Abstract-Golay complementary pairs (GCPs) and complete

complementary codes (CCCs) have found a wide range of practical applications in coding, signal processing and wireless communication due to their ideal correlation properties. In fact, binary CCCs have special advantages in spread spectrum communication due to their simple modulo-2 arithmetic operation. modulation and correlation simplicity, but they are limited in length. In this paper, we present a direct construction of GCPs, mutually orthogonal complementary sets (MOCSs) and binary CCCs of non-power of two lengths to widen their application in the recent field. First, a generalised Boolean function (GBF) based truncation technique has been used to construct GCPs of non-power of two lengths. Then Complementary sets (CSs) and MOCSs of lengths of the form $2^{m-1} + 2^{m-3}$ $(m \ge 5)$ and $2^{m-1}+2^{m-2}+2^{m-4}$ $(m \ge 6)$ are generated by GBFs. Finally, binary CCCs with desired lengths are constructed using the union of MOCSs. The row and column sequence peak to mean envelope power ratio (PMEPR) has been investigated and compared with existing work. The column sequence PMEPR of resultant CCCs can be effectively upper bounded by 2.

Index Terms-Complementary set (CS), complete complementary set (CCC), generalised Boolean function (GBF), Golay complementary pair (GCP), mutually orthogonal complementary set (MOCS)

I. INTRODUCTION

THE Golay complementary pairs (GCPs) were first introduced by Golay [1]. The aperiodic auto-correlation sum (AACS) of a GCP diminishes to zero for all time shifts except at zero. The sequences in a GCP are known as Golay sequences. The idea of GCP is further extended to the complementary set (CS) by Tseng and Liu [2]. A CS is a set of $M(\geq 2)$ sequences of length N with the property that their AACS sum is zero for all non-zero time shifts. Tseng and Liu also proposed the concept of (K, M, N)mutually orthogonal complementary set (MOCS), which is a collection of K CSs each of having M sequences of length N, such that any two distinct CSs are orthogonal to each other, and follows the property $K \leq M$ [3]. For a special case, when the set size of MOCS achieves its upper bound, i.e., K = M, it is known as a set of complete complementary code (CCC) and is denoted by (K, K, N)-CCC [4]. Due to the ideal correlation properties and optimal set size, CCCs have found their application in next-generation multi-carrier code division multiple access (MC-CDMA) [5]-[9]. Apart from this, CCCs

Praveen Kumar and Subhabrata Paul are with the Department of Mathematics, IIT Patna, Bihta, Patna, 801106, Bihar, India (e-mail: praveen_2021ma03@iitp.ac.in; subhabrata@iitp.ac.in).

are utilized in optimal channel estimation in multiple-input and multiple-output (MIMO) frequency-selective fading channels [10], MIMO radar [11], [12], cell search in orthogonal frequency division multiplexing (OFDM) systems [13], and data hiding [14]. In spread spectrum communication, the binary CCC is preferred compared to non-binary CCC due to its simple modulo-2 arithmetic operation, modulation and correlation simplicity.

1

Due to modulo-2 arithmetic operation, binary sequences are easy to implement electronically. The modulo-2 arithmetic is isomorphic with the use of $\{\pm 1\}$ which simplifies both the modulation and correlation processes. However, it is difficult in many cases to get flexible lengths for binary sequences. It has been proved in [15] that binary GCPs exist for only even length. Binary Z-complementary pairs (ZCPs) were introduced by Fan et al. in [16] and they also proved that ZCPs exists for all possible lengths. Several constructions of binary ZCPs of different lengths are proposed in [17], [18]. Construction of binary CSs of non-power of two lengths can be found in [19].

In the year 1999, Davis and Jedwab have proposed a direct construction of 2^h -ary $(h \in \mathbb{N})$ GCPs of length $2^m (m \in \mathbb{N})$ using generalised Boolean functions (GBFs) [20]. Paterson extended the idea of 2^{h} -ary GCPs to q-ary (for even q) GCPs [21]. The construction of GCPs of length $2^{\alpha}10^{\beta}26^{\gamma}$ $(\alpha, \beta, \gamma \in \mathbb{N})$ is provided by using repeated application of Turyn's construction [22]. In [21], Paterson has also proposed a GBFs based construction of CSs of length 2^m . In the recent development GBFs based construction of CSs with more flexible lengths have been proposed in [23]–[27]. CSs with flexible lengths are of interest to OFDM systems where numbers of subcarriers are varied, i.e., non-power of two adopted by the LTE system. A direct and generalised construction of polyphase CSs is proposed in [28] and it has low peak to mean envelope power ratio (PMEPR).

In [29], Rathinakumar and Chaturvedi proposed a direct construction of CCCs of length 2^m by extending the Paterson's idea of CSs generation. A number of direct constructions of CCCs with lengths 2^m are presented in [8], [30]–[32]. Several GBFs based constructions of Z-complementary code sets (ZCCSs) of non-power of two lengths are proposed in the literature [33]-[38], to extend the number of users in ZCCS based MC-CDMA system compared to that of CCC based MC-CDMA system. Apart from the GBFs based construction, MOCSs with non-power of two lengths can be constructed by using different systematic methods, which include reversals, negations, interleaving, concatenations etc. [2], [39]. In the same way, Das et al. presented the construction of MOCSs and binary CCCs of different lengths by using paraunitary (PU)

Praveen Kumar, Sudhan Majhi, Subhabrata Paul

Sudhan Majhi is with the Department of Electrical Communication Engineering, IISC Bangalore, CV Raman Rd, Bengaluru, 560012, Karnataka, India (email:smajhi@iisc.ac.in)

matrices [40]-[42]. PU matrix based construction of ZCCSs has been proposed in [43]. However, the sequence or code generated through these indirect methods may not be friendly for hardware generation due to their large space and time requirements. In [44], [45] direct construction of MOCSs with non-power of two lengths have been proposed, where the set size is upper bounded by half of the number of constituent sequences in a CS, i.e., $K \leq M/2$. In addition, the authors presented an open problem of direct construction of CCCs with non-power of two lengths in [44]. Recently, Sarkar et al. has proposed the construction of CCCs of lengths $p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}$ (where p_i 's are prime and m_i 's are positive integers), using multivariable functions (MVFs) [46]. This direct construction can generate q-ary CCCs of all possible lengths. However, in the case of q = 2, only binary CCC of length in the form 2^m $(m \in \mathbb{Z})$ can be constructed [46]. So, the direct construction of GCPs and binary CCCs of non-power of two lengths is still an open problem.

By motivation of the open problem in [44], [46], in this paper, direct construction of GCPs, MOCSs and binary CCCs of length $2^{m-1} + 2^{m-3}$ $(m \ge 5)$ and $2^{m-1} + 2^{m-2} + 2^{m-4}$ $(m \ge 6)$ have been proposed. Using the idea of graphs corresponding to the quadratic part of GBFs, GCPs of nonpower of two lengths are constructed. For obtaining a GCP, the graph of the quadratic part of f has the property that deleting some vertices and all of their corresponding edges of the graph results in a path. In order to obtain sequences of non-power of two lengths, GBF based truncation technique is used. The idea has been further extended to generate CSs of non-power of two lengths. Using deleted vertices of the quadratic part of GBF, we rearrange the GBFs corresponding to the CS, and different GBF arrangements result in MOCSs of lengths non-power of two. Finally, binary CCC of non-power of two lengths has been constructed using the union of two MOCSs.

The remaining paper is organized as follows. Basic notations and definitions are provided in Section II. In sections III, IV and V the constructions of GCPs, MOCSs and binary CCCs of length non-power of two are given respectively. Section VI describes how to build additional non-power two-length GCPs, MOCSs, and CCCs. Section VII provides the row and column sequence PMEPR of the proposed CCCs. Finally, concluding remarks are provided in Section VIII.

II. NOTATIONS AND DEFINITIONS

In this section, the preliminaries, notations, and immediate results required for our proposed construction are discussed.

Definition 1: Let $\mathbf{d} = (d_0, d_1, \dots, d_{N-1})$ and $\mathbf{e} = (e_0, e_1, \dots, e_{N-1})$ be two length N complex-valued sequences then the aperiodic cross-correlation function (ACCF) between \mathbf{d} and \mathbf{e} at a shift s ($s \in \mathbb{Z}$) can be defined as

$$\mathcal{C}(\mathbf{d}, \mathbf{e})(s) = \begin{cases} \sum_{k=0}^{N-1-s} d_k \cdot e_{k+s}^*, & 0 \le s \le N-1, \\ \sum_{k=0}^{N-1-s} d_{k+s} \cdot e_k^*, & -N+1 \le s \le -1, \\ 0, & |s| \ge N, \end{cases}$$
(1)

where ()* is the complex conjugate operator. When d and e are equal, it is known as aperiodic auto-correlation function (AACF) of e and is denoted by $\mathcal{A}(e)(s)$.

We can also define the ACCF and AACF of \mathbb{Z}_q valued sequences by defining a one-one correspondence between \mathbb{Z}_q valued sequence $\mathbf{e}=(e_0, e_1, \ldots, e_{N-1})$ and the complexvalued sequence $\mathbf{e}'=(e'_0, e'_1, \ldots, e'_{N-1})$, where $e'_i = \omega^{e_i}$ and $\omega = \exp\left(2\pi\sqrt{-1}/q\right)$ is *q*th root of unity. So if **d** and **e** are \mathbb{Z}_q valued sequences then we define their ACCF $\mathcal{C}(\mathbf{d}, \mathbf{e})(s)$ and AACF $\mathcal{A}(\mathbf{e})(s)$ respectively as ACCF and AACF of the corresponding complex-value sequence **d**' and **e**'.

Definition 2: A set of M sequences $e^0, e^1, \ldots, e^{M-1}$, each of length N, is said to be a CS if

$$\mathcal{A}\left(\mathbf{e}^{0}\right)(s) + \mathcal{A}\left(\mathbf{e}^{1}\right)(s) + \dots + \mathcal{A}\left(\mathbf{e}^{M-1}\right)(s)$$
$$= \begin{cases} MN, & s = 0, \\ 0, & \text{otherwise} \end{cases}.$$

For M = 2, it is known as a GCP.

Definition 3: Consider a set $\mathcal{E} = \{E^0, E^1, \dots, E^{K-1}\}$, where each set E^p consists of M sequences, i.e., $E^p = \{\mathbf{e}_0^p, \mathbf{e}_1^p, \dots, \mathbf{e}_{M-1}^p\}$, and length of each sequence \mathbf{e}_l^p is N, where $0 \le p \le K-1$ and $0 \le l \le M-1$. The set \mathcal{E} is called an MOCS, denoted by (K, M, N)-MOCS, if the ACCF of E^p and $E^{p'}$ satisfies

$$\mathcal{C}\left(E^{p}, E^{p'}\right)(s) = \sum_{n=0}^{M-1} \mathcal{C}\left(\mathbf{e}_{n}^{p}, \mathbf{e}_{n}^{p'}\right)(s)$$

$$= \begin{cases} MN, \quad s = 0, p = p', \\ 0, \quad \text{otherwise}, \end{cases}$$
(2)

where $0 \le p, p' \le K - 1$; K, M and N are known as the set size, flock size and sequence length respectively. For a (K, M, N)-MOCS, the set size is always smaller than the flock size, i.e., $K \le M$. For the special case when K = M, the MOCS is called a CCC of order K and length N, and is denoted by (K, K, N)-CCC.

