perp \in (T_p\mathcal{O}_i)^\perp$ the orthogonal projection of $u \in T_pN_i$ on $(T_p\mathcal{O}_i)^\perp$. We claim that:

(a) $(k_i)_{\mathbf{J}_i(p)}((\mathbf{J}_i)_{*_p} u, (\mathbf{J}_i)_{*_p} v) = (g_i)_p(u^\perp, v^\perp)$ for all $p \in N_i^\circ$ and all $u, v \in T_pN_i$.

(b) $\big(G_{*_p}(u)\big)^\perp = G_{*_p}(u^\perp)$ for all $p \in N_1^\circ$ and all $u \in T_pN_1$.

The fist item is a consequence of the fact that $\mathbf{J}_i : N_i^\circ \to \mathbf{J}_i(N_i^\circ)$ is a Riemannian submersion whose fibers are $\mathbb{T}^n$-orbits. To see (b), let $u \in T_pN_1^\circ$ be arbitrary. Since the tangent space of an orbit at a given point is spanned by the fundamental vector fields at that point, there exists $\xi \in \mathrm{Lie}(\mathbb{T}^n)$ such that $u = \xi_{N_1}(p) + u^\perp$. It follows from this and (6.2) that

$$G_{*_p}(u) = (L(\xi))_{N_2}(G(p)) + G_{*_p}(u^\perp).$$

Projecting onto $\big(T_{G(p)}\mathcal{O}\big)^\perp$, where $\mathcal{O}$ is the $\mathbb{T}^n$-orbit of $G(p)$, we get $(G_{*_p}(u))^\perp = \big(G_{*_p}(u^\perp)\big)^\perp$. Thus, it suffices to show that $\big(G_{*_p}(u^\perp)\big)^\perp = G_{*_p}(u^\perp)$, or equivalently, that $G_{*_p}(u^\perp)$ is orthogonal to $T_{G(p)}\mathcal{O}$. Since $\rho : \mathbb{T}^n \to \mathbb{T}^n$ is a Lie group isomorphism, $L = \rho_{*_e}$ is a linear bijection and hence $T_{G(p)}\mathcal{O}$ is spanned by elements of the form $(L(\xi))_{N_2}(G(p))$, where $\xi \in \mathbb{R}^n = \mathrm{Lie}(\mathbb{T}^n)$. Given $\xi \in \mathrm{Lie}(\mathbb{T}^n)$, we compute:

$$\begin{aligned}
&(g_2)_{G(p)}((L(\xi))_{N_2}(G(p)), G_{*_p}(u^\perp)) \\
&= (g_2)_{G(p)}(G_{*_p}\xi_{N_1}(G(p)), G_{*_p}(u^\perp)) \\
&= (G^*g_2)_p(\xi_{N_1}, u^\perp) = (g_1)_p(\xi_{N_1}, u^\perp) = 0,
\end{aligned}$$

where we have used (6.2). This shows that $G_{*_p}(u^\perp) \in \big(T_{G(p)}\mathcal{O}\big)^\perp$ and concludes the proof of the claim.

Next we prove that $f = (L^{-1})^* : \mathbf{J}_1(N_1^\circ) \to \mathbf{J}_2(N_2^\circ)$ is an isomorphism of dually flat spaces. The injectivity of f is a consequence of the injectivity of $(L^{-1})^*$. The fact that f is surjective is a consequence of the formula $\mathbf{J}_2 = (L^{-1})^* \circ \mathbf{J}_1 \circ G^{-1}$. Thus f is a bijection from $\mathbf{J}_1(N_1^\circ)$ to $\mathbf{J}_2(N_2^\circ)$. Let $p \in N_1^\circ$ and $u, v \in T_pN_1$. We compute:

$$\begin{aligned}
&(f^*k_2)_{\mathbf{J}_1(p)}\big((\mathbf{J}_1)_{*_p}(u), (\mathbf{J}_1)_{*_p}(v)\big) \\
&= (k_2)_{(L^{-1})^*(\mathbf{J}_1(p))}\big((L^{-1})^*(\mathbf{J}_1)_{*_p}(u), (L^{-1})^*(\mathbf{J}_1)_{*_p}(v)\big) \\
&= (k_2)_{(\mathbf{J}_2\circ G)(p)}\big((\mathbf{J}_2 \circ G)_{*_p}(u), (\mathbf{J}_2 \circ G)_{*_p}(v)\big) \\
&= (g_2)_{G(p)}\big(G_{*_p}(u^\perp), G_{*_p}(v^\perp)\big) \quad \text{(see (a) and (b))} \\
&= (G^*g_2)_p(u^\perp, v^\perp) = (g_1)_p(u^\perp, v^\perp) \\
&= (k_1)_{\mathbf{J}_1(p)}((\mathbf{J}_1)_{*_p}(u), (\mathbf{J}_1)_{*_p}(v)) \quad \text{(see (a))}
\end{aligned}$$

where we have used $\mathbf{J}_2 \circ G = (L^{-1})^* \circ \mathbf{J}_1$. This shows that $f^*k_2 = k_1$, that is, f is an isometry. Clearly, f is affine from $(\mathbf{J}(N_1^\circ), \nabla^{\mathrm{flat}})$ to $(\mathbf{J}(N_2^\circ), \nabla^{\mathrm{flat}})$, since it is the restriction of a linear map. By Lemma 2.15, it is also affine from $(\mathbf{J}(N_1^\circ), \nabla^{k_1})$ to $(\mathbf{J}(N_2^\circ), \nabla^{k_2})$. It follows that f is an isomorphism of dually flat spaces.

By Proposition 6.12, there are isomorphisms of dually flat spaces $\phi_1 : M_1 \to \mathbf{J}(N_1^\circ)$ and $\phi_2 : M_2 \to \mathbf{J}(N_2^\circ)$. Therefore $\phi_2^{-1} \circ f \circ \varphi_1$ is an isomorphism of dually flat spaces between (M_1, h_1, ∇_1) and (M_2, h_2, ∇_2). ☐

7 The canonical example (complex point of view)

In this section, we re-examine some well-known results of symplectic toric geometry (complex and symplectic point of views, Legendre transform) by using systematically the point of view of torification and the language of dually flat spaces.

We continue to use the notation $x = (x_1, ..., x_n)$ to denote the standard coordinates on $\mathbb{R}^n$. The flat connection on $\mathbb{R}^n$ is denoted by ∇^{flat} (see (2.3)).

The theorem below is the main result of this section.

Theorem 7.1. Let (N, g, J) be a connected Kähler manifold of real dimension $2n$ and let $\Phi : \mathbb{T}^n \times N \to N$ be an effective, Hamiltonian and holomorphic torus action. Suppose that for every $\xi \in \text{Lie}(\mathbb{T}^n)$, the vector field $-J\xi_N$ is complete, where ξ_N is the fundamental vector field of ξ; let $\varphi^\xi : \mathbb{R} \times N \to N$ be the corresponding flow, that is, $\varphi^\xi(0, q) = q$ and $\frac{d}{dt}\varphi^\xi(t, q) = -J\xi_N(\varphi^\xi(t, q))$ for all $(t, q) \in \mathbb{R} \times N$. Fix $p \in N^\circ$ and consider the correspondence h that associates to $x \in \mathbb{R}^n$ the bilinear form h_x on $T_x\mathbb{R}^n = \mathbb{R}^n$ defined by

$$h_x(u, v) = g_{\varphi^x(1,p)}(u_N, v_N), \qquad (x, u, v \in \mathbb{R}^n = \text{Lie}(\mathbb{T}^n)).$$

Then,

(1) $(\mathbb{R}^n, h, \nabla^{\text{flat}})$ is a dually flat manifold.

(2) $\Phi : \mathbb{T}^n \times N \to N$ is a torification of $(\mathbb{R}^n, h, \nabla^{\text{flat}})$.

(3) Given a momentum map $\mathbf{J} : N \to \mathbb{R}^n$, there are coordinates $y = (y_1, ..., y_n)$ on $\mathbb{R}^n$ such that

  (a) (x, y) is a global pair of dual coordinate systems on $(\mathbb{R}^n, h, \nabla^{\text{flat}})$ (in particular, y is affine with respect to the dual connection of ∇^{flat} with respect to h).

  (b) $-y$ is an isomorphism of dually flat spaces between $(\mathbb{R}^n, h, \nabla^{\text{flat}})$ and the canonical dually flat space $(\mathbf{J}(N^\circ), k, \nabla^k)$.

  If in addition there is no $C \in \mathbb{R}^n - \{0\}$ such that $\mathbf{J}(N^\circ) = \mathbf{J}(N^\circ) + C$ (for example if N is compact), then y satisfying (a) and (b) is unique.

The proof of Theorem 7.1 proceeds in a series of lemmas. We begin with a discussion of the holomorphic extension of a torus action. This is a standard construction, which, to our knowledge, was first discribed in [GS82b]. Our presentation follows closely [IK12].

Let $\Phi : \mathbb{T}^n \times N \to N$ be a Lie group action of the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ on a complex manifold N by holomorphic diffeomorphisms. We will identify the Lie algebra of the torus $\mathbb{T}^n$ with $\mathbb{R}^n$ via the derivative at $0 \in \mathbb{R}^n$ of the quotient map $\mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$, $t \mapsto [t]$. Under this identification, the exponential map $\exp : \text{Lie}(\mathbb{T}^n) \to \mathbb{T}^n$ is just the quotient map $\mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$. Given $\xi \in \mathbb{R}^n = \text{Lie}(\mathbb{T}^n)$, we will denote by $\xi_N \in \mathfrak{X}(N)$ the corresponding fundamental vector field on N. We will assume that $J\xi_N$ is complete for all $\xi \in \text{Lie}(\mathbb{T}^n)$, where $J : TN \to TN$ is the complex structure of N.

As a matter of notation, we will denote by φ_t^X the flow of a vector field X. Thus $\frac{d}{dt}\varphi_t^X(p) = X(\varphi_t^X(p))$. For later use, recall that if X and Y are complete commuting vector fields, then $\varphi_t^X = \varphi_1^{tX}$ and $\varphi_t^X \circ \varphi_t^Y = \varphi_t^{X+Y}$ for all $t \in \mathbb{R}$ (in particular, $X + Y$ is complete).

Let $\{e_1, ..., e_k\}$ be the canonical basis of $\mathbb{R}^n = \text{Lie}(\mathbb{T}^n)$. Define $T : \mathbb{C}^n \times N \to N$ by

$$T(z, p) = \left(\varphi_{x_1}^{-J(e_1)_N} \circ \cdots \circ \varphi_{x_n}^{-J(e_n)_N} \circ \varphi_{y_1}^{(e_1)_N} \circ \cdots \circ \varphi_{y_n}^{(e_n)_N}\right)(p), \tag{7.1}$$

where $z = (z_1, ..., z_n) \in \mathbb{C}^n$, $z_k = x_k + iy_k$, $x_k, y_k \in \mathbb{R}$, $k = 1, ..., n$ and $p \in N$.

Lemma 7.2.

(1) T is a Lie group action of $\mathbb{C}^n$ on N.

(2) $T(tz,p) = \varphi_t^{-Jx_N+y_N}(p) = \varphi_t^{-Jx_N}(\Phi([y],p))$ for all $t \in \mathbb{R}$, $z = x+iy \in \mathbb{R}^n + i\mathbb{R}^n$, $p \in N$.

(3) The derivative of T at $(z,p) \in \mathbb{C}^n \times N$ in the direction $(w,u) \in \mathbb{C}^n \times T_pN$ is given by

$$T_{*_{(z,p)}}(w,u) = (-Ja_N + b_N)\big(T(z,p)\big) + (T_z)_{*_p}u,$$

where $w = a+ib$, $a,b \in \mathbb{R}^n = \mathrm{Lie}(\mathbb{T}^n)$ and $T_z : N \to N$, $p \mapsto T(z,p)$.

(4) The map $T : \mathbb{C}^n \times N \to N$ is holomorphic.

Proof. (1) Obviously, $T(0,p) = p$ for every $p \in N$. To prove the formula $T(z,T(z',p)) = T(z+z',p)$, it suffices to show that all the flows in (7.1) commute, which is equivalent to show that for any pair $\xi,\eta \in \mathbb{R}^n = \mathrm{Lie}(\mathbb{T}^n)$,

$$[\xi_N,\eta_N] = [J\xi_N,\eta_N] = [J\xi_N,J\eta_N] = 0. \tag{7.2}$$

The vanishing of $[\xi_N,\eta_N]$ and $[J\xi_N,\eta_N]$ is a consequence of Lemma 7.3 (see below) together with the fact that $\mathbb{R}^n \to \mathfrak{X}(N)$, $\xi \mapsto \xi_N$, is an antihomomorphism of commutative Lie algebras. The vanishing of $[J\xi_N,J\eta_N]$ follows from the vanishing of the other two Lie brackets together with the vanishing of the Nijenhuis tensor:

$$0 = [\xi_N,\eta_N] + J[J\xi_N,\eta_N] + J[\xi_N,J\eta_N] - [J\xi_N,J\eta_N] = -[J\xi_N,J\eta_N].$$

It follows that T is a well defined Lie group action.

(2) This follows easily from the fact that all the flows in (7.1) commute (see above).

(3) By a direct calculation.

(4) First we show that for every $s \in \mathbb{R}$ and every $k = 1,...,n$, the diffeomorphism $\varphi_s^{-J(e_k)_N} : N \to N$ is holomorphic. In view of Lemma 7.3, this is equivalent to show that $[J(e_k)_N, JY] = J[J(e_k)_N, Y]$ for all vector fields Y on N. So let Y be an arbitrary vector field on N. Because the flow of $(e_k)_N$ consists of holomorphic transformations, $[(e_k)_N, JY] = J[(e_k)_N, Y]$. It follows from this and the vanishing of the Nijenhuis tensor that

$$\begin{aligned} 0 &= [J(e_k)_N, JY] - J[(e_k)_N, JY] - J[J(e_k)_N, Y] - [(e_k)_N, Y] \\ &= [J(e_k)_N, JY] + [(e_k)_N, Y] - J[J(e_k)_N, Y] - [(e_k)_N, Y] \\ &= [J(e_k)_N, JY] - J[J(e_k)_N, Y]. \end{aligned}$$

Thus $\varphi_s^{-J(e_k)_N}$ is holomorphic. Now, a straightforward computation using (3) and the fact that the flows $\varphi_{x_k}^{-J(e_k)_N}$ and $\varphi_{y_k}^{(e_k)_N}$ are all holomorphic transformations of N, shows that $T_{*(z,p)}(iw,Ju) = JT_{*_{(z,p)}}(w,u)$, which means that T is holomorphic. ∎

Lemma 7.3. Let X be a vector field on a complex manifold W. The following are equivalent.

(1) The flow of X consists of holomorphic transformations of W.

(2) $[X, JY] = J[X, Y]$ for all vector field Y on W.

Proof. See [Mor07], Lemma 2.7. □

Let $(\mathbb{C}^*)^n$ be the algebraic torus and let $\varepsilon : \mathbb{C}^n \to (\mathbb{C}^*)^n$ be the map defined by

$$\varepsilon(z) = (e^{2\pi z_1}, ..., e^{2\pi z_n}),$$

where $z = (z_1, ..., z_n) \in \mathbb{C}^n$. Then ε is a surjective Lie group homomorphism and a holomorphic covering map with Deck transformation group $i\mathbb{Z}^n$. If $\varepsilon(z) = \varepsilon(w)$ for some $z, w \in \mathbb{C}^n$, then there exists $k \in \mathbb{Z}^n$ such that $z = w + ik$, and so

$$T(z, p) = T(w + ik, p) = T(w, T(ik, p)) = T\big(w, \varphi_1^{-J(0)_N}(\Phi([k], p))\big) = T(w, p),$$

where we have used Proposition 7.2(2). Therefore there exists a unique map $\Phi^{\mathbb{C}} : (\mathbb{C}^*)^n \times N \to N$ such that

$$\Phi^{\mathbb{C}}(\varepsilon(z), p) = T(z, p)$$

for all $z \in \mathbb{C}^n$ and $p \in N$. It follows from the properties of ε and T that $\Phi^{\mathbb{C}} : (\mathbb{C}^*)^n \times N \to N$ is a Lie group action and a holomorphic map. Moreover, $\Phi^{\mathbb{C}}((e^{2i\pi t_1}, ..., e^{2i\pi t_n}), p) = \Phi^{\mathbb{C}}(\varepsilon(it), p) = \Phi([t], p)$ for all $t = (t_1, ..., t_n) \in \mathbb{R}^n$ and all $p \in N$. Therefore $\Phi^{\mathbb{C}}$ is an extension of Φ, provided $\mathbb{T}^n$ is identified with $\{(e^{i2\pi t_1}, ..., e^{i2\pi t_n}) \in \mathbb{C}^n \mid t_1, ..., t_n \in \mathbb{R}\} \subset \mathbb{C}^n$.

We now specialize to the case when N is a connected Kähler manifold of real dimension $2n$ and $\Phi : \mathbb{T}^n \times N \to N$ is effective, Hamiltonian and holomorphic. We continue to use the notation $\Phi^{\mathbb{C}}$ to denote the holomorphic extension of Φ.

Let N° be the dense connected open subset of N of points $p \in N$ where the action Φ is free.

Lemma 7.4. The following hold.

(1) The action Φ is free at $p \in N$ if and only if $\Phi^{\mathbb{C}}$ is free at p.

(2) N° is a $(\mathbb{C}^*)^n$-orbit.

(3) Given $p \in N^\circ$, the orbit map $\Phi^{\mathbb{C}}_p : (\mathbb{C}^*)^n \to N^\circ$ is a holomorphic diffeomorphism, which is equivariant in the sense that $\Phi^{\mathbb{C}}_p(zw) = (\Phi^{\mathbb{C}}_z \circ \Phi^{\mathbb{C}}_p)(w)$ for all $z, w \in (\mathbb{C}^*)^n$.

Proof. (1) Let $p \in N$ be arbitrary. Let $K_1 \subseteq \mathbb{T}^n$ and $K_2 \subseteq (\mathbb{C}^*)^n$ be the corresponding stabilizers. We must show that K_1 is trivial if and only if K_2 is trivial. Suppose K_2 trivial. Because $\Phi^{\mathbb{C}}$ is an extension of Φ, we have $K_1 = \mathbb{T}^n \cap K_2 \subset K_2 = \{e\}$ and hence K_1 is trivial. Conversely, suppose K_1 trivial. Let $\varepsilon(z) = \varepsilon(x + iy) \in K_2$ be arbitrary, where $x, y \in \mathbb{R}^n$. Consider the curve $\alpha : \mathbb{R} \to N$, $t \mapsto \Phi^{\mathbb{C}}(\varepsilon(tx), p) = T(tx, p)$. Note that $\alpha(0)$ and $\alpha(1)$ belong to the same $\mathbb{T}^n$-orbit. Indeed,

$$\alpha(0) = p = \Phi^{\mathbb{C}}_{\varepsilon(z)}(p) = T(z, p) = T(iy, T(x, p)) = \Phi_{[y]}(\alpha(1)),$$

where we have used Lemma 7.2(2). By Lemma 7.2(2), α is an integral curve of $-Jx_N$. Let ω be the symplectic form on N and let $\mathbf{J} : N \to \mathbb{R}^n$ be a momentum map for the action Φ. We denote by $\mathbf{J}^x : N \to \mathbb{R}$ the map defined by $\mathbf{J}^x(q) = \langle \mathbf{J}(q), x\rangle$, where $\langle\, ,\, \rangle$ is the Euclidean pairing on $\mathbb{R}^n$. By definition of the momentum map, the derivative of $\mathbf{J}^x$ along α is given by

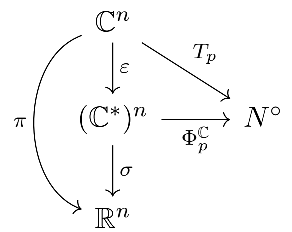
$$\frac{d}{dt}\mathbf{J}^x\big(\alpha(t)\big) = \omega\big(x_N, \dot{\alpha}(t)\big) = -\omega_{\alpha(t)}(x_N, Jx_N) = -g_{\alpha(t)}(x_N, x_N) \leq 0.$$

Thus $\mathbf{J}^x$ is nonincreasing along α. Since $\mathbf{J}^x$ is $\mathbb{T}^n$-invariant and $\alpha(0), \alpha(1)$ belong to the same $\mathbb{T}^n$-orbit, it follows that $\mathbf{J}^x \circ \alpha$ is constant on $[0,1]$. In particular, $0 = \frac{d}{dt}\big|_0 \mathbf{J}^x(\alpha(t)) = -g_p(x_N, x_N)$ and hence $x_N(p) = 0$. Therefore $x \in \mathrm{Lie}(K_1) = \{\xi \in \mathrm{Lie}(\mathbb{T}^n) \mid \xi_N(p) = 0\}$. Since K_1 is trivial, $x = 0$. It follows that $\varepsilon(z) = \varepsilon(iy) = (e^{2i\pi y_1}, ..., e^{2i\pi y_n}) \in \mathbb{T}^n$. Thus $\varepsilon(z) \in \mathbb{T}^n \cap K_2 = K_1 = \{e\}$. It follows that K_2 is trivial and concludes the proof of (1).

