A Tighter Upper Bound of the Expansion Factor for Universal Coding of Integers and Its Code Constructions

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Abstract

In entropy coding, universal coding of integers (UCI) is a binary universal prefix code, such that the ratio of the expected codeword length to max{1, H(P)} is less than or equal to a constant expansion factor K_c for any probability distribution P, where H(P) is the Shannon entropy of P. K_c^* is the infimum of the set of expansion factors. The optimal UCI is defined as a class of UCI possessing the smallest K_c^* . Based on prior research, the range of K_c^* for the optimal UCI is $2 \le K_c^* \le 2.75$. Currently, the code constructions achieve $K_c = 2.75$ for UCI and $K_c = 3.5$ for asymptotically optimal UCI. In this paper, we propose a class of UCI, termed ι code, to achieve $K_c = 2.5$. This further narrows the range of K_c^* to $2 \le K_c^* \le 2.5$. Next, a family of asymptotically optimal UCIs is presented, where their expansion factor infinitely approaches 2.5. Finally, a more precise range of K_c^* for the classic UCIs is discussed.

I. INTRODUCTION

In entropy coding, when the probability distribution of sources is unknown and difficult to measure, some entropy coding, such as arithmetic coding [1, 2] and Huffman coding [3], cannot be applied to compress the source. In this case, universal source coding [4] is a common way to encode the data, and LZ series algorithms [5–7] is one of the well-known algorithms of universal source coding. However, there is no universal source coding for infinite alphabet and discrete memoryless sources [8]. Universal coding of integers (UCI) is a universal code for infinite alphabet and discrete memoryless sources. UCIs have been applied in widespread applications,

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such as unbounded search problems [9, 10], inverted file index [11], inductive inference [12] and biological sequencing data compression [13, 14].

Prefix coding is a class of variable-length code that no codeword is a prefix of any other codeword. Binary coding means that the coding alphabet is $\{0, 1\}$. Elias [15] defined UCI as a binary universal prefix code, such that the ratio of the expected codeword length to $\max\{1, H(P)\}$ is less than or equal to a constant expansion factor K_c for any probability distribution P, where H(P) is the Shannon entropy of P. Many UCIs have been proposed and most of they can be divided into the following two categories [16, 17] (For example, group strategy [18] is the exception).

- message length strategy: This strategy is to encode a positive integer n into two parts. The suffix part of length L represents n, and the prefix part standing for L (The prefix part can be further subdivided). The coding of this strategy was proposed in [15, 19–23].
- flag strategy: This strategy is to select a special sequence, called flag, to determine the end of a codeword. The flag is not allowed to appear within a codeword. The coding of this strategy was proposed in [17, 24–27].

Recently, Yan and Lin [23] first studied the range of K_c . First, the authors defined *optimal* UCI, which is a class of UCI with the smallest $K_c^* \triangleq \inf\{K_c\}$. It is showed that the optimal UCI is in the range $2 \le K_c^* \le 2.75$, where $K_c^* = 2.75$ is achieved by η code [23]. In particular, for the asymptotically optimal UCI, the smallest expansion factor is $K_c = 3.5$, which is achieved by θ code [23] and Elias ω code [15].

In this paper, we further narrow the range of $K_{\mathcal{C}}^*$ of the optimal UCI. The contributions of this paper are listed below.

- 1) A class of UCI, but not asymptotically optimal, with $K_c = 2.5$ is presented. This reduces the upper bound of K_c^* from 2.75 to 2.5.
- 2) A family of asymptotically optimal UCIs is proposed, where $K_{\mathcal{C}}$ infinitely approaches 2.5.
- 3) The range of $K_{\mathcal{C}}^*$ for some classic UCIs is discussed (see Table IV).

In the rest of this paper, Section II introduces some background knowledge. Section III presents the main theorem of this paper. Section IV proposes a class of UCI to achieve $K_c = 2.5$. A family of asymptotically optimal UCIs is proposed in Section V. Section VI gives a more precise range of K_c^* for the classic UCIs. Section VII concludes this work.

II. PRELIMINARIES

A. The definitions of UCI and asymptotically optimal UCI

Elias [15] treated the coding problem as follows. Let C be a given binary prefix coding of the positive integers $\mathcal{N} \triangleq \{1, 2, \dots, m, \dots\}$. Let $L_{\mathcal{C}}(\cdot)$ denote the length function of C(i.e., $L_{\mathcal{C}}(m) = |\mathcal{C}(m)|$, for all $m \in \mathcal{N}$). Let P denote any probability distribution of \mathcal{N} (i.e., $\sum_{n=1}^{\infty} P(n) = 1$, and $P(m) \ge 0$, for all $m \in \mathcal{N}$). In UCI, the source meets the probability distribution

$$P(m) \ge P(m+1),\tag{1}$$

for all $m \in \mathcal{N}$. Let $E_P(L_{\mathcal{C}}) = \sum_{n=1}^{\infty} L_{\mathcal{C}}(n)P(n)$ be the expected codeword length for \mathcal{C} , and let $H(P) = -\sum_{n=1}^{\infty} P(n) \log_2 P(n)$ denote the entropy of P. Elias [15] defined \mathcal{C} to be *universal* if there is a constant $K_{\mathcal{C}}$ such that

$$\frac{E_P(L_c)}{\max\{1, H(P)\}} \le K_c,\tag{2}$$

for all P with finite entropy, where $K_{\mathcal{C}}$ is the expansion factor. Furthermore, \mathcal{C} is called *asymptotically optimal* if \mathcal{C} is universal and a function $R_{\mathcal{C}}(\cdot)$ exists such that

$$\lim_{H(P)\to+\infty} R_{\mathcal{C}}(H(P)) = 1,$$
(3)

and

$$\frac{E_P(L_C)}{\max\{1, H(P)\}} \le R_C(H(P)),\tag{4}$$

for all P with finite entropy.