A. Generalised Boolean function

A GBF f in m binary variables $y_0, y_1, \ldots, y_{m-1}$ is a function from $\{0,1\}^m$ to \mathbb{Z}_q , where $q \ge 2$ is an even integer. A monomial of degree r is defined as the product of any r variables among $y_0, y_1, \ldots, y_{m-1}$. So there are $\sum_{r=0}^m {m \choose r} = 2^m$ monomials, namely $1, y_0, y_1, \ldots, y_{m-1}, y_0y_1, y_0y_2, \ldots, y_{m-2}y_{m-1}, \ldots, y_0y_1 \cdots y_{m-1}$. With the linear combination of these 2^m monomials and by taking coefficient from \mathbb{Z}_q , a GBF can be expressed uniquely. In the expression of a GBF of order r, there exist at least one highest-degree monomial of order r with non-zero coefficient. Corresponding to a GBF f of m variables $y_0, y_1, \ldots, y_{m-1}$, length $2^m \mathbb{Z}_q$ -valued vector is expressed as

$$\mathbf{f} = (f_0, f_1, \dots, f_{2^m - 1}), \qquad (3)$$

where $f_i = f(i_0, i_1, \dots, i_{m-1})$ and $(i_0, i_1, \dots, i_{m-1})$ is the binary vector representation of *i*. A complex-valued vector \mathbf{f}' is associated with every \mathbf{f} by $f'_i = \omega^{f_i}$. When it is clear from the context, only *f* is used to refer to both. Corresponding to a GBF *f* with *m* variables the sequence \mathbf{f} is of length 2^m .

We can restrict the domain of GBF to get sequences of length non-power of two. Let us define a set A which is a

subset of $\{0,1\}^m$. So, depending upon the domain A we can get different length sequences corresponding to GBF f. By \mathbf{a}_m we mean the binary vector representation of positive integer a in m components.

Example 1: Let $f : A \to \mathbb{Z}_2$ be defined as $f(y_0, y_1, y_2) = y_0y_1 + y_2$, where $A = \{\mathbf{0}_3, \mathbf{1}_3, \dots, \mathbf{5}_3\}$, then the sequence corresponding to f is (1, 1, 1, -1, -1, -1), which is of length 6. Similarly, if we define $A = \{\mathbf{3}_3, \mathbf{4}_3, \dots, \mathbf{7}_3\}$, then we get the sequence (-1, -1, -1, -1, 1), which is of length 5.

B. Graph of Quadratic form of GBF

Let $Q: \{0,1\}^m \to \mathbb{Z}_q$ be a GBF of order 2 defined by

$$Q(y_0, y_1, \dots, y_{m-1}) = \sum_{0 \le i < j < k} q_{ij} y_i y_j,$$
(4)

where $k \leq m$ and $q_{ij} \in \mathbb{Z}_q$. We associate a labeled graph G(Q) corresponding to the GBF Q, on k vertices by representing the vertices of G(Q) by $0, 1, \ldots, m-1$ and joining two vertices i and j by an edge labeled q_{ij} if and only if $q_{ij} \neq 0$. In the case, q = 2, q_{ij} can only take values either 0 or 1, so every edge is labelled 1 and by convention, edge labels are omitted in this case. From any given graph G(Q) of this type, the quadratic form Q can be easily and uniquely recovered. A graph G(Q) is called a path on k vertices if the number of edges is labelled q/2. For k = 1, this is a trivial path and for $k \geq 2$, this type of path is known as the Hamiltonian path. For $2 \leq k < m$, a path on k vertices

$$\frac{q}{2} \cdot \sum_{\alpha=1}^{k-1} y_{\pi(\alpha-1)} y_{\pi(\alpha)},\tag{5}$$

where π is a permutation of the set $\{0, 1, \ldots, k-1\}$.

C. Restricted Boolean function

Let $f : A \subseteq \{0,1\}^m \to \mathbb{Z}_q$ be a GBF in variables $y_0, y_1, \ldots, y_{m-1}$ and $\mathbf{y} = (y_{p_0}y_{p_1}\cdots y_{p_{k-1}})$ where $0 \le p_0 < p_1 < \cdots < p_{k-1} < m$. Let $\mathbf{c} = (c_0c_1\cdots c_{k-1})$ be a binary word of length k, i.e., $c_i \in \{0,1\}$. Then the vector $f|_{\mathbf{y}=\mathbf{c}}$ is defined to be the complex-valued vector with component $i = \sum_{j=0}^{m-1} i_j 2^j$ equal to $\omega^{f(i_0,i_1,\ldots,i_{m-1})}$ if $i_{j_{\alpha}} = c_{\alpha}$ for each $0 \le \alpha < k$, and equal to 0 otherwise. As a convention, if \mathbf{y} and \mathbf{c} are null (i.e., of length 0), then $f|_{\mathbf{y}=\mathbf{c}}$ represents the complex-valued vector associated with f.

Lemma 1 ([29]): Let $f, g : A \subseteq \{0, 1\}^m \to \mathbb{Z}_q$ be GBFs in variables $y_0, y_1, \ldots, y_{m-1}$. Let $\mathbf{y} = (y_{p_0}y_{p_1}\cdots y_{p_{k-1}})$ where $0 \leq p_0 < p_1 < \cdots < p_{k-1} < m$ and $\mathbf{c} = (c_0c_1\cdots c_{k-1})$ be a binary word of length k. Further let us denote $\mathbf{z} = (z_{i_0}z_{i_1}\cdots z_{i_{l-1}})$ where $0 \leq i_1 < i_2 < \cdots < i_{l-1} < m$ be a set of indices not in $\{p_0, p_1, \ldots, p_{k-1}\}$. Then for a binary vector $\mathbf{n} = (n_0n_1\cdots n_{k-1})$, the following equality holds

$$\mathcal{C}\left(f|_{\mathbf{y}=\mathbf{c}}, g|_{\mathbf{y}=\mathbf{n}}\right)(s) = \sum_{\mathbf{c}_{1}, \mathbf{c}_{2}} \mathcal{C}\left(f|_{\mathbf{y}\mathbf{z}=\mathbf{c}\mathbf{c}_{1}}, g|_{\mathbf{y}\mathbf{z}=\mathbf{n}\mathbf{c}_{2}}\right)(s).$$
(6)

Lemma 2 ([21]): Let $f : A \subseteq \{0,1\}^m \to \mathbb{Z}_q$ be a GBF in variables y_0, y_1, \dots, y_{m-1} . Let y and c are as defined in Lemma 1, then AACF is given by

$$\mathcal{A}(f)(s) = \sum_{\mathbf{c}} \mathcal{A}\left(f|_{\mathbf{y}=\mathbf{c}}\right)(s) + \sum_{\mathbf{c}_1 \neq \mathbf{c}_2} \mathcal{C}\left(f|_{\mathbf{y}=\mathbf{c}_1}, f|_{\mathbf{y}=\mathbf{c}_2}\right)(s)$$
(7)

III. PROPOSED CONSTRUCTION OF GCPs

In this section, we provide a GBFs based construction of GCPs for non-power of two lengths. Unless otherwise stated, this section and subsequent sections assume $m \ge 5$.

Suppose $Q: \{0,1\}^{m-4} \to \mathbb{Z}_q$ is the quadratic form in variables $z_0, z_1, \ldots, z_{m-5}$, i.e.,

$$Q(z_0, z_1, \dots, z_{m-5}) = \sum_{0 \le i < j < m-4} q_{ij} z_i z_j.$$
 (8)

For any $c, c_i \in \mathbb{Z}_q$, we define a GBF

$$f_1 = Q + \sum_{i=0}^{m-5} c_i z_i + c.$$
(9)

Using the notation $\bar{z}_i = 1 - z_i$ and f_1 defined in (9), the proposed GBF $f : A \to \mathbb{Z}_q$ is defined as

$$f = f_1 + \frac{q}{2}\bar{z}_{m-1} \left(\bar{z}_{m-4} \left(z_{m-3} + z_{m-2}\right) + z_{m-2}z_{m-3}\right) + \frac{q}{2}z_{\beta_1} \left(\bar{z}_{m-1} \left(z_{m-2}\bar{z}_{m-3}\bar{z}_{m-4} + z_{m-2}z_{m-3}\right) + z_{m-1}\bar{z}_{m-2}\bar{z}_{m-3}\right),$$
(10)

where $A = \{\mathbf{0}_m, \mathbf{1}_m, \dots, (\mathbf{2^{m-1} + 2^{m-3} - 1})_m\}.$

We will first prove a special case when the quadratic part of f given in (10) is zero.

Lemma 3: Let the quadratic part Q of $f|_{\mathbf{z}=\mathbf{c}}$ be identically equal to zero and $G(Q|_{\mathbf{z}=\mathbf{c}})$ has a single vertex labeled β , where $\mathbf{z} = (z_{p_0}, z_{p_1}, \dots, z_{p_{m-6}})$ and $\mathbf{c} = (c_0c_1\cdots c_{m-6})$ be a (m-5) length binary vector. Then

$$\left(f|_{\mathbf{z}=\mathbf{c}}, \left(f + \frac{q}{2}z_{\beta} + c'\right)|_{\mathbf{z}=\mathbf{c}}\right), \tag{11}$$

forms a GCP of length $2^{m-1}+2^{m-3}$, with exactly 20 non-zero elements.

Proof: Since $f_1|_{\mathbf{z}=\mathbf{c}}$ is a function containing only one variable z_{β} , so $f_1|_{\mathbf{z}=\mathbf{c}}$ gives exactly 2 non-zero elements in the sequence. The binary variables $z_{m-4}, z_{m-3}, z_{m-2}$ and z_{m-1} remain unaffected by $\mathbf{z} = \mathbf{c}$, and since the length of the sequence is $10 \times 2^{m-4}$, so the function $f|_{\mathbf{z}=\mathbf{c}}$ takes nonzero values in exactly 20 components numbered $k2^{m-4} + \sum_{j\neq\beta} c_j 2^j$, $0 \le k \le 9$ and $2^{\beta} + k2^{m-4} + \sum_{j\neq\beta} c_j 2^j$, $0 \le k \le 9$. These non-zero terms are placed in increasing order as follows

$$\begin{aligned} \left\{ \omega^{\gamma}, \omega^{\delta}, \omega^{\gamma}, \omega^{\delta}, -\omega^{\gamma}, -\omega^{\delta}, \omega^{\gamma}, \omega^{\delta}, -\omega^{\gamma}, \omega^{\delta} \\ , \omega^{\gamma}, \omega^{\delta}, -\omega^{\gamma}, \omega^{\delta}, -\omega^{\gamma}, \omega^{\delta}, \omega^{\gamma}, -\omega^{\delta}, \omega^{\gamma}, -\omega^{\delta} \right\}, \end{aligned}$$

where γ and δ are the values taken by the function $f_1|_{\mathbf{z}=\mathbf{c}}$ at $\sum_{j\neq\beta} c_j 2^j$ and $2^{\beta} + \sum_{j\neq\beta} c_j 2^j$ respectively.

The function $(f_1 + \frac{q}{2}z_{\beta} + c')|_{\mathbf{z}=\mathbf{c}}$ takes values $\gamma + c'$ and $\delta + \frac{q}{2} + c'$ at positions $\sum_{j\neq\beta} c_j 2^j$ and $2^{\beta} + \sum_{j\neq\beta} c_j 2^j$ respectively. So the 20 non-zero components of the function $(f + \frac{q}{2}z_{\beta} + c')|_{\mathbf{z}=\mathbf{c}}$ at positions mentioned above are placed in increasing order as follows

$$\begin{split} & \left\{ \omega^{\gamma+c'}, -\omega^{\delta+c'}, \omega^{\gamma+c'}, -\omega^{\delta+c'}, -\omega^{\gamma+c'}, \omega^{\delta+c'}, \omega^{\gamma+c'} \\ &, -\omega^{\delta+c'}, -\omega^{\gamma+c'}, -\omega^{\delta+c'}, \omega^{\gamma+c'}, -\omega^{\delta+c'}, -\omega^{\gamma+c'} \\ &, -\omega^{\delta+c'}, -\omega^{\gamma+c'}, -\omega^{\delta+c'}, \omega^{\gamma+c'}, \omega^{\delta+c'}, \omega^{\gamma+c'}, \omega^{\delta+c'} \right\}. \end{split}$$

The non-zero value of the AACF of the vectors corresponding to $f|_{\mathbf{z}=\mathbf{c}}$ and $\left(f + \frac{q}{2}z_{\beta} + c'\right)|_{\mathbf{z}=\mathbf{c}}$ occurs only at shifts $s = k2^{m-4} + 2^{\beta}$, $0 \le k \le 9$ and $s = k2^{m-4} - 2^{\beta}$, $1 \le k \le 9$. For $s = k2^{m-4} + 2^{\beta}$ and $0 \le k \le 9$, the AACF of the above two functions are expressed as

$$\mathcal{A}\left(f|_{\mathbf{z}=\mathbf{c}}\right)(s) = \mathbf{t}_k \omega^{\gamma} \left(\omega^{\delta}\right)^* = \mathbf{t}_k \omega^{\gamma-\delta}, \qquad (12)$$

and

$$\mathcal{A}\left(\left(f + \frac{q}{2}z_{\beta} + c'\right)|_{\mathbf{z}=\mathbf{c}}\right)(s) = t_{k}\omega^{\gamma+c'}\left(-\omega^{\delta+c'}\right)^{*}$$

= $-t_{k}\omega^{\gamma-\delta}$, (13)

where t_k is some constant.