(2) Let $\mathcal{O}^{\mathbb{C}}$ be the $(\mathbb{C}^*)^n$-orbit of an arbitrary point $p \in N^\circ$. By (1), $\Phi^{\mathbb{C}}$ is free at p. Since $(\mathbb{C}^*)^n$ is commutative, the stabilizer subgroups associated to $\Phi^{\mathbb{C}}$ are constant along $(\mathbb{C}^*)^n$-orbits. Thus $\Phi^{\mathbb{C}}$ is free at every point $q \in \mathcal{O}^{\mathbb{C}}$. By (1), this implies $\mathcal{O}^{\mathbb{C}} \subseteq N^\circ$. To prove that $\mathcal{O}^{\mathbb{C}} = N^\circ$, it suffices to show that $\mathcal{O}^{\mathbb{C}}$ is open in N°, because then N° would be a disjoint union of open $(\mathbb{C}^*)^n$-orbits and N° is connected. To see that $\mathcal{O}^{\mathbb{C}}$ is open, recall that $\mathcal{O}^{\mathbb{C}}$ is the image of the orbit map $\Phi_p^{\mathbb{C}} : (\mathbb{C}^*)^n \to N$. Since $\Phi_p^{\mathbb{C}} \circ \varepsilon = T_p$ and ε is a submersion, the image of $(\Phi_p^{\mathbb{C}})_{*_e}$ coincides with the image of $(T_p)_{*_0}$. By Lemma 7.2(3), the image of $(T_p)_{*_0}$ is $T_p\mathcal{O} + JT_p\mathcal{O}$, where $\mathcal{O}$ is the $\mathbb{T}^n$-orbit of p. By Lemma 6.9, $T_p\mathcal{O} + JT_p\mathcal{O} = T_pN$. Thus $(T_p)_{*_0}$ is surjective. It follows that $\Phi_p^{\mathbb{C}}$ is a submersion at e. Because orbit maps have constant rank, $\Phi_p^{\mathbb{C}}$ is a submersion, which implies that its image is open. This shows (2).

(3) We already know that $\Phi_p^{\mathbb{C}} : (\mathbb{C}^*)^n \to N$ is an holomorphic map and a submersion (see above). Because $\dim((\mathbb{C}^*)^n) = \dim(N)$, $\Phi_p^{\mathbb{C}}$ is a local diffeomorphism. Since $\Phi^{\mathbb{C}}$ is free at p, it is also injective. Thus $\Phi_p^{\mathbb{C}}$ is an holomorphic diffeomorphism. The equivariance property comes from the fact the $\Phi_p^{\mathbb{C}}$ is an orbit map. $\square$

Let $p \in N^\circ$ be fixed. Consider the following commutative diagram:



- $\varepsilon(z_1, ..., z_n) = (e^{2\pi z_1}, ..., e^{2\pi z_n})$,
- $\sigma(z_1, ..., z_n) = \big(\frac{\ln(|z_1|)}{2\pi}, ..., \frac{\ln(|z_n|)}{2\pi}\big)$,
- $\pi(z_1, ..., z_n) = \big(\mathrm{Real}(z_1), ..., \mathrm{Real}(z_n)\big)$.

Let g be the Kähler metric on N. Because ε is a covering map and $\Phi_p^{\mathbb{C}}$ is a diffeomorphism, T_p is a covering map, and so the pullback $\widetilde{g} = (T_p)^*g$ is a metric on $\mathbb{C}^n$. Let J be the canonical complex structure on $\mathbb{C}^n$ (J = multiplication by i). Given $x \in \mathbb{R}^n$, let h_x be the bilinear form on $T_x\mathbb{R}^n = \mathbb{R}^n$ defined by $h_x(u, v) = g_{T_p(x)}(u_N, v_n)$. Note that $T_p(x) = \varphi^{-Jx_N}(1, p)$ by Lemma 7.2(2). Thus

$$h_x(u, v) = g_{\varphi^{-Jx_N}(1,p)}(u_N, v_N).$$

Lemma 7.5.

(a) $(\mathbb{C}^n, \widetilde{g}, J)$ is a Kähler manifold.

(b) Given $t \in \mathbb{R}^n$, the map $f_t : \mathbb{C}^n \to \mathbb{C}^n$, $z \mapsto z + it$, is an isometry, that is, $f_t^* \widetilde{g} = \widetilde{g}$.

(c) $\widetilde{g}_z(w, w') = \widetilde{g}_{\pi(z)}(w, w')$ for all $z, w, w' \in \mathbb{C}^n$,

(d) $\widetilde{g}_z(ia, b) = 0$ for all $z \in \mathbb{C}^n$ and all $a, b \in \mathbb{R}^n$.

(e) $\widetilde{g}_z(ia, ib) = \widetilde{g}_z(a, b)$ for all $z \in \mathbb{C}^n$ and all $a, b \in \mathbb{R}^n$.

(f) The correspondence $x \in \mathbb{R}^n$, $x \mapsto h_x$ is a Riemannian metric on $\mathbb{R}^n$.

(g) $\widetilde{g}_{x+iy}(a + ib, a' + ib') = h_x(a, a') + h_x(b, b')$ for all $x, y, a, a', b, b' \in \mathbb{R}^n$.

Proof. (a) By Lemma 7.2(4), $T_p : \mathbb{C}^n \to N$ is holomorphic, and since T_p is an isometric covering map, it is locally a holomorphic isometry. This forces $(\mathbb{C}^n, \widetilde{g}, J)$ to be a Kähler manifold.
(b) A simple diagram chase shows that $T_p \circ f_t = \Phi_{[t]} \circ T_p$. Since T_p is a Kähler covering, this implies that f_t is isometric if and only if $\Phi_{[t]}$ is isometric, which is the case.
(c) Follows from (b). To see (d), we use Lemma 7.2(3):

$$\widetilde{g}_z(ia, b) = g_{T_p(z)}((T_p)_* ia, (T_p)_* b) = g_{T_p(z)}(a_N, -Jb_N).$$

By Lemma 7.4(2), the point $T_p(z) = \Phi^{\mathbb{C}}(\varepsilon(z), p)$ belongs to N° and hence the decomposition $T_{T_p(z)}\mathcal{O} \oplus J(T_{T_p(z)}\mathcal{O})$ is orthogonal relative to the Kähler metric g, where $\mathcal{O}$ denotes the $\mathbb{T}^n$-orbit of $T_p(z)$ in N (see Lemma 6.9). It follows that $g_{T_p(z)}(a_N, -Jb_N) = 0$, and thus $\widetilde{g}_z(ia, b) = 0$. This shows (d). Analogously, one shows (e). (f) Let $j : \mathbb{R}^n \to \mathbb{C}^n$ be the canonical injection. Given $x, u, v \in \mathbb{R}^n$, we compute:

$$\begin{aligned} (j^*\widetilde{g})_x(u, v) &= \widetilde{g}_x(u, v) = \big((T_p)^* g\big)_x(u, v) = g_{T_p(x)}\big((T_p)_* u, (T_p)_* v\big) \\ &= g_{T_p(x)}(-Ju_N, -Jv_N) = g_{T_p(x)}(u_N, v_N) = h_x(u, v), \end{aligned}$$

where we have used Lemma 7.2(3). Thus $h = j^*\widetilde{g}$. This shows that h is a Riemannian metric on $\mathbb{R}^n$. (g) By a direct calculation using (c),(d) and (e). This concludes the proof. ☐

Proof of Theorem 7.1. (1) To prove that $(\mathbb{R}^n, h, \nabla^{\text{flat}})$ is dually flat, it suffices to show that the almost Hermitian manifold associated to $(h, \nabla^{\text{flat}})$ via Dombrowski's construction is Kähler (see Proposition 2.16). By Corollary 2.18 and Lemma 7.5(g), $(\mathbb{C}^n, \widetilde{g}, J)$ is the almost Hermitian manifold associated to $(h, \nabla^{\text{flat}})$ via Dombrowski's construction. By Lemma 7.5(1), it is a Kähler manifold.
(2) Apply Proposition 6.5 to the covering map $T_p : \mathbb{C}^n \to N^\circ$ and the parallel lattice $L \subset T\mathbb{R}^n$ generated by the vector fields $\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}$.
(3) Let $\mathbf{J} : N \to \mathbb{R}^n$ be a momentum map. Since $\mathbf{J} \circ \Phi_p^{\mathbb{C}}$ is $\mathbb{T}^n$-invariant and σ is a principal $\mathbb{T}^n$-bundle, there is a smooth map $f = (f^1, ..., f^n) : \mathbb{R}^n \to \mathbb{R}^n$ such that $f \circ \sigma = \mathbf{J} \circ \Phi_p^{\mathbb{C}}$. We claim that $\frac{\partial f^i}{\partial x_j}(x) = -h_{ij}(x) := -h_x(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ for every $x \in \mathbb{R}^n$ and every $i, j = 1, ..., n$. To see this, let

$x \in \mathbb{R}^n$ be arbitrary. Since the coordinate functions of the curve $t \mapsto x + te_j$ are real-valued, we have $x + te_j = \pi(x + te_j) = (\sigma \circ \varepsilon)(x + te_j)$ (see the diagram before Lemma 7.5), and so

$$\begin{aligned}\frac{\partial f^i}{\partial x_j}(x) &= \frac{d}{dt}\Big|_0 f^i(x + te_j) = \frac{d}{dt}\Big|_0 (f^i \circ \sigma \circ \varepsilon)(x + te_j) \\ &= \frac{d}{dt}\Big|_0 \big(\mathbf{J}^{e_i} \circ \Phi_p^{\mathbb{C}} \circ \varepsilon\big)(x + te_j)\big) = \frac{d}{dt}\Big|_0 (\mathbf{J}^{e_i} \circ T_p)(x + te_j),\end{aligned}$$

where we have used $\Phi_p^{\mathbb{C}} \circ \varepsilon = T_p$. Since $(T_p)_{*x}(e_j) = -J(e_j)_N(T_p(x))$ by Lemma 7.2(3), we get

$$\begin{aligned}\frac{\partial f^i}{\partial x_j}(x) &= -(\mathbf{J}^{e_i})_{*T_p(x)} J(e_j)_N = -\omega_{T_p(x)}\big((e_i)_N, J(e_j)_N\big) \\ &= -g_{T_p(x)}\big((e_i)_N, (e_j)_N\big) = -h_x(e_i, e_j) = -h_{ij}(x).\end{aligned}$$

This concludes the proof of the claim. Consider the 1-form θ on $\mathbb{R}^n$ defined by $\theta = \sum_{k=1}^n f^k dx_k$. The form θ is closed ($d\theta = 0$) if and only if $\frac{\partial f^i}{\partial x_j} - \frac{\partial f^j}{\partial x_i} = 0$ on $\mathbb{R}^n$ for all $i, j = 1, ..., n$, which is the case since $\frac{\partial f^j}{\partial x_i} = -h_{ij}$. Therefore there exists a smooth function $\psi : \mathbb{R}^n \to \mathbb{R}$ such that $\theta = -d\psi$. Clearly $\frac{\partial \psi}{\partial x_i} = -f^i$ and hence $\frac{\partial^2 \psi}{\partial x_i \partial x_j} = -\frac{\partial f^i}{\partial x_j} = h_{ij}$ for all $i, j = 1, ..., n$. Thus h is the Hessian of ψ. Let $y = \text{grad}(\psi) = \big(\frac{\partial \psi}{\partial x_1}, ..., \frac{\partial \psi}{\partial x_n}\big)$ be the gradient map of ψ. By Lemma 2.13, $y = (y_1, ..., y_n)$ are coordinates on $\mathbb{R}^n$ and (x, y) is a global pair of dual coordinate systems on $(\mathbb{R}^n, h, \nabla^{\text{flat}})$. This shows (a). Next we prove (b). To see that $-y : \mathbb{R}^n \to \mathbf{J}(N^\circ)$ is an isometry, let $\pi = \sigma \circ \varepsilon : \mathbb{C}^n \to \mathbb{R}^n$ be the canonical projection of the tangent bundle $T\mathbb{R}^n = \mathbb{C}^n$. Since $-y = f$ and $f \circ \sigma = \mathbf{J} \circ \Phi_p^{\mathbb{C}}$, we have

$$-y \circ \pi = f \circ \sigma \circ \varepsilon = \mathbf{J} \circ \Phi_p^{\mathbb{C}} \circ \varepsilon = \mathbf{J} \circ T_p$$

and hence

$$\begin{aligned}(-y)^* k = h \quad &\Leftrightarrow \quad \pi^*(-y)^* k = \pi^* h \\ &\Leftrightarrow \quad \big((-y) \circ \pi\big)^* k = \pi^* h \\ &\Leftrightarrow \quad \big(\mathbf{J} \circ T_p)^* k = \pi^* h.\end{aligned}$$

Thus it suffices to show that $\big(\mathbf{J} \circ T_p)^* k = \pi^* h$. Let $z = x + iy, w = a + ib, w' = a' + ib' \in \mathbb{C}^n$ be arbitrary, where $x, y, a, a', b, b' \in \mathbb{R}^n$. By Lemma 7.2(3) and the fact that $\mathbf{J}$ is constant along $\mathbb{T}^n$-orbits, we have $\mathbf{J}_*(T_p)_{*z} w = -\mathbf{J}_* J a_N(T_p(z))$, and so

$$\begin{aligned}\Big(\big(\mathbf{J} \circ T_p)^* k\Big)_z (w, w') &= k_{(\mathbf{J} \circ T_p)(z)}(\mathbf{J}_*(T_p)_{*z} w, \mathbf{J}_*(T_p)_{*z} w') \\ &= k_{(\mathbf{J} \circ T_p)(z)}(\mathbf{J}_* J a_N(T_p(z)), \mathbf{J}_* J a'_N(T_p(z))) \\ &= g_{T_p(z)}(a_N(T_p(z)), a'_N(T_p(z))).\end{aligned}$$

Using again Lemma 7.2(3), we see that $a_N(T_p(z)) = (T_p)_{*z} ia$, and so

$$\Big(\big(\mathbf{J} \circ T_p)^* k\Big)_z (w, w') = g_{T_p(z)}((T_p)_{*z} ia, (T_p)_{*z} ia') = \widetilde{g}_z(ia, ia'),$$

where $\widetilde{g} = (T_p)^* g$. It follows from this and Lemma 7.5(g), that

$$\Big((\mathbf{J} \circ T_p)^* k\Big)_z (w, w') = h_x(a, a'). \tag{7.3}$$

On the other hand, it follows from the linearity of the map $\pi : \mathbb{C}^n \to \mathbb{R}^n$, $(z_1, ..., z_n) \mapsto (\mathrm{Real}(z_1), ..., \mathrm{Real}(z_n))$ that

$$(\pi^* h)_z(w, w') = h_{\pi(z)}(\pi(w), \pi(w')) = h_x(a, a'). \tag{7.4}$$

Comparing (7.3) and (7.4) we get $\big(\mathbf{J} \circ T_p)^* k = \pi^* h$. This concludes the proof that $-y$ is an isometry. Obviously $-y$ is affine from $(\mathbb{R}^n, (\nabla^{\text{flat}})^*)$ to $(\mathbf{J}(N^\circ), \nabla^{\text{flat}})$ (since $y = (y_1, ..., y_n)$ are $(\nabla^{\text{flat}})^*$-affine coordinates). The fact that it is affine from $(\mathbb{R}^n, \nabla^{\text{flat}})$ to $(\mathbf{J}(N^\circ), \nabla^k)$ follows from Lemma 2.15. This concludes the proof of (b). Finally, suppose there is no $C \in \mathbb{R}^n - \{0\}$ such that $\mathbf{J}(N^\circ) = \mathbf{J}(N^\circ) + C$. Let $y' = (y'_1, ..., y'_n)$ be another system of coordinates on $\mathbb{R}^n$ satisfying (a) and (b). By Proposition 2.12, there is a smooth function $\psi' : \mathbb{R}^n \to \mathbb{R}$ such that $y' = \mathrm{grad}(\psi') = (\frac{\partial \psi'}{\partial x_1}, ..., \frac{\partial \psi'}{\partial x_n})$ and $\frac{\partial^2 \psi'}{\partial x_i \partial x_j} = h(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ for all $i, j = 1, ..., n$. Thus $\frac{\partial^2}{\partial x_i \partial x_j}(\psi - \psi') = 0$ for all $i, j = 1, ..., n$, where ψ is the function defined above, in the proof of (a). Thus there are real numbers $c_0, ..., c_n$ such that $\psi(x) = \psi'(x) + c_0 + c_1 x_1 + ... + c_n x_n$ for all $x = (x_1, ..., x_n) \in \mathbb{R}^n$. Taking the gradient yields $y = y' + C$, where $C = (c_1, ..., c_n)$. It follows from this and (b) that $\mathbf{J}(N^\circ) = \mathbf{J}(N^\circ) + C$. By our assumption, $C = 0$. Thus $y = y'$. □

Combining the results of Proposition 6.8 and Theorem 7.1, we obtain the following corollary.

Corollary 7.6. Let the hypotheses be as in Theorem 7.1. If $\mathbf{J} : N \to \mathbb{R}^n$ is a momentum map, then $\mathbf{J}(N^\circ)$ is an open convex subset of $\mathbb{R}^n$. If in addition $\mathbf{J}$ is proper, that is, if $\mathbf{J}^{-1}(K)$ is compact whenever $K \subseteq \mathbb{R}^n$ is compact, then $\mathbf{J}(N) \subset \mathbb{R}^n$ is convex.

Remark 7.7. Stronger and/or more general results on the convexity properties of the momentum map are available in the litterature [Ati82, BOR08, BOR09, CDM88, GS82a, HNP94, Kir84, Sja98]. With the notable exception of [Sja98], they are all based on either Morse theoretical techniques [Ati82, GS82a, Kir84] or some “Lokal-global-Prinzip” [BOR08, BOR09, CDM88, HNP94]. Our approach is interesting because it relies only on classical convexity theory and Legendre transform (see the appendix).

The following corollary is immediate.

Corollary 7.8. Let the hypotheses be as in Theorem 7.1. Then $\Phi : \mathbb{T}^n \times N \to N$ is a torification of $(\mathbf{J}(N^\circ), k, \nabla^k)$.

Proposition 7.9 (Complement of Theorem 7.1). Let the hypotheses be as in Theorem 7.1. Let $L \subset T\mathbb{R}^n$ be the parallel lattice generated by $X = (\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n})$. Under the identifications $(\mathbb{C}^*)^n = \mathbb{C}^n / i\mathbb{Z}^n = T\mathbb{R}^n / \Gamma(L)$, the triple $(L, X, \Phi_p^{\mathbb{C}})$ is a toric parametrization. The induced toric factorization is $\big(T_p, \sigma \circ (\Phi_p^{\mathbb{C}})^{-1}\big)$.

Proof. By inspection of the proof of Theorem 7.1. □

Given a smooth function $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$, we will denote by $\mathrm{grad}(f) : U \to \mathbb{R}^n$ its gradient map. Thus $\mathrm{grad}(f)(x) = \big(\frac{\partial f}{\partial x_1}(x), ..., \frac{\partial f}{\partial x_n}(x)\big)$, where $(x_1, ..., x_n)$ are standard coordinates on $\mathbb{R}^n$.

Definition 7.10. In the situation of Theorem 7.1, we say that a smooth function $\psi : \mathbb{R}^n \to \mathbb{R}$ is a *potential* if $\frac{\partial^2 \psi}{\partial x_i \partial x_j} = h_{ij} = h(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ for all $i, j = 1, ..., n$. We sall say that a potential $\psi : \mathbb{R}^n \to \mathbb{R}$ is *compatible* with the momentum map $\mathbf{J} : N \to \mathbb{R}^n$ if the gradient map $y = \mathrm{grad}(\psi)$ satisfies (a) and (b) of Theorem 7.1.

Note that (see Proposition 2.12 and Lemma 2.13):

- If $y = (y_1, ..., y_n)$ is a system of coordinates on $\mathbb{R}^n$ satisfying (a) of Theorem 7.1, then $y = \mathrm{grad}(\psi)$ for some smooth function $\psi : \mathbb{R}^n \to \mathbb{R}$ and ψ is automatically a potential.
- Given a momentum map $\mathbf{J} : N \to \mathbb{R}^n$, there exists a potential ψ compatible with $\mathbf{J}$.
- If $\psi : \mathbb{R}^n \to \mathbb{R}$ is a potential, then $(x, \mathrm{grad}(\psi))$ is a global pair of dual coordinate systems on $(\mathbb{R}^n, h, \nabla^{\mathrm{flat}})$, but $-y = -\mathrm{grad}(\psi)$ may fail to satisfy (b) of Theorem 7.1.