Similarly for $s = k2^{m-4} - 2^{\beta}$ and $1 \le k \le 9$, the above can be written as

$$\mathcal{A}\left(f|_{\mathbf{z}=\mathbf{c}}\right)(s) = \mathbf{t}'_{k}\omega^{\delta}\left(\omega^{\gamma}\right)^{*} = \mathbf{t}'_{k}\omega^{\delta-\gamma},\qquad(14)$$

and

$$\mathcal{A}\left(\left(f + \frac{q}{2}z_{\beta} + c'\right)|_{\mathbf{z}=\mathbf{c}}\right)(s) = \mathbf{t}'_{k}(-\omega^{\delta+c'})\left(-\omega^{\gamma+c'}\right)^{*} = -\mathbf{t}'_{k}\omega^{\delta-\gamma},$$
(15)

where t'_k is some constant. So the AACS is zero for all $s \neq 0$, and hence the result follows. Some notations are defined below for proving the general case of construction of GCPs of non-power of two length. Let $0 \leq p_0 < p_1 < \cdots < p_{k-1} < m-4$, be a list of k indices, where $0 \leq k \leq m-5$ and $\mathbf{z} = (z_{p_0}, z_{p_1}, \dots, z_{p_{k-1}})$. Let the remaining m - 4 - k indices between 0 to m - 5 be $0 \leq i_0 < i_1 < \cdots < i_{m-k-5} < m-4$. Let $\mathbf{c} = (c_0c_1 \cdots c_{k-1})$ be a k length binary vector.

Theorem 1: Let us consider the restricted function $f|_{\mathbf{z}=\mathbf{c}}$ that is obtained by restricting the variables $z_{p_{\alpha}}, 0 \leq \alpha \leq k \leq m-5$, of GBF f in (10) with the property that $G(Q|_{\mathbf{z}=\mathbf{c}})$ is a path. Let β_1 and β_2 be the two end vertices of the path $G(Q|_{\mathbf{z}=\mathbf{c}})$ when $0 \leq k < m-5$. In case of k = m-5, $G(Q|_{\mathbf{z}=\mathbf{c}})$ has only a single vertex labeled $\beta = \beta_1 = \beta_2$. Then for any $c' \in \mathbb{Z}_q$, the complex-valued vectors $f|_{\mathbf{z}=\mathbf{c}}$ and $(f + \frac{q}{2}z_{\beta_2} + c')|_{\mathbf{z}=\mathbf{c}}$ forms a GCP of length $2^{m-1} + 2^{m-3}$.

Proof: We prove the result using induction on k, where the statement of the theorem is taken as an inductive hypothesis. The case when k = m-5, follows directly from *Lemma* 3. Now, let the theorem be true when \mathbf{z} contains k+1 variables, and we consider the case for k variables, where $0 \le k < m-5$. When $G(Q|_{\mathbf{z=c}})$ is a path, the non-zero components of function f are determined by the values of function $f|_{\mathbf{z=c}}$ in variables $(z_{i_0}, z_{i_1}, \ldots, z_{i_{m-k-5}}, z_{m-4}, z_{m-3}, z_{m-2}, z_{m-1})$.

So for some permutation π of $\{0, 1, \ldots, m - k - 5\}$ and $c_0, c_1, \ldots, c_{m-k-5}, c \in \mathbb{Z}_q$, we get the function

$$f|_{\mathbf{z}=\mathbf{c}} \left(z_{i_{0}}, z_{i_{1}}, \dots, z_{i_{m-k-5}}, z_{m-4}, z_{m-3}, z_{m-2}, z_{m-1} \right)$$

$$= \frac{q}{2} \sum_{\alpha=0}^{m-k-6} z_{i_{\pi(\alpha)}} z_{i_{\pi(\alpha+1)}} + \sum_{\alpha=0}^{m-k-5} c_{\alpha} z_{i_{\pi(\alpha)}} + c$$

$$+ \frac{q}{2} z_{i_{\pi(m-k-5)}} (\bar{z}_{m-1} (z_{m-2} \bar{z}_{m-3} \bar{z}_{m-4} + z_{m-2} z_{m-3}) + z_{m-1}$$

$$\bar{z}_{m-2} \bar{z}_{m-3}) + \frac{q}{2} \bar{z}_{m-1} (\bar{z}_{m-4} (z_{m-3} + z_{m-2}) + z_{m-2} z_{m-3}).$$
(16)

The higher order terms in (16) is utilized frequently, so for simplicity, it is denoted by R as follows

$$R = \frac{q}{2} z_{i_{\pi(m-k-5)}} (\bar{z}_{m-1} (z_{m-2} \bar{z}_{m-3} \bar{z}_{m-4} + z_{m-2} z_{m-3}) + z_{m-1}$$

$$\bar{z}_{m-2} \bar{z}_{m-3}) + \frac{q}{2} \bar{z}_{m-1} (\bar{z}_{m-4} (z_{m-3} + z_{m-2}) + z_{m-2} z_{m-3})$$
(17)

Now, the aim is to prove that the sequences $f|_{\mathbf{z}=\mathbf{c}}$ and $\left(f + \frac{q}{2}z_{i_{\pi(0)}} + c'\right)|_{\mathbf{z}=\mathbf{c}}$, where $c' \in \mathbb{Z}_q$ is arbitrary, forms a GCP of length $2^{m-1} + 2^{m-3}$. If $s \neq 0$ is chosen arbitrarily, then the sum of AACF of the sequences is given by

$$\mathcal{A}(f|_{\mathbf{z}=\mathbf{c}})(s) + \mathcal{A}\left(\left(f + \frac{q}{2}z_{i_{\pi(0)}} + c\right)|_{\mathbf{z}=\mathbf{c}}\right)(s) \\ = \mathcal{A}(g_1)(s) + \mathcal{A}(g_2)(s) + \mathcal{C}(g_1, g_2)(s) + \mathcal{C}(g_2, g_1)(s) \\ + \mathcal{A}(g_3)(s) + \mathcal{A}(g_4)(s) + \mathcal{C}(g_3, g_4)(s) + \mathcal{C}(g_4, g_3)(s),$$
(18)

where $g_1 = f|_{\mathbf{z}z_{i_{\pi(0)}} = \mathbf{c}0}, g_3 = \left(f + \frac{q}{2}z_{i_{\pi(0)}} + c'\right)|_{\mathbf{z}z_{i_{\pi(0)}} = \mathbf{c}0}, g_2 = f|_{\mathbf{z}z_{i_{\pi(0)}} = \mathbf{c}1}, g_4 = \left(f + \frac{q}{2}z_{i_{\pi(0)}} + c'\right)|_{\mathbf{z}z_{i_{\pi(0)}} = \mathbf{c}1}.$

The non-zero components of the vector g_1 are derived from a function h_1 by substituting $z_{i_{\pi(0)}} = 0$ in the function $f|_{\mathbf{z}=\mathbf{c}}$ in (16). For $0 \le k \le m-7$, the function h_1 is given by

$$h_1|_{\mathbf{z}=\mathbf{c}} \left(z_{i_{\pi(0)}}, z_{i_{\pi(1)}}, \dots, z_{i_{\pi(m-k-5)}}, z_{m-4}, z_{m-3}, z_{m-2}, z_{m-1} \right)$$
$$= \frac{q}{2} \sum_{\alpha=1}^{m-k-6} z_{i_{\pi(\alpha)}} z_{i_{\pi(\alpha+1)}} + \sum_{\alpha=1}^{m-k-5} c_{\alpha} z_{i_{\pi(\alpha)}} + c + R.$$
(19)

While for k = m - 6, it is given by

$$h_1\left(z_{i_{\pi(0)}}, z_{i_{\pi(1)}}, \dots, z_{i_{\pi(m-k-5)}}, z_{m-4}, z_{m-3}, z_{m-2}, z_{m-1}\right) = c_1 z_{i_{\pi(1)}} + c + R,$$
(20)

Similarly, by substituting $z_{i_{\pi(0)}} = 1$ in the function $f|_{\mathbf{z}=\mathbf{c}}$, function h_2 is obtained which yields the non-zero components of the vector g_2 . The function h_2 is given by

$$h_2\left(z_{i_{\pi(0)}}, z_{i_{\pi(1)}}, \dots, z_{i_{\pi(m-k-5)}}, z_{m-4}, z_{m-3}, z_{m-2}, z_{m-1}\right) = h_1 + z_{i_{\pi(1)}} + c_0.$$
(21)

To easily calculate the AACF of g_2 , we consider the vector g'_2 as

$$g_2' = \left(f + \frac{q}{2}z_{i_{\pi(1)}} + c_0\right)|_{\mathbf{z}z_{i_{\pi(0)}} = \mathbf{c}0} .$$
 (22)

Substituting $\mathbf{z} = \mathbf{c}$ and $z_{i_{\pi}(0)} = 0$ in (22) the function $h_1 + \frac{q}{2}z_{i_{\pi}(1)} + c_0$ is obtained which is identical to h_2 . In component *i*, the value of the vector g_2 is the same as the value of the vector g'_2 in the position $i - 2^{z_{i_{\pi}(0)}}$ (i.e., in non-zero positions,

 g_2 is simply a shift of g'_2). Therefore, the vectors g_2 and g'_2 have identical AACFs. Now, consider the pair

$$g_1 = f|_{\mathbf{z}z_{i_{\pi(0)}} = \mathbf{c}0},\tag{23}$$

and

$$g_2' = \left(f + \frac{q}{2}z_{i_{\pi(1)}} + c_0\right)|_{\mathbf{z}z_{i_{\pi(0)}} = \mathbf{c}0}.$$
 (24)

From the above, it is observed that g_1 corresponds to a GBF h_1 such that the graph of the quadratic part of h_1 is a path on m-k-5 vertices. Additionally, either $i_{\pi(1)}$ is an end vertex of this path, or k = m - 6 and it is the single vertex in the graph. By the inductive hypothesis, g_1 and g'_2 forms a GCP, hence for $s \neq 0$, the sum of AACF of g_1 and g'_2 is

$$\mathcal{A}(g_1)(s) + \mathcal{A}(g'_2)(s) = 0.$$
 (25)

Since, $\mathcal{A}(g_2)(s) = \mathcal{A}(g'_2)(s)$ for every s, the sum of AACF g_1 and g_2 is expressed as

$$\mathcal{A}(g_1)(s) + \mathcal{A}(g_2)(s) = 0.$$
⁽²⁶⁾

From the definitions, we have $g_3 = \omega^{c'} g_1$ and $g_4 = -\omega^{c'} g_2$. It follows that $\mathcal{A}(g_3)(s) = \mathcal{A}(g_1)(s)$ and $\mathcal{A}(g_4)(s) = \mathcal{A}(g_2)(s)$, so from (26), the sum of AACF g_3 and g_4 is

$$\mathcal{A}(g_3)(s) + \mathcal{A}(g_4)(s) = 0.$$
 (27)

Also, the ACCF between g_3 and g_4 is defined as

$$C(g_3, g_4)(s) = C(\omega^{c'} g_1, -\omega^{c'} g_2)(s) = -C(g_1, g_2)(s).$$
(28)

So the sum of ACCFs of g_1, g_2 and g_3, g_4 is

$$\mathcal{C}(g_1, g_2)(s) + \mathcal{C}(g_3, g_4)(s) = \mathcal{C}(g_2, g_1)(s) + \mathcal{C}(g_4, g_3)(s) = 0.$$
(29)

So, from (26)-(29), the sum in (18) is zero. Since $s \neq 0$ has been chosen arbitrary, it follows that $f|_{\mathbf{z}=\mathbf{c}}$ and $\left(f + \frac{q}{2}z_{\beta_2} + c'\right)|_{\mathbf{z}=\mathbf{c}}$ forms a GCP of length $2^{m-1} + 2^{m-3}$.

Example 2: For m = 8 and q = 2, consider the 5th order GBF $f : \{\mathbf{0}_8, \mathbf{1}_8, \dots, \mathbf{159}_8\} \rightarrow \mathbb{Z}_2$ defined as

$$f = z_0 z_1 + z_1 z_2 + z_2 z_3 + z_3 z_0 + z_0 z_2 + z_1 z_3 + z_0 + z_1 + z_2 + z_3 + \bar{z}_7 (\bar{z}_4 (z_5 + z_6) + z_6 z_5) + z_2 (\bar{z}_7 (z_6 \bar{z}_5 \bar{z}_4 + z_6 z_5) + z_7 \bar{z}_6 \bar{z}_5).$$

The graph G(Q) (quadratic part of f) is given in Fig. 1.



Fig. 1: The graph of quadratic part Q of f

By substituting $z_0z_3 = 00$ (deleting vertices z_0, z_3), we get $G(Q|_{z_0z_3=00})$ is a path. So by *Theorem* 1, $f|_{z_0z_3=00}$ and $(f + z_1 + 1)|_{z_0z_3=00}$ forms a GCP of length 160, which is not the form of 2^m .

IV. PROPOSED CONSTRUCTION OF MOCSS

In this section, we have proposed a direct construction of 2^k CSs of length $2^{m-1} + 2^{m-3}$, with the property that any two CSs are mutually orthogonal to each other.

Let Q and f be defined in (8) and (10) respectively (q = 2). For $0 \le t < 2^k$, $0 \le k \le m - 5$, the ordered set S_t (with the natural order induced by the binary vector $(aa_0a_1\cdots a_{k-1})$) is defined as

$$S_{t} = \left\{ f + \sum_{\alpha=0}^{k-1} a_{\alpha} z_{p_{\alpha}} + \sum_{\alpha=0}^{k-1} n_{\alpha} z_{p_{\alpha}} + a z_{\beta_{2}} : a, a_{\alpha} \in \{0, 1\} \right\}$$
(31)

where $t = \sum_{\alpha=0}^{k-1} n_{\alpha} 2^{\alpha}$.

"1" represents a vector all of whose component is one and \oplus denotes addition modulo 2.

Theorem 2: Suppose that G(Q) contains a set of $k \le m-5$ distinct vertices labeled $p_0, p_1, \ldots, p_{k-1}$ with the property that deleting those k vertices and all their edges results in a path. Let β_1 and β_2 be the two end vertices of the path. In case of single vertex let $\beta_1 = \beta_2 = \beta$. Then for any $0 \le t < 2^k$, the set S_t is a CS. Also for the case $t' \ne t$, the sets $S_{t'}$ and S_t are MOCSs.

Proof: Since each S_t for $1 \le t < 2^k$ is a permutation of S_0 , so proving S_0 is a complementary set is sufficient to show that for any $0 \le t < 2^k$, the set S_t is a CS.

Let $\mathbf{z} = (z_{p_0}z_{p_1}\dots z_{p_{k-1}})$ and $\mathbf{a} = (a_0a_1\dots a_{k-1})$. So $\mathbf{a} \cdot \mathbf{z} = \sum_{\alpha=0}^{k-1} a_{\alpha}z_{p_{\alpha}}$. Now from *Lemma* 2, for $s \neq 0$, sum of AACF can be expressed as

$$\sum_{\mathbf{a},a} \mathcal{A} \left(f + \mathbf{a} \cdot \mathbf{z} + a z_{\beta_2} \right) (s) = L_1 + L_2, \qquad (32)$$

where

$$L_{1} = \sum_{\mathbf{a},a} \sum_{\mathbf{c}} \mathcal{A} \left(\left(f + \mathbf{a} \cdot \mathbf{z} + a z_{\beta_{2}} \right) |_{\mathbf{z}=\mathbf{c}} \right) (s), \quad (33)$$

and

(30)

$$L_{2} = \sum_{\mathbf{c_{1}}\neq\mathbf{c_{2}}} \sum_{a} \sum_{\mathbf{a}} C\left(\left(f + \mathbf{a} \cdot \mathbf{z} + az_{\beta_{2}}\right)|_{\mathbf{z}=\mathbf{c}_{1}},\right)$$

$$\left(f + \mathbf{a} \cdot \mathbf{z} + az_{\beta_{2}}\right)|_{\mathbf{z}=\mathbf{c}_{2}}\left(s\right).$$
(34)

The graph of the function $(Q + \mathbf{a} \cdot \mathbf{z})|_{\mathbf{z}=\mathbf{c}}$ is a path for any choice of \mathbf{c} and \mathbf{a} . So from *Theorem* 1, for every \mathbf{c} and \mathbf{a} the vectors $(f + \mathbf{a} \cdot \mathbf{z})|_{\mathbf{z}=\mathbf{c}}$ and $(f + \mathbf{a} \cdot \mathbf{z} + z_{\beta_2})|_{\mathbf{z}=\mathbf{c}}$ forms a GCP of length $2^{m-1} + 2^{m-3}$. Hence the term L_1 in (33) is zero. For fixed values of $\mathbf{c_1}, \mathbf{c_2}$ and a, consider the inner sum of L_2 is expressed as

$$\sum_{\mathbf{a}} \mathcal{C} \left(\left(f + \mathbf{a} \cdot \mathbf{z} + a z_{\beta_2} \right) |_{\mathbf{z} = \mathbf{c}_1}, \left(f + \mathbf{a} \cdot \mathbf{z} + a z_{\beta_2} \right) |_{\mathbf{z} = \mathbf{c}_2} \right) (s).$$
(35)

Now, the vector \mathbf{z} contains all the terms of $\mathbf{a} \cdot \mathbf{z}$. So for the fixed values of $\mathbf{c_1}, \mathbf{c_2}$ and a we have,

$$(f + \mathbf{a} \cdot \mathbf{z} + az_{\beta_2})|_{\mathbf{z}=\mathbf{c}_j}$$

=
$$\begin{cases} \mathbf{e}_{\mathbf{j}} = (f + az_{\beta_2})|_{\mathbf{z}=\mathbf{c}_{\mathbf{j}}}, & \text{when } \mathbf{a} \cdot \mathbf{c}_{\mathbf{j}} = 0 \pmod{2}, \\ -\mathbf{e}_{\mathbf{j}}, & \text{when } \mathbf{a} \cdot \mathbf{c}_{\mathbf{j}} = 1 \pmod{2}. \end{cases}$$
(36)

Therefore for the fixed values of c_1, c_2 and a, from (35) and (36) ACCF values are obtained as

$$\mathcal{C}\left(\left(f + \mathbf{a} \cdot \mathbf{z} + az_{\beta_2}\right) |_{\mathbf{z}=\mathbf{c}_1}, \left(f + \mathbf{a} \cdot \mathbf{z} + az_{\beta_2}\right) |_{\mathbf{z}=\mathbf{c}_2}\right)(s)$$

$$= \begin{cases} \mathcal{C}(\mathbf{e}_1, \mathbf{e}_2), & \text{when } \mathbf{a} \cdot \mathbf{c}_1 = \mathbf{a} \cdot \mathbf{c}_2 = 0 \pmod{2}, \\ \mathcal{C}(-\mathbf{e}_1, -\mathbf{e}_2), & \text{when } \mathbf{a} \cdot \mathbf{c}_1 = \mathbf{a} \cdot \mathbf{c}_2 = 1 \pmod{2}, \\ \mathcal{C}(\mathbf{e}_1, -\mathbf{e}_2), & \text{when } \mathbf{a} \cdot \mathbf{c}_1 = 0 \pmod{2}, \mathbf{a} \cdot \mathbf{c}_2 = 1 \pmod{2}, \\ \mathcal{C}(-\mathbf{e}_1, \mathbf{e}_2), & \text{when } \mathbf{a} \cdot \mathbf{c}_1 = 1 \pmod{2}, \mathbf{a} \cdot \mathbf{c}_2 = 0 \pmod{2}. \end{cases}$$
(37)

Since, $C(\mathbf{e_1}, \mathbf{e_2}) = C(-\mathbf{e_1}, -\mathbf{e_2})$ and $C(\mathbf{e_1}, -\mathbf{e_2}) = C(-\mathbf{e_1}, \mathbf{e_2})$, the above can be re-expressed as

$$\mathcal{C}\left(\left(f + \mathbf{a} \cdot \mathbf{z} + az_{\beta_2}\right)|_{\mathbf{z}=\mathbf{c}_1}, \left(f + \mathbf{a} \cdot \mathbf{z} + az_{\beta_2}\right)|_{\mathbf{z}=\mathbf{c}_2}\right)(s)$$

$$= \begin{cases} \mathcal{C}(\mathbf{e}_1, \mathbf{e}_2), & \text{when } \mathbf{a} \cdot \mathbf{c}_1 = \mathbf{a} \cdot \mathbf{c}_2 \pmod{2}, \\ \mathcal{C}(\mathbf{e}_1, -\mathbf{e}_2), & \text{when } \mathbf{a} \cdot \mathbf{c}_1 \neq \mathbf{a} \cdot \mathbf{c}_2 \pmod{2}. \end{cases}$$
(38)

$$\sum_{\mathbf{a}} \mathcal{C} \left((f + \mathbf{a} \cdot \mathbf{z} + az_{\beta_2}) |_{\mathbf{z} = \mathbf{c}_1}, (f + \mathbf{a} \cdot \mathbf{z} + az_{\beta_2}) |_{\mathbf{z} = \mathbf{c}_2} \right) (s)$$

$$= \sum_{\mathbf{a} \cdot \mathbf{c}_1 = \mathbf{a} \cdot \mathbf{c}_2 \pmod{2}} \mathcal{C}(\mathbf{e}_1, \mathbf{e}_2)(s) - \sum_{\mathbf{a} \cdot \mathbf{c}_1 \neq \mathbf{a} \cdot \mathbf{c}_2 \pmod{2}} \mathcal{C}(\mathbf{e}_1, \mathbf{e}_2)(s).$$
(39)

Due to the fact that $\mathbf{c_1} \neq \mathbf{c_2}$, $\mathbf{c_1} + \mathbf{c_2} \neq \mathbf{0} \pmod{2}$, and so the linear functional $\mathbf{a} \cdot (\mathbf{c_1} + \mathbf{c_2}) \pmod{2}$ takes each value 0 and 1 precisely 2^{k-1} times, i.e., an equal number of times. So, from (39) the inner sum of L_2 is zero and so is L_2 . Hence it is proved that S_t is a CS of size 2^k .