Lemma 7.11. Let the hypotheses be as in Theorem 7.1. Let $\mathbf{J} : N \to \mathbb{R}^n$ be a momentum map. Suppose there is no $C \in \mathbb{R}^n - \{0\}$ such that $\mathbf{J}(N^\circ) = \mathbf{J}(N^\circ) + C$. If $\psi : \mathbb{R}^n \to \mathbb{R}$ a potential satisfying $(-\mathrm{grad}(\psi))(\mathbb{R}^n) = \mathbf{J}(N^\circ)$, then ψ is compatible with $\mathbf{J}$.

Proof. This follows from Proposition 2.12 and the uniqueness part of Theorem 7.1(3). □

Proposition 7.12 (Holomorphic versus symplectic). Let the hypotheses be as in Theorem 7.1. Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a potential compatible with the momentum map $\mathbf{J} : N \to \mathbb{R}^n$. Define $\phi : \mathbf{J}(N^\circ) \to \mathbb{R}$ by

$$\phi(x) = -\big\langle x, (-\mathrm{grad}(\psi))^{-1}(x)\big\rangle - \psi\big((-\mathrm{grad}(\psi))^{-1}(x)\big), \tag{7.5}$$

where $\langle \, , \, \rangle$ is the Euclidean pairing. Then,

(1) $(\mathrm{grad}(\phi), x)$ is a global pair of dual coordinate systems on $(\mathbf{J}(N^\circ), k, \nabla^k)$.

(2) $-\mathrm{grad}(\phi)$ is an isomorphism of dually flat spaces from $(\mathbf{J}(N^\circ), k, \nabla^k)$ to $(\mathbb{R}^n, h, \nabla^{\mathrm{flat}})$, with inverse $(-\mathrm{grad}(\phi))^{-1} = -\mathrm{grad}(\psi)$.

(3) k is the Hessian of ϕ, that is, $\frac{\partial^2 \phi}{\partial x_i \partial x_j} = k(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ for all $i, j = 1, ..., n$.

Proof. First we prove that $-\mathrm{grad}(\phi) = (-\mathrm{grad}(\psi))^{-1}$. Let $b = -\mathrm{grad}(\psi)(a) \in \mathbf{J}(N^\circ)$ be arbitrary. By definition of ϕ, we have

$$\phi((-\mathrm{grad}(\psi))(a)) = -\big\langle(-\mathrm{grad}(\psi))(a), a\big\rangle - \psi(a).$$

Taking the derivative with respect to a in the direction $\frac{\partial}{\partial x_i}$, we get

$$\big\langle\big(\mathrm{grad}(\phi) \circ (-\mathrm{grad}(\psi))\big)(a), -H_i\big\rangle = \big\langle H_i, a\big\rangle + \Big\langle \mathrm{grad}(\psi), \frac{\partial}{\partial x_i}\Big\rangle - \frac{\partial \psi}{\partial x_i} = \big\langle H_i, a\big\rangle,$$

where $H_i = \frac{\partial}{\partial x_i}\mathrm{grad}(\psi) = (\frac{\partial^2 \psi}{\partial x_i \partial x_1}, ..., \frac{\partial^2 \psi}{\partial x_i \partial x_n}) = (h_{1i}, ..., h_{ni})$. Thus

$$\big\langle\big((-\mathrm{grad}(\phi)) \circ (-\mathrm{grad}(\psi))\big)(a) - a, H_i\big\rangle = 0$$

for all $a \in \mathbb{R}^n$ and all $i = 1, ..., n$. Since $[h_{ij}]$ is invertible, this implies $\big((-\mathrm{grad}(\phi))\circ(-\mathrm{grad}(\psi))\big)(a) = a$ for all $a \in \mathbb{R}^n$. Therefore $-\mathrm{grad}(\phi) = (-\mathrm{grad}(\psi))^{-1}$. Since $-\mathrm{grad}(\psi)$ is an isomorphism of dually flat spaces, so does $-\mathrm{grad}(\phi)$. This shows (2). Because (x, y) is a global pair of dual coordinates systems on $\mathbb{R}^n$ and $-y$ is an isomorphism of dually flat spaces, $(x \circ (-y)^{-1}, y \circ (-y)^{-1}) = (-\mathrm{grad}(\phi), -x)$ is a global pair of dual coordinate systems on $\mathbf{J}(N^\circ)$. Obviously, the same is true for $(\mathrm{grad}(\phi), x)$. This shows (1). Finally, (3) is a consequence of (1) and Proposition 2.12. □

Remark 7.13. The function ϕ defined in (7.5) satisfies $\phi(x) = \psi^*(-x)$ for all $x \in \mathbf{J}(N^\circ)$, where ψ^* is the Legendre transform of ψ (see Definition A.1).

Definition 7.14. In the situation of the preceding proposition, we say that ϕ is the *dual* of ψ (see (7.5)).

Example 7.15. Let $\Phi : \mathbb{T}^n \times \mathbb{C}^n \to \mathbb{C}^n$ be defined by $\Phi([t], z) = (e^{2i\pi t_1} z_1, ..., e^{2i\pi t_n} z_n)$. We endow $\mathbb{C}^n$ with the standard flat Kähler structure. Then Φ is isometric and Hamiltonian, with momentum map $\mathbf{J} : \mathbb{C}^n \to \mathbb{R}^n$ given by $\mathbf{J}(z) = -\pi(|z_1|^2, ..., |z_n|^2)$.

Given $\xi \in \mathbb{R}^n = \mathrm{Lie}(\mathbb{T}^n)$, the fundamental vector field $\xi_{\mathbb{C}^n}$ at $z \in \mathbb{C}^n$ is given by

$$\xi_{\mathbb{C}^n}(z) = \tfrac{d}{dt}\Big|_0 \Phi(\exp(t\xi), x) = 2i\pi(\xi_1 z_1, ..., \xi_n z_n).$$

Integral curves of $-i\xi_{\mathbb{C}^n}$ are of the form $\alpha(t) = (\lambda_1 e^{2\pi\xi_1 t}, ..., \lambda_k e^{2\pi\xi_n t})$, where $\lambda_1, ..., \lambda_n \in \mathbb{C}$. Therefore $-i\xi_{\mathbb{C}^n}$ is complete and its flow $\varphi^\xi(t, z)$ is given by $\varphi^\xi(t, z) = (z_1 e^{2\pi\xi_1 t}, ..., z_n e^{2\pi\xi_n t})$. Let $p = (1, 1, ..., 1) \in \mathbb{C}^n$ and let h be the metric on $\mathbb{R}^n$ defined by $h_x(u, v) = g_{\varphi^x(1,p)}(u_{\mathbb{C}^n}, v_{\mathbb{C}^n})$, $x, u, v \in \mathbb{R}^n$, and where g is the Euclidean metric on $\mathbb{C}^n$. Let $\langle z, w\rangle = \overline{z}_1 w_1 + ... + \overline{z}_n w_n$ be the Hermitian product on $\mathbb{C}^n$. We have $g_z(w, w') = \mathrm{Real}\langle w, w\rangle$ for all $z, w,' w \in \mathbb{C}^n$, and so

$$\begin{aligned} h_x(u, v) &= \mathrm{Real}\big\langle u_{\mathbb{C}^n}(\varphi^x(1, p)), v_{\mathbb{C}^n}(\varphi^x(1, p))\big\rangle \\ &= \mathrm{Real}\big\langle 2i\pi(u_1 e^{2\pi x_1}, ..., u_n e^{2i\pi x_n}), 2i\pi(v_1 e^{2\pi x_1}, ..., v_n e^{2i\pi x_n})\big\rangle \\ &= 4\pi^2\big(u_1 v_1 e^{4\pi x_1} + ... + u_n v_n e^{4\pi x_n}\big). \end{aligned}$$

Therefore the matrix representation of h at $x \in \mathbb{R}^n$ is given by

$$4\pi^2 \begin{bmatrix} e^{4\pi x_1} & \cdots & 0 \\ & \ddots & \\ 0 & & e^{4\pi x_n} \end{bmatrix}.$$

It is the Hessian of the function $\psi : \mathbb{R}^n \to \mathbb{R}$, $x \mapsto \frac{1}{4}(e^{4\pi x_1} + ... + e^{4\pi x_n})$. The image of $-\mathrm{grad}(\psi) = -\pi(e^{4\pi x_1}, ..., e^{4\pi x_n})$ is the "negative quadrant" $Q = \{(x_1, ..., x_n) \in \mathbb{R}^n \bigm| x_k < 0 \text{ for all } k = 1\}$. It is immediate to verify that $Q = \mathbf{J}((\mathbb{C}^n)^\circ)$ and hence the potential ψ is compatible with $\mathbf{J}$. Let $\phi : Q \to \mathbb{R}$ be the dual of ψ. A direct computation shows that

$$\phi(x_1, ..., x_n) = -\frac{1}{4\pi}\sum_{k=1}^{n}\Big[x_k \ln\Big(\frac{-x_k}{\pi}\Big) - x_k\Big].$$

Taking the Hessian yields the matrix representation of the Riemannian metric k induced on Q at $(x_1, ..., x_n) \in Q$:

$$-\frac{1}{4\pi}\begin{bmatrix} \frac{1}{x_1} & \cdots & 0 \\ & \ddots & \\ 0 & & \frac{1}{x_n} \end{bmatrix}.$$

A slightly more involved example is the complex projective space. Let $\mathbb{P}_n(c)$ denote the complex projective space of complex dimension n, endowed with the Fubini-Study metric g_c normalized in such a way that the holomorphic sectional curvature is c.

Let $\Phi : \mathbb{T}^n \times \mathbb{P}_n(c) \to \mathbb{P}_n(c)$ be the torus action defined by

$$\Phi([t], [z]) = \left[e^{2i\pi t_1} z_1, ..., e^{2i\pi t_n} z_n, z_{n+1}\right],$$

where $[t] = [t_1, ..., t_n] \in \mathbb{T}^n$ and $[z] = [z_1, ..., z_{n+1}] \in \mathbb{P}_n(c)$ (homogeneous coordinates). Then Φ is isometric and Hamiltonian, with momentum map $\mathbf{J}_c : \mathbb{P}_n(c) \to \mathbb{R}^n$ given by

$$\mathbf{J}_c([z]) = -\frac{4\pi}{c}\left(\frac{|z_1|^2}{\langle z, z\rangle}, ..., \frac{|z_n|^2}{\langle z, z\rangle}\right),$$

where $\langle z, w\rangle = \overline{z}_1 w_1 + ... + \overline{z}_{n+1} w_{n+1}$ is the standard Hermitian product on $\mathbb{C}^{n+1}$. The image of $\mathbb{P}_n(c)^\circ = \{[z_1, ..., z_{n+1}] \mid z_k \neq 0 \ \forall k = 1, ..., n+1\}$ under $\mathbf{J}_c$ is $-\frac{4\pi}{c} S_n$, where $S_n \subset \mathbb{R}^n$ is the set $\{(x_1, ..., x_n) \in \mathbb{R}^n \mid x_k > 0 \text{ for all } k \text{ and } \sum_{k=1}^n x_k < 1\}$.

Proposition 7.16. Let $\Phi : \mathbb{T}^n \times \mathbb{P}_n(c) \to \mathbb{P}_n(c)$ and $\mathbf{J}_c : \mathbb{P}_n(c) \to \mathbb{R}^n$ be as defined above. Let h_c be the Riemannian metric on $\mathbb{R}^n$ associated to $p = [1, ..., 1] \in \mathbb{P}_n(c)^\circ$. Then the function $\psi_c : \mathbb{R}^n \to \mathbb{R}$, defined by

$$\psi_c(x) = \frac{1}{c}\ln\left(1 + e^{4\pi x_1} + ... + e^{4\pi x_n}\right),$$

is a potential compatible with $\mathbf{J}_c$. Its dual $\phi_c : \mathbf{J}(\mathbb{P}_n(c)^\circ) \to \mathbb{R}$ is given by:

$$\phi_c(x) = -\frac{1}{4\pi}\sum_{k=1}^n x_k \ln(-x_k) + \frac{1}{4\pi}\left(\frac{4\pi}{c} + \sum_{k=1}^n x_k\right)\ln\left(\frac{4\pi}{c} + \sum_{k=1}^n x_k\right) - \frac{1}{c}\ln\left(\frac{4\pi}{c}\right).$$

Proof. Let Φ' denote the action of $\mathbb{T}^n$ on $\mathbb{C}^{n+1} - \{0\}$ defined by $\Phi'([t], z) = (e^{2i\pi t_1} z_1, ..., e^{2i\pi t_n} z_n, z_{n+1})$. Let $f : \mathbb{C}^{n+1} - \{0\} \to \mathbb{P}_n(c)$, $z \mapsto [z]$. Because f is $\mathbb{T}^n$-equivariant and holomorphic, we have

- $\xi_{\mathbb{P}_n(c)}([z]) = f_{*_z}\xi_{\mathbb{C}^{n+1}-\{0\}}(z)$ and
- $J\xi_{\mathbb{P}_n(c)}([z]) = f_{*_z} i\xi_{\mathbb{C}^{n+1}-\{0\}}(z)$

for all $\xi \in \mathbb{R}^n = \mathrm{Lie}(\mathbb{T}^n)$ and all $z \in \mathbb{C}^{n+1} - \{0\}$, where J denotes the complex structure on $\mathbb{P}_n(c)$. Clearly

$$\xi_{\mathbb{C}^{n+1}-\{0\}}(z) = \frac{d}{dt}\bigg|_0 \Phi'([t\xi], z) = 2i\pi(\xi_1 z_1, ..., \xi_n z_n, 0),$$

and so $-J\xi_{\mathbb{P}_n(c)}([z]) = 2\pi f_{*_z}(\xi_1 z_1, ..., \xi_n z_n, 0)$. It is then easy to verify that

$$\varphi_t^{-J\xi_{\mathbb{P}_n(c)}}([z]) = \left[e^{2\pi t\xi_1} z_1, ..., e^{2\pi t\xi_n} z_n, z_{n+1}\right]$$

is the flow of $-J\xi_{\mathbb{P}_n(c)}$.

Let h_c be the Riemannian metric on $\mathbb{R}^n$ associated to $p = [1, ..., 1] \in \mathbb{P}_n(c)^\circ$. Thus $(h_c)_x(u, v) = (g_c)_{\varphi_1^{-Jx_{\mathbb{P}_n(c)}}(p)}(u_{\mathbb{P}_n(c)}, v_{\mathbb{P}_n(c)})$, where $x, u, v \in \mathbb{R}^n = \text{Lie}(\mathbb{T}^n)$. To compute the matrix representation of h_c, we use the following facts:

(1) The Hopf fibration $\pi_H = f|_{S^{2n+1}} : S^{2n+1} \to \mathbb{P}_n(4)$ is a Riemannian submersion and $g_c = \frac{4}{c} g_4$.

(2) $u_{\mathbb{C}^{n+1}-\{0\}}$ is tangent to S^{2n+1} and for every $z \in S^{2n+1}$, $(\pi_H)_{*_z} u_{\mathbb{C}^{n+1}-\{0\}}(z) = u_{\mathbb{P}_n(c)}([z])$.

(3) Given $z \in S^{2n+1}$, $T_z S^{2n+1} = \mathbb{R}iz \oplus z^\perp$, where $z^\perp = \{w \in \mathbb{C}^{n+1} \,|\, \langle w, z\rangle = 0\}$. In this decomposition, $\mathbb{R}iz$ is the tangent space of the fiber $\pi_H^{-1}(\pi_H(z))$ and $z^\perp$ is its orthogonal complement in $T_z S^{2n+1}$.

Given $w \in T_z S^{2n+1}$, we will denote by $w^\perp$ the unique element in $z^\perp$ such that $w - w^\perp \in \mathbb{R}iz$. Clearly, $w^\perp = w + \langle w, z\rangle z$. It follows from the facts above that

$$(g_c)_{[z]}(u_{\mathbb{P}_n(c)}, v_{\mathbb{P}_n(c)}) = \frac{4}{c}\,\text{Real}\Big\langle \Big(u_{\mathbb{C}^{n+1}-\{0\}}(z)\Big)^\perp, \Big(v_{\mathbb{C}^{n+1}-\{0\}}(z)\Big)^\perp \Big\rangle, \tag{7.6}$$

where $z \in S^{2n+1}$ and $u, v \in \mathbb{R}^n = \text{Lie}(\mathbb{T}^n)$. A direct calculation using (7.6) then shows that

$$(g_c)_{[z]}(u_{\mathbb{P}_n(c)}, v_{\mathbb{P}_n(c)}) = \frac{(4\pi)^2}{c}\left[\sum_{k=1}^n u_k v_k |z_k|^2 - \Big(\sum_{a=1}^n u_a|z_a|^2\Big)\Big(\sum_{b=1}^n v_b |z_b|^2\Big)\right],$$

where $z \in S^{2n+1}$. Taking $[z] = \varphi_1^{-Jx_{\mathbb{P}_n(c)}}(p) = [e^{2\pi x_1}, ..., e^{2\pi x_n}, 1]$ and normalizing appropriately, one finds the matrix representation $[(h_c)_{ij}]$ of h_c at $x \in \mathbb{R}^n$:

$$(h_c)_{ij} = \frac{(4\pi)^2}{c}\left(\frac{\delta_{ij} e^{4\pi x_i}}{1 + e^{4\pi x_1} + ... + e^{4\pi x_n}} - \frac{e^{4\pi(x_i + x_j)}}{(1 + e^{4\pi x_1} + ... + e^{4\pi x_n})^2}\right).$$

It is the Hessian of the function $\psi_c(x) = \frac{1}{c}\ln\left(1 + e^{4\pi x_1} + ... + e^{4\pi x_n}\right)$. A simple verification shows that the image of $-\text{grad}(\psi_c)$ is $-\frac{4\pi}{c} S_n = \mathbf{J}_c(\mathbb{P}_n(c)^\circ)$ and hence ψ_c is compatible with $\mathbf{J}_c$ (see Lemma 7.11). The dual of ψ_c is obtained by a direct calculation. □

8 Examples from Information Theory

For an introduction to information geometry, see for example [AJLS17, AN00, MR93].

Definition 8.1. A *statistical manifold* is a pair (S, j), where S is a manifold and where j is an injective map from S to the space of all probability density functions p defined on a fixed measure space (Ω, dx):

$$j : S \hookrightarrow \Big\{p : \Omega \to \mathbb{R} \;\Big|\; p \text{ is measurable, } p \geq 0 \text{ and } \int_\Omega p(x)dx = 1\Big\}.$$

If $\xi = (\xi_1, ..., \xi_n)$ is a coordinate system on a statistical manifold S, then we shall indistinctly write $p(x; \xi)$ or $p_\xi(x)$ for the probability density function determined by ξ.

Given a "reasonable" statistical manifold S, it is possible to define a metric h_F and a family of connections $\nabla^{(\alpha)}$ on S ($\alpha \in \mathbb{R}$) in the following way: for a chart $\xi = (\xi_1, ..., \xi_n)$ of S, define

$$\begin{aligned} \big(h_F\big)_\xi\big(\partial_i, \partial_j\big) &:= \mathbb{E}_{p_\xi}\big(\partial_i \ln\big(p_\xi\big)\cdot \partial_j \ln\big(p_\xi\big)\big), \\ \Gamma^{(\alpha)}_{ij,k}\big(\xi\big) &:= \mathbb{E}_{p_\xi}\big[\big(\partial_i\partial_j \ln\big(p_\xi\big) + \tfrac{1-\alpha}{2}\partial_i \ln\big(p_\xi\big)\cdot \partial_j \ln\big(p_\xi\big)\big), \partial_k \ln\big(p_\xi\big)\big], \end{aligned}$$

where $\mathbb{E}_{p_\xi}$ denotes the mean, or expectation, with respect to the probability $p_\xi dx$, and where ∂_i is a shorthand for $\frac{\partial}{\partial \xi_i}$. It can be shown that if the above expressions are defined and smooth for every chart of S, then h_F is a well defined metric on S called the *Fisher metric*, and that the $\Gamma^{(\alpha)}_{ij,k}$'s define a connection $\nabla^{(\alpha)}$ via the formula $\Gamma^{(\alpha)}_{ij,k}(\xi) = (h_F)_\xi(\nabla^{(\alpha)}_{\partial_i}\partial_i, \partial_j)$, which is called the *$\alpha$-connection*.