Now, let $\mathbf{n} = (n_0 n_1 \cdots n_{k-1})$, $t = \sum_{\alpha=0}^{k-1} n_{\alpha} 2^{\alpha}$ and $t' = \sum_{\alpha=0}^{k-1} n'_{\alpha} 2^{\alpha}$. It needs to proven that for $t \neq t'$, S_t and $S_{t'}$ are mutually orthogonal. From *Lemma* 1, the sum of ACCF can be written as

$$\sum_{\mathbf{a},a} \mathcal{C} \left(f + (\mathbf{a} + \mathbf{n}) \cdot \mathbf{z} + a z_{\beta_2}, \right.$$

$$f + (\mathbf{a} + \mathbf{n}') \cdot \mathbf{z} + a z_{\beta_2} \right) (s) = M_1 + M_2,$$
(40)

where

$$M_{1} = \sum_{\mathbf{a},a} \sum_{\mathbf{c}_{1} \neq \mathbf{c}_{2}} \mathcal{C} \left(\left(f + (\mathbf{a} + \mathbf{n}) \cdot \mathbf{z} + az_{\beta_{2}} \right) |_{\mathbf{z} = \mathbf{c}_{1}}, \right.$$

$$\left(f + (\mathbf{a} + \mathbf{n}') \cdot \mathbf{z} + az_{\beta_{2}} \right) |_{\mathbf{z} = \mathbf{c}_{2}} \right) (s),$$
(41)

and

$$M_{2} = \sum_{\mathbf{a},a} \sum_{\mathbf{c}} \mathcal{C} \left(\left(f + (\mathbf{a} + \mathbf{n}) \cdot \mathbf{z} + a z_{\beta_{2}} \right) \big|_{\mathbf{z}=\mathbf{c}}, \right.$$

$$\left. \left(f + (\mathbf{a} + \mathbf{n}') \cdot \mathbf{z} + a z_{\beta_{2}} \right) \big|_{\mathbf{z}=\mathbf{c}} \right) (s).$$
(42)

For the fixed c_1, c_2 and a, we consider the following sum of M_1

$$\sum_{\mathbf{a}} \mathcal{C} \left(\left(f + (\mathbf{a} + \mathbf{n}) \cdot \mathbf{z} + a z_{\beta_2} \right) |_{\mathbf{z} = \mathbf{c}_1}, \right. \\ \left. \left(f + (\mathbf{a} + \mathbf{n}') \cdot \mathbf{z} + a z_{\beta_2} \right) |_{\mathbf{z} = \mathbf{c}_2} \right) (s)$$

$$= \sum_{\mathbf{a}} \mathcal{C} \left(\left(f + (\mathbf{a} + \mathbf{n}) \cdot \mathbf{c_1} + az_{\beta_2} \right) |_{\mathbf{z}=\mathbf{c}_1}, \\ \left(f + (\mathbf{a} + \mathbf{n}') \cdot \mathbf{c_2} + az_{\beta_2} \right) |_{\mathbf{z}=\mathbf{c}_2} \right) (s)$$

$$= \sum_{\mathbf{a}} (-1)^{\mathbf{a} \cdot (\mathbf{c}_1 \oplus \mathbf{c}_2)} \mathcal{C} \left(\left(f + \mathbf{n} \cdot \mathbf{z} + az_{\beta_2} \right) |_{\mathbf{z}=\mathbf{c}_1}, \\ \left(f + \mathbf{n}' \cdot \mathbf{z} + az_{\beta_2} \right) |_{\mathbf{z}=\mathbf{c}_2} \right) (s)$$

$$= \mathcal{C} \left(\left(f + \mathbf{n} \cdot \mathbf{z} + az_{\beta_2} \right) |_{\mathbf{z}=\mathbf{c}_1}, \\ \left(f + \mathbf{n}' \cdot \mathbf{z} + az_{\beta_2} \right) |_{\mathbf{z}=\mathbf{c}_2} \right) (s) \sum_{\mathbf{a}} (-1)^{\mathbf{a} \cdot (\mathbf{c}_1 \oplus \mathbf{c}_2)}.$$
(43)

Since $\mathbf{c_1} \neq \mathbf{c_2}$, so the function $\mathbf{a} \cdot (\mathbf{c_1} \oplus \mathbf{c_2})$ in (43) takes values 0 and 1 equal number of times and hence (43) vanishes for all s.

Now for the fixed **a** and **c** consider the following sum of M_2

$$\sum_{a} \mathcal{C} \left(\left(f + (\mathbf{a} + \mathbf{n}) \cdot \mathbf{z} + az_{\beta_2} \right) |_{\mathbf{z}=\mathbf{c},} \right)$$

$$\left(f + (\mathbf{a} + \mathbf{n}') \cdot \mathbf{z} + az_{\beta_2} \right) |_{\mathbf{z}=\mathbf{c}} \right) (s)$$

$$= \sum_{a} \mathcal{C} \left(\left(f + (\mathbf{a} + \mathbf{n}) \cdot \mathbf{c} + az_{\beta_2} \right) |_{\mathbf{z}=\mathbf{c}}, \right) (s)$$

$$= \sum_{a} \mathcal{C} \left(\left(f + (\mathbf{a} + \mathbf{n}') \cdot \mathbf{c} + az_{\beta_2} \right) |_{\mathbf{z}=\mathbf{c}}, \right) (s)$$

$$= \sum_{a} \mathcal{C} \left(\left(f + (\mathbf{n} + \mathbf{n}') \cdot \mathbf{c} + az_{\beta_2} \right) |_{\mathbf{z}=\mathbf{c}}, \right) (s)$$

$$\left(-1 \right)^{(\mathbf{n} \oplus \mathbf{n}') \cdot c} \sum_{a} \mathcal{C} \left(\left(f + az_{\beta_2} \right) |_{\mathbf{z}=\mathbf{c}}, \left(f + az_{\beta_2} \right) |_{\mathbf{z}=\mathbf{c}} \right) (s)$$

$$= (-1)^{(\mathbf{n} \oplus \mathbf{n}') \cdot c} \mathcal{A}(f|_{\mathbf{z}=\mathbf{c}})(s) + \mathcal{A}((f+z_{\beta_2})|_{\mathbf{z}=\mathbf{c}})(s).$$
(44)

=

From *Theorem* 1, the above sum in (44) is zero for all $s \neq 0$. For s = 0, AACF is given by

$$\mathcal{A}\left(f|_{\mathbf{z}=\mathbf{c}}\right)\left(s\right) = \mathcal{A}\left(\left(f+z_{\beta_{2}}\right)|_{\mathbf{z}=\mathbf{c}}\right)\left(s\right) = 2^{m-k-4},\quad(45)$$

for $\mathbf{c} \in \mathbb{Z}_2^k$, substituting this back in (44), we get the sum of ACCF as

$$\sum_{a} \mathcal{C} \left(\left(f + (\mathbf{n} + \mathbf{n}') \cdot \mathbf{c} + a z_{\beta_2} \right) |_{\mathbf{z} = \mathbf{c}}, \right.$$

$$\left(f + a z_{\beta_2} \right) |_{\mathbf{z} = \mathbf{c}} \left(0 \right) = (-1)^{(\mathbf{n} \oplus \mathbf{n}') \cdot c} \cdot 2^{m-k-3}.$$
(46)

Here $t \neq t'$ is considered, which implies $\mathbf{n} \neq \mathbf{n}'$, and hence $\mathbf{n} \oplus \mathbf{n}' \neq 0$. So the linear functional $(\mathbf{n} \oplus \mathbf{n}') \cdot \mathbf{c}$ (regarded as a function of \mathbf{c}) is not equivalent to the zero function. As a result, it is balanced, i.e., the values 0 and 1 are taken equal number of times by the function as \mathbf{c} varies. Hence the sum

$$\sum_{\mathbf{c}} (-1)^{\left(\mathbf{n} \oplus \mathbf{n}'\right) \cdot \mathbf{c}} \cdot 2^{m-k+1} = 0.$$
(47)

Remark 1: [23, Th. 4] generates CSs of length $2^{m-1}+2^{m-3}$ and set size 4 for $\nu = m-3$, which is covered by *Theorem* 2 of our proposed construction by taking k = 2.

Remark 2: By taking $\nu = m-3$, [24, Th. 4] and t = m-3, [27, Th. 3] generates CSs of length $2^{m-1}+2^{m-3}$ and set size 2^{k+1} . The proposed construction of CSs in *Theorem 2* covers these special cases of [23], [27].

TABLE I: Comparison of the proposed MOCS construction with [44], [45]

[Ref.	Parameters	Based on	Length(N)	Constraint
	[44]	$(\boldsymbol{2}^{k'},\boldsymbol{2}^{k+1},N)$	GBF of order 2	$2^{m-1} + 2^t$	$m, k, t \in \mathbb{Z}^+, m \ge 2, k \le m,$ $0 \le k' \le t \le m - 1, k' \le k - 1$
	[45]	$(2^k, 2^{k+1}, N)$	GBF of order 2	$2^{m} + 2^{t}$	$m, k, t \in \mathbb{Z}^+, \ 0 \le t < k \le m$
	Theorem 2	$\left(2^{k+1},2^{k+1},N\right)$	GBF of order > 2	$2^{m-1} + 2^{m-3}$	$m, k \in \mathbb{Z}^+, m \ge 5$
				$2^{m-1} + 2^{m-2} + 2^{m-4}$	$m, k \in \mathbb{Z}^+, m \ge 6$

TABLE I compares the proposed constructions of MOCSs with the existing direct constructions of [44], [45].

Example 3: Let us consider the same GBF as given in *Example 2*, and the deleted vertices are also same, i.e., z_0, z_3 . Then the set,

$$S_{0} = \{f, f + z_{1}, f + z_{0}, f + z_{0} + z_{1}, f + z_{3} , f + z_{3} + z_{1}, f + z_{3} + z_{0}, f + z_{3} + z_{1} + z_{0}\},$$
(48)

is a CS of size 8 and sequence length 160, which is not of the form of 2^m . Similarly the sets S_t for $0 \le t < 4$, which are the permutations of the set S_0 , are also CS of size 8, with the property that any two different CSs are mutually orthogonal to each other.

V. PROPOSED CONSTRUCTION OF CCCS

In this section first we construct a mate of the MOCSs proposed in section IV. Then binary CCCs of length $2^{m-1}+2^{m-3}$ are constructed by union of these two MOCSs through GBFs. For a given GBF f in (10), GBF $\overline{f} : B \to \mathbb{Z}_2$ is defined as

$$\bar{f}(z_0, z_1, \dots, z_{m-1}) = f(\bar{z}_0, \bar{z}_1, \dots, \bar{z}_{m-1}),$$
 (49)

where $B = \{0, 1\}^m \setminus \{\mathbf{0}_m, \mathbf{1}_m, \dots, (\mathbf{2^{m-2} + 2^{m-3} - 1})_m\}.$

Lemma 4: Let us assume a set of $k \le m-5$ distinct vertices labelled with the property that deleting that set of vertices and all the edges transform G(Q) into a path. Let β_1 and β_2 be the two end vertices of this path. In case of k = m-5, the single vertex of the graph is denoted by $\beta_1 = \beta_2 = \beta$. Then for each $0 \le t < 2^k$, the ordered set \overline{S}_t given by

$$\left\{ \bar{f} + \sum_{\alpha=0}^{k-1} a_{\alpha} \bar{z}_{p_{\alpha}} + \sum_{\alpha=0}^{k-1} n_{\alpha} \bar{z}_{p_{\alpha}} + \bar{a} z_{\beta_{2}} : a, a_{\alpha} \in \{0, 1\} \right\},$$
(50)

is a CS of size 2^{k+1} , where \overline{f} is defined in (49). Further, for $t' \neq t$, $\overline{S}_{t'}$ and \overline{S}_t are MOCSs, where the natural order is induced from the binary vector $(aa_0a_1\cdots a_{k-1})$.

The next theorem gives CCCs of length $2^{m-1} + 2^{m-3}$.

Theorem 3: Let the sets S_t and \bar{S}_t be defined in Theorem 2 and Lemma 4 respectively, then

$$\left\{S_t: 0 \le t < 2^k\right\} \cup \left\{\bar{S}_t: 0 \le t < 2^k\right\},\tag{51}$$

forms a $(2^{k+1}, 2^{k+1}, 2^{m-1} + 2^{m-3})$ -CCC.