Among the α-connections, the (± 1)-connections are particularly important; the 1-connection is usually referred to as the *exponential connection*, also denoted by $\nabla^{(e)}$, while the (-1)-connection is referred to as the *mixture connection*, denoted by $\nabla^{(m)}$.

In this paper, we will only consider statistical manifolds S for which the Fisher metric and α-connections are well defined.

Proposition 8.2. Let S be a statistical manifold. Then, $(h_F, \nabla^{(\alpha)}, \nabla^{(-\alpha)})$ is a dualistic structure on S. In particular, $\nabla^{(-\alpha)}$ is the dual connection of $\nabla^{(\alpha)}$.

Proof. See [AN00]. □

We now recall the definition of an exponential family.

Definition 8.3. An *exponential family* $\mathcal{E}$ on a measure space (Ω, dx) is a set of probability density functions $p(x; \theta)$ of the form

$$p(x; \theta) = \exp\bigg\{C(x) + \sum_{i=1}^{n} \theta_i F_i(x) - \psi(\theta)\bigg\},$$

where $C, F_1 ..., F_n$ are measurable functions on Ω, $\theta = (\theta_1, ..., \theta_n)$ is a vector varying in an open subset Θ of $\mathbb{R}^n$ and where ψ is a function defined on Θ.

In the above definition it is assumed that the family of functions $\{1, F_1, ..., F_n\}$ is linearly independent, so that the map $p(x, \theta) \mapsto \theta$ becomes a bijection, hence defining a global chart for $\mathcal{E}$. The parameters $\theta_1, ..., \theta_n$ are called the *natural* or *canonical parameters* of the exponential family $\mathcal{E}$. The function ψ is called *cumulant generating function*.

Example 8.4 (Poisson distribution). A Poisson distribution is a distribution over $\Omega = \mathbb{N} = \{0, 1, ...\}$ of the form

$$p(k; \lambda) = e^{-\lambda}\frac{\lambda^k}{k!},$$

where $k \in \mathbb{N}$ and $\lambda > 0$. Let $\mathscr{P}$ denote the set of all Poisson distributions $p(\,.\,; \lambda)$, $\lambda > 0$. The set $\mathscr{P}$ is an exponential family, because $p(k, \lambda) = \exp\big(C(k) + F(k)\theta - \psi(\theta)\big)$, where

$$C(k) = -\ln(k!), \quad F(k) = k, \quad \theta = \ln(\lambda), \quad \psi(\theta) = \lambda = e^\theta.$$

Example 8.5 (**Categorical distribution**). Given a finite set $\Omega = \{x_1, ..., x_n\}$, define

$$\mathcal{P}_n^\times = \left\{ p : \Omega \to \mathbb{R} \,\middle|\, p(x) > 0 \text{ for all } x \in \Omega \text{ and } \sum_{k=1}^{n} p(x_k) = 1 \right\}.$$

Elements of $\mathcal{P}_n^\times$ can be parametrized as follows: $p(x;\theta) = \exp\big\{ \sum_{i=1}^{n-1} \theta_i F_i(x) - \psi(\theta) \big\}$, where

$$\theta = (\theta_1, ..., \theta_{n-1}) \in \mathbb{R}^{n-1}, \quad F_i(x_j) = \delta_{ij} \quad \text{(Kronecker delta)},$$
$$\psi(\theta) = \ln\left(1 + \sum_{i=1}^{n-1} e^{\theta_i}\right).$$

Therefore $\mathcal{P}_n^\times$ is an exponential family of dimension $n-1$.

Example 8.6 (**Binomial distribution**). The set of binomial distributions defined over $\Omega := \{0, ..., n\}$,

$$p(k) = \binom{n}{k} q^k \big(1-q\big)^{n-k}, \quad (k \in \Omega,\ q \in (0,1)),$$

where $\binom{n}{k} = \frac{n!}{(n-k)!k!}$, is a 1-dimensional statistical manifold, denoted by $\mathcal{B}(n)$, parametrized by $q \in \big(0,1\big)$. It is an exponential family, because $p(k) = \exp\big\{C(k) + \theta F(k) - \psi(\theta)\big\}$, where

$$\theta = \ln\Big(\frac{q}{1-q}\Big), \quad C\big(k\big) = \ln\binom{n}{k}, \quad F(k) = k,$$
$$\psi(\theta) = n\ln\big(1 + e^\theta\big).$$

Example 8.7 (**Multinomial distribution**). Let $A_{m,n} = \{(x_1, ..., x_m) \in \mathbb{R}^m \mid x_i \geq 0 \text{ for all}\, i = 1, ..., m \text{ and } \sum_{i=1}^m x_i = n\}$. Let $\Omega_{m,n} = A_{m,n} \cap \mathbb{Z}^m$. Given $k \in \Omega_{m,n}$ and $\pi \in A_{m,1}$, let

$$p(k;\pi) = \frac{n!}{k_1!...k_m!} \pi_1^{k_1} ... \pi_m^{k_m},$$

where $k = (k_1, ..., k_m)$ and $\pi = (\pi_1, ..., \pi_m)$. Let $\mathcal{M}(m,n)$ be the set of all maps $\Omega_{m,n} \to \mathbb{R}, k \mapsto p(k,\pi)$, where $\pi \in A_{m,1}$. Each element in $\mathcal{M}(m,n)$ is called a *multinomial distribution.* They form an exponential familly, because $p(k,\pi) = \exp(C(k) + \sum_{i=1}^{m-1} F_i(k)\theta_i - \psi(\theta))$, where

$$\theta_i = \ln\Big(\frac{\pi_i}{1 - \sum_{j=1}^{m-1} \pi_j}\Big), \quad C\big(k\big) = \ln\left(\frac{n!}{k_1!...k_m!}\right), \quad F_i(k) = k_i, \qquad i = 1, ..., m-1,$$
$$\psi(\theta) = n\ln\big(1 + e^{\theta_1} + ... + e^{\theta_{m-1}}\big).$$

Example 8.8 (**Negative Binomial distribution**). Let $\Omega = \mathbb{N} = \{0, 1, ...\}$ and $r \in \Omega$, $r \geq 1$. Let $\mathcal{NB}(r)$ denote the set of functions $p : \Omega \to \mathbb{R}$ of the form

$$p(k;q) = \binom{k+r-1}{r-1}(1-q)^k q^r, \qquad (k = 0, 1, 2, ...)$$

where $q \in (0,1)$ and $\binom{k+r-1}{r-1} = \frac{(k+r-1)!}{k!(r-1)!}$. Using the fact that $\frac{1}{(1-t)^r} = \sum_{k\geq 0}\binom{k+r-1}{r-1}t^k$ for $|t| < 1$, one sees that $\sum_{k\geq 0} p(k;q) = 1$. Each element of $\mathcal{NB}(r)$ is a called a *negative Binomial distribution.* The set $\mathcal{NB}(r)$ is an exponential family, because $p(k;q) = p(k;\theta) = \exp\big\{C(k) + \theta F(k) - \psi(\theta)\big\}$, where

$$\theta = \ln(1-q) \in (-\infty, 0), \quad C\big(k\big) = \ln\begin{pmatrix} k+r-1 \\ r-1 \end{pmatrix}, \quad F(k) = k,$$
$$\psi(\theta) = -r\ln\big(1-e^{\theta}\big).$$

Proposition 8.9. Let $\mathcal{E}$ be an exponential family such as in Definition 8.3. Then $(\mathcal{E}, h_F, \nabla^{(e)})$ is a dually flat manifold and the natural parameters $\theta = (\theta_1, ..., \theta_n)$ form a global $\nabla^{(e)}$-affine coordinate system on $\mathcal{E}$.

Proof. See [AN00]. □

Given a smooth function $\psi : \mathbb{R}^n \to \mathbb{R}$, we will use the notation $\mathrm{Hess}(\psi)$ to denote the Hessian $\big[\frac{\partial^2\psi}{\partial x_i \partial x_j}\big]$ of ψ. When $\mathrm{Hess}(\psi)$ is positive definite at each point of $\mathbb{R}^n$, we regard $\mathrm{Hess}(\psi)$ as a Riemannian metric on $\mathbb{R}^n$.

Under mild assumptions (that we will always assume in this paper), it can be shown that the Hessian of the cumulent generating function $\psi : \Theta \to \mathbb{R}$ of an exponential family $\mathcal{E}$ is the coordinate expression for the Fisher metric h_F in the natural parameters θ_i (see [AN00]). Therefore the natural parameters form an isomorphism of dually flat spaces:

$$(\mathcal{E}, h_F, \nabla^{(e)}) \overset{(\theta_1,...,\theta_n)}{\longrightarrow} (\Theta, \mathrm{Hess}(\psi), \nabla^{\mathrm{flat}}),$$

where ∇^{flat} is the canonical flat connection on $\mathbb{R}^n$.

Lemma 8.10. Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a smooth function whose Hessian is positive definite at each point of $\mathbb{R}^n$ and let $f : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible affine map (thus $f(x) = Ax + B$, where A is an invertible $n \times n$ matrix and $B \in \mathbb{R}^n$). Then

$$f^*\mathrm{Hess}(\psi) = \mathrm{Hess}(\psi \circ f).$$

Proof. By a direct calculation. □

Let $\mathbb{P}_n$ and $\mathbb{D}$ denote the complex projective space of complex dimension n and unit disk in $\mathbb{C}$, respectively (both regarded as complex manifolds). We denote by

- $\mathbb{P}_n(c)$ the complex projective space endowed with the Fubini-Study metric normalized in such a way that the holomorphic sectional curvature is $c > 0$.
- $\mathbb{D}(c)$ the disk $\mathbb{D}$ endowed with the Hyperbolic metric $ds^2 = -\frac{4}{c}\frac{dx^2+dy^2}{(1-x^2-y^2)^2}$ of constant holomorphic sectional curvature $c < 0$.

Let Φ_n be the action of $\mathbb{T}^n$ on $\mathbb{P}_n$ defined by

$$\Phi_n([t],[z]) = [e^{2i\pi t_1}z_1, ..., e^{2i\pi t_n}z_n, z_{n+1}].$$

Proposition 8.11.

(1) $\mathbb{T}^1 \times \mathbb{C} \to \mathbb{C}$, $([t], z) \mapsto e^{2i\pi t}z$ is a torification of $\mathscr{P}$.

(2) $\Phi_n : \mathbb{T}^n \times \mathbb{P}_n(1) \to \mathbb{P}_n(1)$ is a torification of $\mathcal{P}_{n+1}^{\times}$.

(3) $\Phi_1 : \mathbb{T}^1 \times \mathbb{P}_1(\frac{1}{n}) \to \mathbb{P}_1(\frac{1}{n})$ is a torification of $\mathcal{B}(n)$.

(4) $\Phi_m : \mathbb{T}^m \times \mathbb{P}_m(\frac{1}{n}) \to \mathbb{P}_m(\frac{1}{n})$ is a torification of $\mathcal{M}(m+1, n)$.

(5) $\mathbb{T}^1 \times \mathbb{D}(-\frac{1}{r}) \to \mathbb{D}(-\frac{1}{r})$, $([t], z) \mapsto e^{2i\pi t}z$ is a torification of $\mathcal{NB}(r)$.

Proof. (1) Let $\psi_1, \psi_2 : \mathbb{R} \to \mathbb{R}$ be defined by $\psi_1 = \frac{1}{4}e^{4\pi x}$ and $\psi_2(x) = e^x$, respectively. Then $\mathbb{T}^1 \times \mathbb{C} \to \mathbb{C}$ is a torification of $(\mathbb{R}, \mathrm{Hess}(\psi_1), \nabla^{\mathrm{flat}})$ (see Example 7.15) and $(\mathbb{R}, \mathrm{Hess}(\psi_2), \nabla^{\mathrm{flat}})$ is isomorphic to $(\mathscr{P}, h_F, \nabla^{(e)})$ (see Example 8.4). Let $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto 4\pi x - \ln(4)$. Since $\psi_1 = \psi_2 \circ f$, Lemma 8.10 implies that $f^*\mathrm{Hess}(\psi_2) = \mathrm{Hess}(\psi_1)$. It follows that f can be regarded as an isomorphism of dually flat spaces from $(\mathbb{R}, \mathrm{Hess}(\psi_1), \nabla^{\mathrm{flat}})$ to $(\mathscr{P}, h_F, \nabla^{(e)})$. By Proposition 6.7, $\mathbb{T}^1 \times \mathbb{C} \to \mathbb{C}$ is a torification of $(\mathscr{P}, h_F, \nabla^{(e)})$. This concludes the proof of (1).
(2)–(4) The proof is entirely analogous: for each exponential family, just compare the cumulant generating function to the potential ψ_c described in Proposition 7.16, and use Lemma 8.10.
(5) Apply Proposition 6.5 to the Kähler covering map $T\mathcal{NB}(r) \cong \{z \in \mathbb{C} \mid \mathrm{Real}(z) < 0\} \to \mathbb{D}(-\frac{1}{r})^{\circ}$, $z \mapsto e^{z/2}$. □

In what follows, we will identify the tangent bundle of $\mathcal{E} = \mathscr{P}, \mathcal{P}_{n+1}^{\times}, \mathcal{B}(n), \mathcal{M}(m+1, n), \mathcal{NB}(r)$ with $\mathbb{C}^{\dim(\mathcal{E})}$ via the map $T\mathcal{E} \to \mathbb{C}^{\dim(\mathcal{E})}$, $\sum \dot{\theta}_k \frac{\partial}{\partial \theta_k}\big|_{p(.,\theta)} \mapsto (z_1, ..., z_{\dim(\mathcal{E})})$, where the θ_k's are natural parameters and $z_k = \theta_k + i\dot{\theta}_k$.

Proposition 8.12 (Complement of Proposition 8.11). In each case below, the pair (τ, κ) is a toric factorization of the indicated exponential family $\mathcal{E}$.

1. $\mathcal{E} = \mathscr{P}$, $\quad \mathbb{C} = T\mathscr{P} \xrightarrow{\tau} \mathbb{C}^* \xrightarrow{\kappa} \mathscr{P}$,

$$\tau(z) = 2e^{z/2}, \qquad \kappa(z)(k) = e^{-|z|^2/4}\frac{1}{k!}\left(\frac{|z|^2}{4}\right)^k, \quad k = 0, 1, 2...$$

2. $\mathcal{E} = \mathcal{P}_{n+1}^{\times}$, $\quad \mathbb{C}^n = T\mathcal{P}_{n+1}^{\times} \xrightarrow{\tau} \mathbb{P}_n(1)^{\circ} \xrightarrow{\kappa} \mathcal{P}_{n+1}^{\times}$,

$$\tau(z) = \left[e^{z_1/2}, ..., e^{z_n/2}, 1\right], \qquad \kappa([z])(x_k) = \frac{|z_k|^2}{|z_1|^2 + ... + |z_{n+1}|^2}, \quad k = 1, 2..., n+1.$$

3. $\mathcal{E} = \mathcal{B}(n)$, $\quad \mathbb{C} = T\mathcal{B}(n) \xrightarrow{\tau} \mathbb{P}_1(\frac{1}{n})^{\circ} \xrightarrow{\kappa} \mathcal{B}(n)$,

$$\tau(z) = \left[e^{z/2}, 1\right], \qquad \kappa([z_1, z_2])(k) = \binom{n}{k}\frac{|z_1^k z_2^{n-k}|^2}{(|z_1|^2 + |z_2|^2)^n}, \quad k = 0, 1, 2..., n.$$

4. $\mathcal{E} = \mathcal{M}(m+1,n)$, $\quad \mathbb{C}^m = T\mathcal{M}(m+1,n) \xrightarrow{\tau} \mathbb{P}_m(\frac{1}{n})^\circ \xrightarrow{\kappa} \mathcal{M}(m+1,n)$,

$$\tau(z) = \left[e^{z_1/2}, ..., e^{z_m/2}, 1\right], \qquad \kappa([z])(k) = \begin{cases} \dfrac{n!}{k_1!...k_{m+1}!} \dfrac{|z_1^{k_1} ... z_{m+1}^{k_{m+1}}|^2}{(|z_1|^2 + ... + |z_{m+1}|^2)^n}, \\ k = (k_1, ..., k_{m+1}) \in \mathbb{N}^{m+1}, \\ k_1 + ... + k_{m+1} = n. \end{cases}$$

5. $\mathcal{E} = \mathcal{NB}(r)$, $\quad \mathbb{C} = T\mathcal{NB}(r) \xrightarrow{\tau} \mathbb{D}(-\frac{1}{r})^\circ \xrightarrow{\kappa} \mathcal{NB}(r)$,

$$\tau(z) = e^{z/2}, \qquad \kappa(z)(k) = \binom{k+r-1}{r-1}|z|^{2k}(1-|z|^2)^r, \;\; k = 0, 1, 2...$$

Sketch of proof. (1) Let $\Phi : \mathbb{T}^1 \times \mathbb{C} \to \mathbb{C}$, $([t], z) \mapsto e^{2i\pi t}z$. Recall that the dually flat manifold associated to Φ and the point $p = 1 \in \mathbb{C}$ is $(\mathbb{R}, \text{Hess}(\psi_1), \nabla^{\text{flat}})$, where $\psi_1 : \mathbb{R} \to \mathbb{R}$, $x \mapsto \frac{1}{4}e^{4\pi x}$ (see Example 7.15). Let $\Phi^{\mathbb{C}} : \mathbb{C}^* \times \mathbb{C} \to \mathbb{C}$ be the holomorphic extension of Φ. A direct calculation shows that $\Phi^{\mathbb{C}}(z, w) = zw$. Then, using Proposition 7.9, it is easy to check that $\tau : \mathbb{C} \to \mathbb{C}^*$, $z \mapsto e^{2\pi z}$ and $\kappa : \mathbb{C}^* \to \mathbb{R}$, $z \mapsto \frac{\ln(|z|)}{2\pi}$ form a toric factorization. Now let $\psi_2 : \mathbb{R} \to \mathbb{R}$, $x \mapsto e^x$ and $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto 4\pi x - \ln(4)$. As we saw in the proof of Proposition 8.11, f is an isomorphism of dually flat spaces from $(\mathbb{R}, \text{Hess}(\psi_1), \nabla^{\text{flat}})$ to $(\mathbb{R}, \text{Hess}(\psi_2), \nabla^{\text{flat}}) \cong (\mathscr{P}, h_F, \nabla^{(e)})$. By Proposition 6.7, $(\tau', \kappa') = (\tau \circ (f_*)^{-1}, f \circ \kappa)$ is a toric factorization of $(\mathscr{P}, h_F, \nabla^{(e)})$. Under the identification $T\mathscr{P} = \mathbb{C}$, we have $f_* z = 4\pi z - \ln(4)$ and hence $\tau'(z) = \tau(\frac{1}{4\pi}(z + \ln(4))) = 2e^{z/2}$ and $\kappa'(z) = (f \circ \kappa)(z) = f(\ln(|z|)/2\pi) = \ln(|z|^2) - \ln(4) =: \theta$. The corresponding distribution $p(.;\theta)$ is a Poisson distribution with parameter $\lambda = e^\theta = e^{\ln(|z|^2) - \ln(4)} = \frac{1}{4}|z|^2$ (see Example 8.4). This concludes the proof of (1). The other cases are analogous, or simpler. □

9 Lifting isometric affine maps

Let $\Phi : \mathbb{T}^n \times N \to N$ and $\Phi' : \mathbb{T}^d \times N' \to N'$ be torifications of the dually flat manifolds (M, h, ∇) and (M', h', ∇'), respectively.

Definition 9.1. Let $f : M \to M'$ and $m : N \to N'$ be smooth maps. We say that m is a *lift of* f if there are compatible covering maps $\tau : TM \to N^\circ$ and $\tau' : TM' \to (N')^\circ$ such that $m \circ \tau = \tau' \circ f_*$. In this case, we say that *m is a lift of f with respect to* τ and τ'.

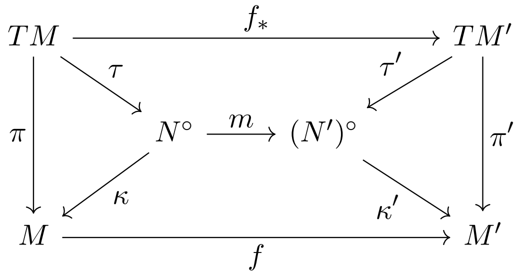
The terminology is motivated by the following lemma.

Lemma 9.2. Let $\pi = \kappa \circ \tau : TM \to M$ and $\pi' = \kappa' \circ \tau' : TM' \to M'$ be toric factorizations. If m is a lift of $f : M \to M'$ with respect to τ and τ', then $m(N^\circ) \subseteq (N')^\circ$ and $\kappa' \circ m = f \circ \kappa$.