Proof: It will be shown that CSs S_{t_1} and \overline{S}_{t_2} are mutually orthogonal to each other. The sum of ACCF of these CSs can be expressed as

$$\sum_{\mathbf{a}} \mathcal{C} \left(f + (\mathbf{a} + \mathbf{n}) \cdot \mathbf{z} + z_{\beta_2}, \bar{f} + (\mathbf{a} + \mathbf{n}') \cdot \bar{\mathbf{z}} \right) (s) + \mathcal{C} \left(f + (\mathbf{a} + \mathbf{n}) \cdot \mathbf{z} \right), \left(\bar{f} + (\mathbf{a} + \mathbf{n}') \cdot \bar{\mathbf{z}} + z_{\beta_2} \right) (s) = \sum_{\mathbf{a}} \sum_{\mathbf{c}_1, \mathbf{c}_2} \mathcal{C} \left(\left(f + (\mathbf{a} + \mathbf{n}) \cdot \mathbf{z} + z_{\beta_2} \right) |_{\mathbf{z} = \mathbf{c}_1}, \left(\bar{f} + (\mathbf{a} + \mathbf{n}') \cdot \bar{\mathbf{z}} \right) |_{\mathbf{z} = \mathbf{c}_2} \right) (s) + \mathcal{C} \left(\left(f + (\mathbf{a} + \mathbf{n}) \cdot \mathbf{z} \right) |_{\mathbf{z} = \mathbf{c}_1}, \\\left(\bar{f} + (\mathbf{a} + \mathbf{n}') \cdot \bar{\mathbf{z}} + z_{\beta_2} \right) |_{\mathbf{z} = \mathbf{c}_2} \right) (s) = M(say)$$
(52)

For a given c_1 and c_2 , consider the following sum of the first term in (52)

$$\sum_{\mathbf{a}} \mathcal{C} \left(\left(f + (\mathbf{a} + \mathbf{n}) \cdot \mathbf{z} + z_{\beta_2} \right) |_{\mathbf{z}=\mathbf{c}_1}, \right)$$

$$= \sum_{\mathbf{a}} \mathcal{C} \left(\left(f + (\mathbf{a} + \mathbf{n}') \cdot \overline{\mathbf{z}} \right) |_{\mathbf{z}=\mathbf{c}_2} \right) (s)$$

$$= \sum_{\mathbf{a}} \mathcal{C} \left(\left(f + (\mathbf{a} + \mathbf{n}) \cdot \mathbf{z} + z_{\beta_2} \right) |_{\mathbf{z}=\mathbf{c}_1}, \right)$$

$$= \mathcal{C} \left(\left(f + z_{\beta_2} \right) |_{\mathbf{z}=\mathbf{c}_1}, \left(\overline{f} |_{\mathbf{z}=\mathbf{c}_2} \right) (s) \right)$$

$$= \mathcal{C} \left(\left(f + z_{\beta_2} \right) |_{\mathbf{z}=\mathbf{c}_1}, \left(\overline{f} |_{\mathbf{z}=\mathbf{c}_2} \right) (s) \right)$$

$$= \mathcal{C} \left(\left(f + z_{\beta_2} \right) |_{\mathbf{z}=\mathbf{c}_1}, \left(\overline{f} |_{\mathbf{z}=\mathbf{c}_2} \right) (s) \right)$$

$$= \mathcal{C} \left(\left(f + z_{\beta_2} \right) |_{\mathbf{z}=\mathbf{c}_1}, \left(\overline{f} |_{\mathbf{z}=\mathbf{c}_2} \right) (s) \right)$$

$$\cdot (-1)^{\mathbf{n} \cdot \mathbf{c}_1 \oplus \mathbf{n}' \cdot \mathbf{c}_2 \oplus \mathbf{n}' \cdot \mathbf{1}} \sum_{\mathbf{a}} (-1)^{\mathbf{a} \cdot (\mathbf{c}_1 \oplus \mathbf{c}_2 \oplus \mathbf{1})}.$$
(53)

The above sum in (53) vanishes whenever $(\mathbf{c}_1 \oplus \mathbf{c}_2) \neq \mathbf{1}$. So, the first correlation term in (52) is zero whenever \mathbf{c}_1 and \mathbf{c}_2 are equal. Thus, summing (53) over all $\mathbf{c}_1 \neq \mathbf{c}_2$, the above term further can be simplified as

$$\sum_{\substack{\mathbf{c}_{1}\neq\mathbf{c}_{2}\\\mathbf{c}_{1}+\mathbf{c}_{2}=\mathbf{1}}} \mathcal{C}\left((f+z_{\beta_{2}})|_{\mathbf{z}=\mathbf{c}_{1}}, (\bar{f}|_{\mathbf{z}=\mathbf{c}_{2}})(s)\right)$$

$$\cdot (-1)^{\mathbf{n}\cdot\mathbf{c}_{1}\oplus\mathbf{n}'\cdot\mathbf{c}_{2}\oplus\mathbf{n}'\cdot\mathbf{1}}2^{k}$$

$$= \sum_{\mathbf{c}} \mathcal{C}\left((f+z_{\beta_{2}})|_{\mathbf{z}=\mathbf{c}}, (\bar{f}|_{\mathbf{z}=(\mathbf{c}\oplus\mathbf{1})})(s)\right)$$

$$\cdot (-1)^{\mathbf{n}'\cdot\mathbf{1}}2^{k} \cdot (-1)^{\mathbf{n}\cdot\mathbf{c}\oplus\mathbf{n}'\cdot(\mathbf{1}\oplus\mathbf{c})}$$

$$= \sum_{\mathbf{c}} \mathcal{C}\left((f+z_{\beta_{2}})|_{\mathbf{z}=\mathbf{c}}, (\bar{f}|_{\mathbf{z}=(\mathbf{c}\oplus\mathbf{1})})(s)2^{k} \cdot (-1)^{(\mathbf{n}\oplus\mathbf{n}')\cdot\mathbf{c}}\right).$$
(54)

From *Lemma* 1, the inner sum of (54) can be further simplified as

$$C\left(\left(f+z_{\beta_{2}}\right)|_{\mathbf{z}=\mathbf{c}},\bar{f}|_{\mathbf{z}=(\mathbf{c}\oplus\mathbf{1})}\right)(s)$$

$$=C\left(\left(f+z_{\beta_{2}}\right)|_{\mathbf{z}z_{\beta_{2}}=\mathbf{c}0},\bar{f}|_{\mathbf{z}z_{\beta_{2}}=(c\oplus1)0}\right)(s)$$

$$+C\left(\left(f+z_{\beta_{2}}\right)|_{\mathbf{z}z_{\beta_{2}}=\mathbf{c}0},\bar{f}|_{\mathbf{z}z_{\beta_{2}}=(c\oplus1)1}\right)(s)$$

$$+C\left(\left(f+z_{\beta_{2}}\right)|_{\mathbf{z}z_{\beta_{2}}=\mathbf{c}1},\bar{f}|_{\mathbf{z}z_{\beta_{2}}=(c\oplus1)0}\right)(s)$$

$$+C\left(\left(f+z_{\beta_{2}}\right)|_{\mathbf{z}z_{\beta_{2}}=\mathbf{c}1},\bar{f}|_{\mathbf{z}z_{\beta_{2}}=(c\oplus1)1}\right)(s)$$

$$= \mathcal{C} \left(f|_{\mathbf{z}z_{\beta_{2}}=\mathbf{c}0}, \bar{f}|_{\mathbf{z}z_{\beta_{2}}=(c\oplus 1)0} \right) (s) + \mathcal{C} \left(f|_{\mathbf{z}z_{\beta_{2}}=\mathbf{c}0}, \bar{f}|_{\mathbf{z}z_{\beta_{2}}=(c\oplus 1)1} \right) (s) - \mathcal{C} \left(f|_{\mathbf{z}z_{\beta_{2}}=\mathbf{c}1}, \bar{f}|_{\mathbf{z}z_{\beta_{2}}=(c\oplus 1)0} \right) (s) - \mathcal{C} \left(f|_{\mathbf{z}z_{\beta_{2}}=\mathbf{c}1}, \bar{f}|_{\mathbf{z}z_{\beta_{2}}=(c\oplus 1)1} \right) (s).$$

$$(55)$$

Similarly, the second term of the correlation in (52) becomes

$$\sum_{\mathbf{d}} \mathcal{C} \left(\left(f + (\mathbf{d} + \mathbf{n}) \cdot \mathbf{z} \right) |_{\mathbf{z}=\mathbf{c}_{1}}, \left(\bar{f} + (\mathbf{a} + \mathbf{n}') \cdot \bar{\mathbf{z}} + z_{\beta_{2}} \right) |_{\mathbf{z}=\mathbf{c}_{2}} \right) (s)$$
$$= \sum_{\mathbf{c}} \mathcal{C} \left(f|_{\mathbf{z}=\mathbf{c}}, \left(\bar{f} + z_{\beta_{2}} \right) |_{\mathbf{z}=(\mathbf{c}\oplus\mathbf{1})} \right) (s) \cdot 2^{k} \cdot (-1)^{\left(\mathbf{n}\oplus\mathbf{n}'\right)\cdot\mathbf{c}}.$$
(56)

The inner sum of (56) can be simplified as

$$\mathcal{C}\left(f|_{\mathbf{z}=\mathbf{c}}, \left(\bar{f} + z_{\beta_{2}}\right)|_{\mathbf{z}=(\mathbf{c}\oplus\mathbf{1})}\right)(s)$$

$$=\mathcal{C}\left(f|_{\mathbf{z}z_{\beta_{2}}=\mathbf{c}0}, \bar{f}|_{\mathbf{z}z_{\beta_{2}}=(c\oplus1)0}\right)(s)$$

$$-\mathcal{C}\left(f|_{\mathbf{z}z_{\beta_{2}}=\mathbf{c}0}, \bar{f}|_{\mathbf{z}z_{\beta_{2}}=(c\oplus1)1}\right)(s)$$

$$+\mathcal{C}\left(f|_{\mathbf{z}z_{\beta_{2}}=\mathbf{c}1}, \bar{f}|_{\mathbf{z}z_{\beta_{2}}=(c\oplus1)0}\right)(s)$$

$$-\mathcal{C}\left(f|_{\mathbf{z}z_{\beta_{2}}=\mathbf{c}1}, \bar{f}|_{\mathbf{z}z_{\beta_{2}}=(c\oplus1)1}\right)(s).$$
(57)

So from (54), (55), (56) and (57) we get the value of M in (52) as

$$M = 2 \sum_{\mathbf{c}} \left(\mathcal{C} \left(f |_{\mathbf{z}z_{\beta_2} = \mathbf{c}0}, \bar{f} |_{\mathbf{z}z_{\beta_2} = (c \oplus 1)0} \right) (s) - \mathcal{C} \left(f |_{\mathbf{z}z_{\beta_2} = \mathbf{c}1}, \bar{f} |_{\mathbf{z}z_{\beta_2} = (c \oplus 1)1} \right) (s) \right) \cdot 2^k \cdot (-1)^{(\mathbf{n} \oplus \mathbf{b}') \cdot \mathbf{c}}.$$
(58)

Since $G(Q|_{\mathbf{z}=\mathbf{c}})$ is a path, so for some permutation π of $\{0, 1, \ldots, m - k - 5\}$ and $c_{\alpha}, c \in \mathbb{Z}_q$ the function $f|_{\mathbf{z}=\mathbf{c}}$ obtained by substituting $\mathbf{z} = \mathbf{c}$ in f should be of the form

$$f|_{\mathbf{z}=\mathbf{c}} = \sum_{\alpha=0}^{m-k-6} z_{\pi(\alpha)} z_{\pi(\alpha+1)} + \sum_{\alpha=0}^{m-k-5} c_{\alpha} z_{\pi(\alpha)} + c$$
$$+ z_{\pi(m-k-5)} \left(\bar{z}_{m-1} (z_{m-2} \bar{z}_{m-3} \bar{z}_{m-4} + z_{m-2} z_{m-3}) + z_{m-1} \right)$$
$$\bar{z}_{m-2} \bar{z}_{m-3} + \bar{z}_{m-1} \left(\bar{z}_{m-4} \left(z_{m-3} + z_{m-2} \right) + z_{m-2} z_{m-3} \right).$$
(59)

Let h1 and h2 be the function obtained from f by substituting $\mathbf{z} = \mathbf{c}$, $z_{\beta_2} = 0$ and $\mathbf{z} = \mathbf{c}$, $z_{\beta_2} = 1$ respectively. Further without loss of generality let $\beta_2 = \pi(0)$. Then both the function can be expressed as

$$h_1 = \sum_{\alpha=1}^{m-k-6} z_{\pi(\alpha)} z_{\pi(\alpha+1)} + \sum_{\substack{\alpha=0\\\pi(\alpha)\neq 0}}^{m-k-5} c_\alpha z_{\pi(\alpha)} + c + R, \quad (60)$$

and
$$h_2 = h_1 + z_{\pi(1)} + c_0.$$
 (61)

The functions h1 and h2 give non-zero components of the complex vectors $\mathbf{e_1} = f|_{\mathbf{z}z_{\beta_2}=\mathbf{c}0}$ and $\mathbf{e_2} = f|_{\mathbf{z}z_{\beta_2}=\mathbf{c}1}$ respectively. Similarly, $\bar{h_2}$ and h_1 give the non-zero components of the vector $\bar{f}|_{\mathbf{z}z_{\beta_2}=(\mathbf{c}\oplus\mathbf{1})0}$ and $\bar{f}|_{\mathbf{z}z_{\beta_2}=(\mathbf{c}\oplus\mathbf{1})1}$ respectively. For any complex-valued sequences $\mathbf{e_1}$ and $\mathbf{e_2}$ the following identity holds

$$\mathcal{C}\left(\mathbf{e_{1}}, \bar{\mathbf{e}}_{2}\right)(s) = \mathcal{C}\left(\mathbf{e_{2}}, \bar{\mathbf{e}}_{1}\right)(s).$$
(62)

Using the above identity, we get,

$$\mathcal{C}\left(f|_{\mathbf{z}z_{\beta_{2}}=\mathbf{c}0}, \bar{f}|_{\mathbf{z}z_{\beta_{2}}=(c\oplus1)0}\right)(s)
= \mathcal{C}\left(f|_{\mathbf{z}z_{\beta_{2}}=\mathbf{c}1}, \bar{f}|_{\mathbf{z}z_{\beta_{2}}=(c\oplus1)1}\right)(s),$$
(63)

which shows that M in (58) is zero, and hence the result follows from (52).