Proof. The inclusion $m(N^\circ) \subseteq (N')^\circ$ is an immediate consequence of the formula $m \circ \tau = \tau' \circ f_*$. It follows from the same formula and the definition of a toric factorization that

$$\kappa' \circ m \circ \tau = \kappa' \circ \tau' \circ f_* = \pi' \circ f_* = f \circ \pi = f \circ \kappa \circ \tau,$$

and thus $\kappa' \circ m \circ \tau = f \circ \kappa \circ \tau$. This implies $\kappa' \circ m = f \circ \kappa$. □



Proposition 9.3. Let $m : N \to N'$ be a lift of $f : M \to M'$.

(1) The map m is a Kähler immersion if and only if f is an isometric affine immersion.

(2) If f is an isometric affine immersion, then there exists a unique Lie group homomorphism $\rho : \mathbb{T}^n \to \mathbb{T}^d$ with finite kernel such that $m \circ \Phi_a = \Phi'_{\rho(a)} \circ m$ for all $a \in \mathbb{T}^n$.

The proof of Proposition 9.3 proceeds in a series of lemmas. Let $\pi = \kappa \circ \tau : TM \to M$ and $\pi' = \kappa' \circ \tau' : TM' \to M'$ be fixed toric factorizations.

Lemma 9.4. Let $f : M \to M'$ be an isometric affine immersion. Given $i = 1, 2$, let $m_i : N \to N'$ be a Kähler immersion satisfying $m_i(N^\circ) \subseteq (N')^\circ$ and $\kappa' \circ m_i = f \circ \kappa$ on N°. Then there exists a unique $a \in \mathbb{T}^d$ such that $m_1 = \Phi'_a \circ m_2$.

Proof. Because $\kappa' \circ m_1 = \kappa' \circ m_2$ on N° and $\kappa' : (N')^\circ \to M'$ is a trivial principal fiber bundle with respect to the torus action Φ', there exists a unique smooth function $\phi : N^\circ \to \mathbb{T}^d$ such that $m_1(p) = \Phi'(\phi(p), m_2(p))$ for all $p \in N^\circ$. Taking the derivative with respect to $p \in N^\circ$ in the direction $u \in T_pN$ we obtain

$$(m_1)_{*_p} u = \big(\Phi'_{m_2(p)}\big)_* \phi_{*_p} u + \big(\Phi'_{\phi(p)}\big)_* (m_2)_{*_p} u.$$

The same formula with Ju in place of u yields

$$(m_1)_{*_p} Ju = \big(\Phi'_{m_2(p)}\big)_* \phi_{*_p} Ju + J'\big(\Phi'_{\phi(p)}\big)_* (m_2)_{*_p} u, \tag{9.1}$$

where we have used the fact that m_2 and $\Phi'_{\phi(p)}$ are holomorphic. Since m_1 is holomorphic, we also have

$$(m_1)_{*_p} Ju = J'(m_1)_{*_p} u = J'\big(\Phi'_{m_2(p)}\big)_* \phi_{*_p} u + J'\big(\Phi'_{\phi(p)}\big)_* (m_2)_{*_p} u. \tag{9.2}$$

Comparing (9.1) and (9.2) we get

$$\big(\Phi'_{m_2(p)}\big)_{*_{\phi(p)}} \phi_{*_p} Ju = J'\big(\Phi'_{m_2(p)}\big)_{*_{\phi(p)}} \phi_{*_p} u. \tag{9.3}$$

Given $a \in \mathbb{T}^d$, let $L_a : \mathbb{T}^d \to \mathbb{T}^d$, $b \mapsto ab$. Then $\Phi'_{m_2(p)} = \Phi'_{\phi(p)} \circ \Phi'_{m_2(p)} \circ L_{\phi(p)^{-1}}$ and hence

$$\big(\Phi'_{m_2(p)}\big)_{*_{\phi(p)}} = (\Phi'_{\phi(p)})_{*_{m_2(p)}} (\Phi'_{m_2(p)})_{*_e} (L_{\phi(p)^{-1}})_{*_{\phi(p)}}.$$

Using this, we can rewrite (9.3) as

$$(\Phi'_{\phi(p)})_{*_{m_2(p)}}\big(\Phi'_{m_2(p)}\big)_{*_e}(L_{\phi(p)^{-1}})_{*_{\phi(p)}}\phi_{*_p}Ju = J'(\Phi'_{\phi(p)})_{*_{m_2(p)}}\big(\Phi'_{m_2(p)}\big)_{*_e}(L_{\phi(p)^{-1}})_{*_{\phi(p)}}\phi_{*_p}u, \quad (9.4)$$

and since $(\Phi'_{\phi(p)})_{*_{m_2(p)}}$ is a linear bijection that commutes with J',

$$\big(\Phi'_{m_2(p)}\big)_{*_e}(L_{\phi(p)^{-1}})_{*_{\phi(p)}}\phi_{*_p}Ju = J'\big(\Phi'_{m_2(p)}\big)_{*_e}(L_{\phi(p)^{-1}})_{*_{\phi(p)}}\phi_{*_p}u. \quad (9.5)$$

Let $\xi := (L_{\phi(p)^{-1}})_{*_{\phi(p)}}\phi_{*_p}Ju$ and $\eta := (L_{\phi(p)^{-1}})_{*_{\phi(p)}}\phi_{*_p}u$. Then $\xi, \eta \in \mathbb{R}^d = \mathrm{Lie}(\mathbb{T}^d)$. We denote by $\xi_{N'}$ and $\eta_{N'}$ the corresponding fundamental vector fields on N' associated to Φ'. With this notation, (9.5) can be rewritten as

$$\xi_{N'}(m_2(p)) = J'\eta_{N'}(m_2(p)).$$

By Lemma 6.9, $\xi_{N'}(m_2(p)) = \eta_{N'}(m_2(p)) = 0$. The action Φ' being free at $m_2(p) \in (N')^\circ$, the orbit map $\Phi'_{m_2(p)} : \mathbb{T}^d \to N'$ is immersive at e and hence the equation $0 = \eta_{N'}(m_2(p)) = (\Phi'_{m_2(p)})_{*_e}\eta$ has a unique solution $\eta = 0$. It follows that $\phi_{*_p}u = 0$. Since $p \in N^\circ$ and $u \in T_pN$ are arbitrary and N° is connected, ϕ is constant on N°, say $\phi \equiv a \in \mathbb{T}^d$. Then $m_1 = \Phi'_a \circ m_2$ on N°. Since N° is dense in N, $m_1 = \Phi'_a \circ m_2$ on N. Uniqueness of a is a consequence of the uniqueness of the function ϕ (see above). The lemma follows. □

Lemma 9.5. Let $f : M \to M'$ be an isometric affine immersion and let $m : N \to N'$ be a Kähler immersion satisfying $m(N^\circ) \subseteq (N')^\circ$ and $\kappa' \circ m = f \circ \kappa$ on N°. Then there exists a unique Lie group homomorphism $\rho : \mathbb{T}^n \to \mathbb{T}^d$ such that

$$m \circ \Phi_a = \Phi'_{\rho(a)} \circ m$$

for all $a \in \mathbb{T}^n$. Moreover, the kernel of ρ is finite.

Proof. Let $a \in \mathbb{T}^n$ be arbitrary. The map $m_a := m \circ \Phi_a$ is a Kähler immersion satisfying $m_a(N^\circ) \subseteq (N')^\circ$ and $\kappa' \circ m_a = f \circ \kappa$. By Lemma 9.4, there exists a unique $\rho(a) \in \mathbb{T}^d$ such that $m_a = m \circ \Phi_a = \Phi'_{\rho(a)} \circ m$. Let $\rho : \mathbb{T}^n \to \mathbb{T}^d$, $a \mapsto \rho(a)$. This is a smooth function, as can be seen by trivializing the principal fiber bundles $\kappa : N^\circ \to M$ and $\kappa' : (N')^\circ \to M'$. Using the uniqueness of ρ, it is easily seen that $\rho(ab) = \rho(a)\rho(b)$ and $\rho(e) = e$. Therefore ρ is a Lie group homomorphism. Let $K \subseteq \mathbb{T}^n$ be the kernel of ρ. Suppose that K is not discrete. Then there exists $\xi \in \mathrm{Lie}(\mathbb{T}^n)$, $\xi \neq 0$, such that $\exp(t\xi) \in K$ for all $t \in \mathbb{R}$, where $\exp : \mathrm{Lie}(\mathbb{T}^n) \to \mathbb{T}^n$ is the exponential map of the torus. Let $p \in N^\circ$ be arbitrary and let ξ_N denote the fundamental vector field of ξ on N. We compute:

$$m_{*_p}\xi_N(p) = \frac{d}{dt}\bigg|_0 (m \circ \Phi_{\exp(t\xi)})(p) = \frac{d}{dt}\bigg|_0 \Phi'(\rho(\exp(t\xi)), m(p)) = \frac{d}{dt}\bigg|_0 m(p) = 0.$$

Thus $m_{*_p}\xi_N(p) = 0$. Because m is an immersion, this implies that $\xi_N(p) = 0$, which means that ξ belongs to the kernel of $(\Phi_p)_{*_e}$. Since Φ is free at p, this kernel is trivial and hence $\xi = 0$, a contradiction. This shows that K is discrete. Since $\mathbb{T}^n$ is compact, K is finite. □

Proof of Proposition 9.3. (1) By definition, there are compatible covering maps $\tilde{\tau}$ and $\tilde{\tau}'$ such that $m \circ \tilde{\tau} = \tilde{\tau}' \circ f_*$. Since compatible covering maps are locally Kähler isomorphisms, we have the following equivalences

$$\begin{aligned} f \text{ is an affine isometric map} \quad &\Leftrightarrow \quad f_* \text{ is a Kähler immersion} \\ &\Leftrightarrow \quad m : N^\circ \to N' \text{ is a Kähler immersion,} \end{aligned}$$

where we have used Proposition 2.19. Therefore we must show that $m : N \to N'$ is a Kähler immersion if and only if $m : N^\circ \to N'$ is a Kähler immersion. One direction is obvious. So assume that $m : N^\circ \to N'$ is a Kähler immersion. To see that m is holomorphic on N, let J and J' denote the complex structures on N and N', respectively. The maps $m_* \circ J$ and $J' \circ m_*$ are continuous on TN and coincide on TN° (since m is holomorphic on N°). Since TN° is dense in TN and TN' is Hausdorff, this implies that $m_* \circ J$ and $J' \circ m_*$ coincide on TN, which means that m is holomorphic on N. Analogously, one proves that $m : N \to N'$ is isometric. This concludes the proof of (1). (2) This follows from Lemma 9.5. □

Now we focus our attention on the existence of lifts. Recall that a Riemannian manifold (M, g) is said to be *real analytic* if M is a real analytic manifold and the metric g is a real analytic tensor.

Theorem 9.6. Let M and M' be real analytic Riemannian manifolds. If M is connected and simply connected and if M' is complete, then every isometric immersion f_U of a connected open subset U of M into M' can be uniquely extended to an isometric immersion f of M into M'.

Proof. See [KN96], Theorem 6.3. □

The preceding theorem motivates the following definition.

Definition 9.7. A Kähler manifold N is *regular* if it is connected, simply connected, complete and if the Kähler metric is real analytic. A torification $\Phi : \mathbb{T}^n \times N \to N$ is *regular* if N is regular.

Note that the requirement that the Kähler metric be real analytic makes sense, because complex manifolds are naturally real analytic manifolds[4].

Proposition 9.8 (Existence of lifts). Suppose $\Phi : \mathbb{T}^n \times N \to N$ and $\Phi' : \mathbb{T}^d \times N' \to N'$ are regular torifications of (M, h, ∇) and (M', h', ∇'), respectively. Let $\tau : TM \to N^\circ$ and $\tau' : TM' \to (N')^\circ$ be compatible covering maps. Then every isometric affine immersion $f : M \to M'$ has a unique lift $m : N \to N'$ with respect to τ and τ'.

Proof. Since τ is an isometric covering map, there is an open cover $\{U_\alpha\}_{\alpha \in A}$ of TM such that for every $\alpha \in A$, U_α is connected and $\tau|_{U_\alpha} : U_\alpha \to V_\alpha := \tau(U_\alpha)$ is an isometry. Given $\alpha \in A$, let $m_\alpha : V_\alpha \to N'$ be defined by

$$m_\alpha = \tau' \circ f_* \circ (\tau|_{U_\alpha})^{-1}.$$

[4]This follows from the following simple observation. If a function $f : U \to \mathbb{C}$ is holomorphic on an open set $U \subseteq \mathbb{C}^n \cong \mathbb{R}^{2n}$, then its real and imaginary parts $u : U \to \mathbb{R}$ and $v : U \to \mathbb{R}$ are harmonic functions on U (this follows from the Cauchy-Riemann equations) and hence they are real analytic.

Then m_α is an isometric immersion from $V_\alpha \subseteq N$ into N', because f_* is a Kähler immersion (Proposition 2.19). By Theorem 9.6, m_α extends uniquely to an isometric immersion $\widetilde{m_\alpha} : N \to N'$.

Let $\alpha, \beta \in A$ be arbitrary. We claim that $\widetilde{m_\alpha} = \widetilde{m_\beta}$. Indeed, since TM is connected, it is well-chained and hence there exist an integer $s \geq 1$ and indices $\alpha_0, ..., \alpha_s \in A$ such that $\alpha_0 = \alpha$, $\alpha_s = \beta$ and $U_{\alpha_{i-1}} \cap U_{\alpha_i} \neq \emptyset$ for all $i = 1, ..., s$. For each $i = 1, ..., s$, let W_i be an open connected subset of $U_{\alpha_{i-1}} \cap U_{\alpha_i}$. Clearly, $\tau(W_i) \subseteq V_{i-1} \cap V_i$ and $(\tau|_{U_{\alpha_{i-1}}})^{-1} = (\tau|_{U_{\alpha_i}})^{-1}$ on $\tau(W_i)$. It follows that $m_{\alpha_{i-1}}$ and m_{α_i} coincide on the connected set $\tau(W_i)$. By the uniqueness part of Theorem 9.6, their global extensions coincide on N, that is, $\widetilde{m_{\alpha_{i-1}}} = \widetilde{m_{\alpha_i}}$. But then $\widetilde{m_\alpha} = \widetilde{m_{\alpha_0}} = \widetilde{m_{\alpha_1}} = ... = \widetilde{m_{\alpha_s}} = \widetilde{m_\beta}$ and hence $\widetilde{m_\alpha} = \widetilde{m_\beta}$. This concludes the proof of the claim.

Set $m = \widetilde{m_\alpha}$, where α is any element in A (this is independent of the choice of α by virtue of the claim). By construction, m is an isometric immersion from N into N' satisfying $m \circ \tau = \tau' \circ f_*$. In particular, m is a lift of f with respect to τ and τ'.

Uniqueness of m follows from the formula $m \circ \tau = \tau' \circ f_*$ and the fact that N° is dense in N. □

As a matter of notation, we will denote by $m_{\tau', \tau}(f)$ the unique lift of the isometric affine immersion $f : M \to M'$ with respect to $\tau : TM \to N^\circ$ and $\tau' : TM' \to (N')^\circ$, and by $\rho_{\tau', \tau} : \mathbb{T}^n \to \mathbb{T}^d$ the corresponding Lie group homomorphism (see Proposition 9.3).

Proposition 9.9. Given $i = 1, 2, 3$, let $\Phi_i : \mathbb{T}^{n_i} \times N_i \to N_i$ be a regular torification of (M_i, h_i, ∇_i) with compatible covering map $\tau_i : TM_i \to N_i^\circ$. Let $f_1 : M_1 \to M_2$ and $f_2 : M_2 \to M_3$ be isometric affine immersions.

(1) $m_{\tau_3\tau_1}(f_2 \circ f_1) = m_{\tau_3\tau_2}(f_2) \circ m_{\tau_2\tau_1}(f_1)$.

(2) $m_{\tau_1\tau_1}(Id_{M_1}) = Id_{N_1}$, where Id_{M_1} and Id_{N_1} are the identity maps on M_1 and N_1, respectively.

(3) $\rho_{\tau_3\tau_1}(f_2 \circ f_1) = \rho_{\tau_3\tau_2}(f_2) \circ \rho_{\tau_2\tau_1}(f_1)$.

(4) $\rho_{\tau_1\tau_1}(Id_{M_1}) = Id_{\mathbb{T}^{n_1}}$, where $Id_{\mathbb{T}^{n_1}}$ is the identity map on $\mathbb{T}^{n_1}$.

Proof. For simplicity, write $m_1 = m_{\tau_2\tau_1}(f_1)$, $m_2 = m_{\tau_3\tau_2}(f_2)$, $\rho_1 = \rho_{\tau_2\tau_1}(f_1)$ and $\rho_2 = \rho_{\tau_3\tau_2}(f_2)$.
(1) Taking into account the definition of a lift, we compute:

$$(m_2 \circ m_1) \circ \tau_1 = m_2 \circ \tau_2 \circ (f_1)_* = \tau_3 \circ (f_2)_* \circ (f_1)_* = \tau_3 \circ (f_2 \circ f_1)_*.$$

This shows that $m_2 \circ m_1$ is a lift of $f_2 \circ f_1$ with respect to τ_1 and τ_3.
(2) Since $(Id_{M_1})_* = Id_{TM_1}$, we have $Id_{N_1} \circ \tau_1 = \tau_1 \circ (Id_{M_1})_*$. This shows that Id_{N_1} is a lift of Id_{M_1} with respect to τ_1 and τ_1.
(3) Let $a \in \mathbb{T}^{n_1}$ be arbitrary. Taking into account the definition of ρ_i, we compute:

$$\begin{aligned} m_{\tau_3\tau_1} \circ (\Phi_1)_a &= m_2 \circ m_1 \circ (\Phi_1)_a = m_2 \circ (\Phi_2)_{\rho_1(a)} \circ m_1 \\ &= (\Phi_3)_{\rho_2(\rho_1(a))} \circ m_2 \circ m_1 = (\Phi_3)_{\rho_2(\rho_1(a))} \circ m_{\tau_3\tau_1}. \end{aligned}$$

This shows that $\rho_{\tau_3\tau_1}(f_2 \circ f_1) = \rho_2 \circ \rho_1$.

(4) Given $a \in \mathbb{T}^{n_1}$, we have

$$m_{\tau_1\tau_1}(Id_{M_1}) \circ (\Phi_1)_a = Id_{N_1} \circ (\Phi_1)_a = (\Phi_1)_a \circ Id_{N_1} = (\Phi_1)_a \circ m_{\tau_1\tau_1}(Id_{M_1}).$$

The result follows. $\square$

Definition 9.10. Suppose $\Phi : \mathbb{T}^n \times N \to N$ and $\Phi' : \mathbb{T}^d \times N' \to N'$ are torifications of (M, h, ∇) and (M', h', ∇'), respectively. We shall say that N and N' are *equivalent* if there exist a Lie group isomorphism $\rho : \mathbb{T}^n \to \mathbb{T}^d$ and a Kähler isomorphism $G : N \to N'$ such that

$$G \circ \Phi_a = \Phi'_{\rho(a)} \circ G$$

for all $a \in \mathbb{T}^n$. In this case, we write $N \sim N'$.

Note that $\sim$ is an equivalence relation.

Theorem 9.11. Regular torifications of the dually flat space (M, h, ∇) are equivalent.

Proof. Let $\Phi : \mathbb{T}^n \times N \to N$ and $\Phi' : \mathbb{T}^d \times N' \to N'$ be regular torifications of (M, h, ∇). Let $\tau : TM \to N^\circ$ and $\tau' : TM' \to (N')^\circ$ be compatible covering maps. By Proposition 9.9, $m = m_{\tau'\tau}(Id_M) : N \to N'$ is a Kähler isomorphism satisfying $m \circ \Phi_a = \Phi_{\rho(a)} \circ m$ for all $a \in \mathbb{T}^n$, where $\rho = \rho_{\tau'\tau}$. By Proposition 9.9, ρ is a Lie group isomorphism. $\square$

Definition 9.12. We shall say that a connected dually flat space (M, h, ∇) is *toric* if it has a regular torification.

Example 9.13. The torifications considered in Proposition 8.11 are all regular. Therefore the exponential families $\mathscr{P}$, $\mathcal{P}_n^\times$, $\mathcal{B}(n)$, $\mathcal{M}(m, n)$, $\mathcal{NB}(r)$ are toric.

From now on, when we deal with a regular torification of a dually flat space (M, h, ∇), we will say "the regular torification of M", and keep in mind that it is only defined up to an equivariant Kähler isomorphism.

Now we continue with some consequences of Proposition 9.9. Let N be a Kähler manifold equipped with a torus action $\Phi : \mathbb{T}^n \times N \to N$, with Kähler metric g. We will use the following notation:

- $\mathrm{Aut}(\mathbb{T}^n)$ is the group of Lie group isomorphisms $\rho : \mathbb{T}^n \to \mathbb{T}^n$.
- $\mathrm{Aut}(N, g)$ is the group of holomorphic and isometric transformations of N.
- $\mathrm{Aut}(N, g)^{\mathbb{T}^n}$ is the subgroup of $\mathrm{Aut}(N, g)$ characterized by the following condition: for each $\varphi \in \mathrm{Aut}(N, g)^{\mathbb{T}^n}$, there exists $\rho \in \mathrm{Aut}(\mathbb{T}^n)$ such that $\varphi \circ \Phi_a = \Phi_{\rho(a)} \circ \varphi$ for all $a \in \mathbb{T}^n$.

Remark 9.14. With the usual identification $\mathbb{R}^n = \mathrm{Lie}(\mathbb{T}^n)$, the exponential map $\exp : \mathbb{R}^n \to \mathbb{T}^n$ is just the quotient map $t \mapsto [t]$. Therefore, if $\rho \in \mathrm{Aut}(\mathbb{T}^n)$, then $\rho([t]) = [\rho_{*_e} t]$ for all $t \in \mathbb{R}^n$. This forces the matrix representation of ρ_{*_e} with respect to the canonical basis to be an element of $\mathrm{GL}(n, \mathbb{Z})$. It is then easy to see that $\mathrm{Aut}(\mathbb{T}^n) \to \mathrm{GL}(n, \mathbb{Z})$, $\rho \mapsto \rho_{*_e}$ is a group isomorphism.

Lemma 9.15. Let $\Phi : \mathbb{T}^n \times N \to N$ be a regular torification of (M, h, ∇), with toric factorization $\pi = \kappa \circ \tau$. Let g be the Kähler metric on N. Two maps $\tau' : TM \to N^\circ$ and $\kappa' : N^\circ \to M$ form a toric factorization if and only if there exists $G \in \mathrm{Aut}(N, g)^{\mathbb{T}^n}$ such that $\tau' = G \circ \tau$ and $\kappa' = \kappa \circ G^{-1}$.

Proof. $(\Rightarrow)$ Suppose $\pi = \kappa \circ \tau = \kappa' \circ \tau'$ are toric factorizations. Let $G := m_{\tau'\tau}(Id_M)$. By Proposition 9.9, $G \in \mathrm{Aut}(N, g)^{\mathbb{T}^n}$ and $G \circ \tau = \tau'$. To see that $\kappa' = \kappa \circ G^{-1}$, let $y = \tau'(x)$ be arbitrary. We have

$$\kappa'(y) = (\kappa' \circ \tau')(x) = \pi(x) = (\kappa \circ \tau)(x) = \kappa(G^{-1}(\tau'(x))) = (\kappa \circ G^{-1})(y). \tag{9.6}$$

$(\Leftarrow)$ Let $G \in \mathrm{Aut}(N, g)^{\mathbb{T}^n}$ be arbitrary. There exists $A \in \mathrm{GL}(n, \mathbb{Z})$ such that $G \circ \Phi_{[t]} = \Phi_{[At]} \circ G$ for all $t \in \mathbb{R}^n$ (see Remark 9.14). The fact that $\pi = \kappa \circ \tau$ is a toric factorization means that there is a toric parametrization (L, X, F) such that $\tau = F \circ q_L$ and $\kappa = \pi_L \circ F^{-1}$. Let $X' \in \mathrm{gen}(L)$ be defined by $X' = (A^T)^{-1}X$. Using Lemma 3.7, it is easy to check that $(L, X', G \circ F)$ is a toric parametrization and that the induced toric factorization is (τ', κ'), where $\tau' = G \circ \tau$ and $\kappa' = \kappa \circ G^{-1}$. ☐

In the next two propositions, $\Phi : \mathbb{T}^n \times N \to N$ and $\Phi' : \mathbb{T}^d \times N' \to N'$ are regular torifications of the dually flat spaces (M, h, ∇) and (M', h', ∇'), respectively. We denote by g and g' the Kälher metrics on N and N', respectively.

Proposition 9.16 (Lifts are conjugate). Let $m : N \to N'$ be a lift of the isometric affine immersion $f : M \to M'$. Given a smooth map $m' : N \to N'$, the following are equivalent:

(1) m' is a lift of f.

(2) There are $G_1 \in \mathrm{Aut}(N, g)^{\mathbb{T}^n}$ and $G_2 \in \mathrm{Aut}(N', g')^{\mathbb{T}^d}$ such that $m' \circ G_1 = G_2 \circ m$.

Proof. (1) $\Rightarrow$ (2) Suppose $m, m' : N \to N'$ are lifts of f. This means that there are compatible covering maps $\tau_1, \tau_2 : TM \to N^\circ$ and $\tau_1', \tau_2' : TM' \to (N')^\circ$ such that $m \circ \tau_1 = \tau_1' \circ f_*$ and $m' \circ \tau_2 = \tau_2' \circ f_*$. Let $G_1 = m_{\tau_2\tau_1}(Id_M)$ and $G_2 = m_{\tau_2'\tau_1'}(Id_{M'})$. By Proposition 9.9, $G_1 \in \mathrm{Aut}(N, g)^{\mathbb{T}^n}$, $G_2 \in \mathrm{Aut}(N', g')^{\mathbb{T}^d}$ and $G_1 \circ \tau_1 = \tau_2$ and $G_2 \circ \tau_1' = \tau_2'$. Thus

$$\begin{aligned} m' \circ \tau_2 = \tau_2' \circ f_* \quad &\Rightarrow \quad m' \circ G_1 \circ \tau_1 = G_2 \circ \tau_1' \circ f_* \\ &\Rightarrow \quad m' \circ G_1 \circ \tau_1 = G_2 \circ m \circ \tau_1 \\ &\Rightarrow \quad m' \circ G_1 = G_2 \circ m \ \text{ on } \ N^\circ. \end{aligned}$$

Since N° is dense in N, $m' \circ G_1 = G_2 \circ m$ on N.

(2) $\Rightarrow$ (1). Suppose that $m : N \to N'$ is a lift of f and that $m' : N \to N'$ is a smooth map satisfying $m' \circ G_1 = G_2 \circ m$ for some $G_1 \in \mathrm{Aut}(N, g)^{\mathbb{T}^n}$ and some $G_2 \in \mathrm{Aut}(N', g')^{\mathbb{T}^d}$. Because m is a lift, there are compatible projections $\tau : TM \to N^\circ$ and $\tau' : TM' \to (N')^\circ$ such that $m \circ \tau = \tau' \circ f_*$, and so

$$\begin{aligned} m' \circ G_1 = G_2 \circ m \quad &\Rightarrow \quad m' \circ G_1 \circ \tau = G_2 \circ m \circ \tau \\ &\Rightarrow \quad m' \circ G_1 \circ \tau = G_2 \circ \tau' \circ f_*. \end{aligned} \tag{9.7}$$

By Lemma 9.14, $G_1 \circ \tau$ and $G_2 \circ \tau'$ are compatible covering maps. It follows from this and (9.7) that m' is a lift of f with respect to $G_1 \circ \tau$ and $G_2 \circ \tau'$. In particular, m' is a lift of f. ☐

Proposition 9.17 (**The derivative of an isometric affine immersion is equivariant**)**.** Let (L, X, F) and (L', X', F') be toric factorizations of N and N', respectively, with induced toric parametrizations $\pi = \kappa \circ \tau : TM \to M$ and $\pi' = \kappa' \circ \tau' : TM' \to M'$. If $f : M \to M'$ is an isometric affine immersion, then its derivative $f_* : TM \to TM'$ satisfies

$$f_* \circ (T_X)_t = (T_{X'})_{(\rho_{\tau'\tau}(f))_{*e} t} \circ f_*$$

for all $t \in \mathbb{R}^n$, where $T_X : \mathbb{R}^n \times TM \to TM$, $(t, u) \mapsto u + \sum_{k=1}^n t_k X_k$ ($T_{X'}$ is defined similarly).

Proof. Let $H : \mathbb{R}^n \times TM \to TM'$ be the map defined by $H(t, u) = \big((T_{X'})_{(\rho_{\tau'\tau}(f))_{*e} t} \circ f_* \circ (T_X)_{-t}\big)(u)$. We claim that

$$\tau'(f_*(u)) = \tau'(H(t, u)) \tag{9.8}$$

for all $(t, u) \in \mathbb{R}^n \times TM$. Indeed, it follows from Proposition 9.9 and the formulas $\tau \circ (T_X)_t = \Phi_{[t]} \circ \tau$ and $\tau' \circ (T_{X'})_{t'} = \Phi'_{[t']} \circ \tau'$ that

$$\begin{aligned}
\tau'(H(t, u)) &= \tau'\Big(\big((T_{X'})_{(\rho_{\tau'\tau}(f))_{*e} t} \circ f_* \circ (T_X)_{-t}\big)(u)\Big) \\
&= \Big(\Phi'_{\rho_{\tau'\tau}(f)([t])} \circ \tau' \circ f_* \circ (T_X)_{-t}\Big)(u) \\
&= \Big(\Phi'_{\rho_{\tau'\tau}(f)([t])} \circ m_{\tau'\tau}(f) \circ \tau \circ (T_X)_{-t}\Big)(u) \\
&= \Big(\Phi'_{\rho_{\tau'\tau}(f)([t])} \circ m_{\tau'\tau}(f) \circ \Phi_{[-t]} \circ \tau\Big)(u) \\
&= \Big(\Phi'_{\rho_{\tau'\tau}(f)([t])} \circ \Phi'_{\rho_{\tau'\tau}(f)([-t])} \circ m_{\tau'\tau}(f) \circ \tau\Big)(u) \\
&= (m_{\tau'\tau}(f) \circ \tau)(u) \\
&= (\tau' \circ f_*)(u).
\end{aligned}$$

This concludes the proof of the claim. Taking the derivative with respect to t in (9.8) yields

$$0 = \frac{\partial}{\partial t}(\tau'(H(t, u))) = \tau'_{*H(t,u)} \frac{\partial}{\partial t} H(t, u),$$

which implies $\frac{\partial}{\partial t} H(t, u) = 0$, since τ'_* is a linear bijection at every point of TM'. Thus $H(t, u)$ is independent of t. It follows that $H(t, u) = H(0, u)$, which proves the proposition. ☐

10 Fundamental lattices and Kähler functions

In this section, we explore the close relationship between parallel lattices on a toric dually flat space M and the space of Kähler functions on TM (see Definition 10.3 below). We begin with the following simple observation.

Lemma 10.1. Let $\Phi : \mathbb{T}^n \times N \to N$ and $\Phi' : \mathbb{T}^n \times N' \to N'$ be regular torifications of the dually flat space (M, h, ∇). If (L, X, F) and (L', X', F') are toric parametrizations of N and N', respectively, then $L = L'$.

Proof. It suffices to show that $\Gamma(L) = \Gamma(L')$. Let $\pi = \kappa \circ \tau$ (resp. $\pi = \kappa' \circ \tau'$) be the toric factorization induced by (L, X, F) (resp. (L', X', F')). Since $\Gamma(L) = \mathrm{Deck}(\tau)$ and $\Gamma(L') = \mathrm{Deck}(\tau')$, we must show that $\mathrm{Deck}(\tau) = \mathrm{Deck}(\tau')$. Let $m = m_{\tau'\tau}(Id_M) : N \to N'$ be the lift of Id_M with respect to τ and τ'. By Proposition 9.9, m is a Kähler isomorphism satisfying $m \circ \tau = \tau' \circ (Id_M)_* = \tau'$, that is, $m \circ \tau = \tau'$. Now, let $\varphi : TM \to TM$ be a diffeomorphism. We have the following equivalences:

$$\varphi \in \mathrm{Deck}(\tau) \quad \Leftrightarrow \quad \tau \circ \varphi = \tau \quad \Leftrightarrow \quad m \circ \tau \circ \varphi = m \circ \tau \quad \Leftrightarrow \quad \tau' \circ \varphi = \tau' \quad \Leftrightarrow \quad \varphi \in \mathrm{Deck}(\tau').$$

Thus $\mathrm{Deck}(\tau) = \mathrm{Deck}(\tau')$. □

In view of this lemma, we are led to introduce the following definition.

Definition 10.2. Let (M, h, ∇) be a toric dually flat space. The *fundamental lattice* of M is the parallel lattice described in Lemma 10.1.

Next we turn our attention to Kähler functions, that we now define.

Definition 10.3. Let N be a Kähler manifold with Kähler structure (g, J, ω). A smooth function $f : N \to \mathbb{R}$ is called a *Kähler function* if it satisfies $\mathscr{L}_{X_f} g = 0$, where X_f is the Hamiltonian vector field associated to f (i.e. $\omega(X_f, \,.\,) = df(.)$) and where $\mathscr{L}_{X_f}$ is the Lie derivative in the direction of X_f.

Following [CMP90], we denote by $\mathscr{K}(N)$ the space of Kähler functions on N. When N has a finite number of connected components, then $\mathscr{K}(N)$ is a finite dimensional[5] Lie algebra for the Poisson bracket $\{f, g\} := \omega(X_f, X_g)$.

We say that $\mathscr{K}(N)$ *separates the points of* N if $f(p) = f(q)$ for all $f \in \mathscr{K}(N)$ implies $p = q$.

The proposition below is the main result of this section.

Proposition 10.4. Let $\Phi : \mathbb{T}^n \times N \to N$ be a regular torification of the dually flat space (M, h, ∇), with fundamental lattice $\mathscr{L} \subset TM$. Then

$$\Gamma(\mathscr{L}) \subseteq \big\{\varphi \in \mathrm{Diff}(TM) \;\big|\; f \circ \varphi = f \;\forall f \in \mathscr{K}(TM)\big\}. \tag{10.1}$$

When $\mathscr{K}(N)$ separates the points of N, (10.1) is an equality.

Before proving Proposition 10.4, we give an application of it.

Corollary 10.5. Let (M, h, ∇) be a connected dually flat space of dimension n. If $\mathscr{K}(TM)$ separates the points of TM, then M is not toric.

Example 10.6. Consider the set $\mathcal{N}(\mu)$ of normal distributions with known variance $\sigma = 1$:

$$p(x;\mu) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(x-\mu)^2}{2}\right\} \quad \big(x \in \mathbb{R}\big).$$

[5] The fact that $\mathscr{K}(N)$ is finite dimensional comes from the following result: if (M, h) is a connected Riemannian manifold, then its space of Killing vector fields $\mathfrak{i}(M) := \{X \in \mathfrak{X}(M) \,|\, \mathscr{L}_X h = 0\}$ is finite dimensional (see for example [Jos02]).

The set $\mathcal{N}(\mu)$ is a 1-dimensional statistical manifold parameterized by the mean $\mu \in \mathbb{R}$. It is an exponential family, because $p(x;\mu) = \exp\big\{C(x) + \theta F(x) - \psi(\theta)\big\}$, where

$$\theta = \mu, \qquad C(x) = \ln\left(\frac{1}{\sqrt{2\pi}}\right) - \frac{x^2}{2}, \qquad F(x) = x, \qquad \psi(\theta) = \frac{\theta^2}{2}.$$

The Hessian of ψ is [1]. It follows from this and Proposition 2.17 that $T\mathcal{N}(\mu)$ is Kähler isomorphic to $\mathbb{C}$ endowed with the flat canonical Kähler structure. Let $f_1 : \mathbb{C} \to \mathbb{R}$ and $f_2 : \mathbb{C} \to \mathbb{R}$ be defined by $f_1(z) = \mathrm{Real}(z)$ and $f_2(z) = \mathrm{Im}(z)$. A simple verification shows that f_1 and f_2 are Kähler functions. Clearly, if $f_k(z) = f_k(w)$, $k = 1, 2$, then $z = w$. This implies that $\mathscr{K}(\mathbb{C})$ separates the points of $\mathbb{C}$. By Corollary 10.5, $\mathcal{N}(\mu)$ is not toric.

Example 10.7. Let $\mathscr{P}$ be the set of Poisson distributions defined over $\Omega = \mathbb{N} = \{0, 1, ...\}$ (see Example 8.4). Then $\mathscr{P}$ is toric with regular torification $\Phi : \mathbb{T} \times \mathbb{C} \to \mathbb{C}$, $([t], z) \to e^{2i\pi t} z$. Let $\mathscr{L} \subset T\mathscr{P}$ be the fundamental lattice of $\mathscr{P}$. As we noted in Example 10.6, $\mathscr{K}(\mathbb{C})$ separates the points of $\mathbb{C}$. Therefore $\Gamma(\mathscr{L})$ coincides with the set of diffeomorphisms $\varphi : T\mathscr{P} \to T\mathscr{P}$ satisfying $f \circ \varphi = f$ for all $f \in \mathscr{K}(T\mathscr{P})$. The coordinate expression for the Fisher metric h_F is the Hessian of the cumulant generating function: $h_F(\theta) = \mathrm{Hess}(\psi) = \big[e^{\theta}\big]$. It follows from this and Proposition 2.17 that $T\mathscr{P}$ is Kähler isomorphic to $\mathbb{C}$ endowed with the Kähler metric

$$g_z(u, v) = e^x \,\mathrm{Real}(\overline{u}v),$$

where $z, u, v \in \mathbb{C}$, $z = x + iy$, $x, y \in \mathbb{R}$. The space of Kähler functions on $T\mathscr{P} = \mathbb{C}$ is spanned by

$$1, \quad e^x, \quad e^{\frac{x}{2}} \cos\left(\tfrac{y}{2}\right), \quad e^{\frac{x}{2}} \sin\left(\tfrac{y}{2}\right)$$

(to see this, use Proposition 2.25 in [Mol14]). Let $\varphi : \mathbb{C} \to \mathbb{C}$ be a diffeomorphism satisfying $f \circ \varphi = f$ for all $f \in \mathscr{K}(\mathbb{C}) = \mathscr{K}(T\mathscr{P})$. Let $f_0 : \mathbb{C} \to \mathbb{C}$, $z \mapsto e^{\frac{z}{2}}$. Because the real and imaginary parts of f_0 are Kähler functions, we have $f_0 \circ \varphi = f_0$. It is well known that the map $\mathbb{C} \to \mathbb{C}^*$, $z \mapsto e^z$ is a covering map whose Deck transformation group is $\mathrm{Deck}(e^z) = 2i\pi\mathbb{Z}$ (here we identify translations in $\mathbb{C}$ with complex numbers). Therefore $f_0 : \mathbb{C} \to \mathbb{C}^*$ is a covering map with Deck transformation group $4i\pi\mathbb{Z}$. Thus $\varphi \in \mathrm{Deck}(f_0) = 4i\pi\mathbb{Z}$ and hence there exists $k \in \mathbb{Z}$ such that $\varphi(z) = z + 4\pi ik$ for all $z \in \mathbb{C}$. Conversely, if $\varphi \in \mathrm{Diff}(\mathbb{C})$ is of the form $\varphi(z) = z + 4\pi ik$ for some $k \in \mathbb{Z}$, then clearly $f \circ \varphi = f$ for all $f \in \mathscr{K}(\mathbb{C})$. It follows that $\Gamma(\mathscr{L})$ is the set of diffeomorphisms $\varphi \in \mathrm{Diff}(\mathbb{C})$ of the form $\varphi(z) = z + 4\pi ik$, where $k \in \mathbb{Z}$. If θ is the natural parameter of $\mathscr{P}$ as described in Example 8.4, then we see that the fundamental lattice $\mathscr{L}$ is generated by $X = 4\pi \frac{\partial}{\partial \theta} \in \mathfrak{X}(\mathscr{P})$.

Now we proceed with the proof of Proposition 10.4. It is based on the following result, which is due to Nomizu [Nom60]:

Theorem 10.8. Let (M, g) be a real analytic Riemannian manifold and let X be a Killing vector field defined on the open set $U \subseteq M$. If M and U are connected and M is simply connected, then X extends uniquely to a global Killing vector field on M.

We also need the following lemma.

Lemma 10.9. Let $\Phi : \mathbb{T}^n \times N \to N$ be a regular torification of a dually flat space (M, h, ∇). Let $\tau : TM \to N^\circ$ be a compatible covering map and $f : TM \to \mathbb{R}$ a smooth function. Then f is Kähler if and only if there exists a Kähler function $\overline{f}$ on N such that $f = \overline{f} \circ \tau$.