Example 4: Let us consider the set S_t for $0 \le t < 4$ as defined in *Example 3.* Now for the same GBF defined in *Example 2*, using *Lemma 4* construct a MOCS \bar{S}_t $(0 \le t < 4)$ of length 160 as

$$\left\{ \bar{f} + a_0 \bar{z}_0 + a_1 \bar{z}_3 + n_0 \bar{z}_0 + n_1 \bar{z}_3 + \bar{a} z_1 : a, a_0, a_1 \in \{0, 1\} \right\},$$
(64)

where $t = n_0 2^0 + n_1 2^1$. Then from *Theorem* 3

$$\{S_t : 0 \le t < 4\} \cup \{\bar{S}_t : 0 \le t < 4\}, \tag{65}$$

is a (8, 8, 160)-CCC.

In *TABLE* II, the proposed construction of CCC is compared with the existing construction CCC on different parameters.

VI. CONSTRUCTION OF SEQUENCES OF LENGTH $2^{m-1} + 2^{m-2} + 2^{m-4}$.

In this section, we have extended our proposed construction to provide GCPs, MOCSs and binary CCCs of length $2^{m-1} + 2^{m-2} + 2^{m-4}$.

Consider an integer $m \ge 6$, for any $c, c_i \in \mathbb{Z}_q$, we define a function

$$f_1(z_0, z_1, \dots, z_{m-6}) = Q + \sum_{i=0}^{m-6} c_i z_i + c, \qquad (66)$$

where Q is the quadratic part in variables $z_0, z_1, \ldots, z_{m-6}$. Now, we define the GBF $f: A \to \mathbb{Z}_q$ as

$$\begin{split} \mathbf{f} &= \mathbf{f}_{1} + \frac{q}{2} z_{\beta_{1}} \\ & (\bar{z}_{m-1} \bar{z}_{m-2} + \bar{z}_{m-1} z_{m-2} \left(\bar{z}_{m-3} + z_{m-3} \bar{z}_{m-4} z_{m-5} \right) \right) \\ & + \frac{q}{2} \left(\bar{z}_{m-1} \bar{z}_{m-2} \left(\bar{z}_{m-3} z_{m-4} z_{m-5} + z_{m-3} \bar{z}_{m-4} + z_{m-4} \left(\bar{z}_{m-3} z_{m-5} \right) \right) \\ & + \bar{z}_{m-1} z_{m-2} \left(\bar{z}_{m-3} \bar{z}_{m-5} + z_{m-3} \bar{z}_{m-4} + z_{m-4} \left(\bar{z}_{m-3} z_{m-5} + z_{m-5} \bar{z}_{m-5} \right) \right) \\ & + z_{m-5} \bar{z}_{m-5} \right) + z_{m-1} \bar{z}_{m-2} \left(\bar{z}_{m-4} \bar{z}_{m-5} + z_{m-3} z_{m-4} + \bar{z}_{m-3} z_{m-4} + \bar{z}_{m-3} \bar{z}_{m-4} + z_{m-4} \bar{z}_{m-5} + z_{m-1} z_{m-2} \bar{z}_{m-3} \bar{z}_{m-4} \right)), \end{split}$$

where $A = \{\mathbf{0}_m, \mathbf{1}_m, \dots, (\mathbf{2^{m-1}} + \mathbf{2^{m-2}} + \mathbf{2^{m-4}} - \mathbf{1})_m\}$. Also we define the GBF $\overline{\mathbf{f}} : B \to \mathbb{Z}_2$ as

$$\overline{\mathbf{f}}(z_0, z_1, \dots, z_{m-1}) = \mathbf{f}(\overline{z}_0, \overline{z}_1, \dots, \overline{z}_{m-1}),$$
 (68)

where $B = \{0, 1\}^m \setminus \{\mathbf{0}_m, \mathbf{1}_m, \dots, (\mathbf{2^{m-3} + 2^{m-4} - 1})_m\}$ and $\bar{z}_i = 1 - z_i$. Now, by replacing the GBF f used in the above *Theorems*, by the function f defined in (67), we can generate GCP, CS and CCC of length $2^{m-1} + 2^{m-2} + 2^{m-4}$ $(m \ge 6)$, from *Theorem* 1, *Theorem* 2, *Theorem* 3, respectively.

Remark 3: The direct construction of MOCSs of length $2^{m-1}+2^{m-3}$ are available in [44] (for $t = 2^{m-3}$), but MOCSs of lengths $2^{m-1} + 2^{m-2} + 2^{m-4}$ $(m \ge 6)$ has never been reported in the literature.

TABLE II: Comparison of the proposed CCC construction with [8], [29]-[32], [46]

Ref.	Parameters	Phase	Based on	Length(N)	Constraints
[8]	$(2^{k+1}, 2^{k+1}, N)$	$q \ge 2, q$ is even	GBF of order > 2	2^{m}	$m, k \in \mathbb{Z}^+, \ m > 1$
[29]	$(2^{k+1}, 2^{k+1}, N)$	$q \ge 2, q$ is even	GBF of order 2	2^{m}	$m, k \in \mathbb{Z}^+, \ m > 1$
[30]	$(2^k, 2^k, N)$	$q \ge 2, q$ is even	GBF of order > 2	2^{m}	$k, m \in \mathbb{Z}^+, \ m \ge 1, k \le m$
[31]	$(2^{k+1}, 2^{k+1}, N)$	$q \geq 2, q$ is even	GBF of order 2	2^{m}	$m, k \in \mathbb{Z}^+, m > 1, 1 \le k \le m - 1$
[32]	$(2^k, 2^k, N)$	$q \ge 2, q$ is even	GBF of order 2	2^m	$m, k \in \mathbb{Z}^+, \ m \ge 3, 1 \le k \le m$
[46]	(M, M, N) $M = p_1 p_2 \dots p_k$	$q = \operatorname{lcm}(p_1, p_2, \dots, p_k, r)$	MVF of order 2	$p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}$	$r, m_i \in \mathbb{Z}^+, p_i \text{ is prime, } 1 \le i \le k,$
Theorem 3	$(2^{k+1}, 2^{k+1}, N)$	2	GBF of order > 2	$\frac{2^{m-1} + 2^{m-3}}{2^{m-1} + 2^{m-2} + 2^{m-4}}$	$\begin{array}{c} m,k\in\mathbb{Z}^+,m\geq5\\ m,k\in\mathbb{Z}^+,m\geq6 \end{array}$

Example 5: For m = 8 and q = 2, let us consider the GBF $f: \{\mathbf{0}_8, \mathbf{1}_8, \dots, \mathbf{208}_8\} \rightarrow \mathbb{Z}_2$ defined as

$$\begin{split} \mathbf{f} &= z_0 z_1 + z_0 z_2 + z_1 z_2 + z_1 \left(\bar{z}_7 \bar{z}_6 + \bar{z}_7 z_6 \left(\bar{z}_5 + z_5 \bar{z}_4 z_3 \right) \right) \\ &+ \left(\bar{z}_7 \bar{z}_6 \left(\bar{z}_5 z_4 z_3 + z_5 \bar{z}_4 \bar{z}_3 \right) + \bar{z}_7 z_6 \left(\bar{z}_5 \bar{z}_3 + z_5 \bar{z}_4 + z_4 \left(\bar{z}_5 z_3 \right) + z_7 \bar{z}_6 \left(\bar{z}_4 \bar{z}_3 + z_5 z_4 + \bar{z}_5 z_4 z_3 + z_7 z_6 \bar{z}_5 \bar{z}_4 \right) \right) . \end{split}$$

In this example, after deleting vertex z_2 , f forms a path, so the sets

$$S_0 = \{f + a_0 z_2 + a z_0 : a, a_0, \in \{0, 1\}\},$$
(70)

and
$$S_1 = \{f + a_0 z_2 + a z_0 + z_2 : a, a_0 \in \{0, 1\}\},$$
 (71)

are MOCSs of length 208. Similarly the sets

$$\bar{S}_0 = \left\{ \bar{f} + a_0 \bar{z}_2 + \bar{a} z_0 : a, a_0, \in \{0, 1\} \right\},$$
(72)

and
$$S_1 = \left\{ f + a_0 \bar{z}_2 + \bar{a} z_0 + \bar{z}_2 : a, a_0, \in \{0, 1\} \right\},$$
 (73)

are MOCSs of length 208 and hence their union i.e., the set $\{S_0, S_1, \overline{S}_0, \overline{S}_1\}$ forms a (4, 4, 208)-CCC.

VII. PMEPR OF MOCSS AND CCCS

In this section, the row and column sequence PMEPR of the sequences generated by *Theorem 2*, *Theorem 3* and MOCSs and CCCs constructed in the section VIare investigated. The PMEPR of the CCC-MC-CDMA system is determined by the column sequences of the complementary matrices when each complementary code is arranged as a matrix [8]. Thus, in this section, the column sequence PMEPR of constructed MOCSs and CCCs is effectively bound by 2.

Since the row sequences of S_t forms a CS of size 2^k , its PMEPR is upper bounded by 2^k . The column sequence PMEPR of the CCC generated from *Theorem* 3 can be bounded above by 2 by adding a suitable constant. For GBFs f, g and constants c, c_1 and c_2 , it can be easily verified that A(f+c) = A(f) and $C(f+c_1, g+c_2) = C(f, g)\omega^{c_1-c_2}$. For a permutation π' of $\{0, 1, \ldots, k-1\}$, the set (matrix) S_t of (31) is redefined by adding the constant $\sum_{\alpha=0}^{k-1} a_{\pi'(\alpha)} a_{\pi'(\alpha+1)}$

$$\left\{ f + \sum_{\alpha=0}^{k-1} a_{\alpha} z_{p_{\alpha}} + \sum_{\alpha=0}^{k-1} n_{\alpha} z_{p_{\alpha}} + a z_{\beta_{2}} + \sum_{\alpha=0}^{k-2} a_{\pi'(\alpha)} a_{\pi'(\alpha+1)} : a, a_{\alpha} \in \{0,1\} \right\},$$
(74)

where $t = \sum_{\alpha=0}^{k-1} n_{\alpha} 2^{\alpha}$. Adding the same constant to the set \bar{S}_t and noting that AACS remains unchanged and ACCS

changes by a constant, so the new set is still a CCC with same parameters. It can be observed from (74) that the *i*th column of S_t can be obtained by fixing $\mathbf{z} = (i_0, i_1, \dots, i_{m-1})$, $0 \le z < 2^{m-1} + 2^{m-3}$. So *i*th column of the matrix S_t is dependent on a function ϕ defined as

$$\phi(\mathbf{a}) = \sum_{\alpha=0}^{k-1} a_{\pi'(\alpha)} a_{\pi'(\alpha+1)} + \sum_{\alpha=0}^{k-1} a_{\alpha} i_{p_{\alpha}} + a z_{\beta_2} + \mathcal{C}, \quad (75)$$

where C is a constant (independent of *a*). Since any column sequence of the matrix S_t is obtained by a GBF, whose graph is a path consisting of *k* vertices. Hence, from [20] the *i*th column of S_t is a Golay sequence, and so its PMEPR is upper bounded by 2. Similarly it can be verified that the column sequence PMEPR of \bar{S}_t is also upper bounded by 2. So the maximum column sequence PMEPR of $(2^{k+1}, 2^{k+1}, 2^{m-1} + 2^{m-3})$ -CCC, can be suitably upper bounded by 2. The same is true for $(2^{k+1}, 2^{k+1}, 2^{m-1} + 2^{m-2} + 2^{m-4})$ -CCC.