Proof. One direction is immediate: if $\overline{f} : N \to \mathbb{R}$ is Kähler, then so is $f = \overline{f} \circ \tau$ (since τ is a Kähler covering map). Conversely, suppose $f : TM \to \mathbb{R}$ is Kähler. Let $\{U_\alpha\}_{\alpha \in A}$ be a cover of TM by connected open sets U_α such that $\tau|_{U_\alpha} : U_\alpha \to V_\alpha := \tau(U_\alpha)$ is a Kähler isomorphism for all $\alpha \in A$. Given $\alpha \in A$, define $f_\alpha : V_\alpha \to \mathbb{R}$ by

$$f_\alpha := f \circ (\tau|_{U_\alpha})^{-1}.$$

Since $\tau|_{U_\alpha}$ is a Kähler isomorphism, f_α is a Kähler function on $V_\alpha \subseteq N$. This implies that the Hamiltonian vector field X_{f_α} of f_α is a Killing vector field on V_α. Since V_α is connected and N is a connected and simply connected real analytic Riemannian manifold, X_{f_α} extends uniquely to a globally defined Killing vector field, say X_α, on N (see Theorem 10.8). Note that $\mathscr{L}_{X_\alpha}\omega = 0$ on V_α, by construction. By an argument entirely analogous to that in the proof of Proposition 9.8, one sees that $X_\alpha = X_\beta$ on N whenever $\alpha, \beta \in A$. Set $X = X_\alpha \in \mathfrak{X}(N)$, where α is any element in A (this is independent of the choice of α). Since $\{V_\alpha\}_{\alpha \in A}$ is a cover of N° and since $\mathscr{L}_X\omega = \mathscr{L}_{X_\alpha}\omega = 0$ on each V_α, we see that $\mathscr{L}_X\omega = 0$ on N°. Since N° is dense in N, $\mathscr{L}_X\omega = 0$ on N. Because of this and because N is simply connected, there exists a smooth function $\overline{f} : N \to \mathbb{R}$ such that $X = X_{\overline{f}}$. The function $\overline{f}$ is Kähler, since X is a Killing vector field. By construction, $X_{\overline{f}} = X_{f_\alpha}$ on V_α for every $\alpha \in A$. Since V_α is connected, there is $C_\alpha \in \mathbb{R}$ such that $\overline{f} = f_\alpha + C_\alpha$ on V_α. This implies $\overline{f} \circ \tau = f_\alpha \circ \tau + C_\alpha$ on U_α for all $\alpha \in A$. Since $f_\alpha = f \circ (\tau|_{U_\alpha})^{-1}$, this implies $\overline{f} \circ \tau = f + C_\alpha$ on U_α for all $\alpha \in A$. Since $\overline{f} \circ \tau - f$ is continuous on TM, $C_\alpha = C_\beta \equiv C$ whenever $\alpha, \beta \in A$. Thus $\overline{f} \circ \tau = f + C$ on TM. Redefining $\overline{f}$ by adding a constant if necessary, we can assume $C = 0$. The lemma follows. □

Proof of Proposition 10.4. Let $\varphi \in \Gamma(\mathscr{L})$ be arbitrary. Let $\tau : TM \to N^\circ$ be a compatible covering map. By Lemma 10.1, $\Gamma(\mathscr{L}) = \mathrm{Deck}(\tau)$ and hence $\tau \circ \varphi = \tau$. It follows that $\overline{f} \circ \tau \circ \varphi = \overline{f} \circ \tau$ for all $\overline{f} \in \mathscr{K}(N)$. By Lemma 10.9, this implies that $f \circ \varphi = f$ for all $f \in \mathscr{K}(TM)$. This shows the inclusion (10.1).

Suppose now that $\mathscr{K}(N)$ separates the points of N. Let $\varphi \in \mathrm{Diff}(TM)$ be such that $f \circ \varphi = f$ for all $f \in \mathscr{K}(TM)$. By Lemma 10.9, $\overline{f} \circ \tau \circ \varphi = \overline{f} \circ \tau$ for all $\overline{f} \in \mathscr{K}(N)$. Because $\mathscr{K}(N)$ separates the points of N, this implies that $\tau \circ \varphi = \tau$, and so $\varphi \in \mathrm{Deck}(\tau) = \Gamma(\mathscr{L})$. This shows the converse inclusion. □

11 Torifications and projective varieties

Throughout this section, $\mathcal{E}$ is an exponential family of dimension n defined over a finite set $\Omega = \{x_0, ..., x_r\}$, with elements of the form $p(x; \theta) = \exp(C(x) + \langle F(x), \theta \rangle - \psi(\theta))$, where $\langle\, ,\, \rangle$ is the Euclidean pairing on $\mathbb{R}^n$ and

$$x \in \Omega, \qquad \theta \in \mathbb{R}^n, \qquad C : \Omega \to \mathbb{R}, \qquad F = (F_1, ..., F_n) : \Omega \to \mathbb{R}^n, \qquad \psi : \mathbb{R}^n \to \mathbb{R}.$$

It is assumed that the functions $1, F_1, ..., F_n : \Omega \to \mathbb{R}$ are independent so that the map $\mathcal{E} \to \mathbb{R}^n$, $p(., \theta) \mapsto \theta$ becomes a bijection. Note that:

- The condition $\sum_{k\in\Omega} p(k;\theta) = 1$ implies that $\psi(\theta) = \ln\big(\sum_{k\in\Omega}\exp(C(k) - \langle F(k),\theta\rangle)\big)$ for all $\theta \in \mathbb{R}^n$.
- $\mathcal{E}$ is a subset of $\mathcal{P}^{\times}_{r+1}$ (see Example 8.5).

We endow $\mathcal{E}$ and $\mathcal{P}^{\times}_{r+1}$ with their canonical dually flat structures (given by the Fisher metric and exponential connection).

Lemma 11.1. The inclusion map $j : \mathcal{E} \hookrightarrow \mathcal{P}^{\times}_{r+1}$ is an affine isometric immersion.

Proof. Let $\theta = (\theta_1, ..., \theta_n)$ and $\theta' = (\theta'_1, ..., \theta'_r)$ be the natural parameters of M and $\mathcal{P}^{\times}_{r+1}$, respectively. Recall that $p(x_i;\theta') = e^{\theta'_{i+1}-\psi'(\theta')}$ if $i = 0, ..., r-1$ and $p(x_r;\theta') = e^{-\psi'(\theta')}$, where $\psi'(\theta') = \ln(1 + \sum_{k=1}^r e^{\theta'_k})$ is the cumulant generating function of $\mathcal{P}^{\times}_{r+1}$ (see Example 8.5). Let $p = p(.,\theta) = p(.,\theta') \in \mathcal{E}$ and $i = 0, ..., r$ be arbitrary. Since $p(x_i,\theta) = p(x_i,\theta')$, we have

$$\exp\bigg(C(x_i) + \sum_{j=1}^n F_j(x_i)\theta_j - \psi(\theta)\bigg) = \begin{cases} \exp\big(\theta'_{i+1} - \psi'(\theta')\big), & \text{if } \ i = 0, ..., r-1, \\ \exp\big(-\psi'(\theta')\big), & \text{if } \ i = r, \end{cases}$$

and thus

$$\begin{cases} C(x_i) + \sum_{j=1}^n F_j(x_i)\theta_j - \psi(\theta) &= \theta'_{i+1} - \psi'(\theta'), \quad i = 0, ..., r-1, \\ C(x_r) + \sum_{j=1}^n F_j(x_r)\theta_j - \psi(\theta) &= -\psi'(\theta'). \end{cases}$$

Therefore

$$\theta'_i = C(x_{i-1}) - C(x_r) + \big\langle \theta, F(x_{i-1}) - F(x_r)\big\rangle \qquad (i = 1, ..., r)$$

that is,

$$\begin{bmatrix} \theta'_1 \\ \vdots \\ \theta'_r \end{bmatrix} = \begin{bmatrix} F_1(x_0) - F_1(x_r) & \cdots & F_n(x_0) - F_n(x_r) \\ \vdots & & \vdots \\ F_1(x_{r-1}) - F_1(x_r) & \cdots & F_n(x_{r-1}) - F_n(x_r) \end{bmatrix} \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_n \end{bmatrix} + \begin{bmatrix} C(x_0) - C(x_r) \\ \vdots \\ C(x_{r-1}) - C(x_r) \end{bmatrix}. \tag{11.1}$$

Formula (11.1) is the coordinate expression for j in the natural parameters. This shows that j is affine.

Next we prove that j is isometric. Let h and h' be the Fisher metrics on $\mathcal{E}$ and $\mathcal{P}^{\times}_{r+1}$, respectively, and let g be the Euclidean metric on $\mathbb{R}^{r+1}$. Let $f : M \to \mathbb{R}^{r+1}$, $p \mapsto (\sqrt{p(x_0)}, ..., \sqrt{p(x_r)})$. Given $1 \leq i, j \leq n$, a simple calculation shows that

$$g_{f(p)}\bigg(f_{*_p}\frac{\partial}{\partial\theta_i}, f_{*_p}\frac{\partial}{\partial\theta_j}\bigg) = \frac{1}{4}\sum_{k=0}^r \bigg(F_i(x_k) - \frac{\partial\psi}{\partial\theta_i}\bigg)\bigg(F_j(x_k) - \frac{\partial\psi}{\partial\theta_j}\bigg)p(x_k). \tag{11.2}$$

In [AN00], Formula 3.59, it is observed that the coordinate expression for h in the natural parameters is given by

$$h_{ij}(\theta) = \sum_{k=0}^r \bigg(F_i - \frac{\partial\psi}{\partial\theta_i}\bigg)\bigg(F_j - \frac{\partial\psi}{\partial\theta_j}\bigg)p(x_k;\theta). \tag{11.3}$$

Comparing (11.2) and (11.3) we obtain $f^*g = \frac{1}{4}h$. Now let $\tilde{f} : \mathcal{P}_{r+1}^{\times} \to \mathbb{R}^{r+1}$, $p \mapsto (\sqrt{p(x_0)}, ..., \sqrt{p(x_r)})$. Note that $f = \tilde{f} \circ j$. By the same argument as above with $\mathcal{E}$ replaced by $\mathcal{P}_{r+1}^{\times}$, we get $\tilde{f}^*g = \frac{1}{4}h'$ and hence

$$h = 4f^*g = 4(\tilde{f} \circ j)^*g = 4j^*\tilde{f}^*g = 4j^*\tfrac{1}{4}h' = j^*h'.$$

Therefore $h = j^*h'$. □

Given $n \geq 1$, recall the notation $\Phi_n : \mathbb{T}^n \times \mathbb{P}_n(c) \to \mathbb{P}_n(c)$ defined before Proposition 8.11.

Theorem 11.2. Let $\mathcal{E}$ be an exponential family of dimension n defined over a finite set $\Omega = \{x_0, x_1, ..., x_r\}$ (as described in the begining of this section). Suppose $\mathcal{E}$ toric with regular torification $\Phi : \mathbb{T}^n \times N \to N$. Then there is a Kähler immersion $m : N \to \mathbb{P}_r(1)$ and a Lie group homomorphism $\rho : \mathbb{T}^n \to \mathbb{T}^r$ with finite kernel such that $m \circ \Phi_a = (\Phi_r)_{\rho(a)} \circ m$ for all $a \in \mathbb{T}^n$.

Proof. Since the inclusion map $j : \mathcal{E} \to \mathcal{P}_{r+1}^{\times}$ is an isometric affine immersion between toric dually flat spaces, it has a lift m with the desired properties. □

We now look at some examples. Let $\mathcal{B}(n)$ be the set of Binomial distributions defined over $\Omega = \{0, 1, ..., n\}$ (see Example 8.6). Recall that $\Phi_1 : \mathbb{T}^1 \times \mathbb{P}_1(\frac{1}{n}) \to \mathbb{P}_1(\frac{1}{n})$ is the regular torification of $\mathcal{B}(n)$ (see Proposition 8.11).

Proposition 11.3. The lift of the inclusion map $\mathcal{B}(n) \hookrightarrow \mathcal{P}_{n+1}^{\times}$ is the Veronese embedding:

$$\begin{array}{rccl} m: & \mathbb{P}_1(\frac{1}{n}) & \to & \mathbb{P}_n(1) \\ & [z_1, z_2] & \mapsto & \left[z_1^n, ..., \binom{n}{k}^{1/2} z_1^{n-k} z_2^k, ..., z_2^n \right]. \end{array}$$

The corresponding Lie group homomorphism $\rho : \mathbb{T}^1 \to \mathbb{T}^n$ is given by

$$\rho([t]) = \big[nt, ..., (n-k)t, ..., t\big].$$

Proof. Let $j : \mathcal{B}(n) \hookrightarrow \mathcal{P}_{n+1}^{\times}$ be the inclusion map. In the proof of Lemma 11.1, we computed the coordinate expression for j in the natural parameters. With $F(k) = k$ and $C(k) = \ln \binom{n}{k}$, $k = 0, ..., n$ (see Example 8.6), this yields

$$j(\theta) = -\theta(n, n-1, ..., 1) + \big(\ln \tbinom{n}{0}, ..., \ln \tbinom{n}{n-1}\big)$$

for all $\theta \in \mathbb{R}$. Under the usual identifications $T\mathcal{B}(n) = \mathbb{C}$ and $T\mathcal{P}_{n+1}^{\times} = \mathbb{C}^n$, we get

$$j_*(z) = -z(n, n-1, ..., 1) + \big(\ln \tbinom{n}{0}, ..., \ln \tbinom{n}{n-1}\big)$$

for all $z \in \mathbb{C}$. Let $\tau : \mathbb{C} = T\mathcal{B}(n) \to \mathbb{P}_1(\frac{1}{n})^\circ$, $z \mapsto [e^{z/2}, 1]$ and $\tau' : \mathbb{C}^n = T\mathcal{P}_{n+1}^{\times} \to \mathbb{P}_n(1)^\circ$, $(z_1, ..., z_n) \mapsto [e^{z_1/2}, ..., e^{z_n/2}, 1]$. By Proposition 8.12, τ and τ' are compatible covering maps. We compute:

$$\begin{aligned} (\tau' \circ j_*)(z) &= \tau'\big(- z(n, n-1, ..., 1) + \big(\ln \tbinom{n}{0}, ..., \ln \tbinom{n}{n-1}\big)\big) \\ &= \big[\tbinom{n}{0}^{1/2} (e^{-z/2})^n, ..., \tbinom{n}{k}^{1/2} (e^{-z/2})^{n-k}, ..., 1\big] \\ &= \big[\tbinom{n}{0}^{1/2}, ..., \tbinom{n}{k}^{1/2} (e^{z/2})^k, ..., (e^{z/2})^n\big], \end{aligned} \tag{11.4}$$

where, in the last line, we have multiplied every entry by $(e^{z/2})^n$. Let $\widetilde{m} : \mathbb{P}_1(\frac{1}{n}) \to \mathbb{P}_n(1)$ be defined by $\widetilde{m}([z_1, z_2]) = \big[z_2^n, ..., \binom{n}{k}^{1/2} z_2^{n-k} z_1^k, ..., z_1^n\big]$. We compute:

$$(\widetilde{m} \circ \tau)(z) = \widetilde{m}([e^{z/2}, 1]) = \big[1, ..., \tbinom{n}{k}^{1/2}(e^{z/2})^k, ..., (e^{z/2})^n\big] \tag{11.5}$$

for all $z \in \mathbb{C}$. Comparing (11.4) and (11.5), we see that $\widetilde{m} \circ \tau = \tau' \circ j_*$. Therefore $\widetilde{m}$ is the lift of j with respect to τ and τ'. Let G be the holomorphic isometry of $\mathbb{P}_1(\frac{1}{n})$ defined by $G([z_1, z_2]) = [z_2, z_1]$. Since $G \circ (\Phi_1)_{[t]} = (\Phi_1)_{[-t]} \circ G$ for all $t \in \mathbb{R}$, Proposition 9.16 implies that $m = \widetilde{m} \circ G$ is a lift of j. The second formula is obtained by a direct computation. □

Let $\mathcal{M}(m+1, n)$ be the set of multinomial distributions defined over $\Omega_{m+1,n} = \{(k_1, ..., k_{m+1}) \in \mathbb{N}^{m+1} \mid k_1 + ... + k_{m+1} = n\}$ (see Example 8.7). Recall that $\mathcal{M}(m+1, n)$ is toric with regular torification $\Phi_m : \mathbb{T}^m \times \mathbb{P}_m(\frac{1}{n}) \to \mathbb{P}_m(\frac{1}{n})$ (see Proposition 8.11). Since $\mathrm{Card}(\Omega_{m+1,n}) = \binom{m+n}{m} = \frac{(m+n)!}{m!n!}$, $\mathcal{M}(m+1, n)$ is a subset of $\mathcal{P}^\times_{\binom{m+n}{m}}$.

Proposition 11.4. The lift of the inclusion map $\mathcal{M}(m+1, n) \hookrightarrow \mathcal{P}^\times_{\binom{m+n}{m}}$ is the *n-th* Veronese embedding:

$$\begin{array}{ccc} \mathbb{P}_m(\frac{1}{n}) & \to & \mathbb{P}_{\binom{m+n}{m}-1}(1) \\ [z_1, z_2, ..., z_{m+1}] & \mapsto & \left[\sqrt{\dfrac{n!}{k_1!...k_{m+1}!}}\; z_1^{k_1} ... z_{m+1}^{k_{m+1}}\right]_{k \in \Omega_{m+1,n}} \end{array}.$$

Proof. Let $\phi : \{1, 2, ..., \binom{m+n}{m}\} \to \Omega_{m+1,n}$ be a bijection. Let $K := \phi(\binom{m+n}{m})$ and $A = \Omega_{m+1,n} - \{K\}$ (K is just the "last" element of $\Omega_{m+1,n}$). The coordinate expression for the inclusion map $j : \mathcal{M}(m+1, n) \hookrightarrow \mathcal{P}^\times_{\binom{m+n}{m}}$ in the natural parameters is given by $j(\theta_1, ..., \theta_m) = [\theta'_k]_{k \in A}$, where

$$\begin{cases} \theta'_k = \sum_{j=1}^m (F_j(k) - F_j(K))\theta_j + C(k) - C(K), \\ F_j(k) = k_j \quad \text{and} \quad C(k) = \ln\big(\frac{n!}{k_1!....k_{m+1}!}\big) \end{cases}$$

(see Example 8.7 and the proof of Lemma 11.1). Under the usual identifications $T\mathcal{M}(m+1, n) = \mathbb{C}^m$ and $T\mathcal{P}^\times_{\binom{m+n}{m}} = \mathbb{C}^{\binom{m+n}{m}-1}$, we have $j_*(z_1, ..., z_m) = [z'_k]_{k \in A}$, where

$$z'_k = \sum_{j=1}^m (F_j(k) - F_j(K))z_j + C(k) - C(K).$$

Let $\tau : \mathbb{C}^m \to \mathbb{P}_m(\frac{1}{n})$, $(z_1, ..., z_m) \mapsto [e^{z_1/2}, ..., e^{z_m/2}, 1]$ and $\tau' : \mathbb{C}^{\binom{m+n}{m}-1} \to \mathbb{P}_{\binom{m+n}{m}-1}(1)$, $(z_k)_{k \in A} \mapsto [(e^{z_k/2})_{k \in A}, 1]$. By Proposition 8.12, τ and τ' are compatible covering maps. We compute:

$$\begin{aligned}
(\tau' \circ j_*)(z_1, ..., z_m) &= \tau'([z'_k]_{k\in A}) = [(e^{z'_k/2})_{k\in A}, 1] \\
&= \Big[\Big(e^{\frac{1}{2}\left(\sum_{j=1}^m (F_j(k)-F_j(K))z_j + C(k) - C(K)\right)}\Big)_{k\in A}, 1\Big] \\
&= \Big[\Big(e^{\frac{1}{2}\left(\sum_{j=1}^m F_j(k)z_j + C(k)\right)}\Big)_{k\in A}, e^{\frac{1}{2}\left(\sum_{j=1}^m F_j(K)z_j + C(K)\right)}\Big] \\
&= \Big[e^{\left(\frac{1}{2}\sum_{j=1}^m F_j(k)z_j + C(k)\right)}\Big]_{k\in\Omega_{m+1,n}} \\
&= \Bigg[\sqrt{\frac{n!}{k_1!...k_{m+1}!}}\,(e^{z_1/2})^{k_1}...(e^{z_m/2})^{k_m}\Bigg]_{k\in\Omega_{m+1,n}} \\
&= (m\circ\tau)(z_1, ..., z_m),
\end{aligned}$$

where m is the *n-th* Veronese embedding. This shows that m is the lift of j. $\square$

Now we show how to construct new examples from old ones.

Proposition 11.5 (Torification of products). Given $i = 1, 2$, let $\psi_i : \mathbb{R}^{n_i} \to \mathbb{R}$ be a smooth function whose Hessian is positive definite at each point of $\mathbb{R}^n$. Suppose that $\Phi_i : \mathbb{T}^{n_i} \times N_i \to N_i$ is a torification of $(\mathbb{R}^{n_i}, \mathrm{Hess}(\psi_i), \nabla^{\mathrm{flat}})$, $i = 1, 2$. Let $\Phi = \Phi_1 \times \Phi_2 : \mathbb{T}^{n_1+n_2} \times (N_1 \times N_2) \to N_1 \times N_2$ be the torus action defined by

$$\Phi\big((a, b), (x, y)\big) = \big(\Phi_1(a, x), \Phi_2(b, y)\big).$$

(1) $N_1 \times N_2$, together with the torus action Φ, is a torification of $(\mathbb{R}^{n_1+n_2}, \mathrm{Hess}(\psi), \nabla^{\mathrm{flat}})$, where $\psi : \mathbb{R}^{n_1+n_2} = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$, $(x, y) \mapsto \psi_1(x) + \psi_2(y)$.

(2) If $\tau_i : T\mathbb{R}^{n_i} \to N_i^\circ$ is a compatible covering map, $i = 1, 2$, then the map $\tau : T\mathbb{R}^{n_1+n_2} = T\mathbb{R}^{n_1} \times T\mathbb{R}^{n_2} \to N_1^\circ \times N_2^\circ$, defined by $\tau(u, v) = \big(\tau_1(u), \tau_2(v)\big)$, is a compatible covering map.

(3) If N_1 and N_2 are regular, then $N_1 \times N_2$ is regular.

Proof. (1) Let (L_i, X_i, F_i) be a toric factorization of $\Phi_i : \mathbb{T}^{n_i} \times N_i \to N_i$, $i = 1, 2$. Write $X_i = ((X_i)_1, ..., (X_i)_{n_i})$, where $(X_i)_j$ is a vector field on $\mathbb{R}^{n_i}$. Given $1 \le i \le n_1$ and $1 \le j \le n_2$ define the vector fields $(\widetilde{X}_1)i$ and $(\widetilde{X}_2)_j$ on $\mathbb{R}^{n_1+n_2} = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ by letting

$$(\widetilde{X}_1)_i(p_1, p_2) = ((X_1)_i(p_1), 0) \qquad \text{and} \qquad (\widetilde{X}_2)_j(p_1, p_2) = (0, (X_2)_j(p_2)).$$

Clearly the vector fields $(\widetilde{X}_i)_j$ are pointwise linearly independent and parallel with respect to the flat connection (since they are constant). It follows that $\widetilde{X} = ((\widetilde{X}_1)_1, ..., (\widetilde{X}_1)_{n_1}, (\widetilde{X}_2)_1, ..., (\widetilde{X}_2)_{n_2})$ is the generator of a parallel lattice $L \subset T\mathbb{R}^{n_1+n_2}$.

Under the natural identifications $\Gamma(L) = \mathbb{Z}^{n_1+n_2}$ and $\Gamma(L_i) = \mathbb{Z}^{n_i}$, the map $\mathbb{Z}^{n_1+n_2} \to \mathbb{Z}^{n_1} \times \mathbb{Z}^{n_2}$, $(k_1, ..., k_{n_1+n_2}) \mapsto ((k_1, ..., k_{n_1}), (k_{n_1+1}, ..., k_{n_1+n_2}))$ induces a group isomorphism $\phi : \Gamma(L) \to \Gamma(L_1) \times \Gamma(L_2)$. Let $\pi_i : \mathbb{R}^{n_1+n_2} = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_i}$ be the projection onto $\mathbb{R}^{n_i}$ and let $G : T\mathbb{R}^{n_1+n_2} \to T\mathbb{R}^{n_1} \times T\mathbb{R}^{n_2}$ be the diffeomorphism defined by $G(u) = ((\pi_1)_* u, (\pi_2)_* u)$. A direct verification using $(\pi_k)_{*_u}(\widetilde{X_i})_j = \delta_{ki}(X_i)_j(\pi_k(u))$ (δ_{ki} = Kronecker delta) shows that G is equivariant in the sense that

$$G \circ \gamma = \phi(\gamma) \cdot G$$

for all $\gamma \in \Gamma(L)$, where the action of $\Gamma(L_1) \times \Gamma(L_2)$ on $T\mathbb{R}^{n_1} \times T\mathbb{R}^{n_2}$ is given by $(\gamma_1, \gamma_2) \cdot (u, v) = (\gamma_1(u), \gamma_2(v))$. Moreover, a direct verification using Proposition 2.17 and the formula

$$\text{Hess}(\psi) = \begin{bmatrix} \text{Hess}(\psi_1) & 0 \\ 0 & \text{Hess}(\psi_2) \end{bmatrix}$$

shows that G is a Kähler isomorphism. It follows that G descends to a Kähler isomorphism,

$$\widetilde{G} \ : \ \mathbb{R}^{n_1+n_2}_L \to \mathbb{R}^{n_1}_{L_1} \times \mathbb{R}^{n_2}_{L_2}.$$

Let $\Phi_{\widetilde{X}} : \mathbb{T}^{n_1+n_2} \times \mathbb{R}^{n_1+n_2}_L \to \mathbb{R}^{n_1+n_2}_L$ and $\Phi_{X_i} : \mathbb{T}^{n_i} \times \mathbb{R}^{n_i}_{L_i} \to \mathbb{R}^{n_i}_{L_i}$ be the torus actions associated to the generators $\widetilde{X}$ and X_i, respectively (see (3.2)). Let $\Phi_{X_1} \times \Phi_{X_2}$ be the action of $\mathbb{T}^{n_1+n_2} = \mathbb{T}^{n_1} \times \mathbb{T}^{n_2}$ on $\mathbb{R}^{n_1}_{L_1} \times \mathbb{R}^{n_2}_{L_2}$ defined by $(\Phi_{X_1} \times \Phi_{X_2})((a, b), (x, y)) = \big(\Phi_{X_1}(a, x), \Phi_{X_2}(b, y)\big)$. A direct calculation shows that

$$\widetilde{G} \circ \big(\Phi_{\widetilde{X}}\big)_a = (\Phi_{X_1} \times \Phi_{X_2})_a \circ \widetilde{G}$$

for all $a \in \mathbb{T}^{n_1+n_2}$. Thus $\widetilde{G}$ is equivariant.

Now let $F : \mathbb{R}^{n_1}_{L_1} \times \mathbb{R}^{n_2}_{L_2} \to N_1^\circ \times N_2^\circ = (N_1 \times N_2)^\circ$ be the equivariant Kähler isomorphism defined by $F(x, y) = (F_1(x), F_2(y))$. Since the composition $F \circ \widetilde{G}$ is an equivariant Kähler isomorphism from $\mathbb{R}^{n_1+n_2}_L$ to $(N_1 \times N_2)^\circ$, $N_1 \times N_2$ is a torification of $(\mathbb{R}^{n_1+n_2}, \text{Hess}(\psi), \nabla^{\text{flat}})$ with corresponding torus action $\Phi_1 \times \Phi_2$.

(2) Suppose that τ_i is induced by the toric factorization (L_i, X_i, F_i), $i = 1, 2$. This means that $\tau_i = F_i \circ q_{L_i}$, where $q_{L_i} : T\mathbb{R}^{n_i} \to \mathbb{R}^{n_i}_{L_i}$ is the quotient map associated to the action of $\Gamma(L_i)$ on $T\mathbb{R}^{n_i}$. Let $\widetilde{X}, L, G$ and $\widetilde{G}$ be defined as above. It follows from the discussion above that $(L, \widetilde{X}, F \circ \widetilde{G})$ is a toric factorization and that the diagram

$$\begin{array}{ccccc} T\mathbb{R}^{n_1+n_2} & \xrightarrow{G} & T\mathbb{R}^{n_1} \times T\mathbb{R}^{n_2} & & \\ \downarrow{\scriptstyle q_L} & & \downarrow{\scriptstyle q_{L_1} \times q_{L_2}} & \searrow^{\tau_1 \times \tau_2} & \\ \mathbb{R}^{n_1+n_2}_L & \xrightarrow[\widetilde{G}]{} & \mathbb{R}^{n_1}_{L_1} \times \mathbb{R}^{n_2}_{L_2} & \xrightarrow[F]{} & N_1^\circ \times N_2^\circ. \end{array}$$

is commutative. From this we see that $F \circ \widetilde{G} \circ q_L = (\tau_1 \times \tau_2) \circ G$ is a compatible covering map. If $T\mathbb{R}^{n_1+n_2}$ and $T\mathbb{R}^{n_1} \times T\mathbb{R}^{n_2}$ are identified via the map G, then $\tau_1 \times \tau_2$ itself is a compatible covering map.

(3) This is immediate. □

Definition 11.6. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be exponential families defined over the finite sets $\Omega_1 = \{x_1, ..., x_r\}$ and $\Omega_2 = \{y_1, ..., y_s\}$, respectively. The *product* of $\mathcal{E}_1$ and $\mathcal{E}_2$, denoted by $\mathcal{E}_1 \times \mathcal{E}_2$, is the set of all maps $p : \Omega = \Omega_1 \times \Omega_2 \to \mathbb{R}$ of the form $p(x_i, y_j) = p_1(x_i)p_2(y_j)$, where $p_1 \in \mathcal{E}_1$ and $p_2 \in \mathcal{E}_2$.

One can readily check that $\mathcal{E}_1 \times \mathcal{E}_2$ is an exponential family of dimension $\dim(\mathcal{E}_1) + \dim(\mathcal{E}_2)$ defined over $\Omega = \Omega_1 \times \Omega_2$. If $\psi_1 : \mathbb{R}^{n_1} \to \mathbb{R}$ and $\psi_2 : \mathbb{R}^{n_2} \to \mathbb{R}$ are the cumulant generating functions of $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively, then $\psi : \mathbb{R}^{n_1+n_2} = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$, $(\theta, \theta') \mapsto \psi_1(\theta) + \psi_2(\theta')$ is the cumulant generating function of $\mathcal{E}_1 \times \mathcal{E}_2$. It follows from this and Proposition 11.5 that if $\Phi_i : \mathbb{T}^{n_i} \times N_i \to N_i$ is a torification of $\mathcal{E}_i$, then $N_1 \times N_2$ is a torification of $\mathcal{E}_1 \times \mathcal{E}_2$ with torus action $\Phi_1 \times \Phi_2$. Moreover, if $\tau_i : T\mathcal{E}_i \to N_i^\circ$ is a compatible covering map, then $\tau_1 \times \tau_2 : T\mathcal{E}_1 \times T\mathcal{E}_2 \to N_1^\circ \times N_2^\circ$, $(u, v) \mapsto (\tau_1(u), \tau_2(v))$ is a compatible covering map.

Example 11.7. $\mathbb{P}_n(1) \times \mathbb{P}_m(1)$ is the regular torification of $\mathcal{P}_{n+1}^\times \times \mathcal{P}_{m+1}^\times$.

Proposition 11.8. The lift of the inclusion map $\mathcal{P}_{n+1}^\times \times \mathcal{P}_{m+1}^\times \to \mathcal{P}_{(n+1)(m+1)}^\times$ is the Segre embedding:

$$\begin{aligned} \mathbb{P}_n(1) \times \mathbb{P}_m(1) &\to \mathbb{P}_{(n+1)(m+1)-1}(1) \\ ([z_i], [w_j]) &\mapsto [z_i w_j], \end{aligned}$$

where $i = 0, ..., n, j = 0, ..., m$ and lexicographic ordering is adopted.

Sketch of proof. The proof is entirely analogous to the proof for Proposition 11.4. For simplicity, we assume $n = m = 1$. In this case, the Segre embedding $f : \mathbb{P}_1(1) \times \mathbb{P}_1(1) \to \mathbb{P}_3(1)$ reads $f([z_1, z_2], [w_1, w_2]) = [z_1 w_1, z_1 w_2, z_2 w_1, z_2 w_2]$. Let θ be the natural parameter on $\mathcal{P}_2^\times$. Elements of $\mathcal{P}_2^\times$ are parametrized as follows: $p(x_i; \theta) = e^{\delta_{1i}\theta - \psi(\theta)}$, where $x_i \in \Omega = \{x_1, x_2\}$ and $\psi(\theta) = \ln(1 + e^\theta)$ (see Example 8.5). By definition, if $p \in \mathcal{P}_2^\times \times \mathcal{P}_2^\times$, then there are real numbers θ_1 and θ_2 such that $p(x_i, x_j) = p(x_i; \theta_1)p(x_j; \theta_2)$ for all $x_i, x_j \in \Omega$, and hence

$$p(x_i, x_j) = e^{\delta_{1i}\theta_1 + \delta_{1j}\theta_2 - \psi(\theta_1) - \psi(\theta_2)} = e^{H_1(x_i,x_j)\theta_1 + H_2(x_i,x_j)\theta_2 - \phi(\theta_1,\theta_2)},$$

where $H_1(x_i, x_j) = \delta_{1i}$, $H_2(x_i, x_j) = \delta_{1j}$ and $\phi(\theta_1, \theta_2) = \psi(\theta_1) + \psi(\theta_2)$. This shows in particular that $\mathcal{P}_2^\times \times \mathcal{P}_2^\times$ is an exponential family with natural parameters (θ_1, θ_2) and cumulant generating function ϕ.

In order to find the local expression for the inclusion $j : \mathcal{P}_2^\times \times \mathcal{P}_2^\times \to \mathcal{P}_4^\times$, we need to give $\Omega \times \Omega$ an ordering. Let $y_1 = (x_1, x_1)$, $y_2 = (x_1, x_2)$, $y_3 = (x_2, x_1)$ and $y_4 = (x_2, x_2)$. Then $\Omega \times \Omega = \{y_1, y_2, y_3, y_4\}$. Let $(\theta_1', \theta_2', \theta_3')$ be the natural parameters on $\mathcal{P}_4^\times$. Taking into account the proof of Lemma 11.1, we see that the coordinate expression for the inclusion map j in the natural parameters is given by

$$\begin{bmatrix} \theta_1' \\ \theta_2' \\ \theta_3' \end{bmatrix} = \begin{bmatrix} H_1(y_1) - H_1(y_4) & H_2(y_1) - H_2(y_4) \\ H_1(y_2) - H_1(y_4) & H_2(y_2) - H_2(y_4) \\ H_1(y_3) - H_1(y_4) & H_2(y_3) - H_2(y_4) \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}.$$

Since $H_1(y_1) = H_1(y_2) = H_2(y_1) = H_2(y_3) = 1$ and all the other values of H_1 and H_2 are zero, we find

$$\begin{bmatrix} \theta_1' \\ \theta_2' \\ \theta_3' \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}.$$

Thus $j(\theta_1, \theta_2) = (\theta_1 + \theta_2, \theta_1, \theta_2)$. Under the usual identification $T\mathcal{P}_{n+1}^{\times} = \mathbb{C}^n$, the derivative $j_* : \mathbb{C} \times \mathbb{C} \to \mathbb{C}^3$ is given by $j_*(z_1, z_2) = (z_1 + z_2, z_1, z_2)$. Let $\tau' : \mathbb{C}^3 \to \mathbb{P}_3(1)$, $(z_1, z_2, z_3) \mapsto [e^{z_1/2}, e^{z_2/2}, e^{z_3/3}, 1]$. Recall that τ' is a compatible covering map (see Proposition 8.12). We compute:

$$(\tau' \circ j_*)(z_1, z_2) = \tau(z_1 + z_2, z_1, z_2) = [e^{(z_1+z_2)/2}, e^{z_1/2}, e^{z_2/2}, 1]. \tag{11.6}$$

Let $\tau : \mathbb{C} \to \mathbb{P}_1(1)$, $z \mapsto [e^{z/2}, 1]$. By Proposition 11.5, $\tau \times \tau$ is a compatible covering map. We compute:

$$(f \circ (\tau \times \tau))(z_1, z_2) = f([e^{z_1/2}, 1], [e^{z_2/2}, 1]) = [e^{(z_1+z_2)/2}, e^{z_1/2}, e^{z_2/2}, 1]. \tag{11.7}$$

Comparing (11.6) and (11.7), we see that $\tau' \circ j_* = f \circ (\tau \times \tau)$. This shows that the Segre embedding f is a lift of j. □

12 Duality

In this section, we summarize some of the results of this paper in the form of a duality (bijection) similar to Delzant correspondence in symplectic geometry [Del88].

As the literature is not uniform, we give the following definition.

Definition 12.1. A *Kähler toric manifold* is a connected Kähler manifold N of complex dimension n equipped with an effective isometric and holomorphic action $\Phi : \mathbb{T}^n \times N \to N$ of the n-dimension real torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ such that for every $\xi \in \text{Lie}(\mathbb{T}^n)$, the vector vector field $J\xi_N$ is complete, where J is the complex structure on N and ξ_N is the fundamental vector field on N associated to ξ.

Note that the definition of a torification N does not require the vector fields $J\xi_N$ to be complete (see Section 6). Therefore torifications are not necessarily Kähler toric manifolds in the sense of Definition 12.1. Note also that if a Kähler toric manifold is regular, then it is simply connected and hence the torus action is Hamiltonian (this follows, for example, from [OR04], Propositions 4.5.17 and 4.5.19).

Recall that two Kähler toric manifolds $\Phi : \mathbb{T}^n \times N \to N$ and $\Phi' : \mathbb{T}^n \times N' \to N'$ are *equivalent* if there exist a Kähler isomorphism $G : N \to N'$ and a Lie group isomorphism $\rho : \mathbb{T}^n \to \mathbb{T}^n$ such that $G \circ \Phi_a = \Phi_{\rho(a)} \circ G$ for all $a \in \mathbb{T}^n$. In this case, we write $N \sim N'$.

Recall that two dually flat manifolds (M, h, ∇) and (M', h', ∇') are *equivalent* if there is an isomorphism of dually flat spaces between them. In this case, we also write $M \sim M'$.

Equivalence classes are denoted by $[M, h, \nabla]$ and $[\Phi : \mathbb{T}^n \times N \to N]$, or simply $[M]$ and $[N]$.

Theorem 12.2. Let A be the set of toric dually flat manifolds (M, h, ∇) of dimension n admitting a global pair of dual coordinate systems and whose regular torifications are Kähler toric manifolds. Let B be the set of regular Kähler toric manifolds N of complex dimension n. The map

$$\begin{aligned} A/\!\sim \quad &\to \quad B/\!\sim, \\ [M, h, \nabla] \quad &\mapsto \quad [\text{regular torification of } M] \end{aligned}$$

is a bijection.

Proof. Let F be the map in the theorem. The existence of lifts guarantees that F is well defined. Injectivity of F is a direct consequence of Proposition 6.13. Surjectivity of F follows from Theorem 7.1. $\square$

A Legendre transform

Throughout this section, $(x_1, ..., x_n)$ are standard coordinates on $\mathbb{R}^n$ and $\langle\, ,\, \rangle$ is the ordinary inner product in $\mathbb{R}^n$. Given a differentiable function $h : U \to \mathbb{R}$ defined on an open set $U \subseteq \mathbb{R}^n$, we will denote by $\mathrm{grad}(h)$ the corresponding gradient map. Thus $\mathrm{grad}(h)(x) = \big(\frac{\partial h}{\partial x_1}(x), ..., \frac{\partial h}{\partial x_n}(x)\big)$, $x \in U$.

The material presented in this section is taken from [Roc67] (see also [Roc70]).

Definition A.1. Let h be a differentiable real-valued function defined on a non-empty open set U in $\mathbb{R}^n$. Let U^* be the image of U under the gradient map $\mathrm{grad}(h)$. If $\mathrm{grad}(h)$ is injective, then the function

$$h^*(x^*) = \big\langle x^*, (\mathrm{grad}(h))^{-1}(x^*)\big\rangle - h\big((\mathrm{grad}(h))^{-1}(x^*)\big)$$

is well-defined on U^*. The pair (U^*, h^*) is called the *Legendre transform* of (U, h).

Definition A.2. We shall say that a pair (U, h) is a *convex function of Legendre type* on $\mathbb{R}^n$ if the following conditions hold:

(1) U is a non-empty open convex set in $\mathbb{R}^n$.

(2) $h : U \to \mathbb{R}$ is a strictly convex differentiable function on U.

(3) $\lim_{\lambda \to 0^+} \frac{d}{d\lambda} h(\lambda a + (1-\lambda)x) = -\infty$ whenever $a \in U$ and x is a boundary point of U.

Note that the third condition is automatically satisfied when $U = \mathbb{R}^n$ (since there is no boundary point in this case).

Theorem A.3 ([Roc67])**.** Let (U, h) be a convex function of Legendre type on $\mathbb{R}^n$. The Legendre transform (U^*, h^*) is then well-defined. It is another convex function of Legendre type on $\mathbb{R}^n$, and $\mathrm{grad}(h^*) = \big(\mathrm{grad}(h)\big)^{-1}$ on U^*. The Legendre transform of (U^*, h^*) is (U, h) again.

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