Remark 4: There exist PU matrix based construction of CCCs of length non-power of two [40]–[42], but their column sequence PMEPR are high compared to the proposed construction.

VIII. CONCLUSION

In this paper, we have proposed a direct and generalized construction of GCP and binary CCC of non-power of two lengths by using higher-order GBFs. The resultant CCCs can be obtained directly from GBFs without using other tedious sequence operations. The non-power of two length binary CCCs directly constructed using GBFs finds many applications in wireless communication due to its simple modulo-2 arithmetic operation, modulation and good correlation properties. Column sequence PMEPR of the proposed CCC can be effectively reduced to be upper bounded by 2. The construction of MOCSs of non-power of two lengths is also provided in this paper. The proposed work solved the open problem cited in [44], [46]. The work is compared with existing literature.

References

- M. Golay, "Complementary series," *IRE Trans. Inf. Theory*, vol. 7, no. 2, pp. 82–87, Apr. 1961.
- [2] C.-C. Tseng and C. Liu, "Complementary sets of sequences," *IEEE Trans. Inf. Theory*, vol. 18, no. 5, pp. 644–652, 1972.
- [3] H.-H. Chen, S.-W. Chu, and M. Guizani, "On next generation CDMA technologies: The real approach for perfect orthogonal code generation," *IEEE Trans. Veh. Technol.*, vol. 57, no. 5, pp. 2822–2833, 2008.
- IEEE Trans. Veh. Technol., vol. 57, no. 5, pp. 2822–2833, 2008.
 [4] N. Suehiro and M. Hatori, "N-shift cross-orthogonal sequences," IEEE Trans. Inf. Theory, vol. 34, no. 1, pp. 143–146, 1988.

- [5] S.-M. Tseng and M. Bell, "Asynchronous multicarrier DS-CDMA using mutually orthogonal complementary sets of sequences," *IEEE Trans. Commun.*, vol. 48, no. 1, pp. 53–59, 2000.
- [6] H.-H. Chen, J.-F. Yeh, and N. Suehiro, "A multicarrier CDMA architecture based on orthogonal complementary codes for new generations of wideband wireless communications," *IEEE Commun. Mag.*, vol. 39, no. 10, pp. 126–135, 2001.
- H. Chen, *The Next Generation CDMA Technologies*. wiley, Jul. 2007, publisher Copyright: © 2007 John Wiley & Sons Ltd. All Rights Reserved.
- [8] Z. Liu, Y. L. Guan, and U. Parampalli, "New complete complementary codes for peak-to-mean power control in multi-carrier CDMA," *IEEE Trans. Commun.*, vol. 62, no. 3, pp. 1105–1113, 2014.
- [9] Z. Liu, Y. L. Guan, and H.-H. Chen, "Fractional-delay-resilient receiver design for interference-free MC-CDMA communications based on complete complementary codes," *IEEE Trans. Wireless Commun.*, vol. 14, no. 3, pp. 1226–1236, 2015.
- [10] S. Wang and A. Abdi, "MIMO ISI channel estimation using uncorrelated Golay complementary sets of polyphase sequences," *IEEE Trans. Veh. Technol.*, vol. 56, no. 5, pp. 3024–3039, 2007.
- [11] S. Li, J. Chen, and L. Zhang, "Optimisation of complete complementary codes in MIMO radar system," *Electron. Lett.*, vol. 46, pp. 1157–1159, 2010.
- [12] J. Tang, N. Zhang, Z. Ma, and B. Tang, "Construction of doppler resilient complete complementary code in MIMO radar," *IEEE Trans. Signal Process.*, vol. 62, no. 18, pp. 4704–4712, 2014.
- [13] C.-Y. Chen, Y.-J. Min, K.-Y. Lu, and C.-C. Chao, "Cell search for cellbased OFDM systems using quasi complete complementary codes," in *IEEE Int. Conf. Commun.*, 2008, pp. 4840–4844.
- [14] T. Kojima, T. Tachikawa, A. Oizumi, Y. Yamaguchi, and U. Parampalli, "A disaster prevention broadcasting based on data hiding scheme using complete complementary codes," in *Int. Symp. Inf. Theory Appl. (ISITA)*, 2014, pp. 45–49.
- [15] M. Golay, "Complementary series," *IRE Trans. Inf. Theory*, vol. 7, no. 2, pp. 82–87, 1961.
- [16] P. Fan, W. Yuan, and Y. Tu, "Z-complementary binary sequences," *IEEE Signal Process. Lett.*, vol. 14, no. 8, pp. 509–512, Aug. 2007.
- [17] B. Shen, Y. Yang, Z. Zhou, P. Fan, and Y. Guan, "New optimal binary Z-complementary pairs of odd length 2^m + 3," *IEEE Signal Process. Lett.*, vol. 26, no. 12, pp. 1931–1934, 2019.
- [18] C. Xie and Y. Sun, "Constructions of even-period binary Zcomplementary pairs with large ZCZs," *IEEE Signal Process. Lett.*, vol. 25, no. 8, pp. 1141–1145, 2018.
- [19] B. Shen, Y. Yang, and Z. Zhou, "A construction of binary golay complementary sets based on even-shift complementary pairs," *IEEE Access*, vol. 8, pp. 29882–29890, 2020.
- [20] J. A. Davis and J. Jedwab, "Peak-to-mean power control in OFDM, Golay complementary sequences, and Reed-Muller codes," *IEEE Trans. Inf. Theory*, vol. 45, no. 7, pp. 2397–2417, Nov. 1999.
- [21] K. G. Paterson, "Generalized Reed-Muller codes and power control in OFDM modulation," *IEEE Trans. Inf. Theory*, vol. 46, no. 1, pp. 104– 120, Jan. 2000.
- [22] R. Turyn, "Hadamard matrices, Baumert-Hall units, four-symbol sequences, pulse compression and surface wave encodings," J. Combin. Theory (A), vol. 16, pp. 313–333, 1974.
- [23] C.-Y. Chen, "Complementary sets of non-power-of-two length for peakto-average power ratio reduction in OFDM," *IEEE Trans. Inf. Theory*, vol. 62, no. 12, pp. 7538–7545, 2016.
- [24] —, "A new construction of golay complementary sets of non-powerof-two length based on Boolean functions," in *IEEE Wirel. Commun. Netw. Conf. (WCNC)*, 2017, pp. 1–6.
- [25] —, "A novel construction of complementary sets with flexible lengths based on Boolean functions," *IEEE Commun. Lett.*, vol. 22, no. 2, pp. 260–263, 2018.
- [26] A. R. Adhikary and S. Majhi, "New constructions of complementary sets of sequences of lengths non-power-of-two," *IEEE Commun. Lett.*, vol. 23, no. 7, pp. 1119–1122, 2019.
- [27] Y.-J. Lin, Z.-M. Huang, and C.-Y. Chen, "Golay complementary sets and multiple-shift complementary sets with non-power-of-two length and bounded PAPRs," *IEEE Commun. Lett.*, vol. 25, no. 5, pp. 2805– 2809, 2021.
- [28] P. Sarkar, S. Majhi, and Z. Liu, "A direct and generalized construction of polyphase complementary sets with low PMEPR and high code-rate for OFDM system," *IEEE Trans. Commun.*, vol. 68, no. 10, pp. 6245–6262, 2020.

- [29] A. Rathinakumar and A. K. Chaturvedi, "Complete mutually orthogonal Golay complementary sets from Reed-Muller codes," *IEEE Trans. Inf. Theory*, vol. 54, no. 3, pp. 1339–1346, Mar. 2008.
- [30] C.-Y. Chen, C.-H. Wang, and C.-C. Chao, "Complete complementary codes and generalized reed-muller codes," *IEEE Commun. Lett.*, vol. 12, no. 11, pp. 849–851, 2008.
- [31] L. Tian, Y. Li, and C. Xu, "Multiple complete complementary codes with inter-set zero cross-correlation zone," *IEEE Trans. Commun.*, vol. 68, no. 3, pp. 1925–1936, 2020.
- [32] S.-W. Wu and C.-Y. Chen, "Optimal Z-complementary sequence sets with good peak-to-average power-ratio property," *IEEE Signal Process. Lett.*, vol. 25, no. 10, pp. 1500–1504, 2018.
- [33] P. Sarkar, A. Roy, and S. Majhi, "Construction of Z-complementary code sets with non-power-of-two lengths based on generalised Boolean functions," *IEEE Commun. Lett.*, vol. 24, no. 8, pp. 1607–1611, 2020.
- [34] S.-W. Wu, A. Şahin, Z.-M. Huang, and C.-Y. Chen, "Z-complementary code sets with flexible lengths from generalised Boolean functions," *IEEE Access*, vol. 9, pp. 4642–4652, 2021.
- [35] P. Sarkar, S. Majhi, and Z. Liu, "Optimal Z-complementary code set from generalized Reed-Muller codes," *IEEE Trans. Commun.*, vol. 67, no. 3, pp. 1783–1796, 2019.
- [36] B. Shen, Y. Yang, P. Fan, and Z. Zhou, "New Zcomplementary/complementary sequence sets with non-power-of-two length and low PAPR," *Cryptogr. Commun.*, 2022.
- [37] G. Ghosh, S. Majhi, P. Sarkar, and A. K. Upadhyay, "Direct construction of optimal Z-complementary code sets with even lengths by using generalized Boolean functions," *IEEE Signal Process. Lett.*, 2022.
- [38] C. Xie, Y. Sun, and Y. Ming, "Constructions of optimal binary Zcomplementary sequence sets with large zero correlation zone," *IEEE Signal Process. Lett.*, vol. 28, pp. 1694–1698, 2021.
- [39] B. Shen, Y. Yang, and Z.Zhou, "A construction framework for mutually orthogonal complementary sequence sets with flexible lengths," in *Proc. Int. Conf. Sequences Appl.*, 2020.
- [40] S. Das, S. Budišin, S. Majhi, Z. Liu, and Y. L. Guan, "A multiplier-free generator for polyphase complete complementary codes," *IEEE Trans. Signal Process.*, vol. 66, no. 5, pp. 1184–1196, 2018.
- [41] S. Das, S. Majhi, and Z. Liu, "A novel class of complete complementary codes and their applications for APU matrices," *IEEE Signal Process. Lett.*, vol. 25, no. 9, pp. 1300–1304, 2018.
- [42] S. Das, S. Majhi, S. Budišin, and Z. Liu, "A new construction framework for polyphase complete complementary codes with various lengths," *IEEE Trans. Signal Process.*, vol. 67, no. 10, pp. 2639–2648, 2019.
- [43] L. Tian, Y. Li, Z. Zhou, and C. Xu, "Two classes of Z-complementary code sets with good cross-correlation subsets via paraunitary matrices," *IEEE Trans. Commun.*, vol. 69, no. 5, pp. 2935–2947, 2021.
- [44] S.-W. Wu, C.-Y. Chen, and Z. Liu, "How to construct mutually orthogonal complementary sets with non-power-of-two lengths?" *IEEE Trans. Inf. Theory*, vol. 67, no. 6, pp. 3464–3472, 2021.
- [45] L. Tian, X. Lu, C. Xu, and Y. Li, "New mutually orthogonal complementary sets with non-power-of-two lengths," *IEEE Signal Process. Lett.*, vol. 28, pp. 359–363, 2021.
- [46] P. Sarkar, Z. Liu, and S. Majhi, "Multivariable function for new complete complementary codes with arbitrary lengths," 2021.