B. Some classic UCIs

In this subsection, we briefly introduce five classic UCIs, termed γ code, δ code, ω code, η code, and θ code. For the specific structure of classic UCIs, please refer to [15, 23]. First, the codeword lengths and the range of K_c^* of these UCIs are listed in Table I. And the five classic UCIs all satisfy $L_c(1) = 1$. Next, the following theorem can be used to judge whether a UCI is asymptotically optimal.

Theorem 1. [15, 24] Given a UCI C, the function $L_{\mathcal{C}}(\cdot)$ satisfies $L_{\mathcal{C}}(m) \ge c + b \lfloor \log_2 m \rfloor$ for all $m \in \mathcal{N}$, where b is a constant greater than 1 and c is a constant. Then, C is not asymptotically optimal.

Code	The codeword lengths for $2 \leq m \in \mathcal{N}$	The range of $K^*_{\mathcal{C}}$	Asymptotically optimal
γ code	$L_{\gamma}(m) = 1 + 2\lfloor \log_2 m \rfloor$	$K_{\gamma}^* = 3$	No
δ code	$L_{\delta}(m) = 1 + \lfloor \log_2 m \rfloor + 2 \lfloor \log_2(1 + \lfloor \log_2 m \rfloor) \rfloor$	$2.5 \le K_{\delta}^* \le 4$	Yes
ω code	$L_{\omega}(m) = 1 + \sum_{n=1}^{s} (\lambda^{n}(m) + 1)^{-1}$	$2.1 < K_\omega^* \leq 3.5$	Yes
η code	$L_{\eta}(m) = 3 + \lfloor \log_2(m-1) \rfloor + \lfloor \frac{\lfloor \log_2(m-1) \rfloor}{2} \rfloor$	$2.5 \le K_{\eta}^* \le 2.75$	No
θ code	$L_{\theta}(m) = 3 + \lfloor \log_2 m \rfloor + \lfloor \log_2 \lfloor \log_2 m \rfloor \rfloor + \lfloor \frac{\lfloor \log_2 \lfloor \log_2 m \rfloor \rfloor}{2} \rfloor$	$2.5 \le K_{\theta}^* \le 3.5$	Yes

TABLE I: The codeword lengths and ranges of $K^*_{\mathcal{C}}$ of some classic UCIs

 $^{1}\lambda(m) \triangleq \lfloor \log_2 m \rfloor, \lambda^n$ is the *n*-fold compositions of function λ , and $s = s(m) \in \mathcal{N}$ is a uniquely integer satisfying $\lambda^s(m) = 1.$

III. THE MAIN THEOREM

In this section, we present the main theorem of this paper. First, a related lemma is provided, then the theorem is given.

Lemma 1. Given an any probability distribution $P = (P(1), P(2), \dots, P(m), \dots)$, then

(1) $H(P) \ge -\log_2 P(1);$ (2) If H(P) < 1, then $P(1) > \frac{1}{2}$.

Proof. (1)

$$H(P) = \sum_{n=1}^{\infty} P(n) \log_2 \frac{1}{P(n)}$$

$$\geq \sum_{n=1}^{\infty} P(n) \log_2 \frac{1}{P(1)}$$

$$= -\log_2 P(1).$$
(5)

(2) When H(P) < 1, then

$$-\log_2 P(1) \le H(P) < 1 \Rightarrow P(1) > \frac{1}{2}.$$
 (6)

Theorem 2. Given a prefix code C, the function $L_{\mathcal{C}}(\cdot)$ satisfies $L_{\mathcal{C}}(1) = 1$ and $L_{\mathcal{C}}(m) \leq b + 1 + b \lfloor \log_2 m \rfloor$ for all $2 \leq m \in \mathcal{N}$, where the constant b is in the range $1 \leq b \leq \frac{9}{4}$. Then,

$$\frac{E_P(L_C)}{\max\{1, H(P)\}} \le b+1; \tag{7}$$

that is, C is a UCI and $K_{C}^{*} \leq b + 1$.

Proof. Due to

$$mP(m) \le \sum_{n=1}^{m} P(n) \le \sum_{n=1}^{\infty} P(n) = 1,$$
 (8)

we have $m \leq \frac{1}{P(m)}$ for all $m \in \mathcal{N}$. Thus, we obtain

$$\sum_{n=2}^{\infty} P(n) \log_2 n \le \sum_{n=2}^{\infty} P(n) \log_2 \frac{1}{P(n)}$$

$$= H(P) + P(1) \log_2 P(1).$$
(9)

The expected codeword length is

$$E_{P}(L_{\mathcal{C}}) \leq P(1) + \sum_{n=2}^{\infty} P(n)(b+1+b\lfloor \log_{2} n \rfloor)$$

= $b+1 - bP(1) + b\sum_{n=2}^{\infty} P(n)\lfloor \log_{2} n \rfloor$
 $\leq b+1 - bP(1) + b\sum_{n=2}^{\infty} P(n)\log_{2} n$
 $\leq b+1 - bP(1) + bH(P) + bP(1)\log_{2} P(1).$ (10)

We consider three cases below.

1) Case H(P) < 1: In this case, we obtain $P(1) > \frac{1}{2}$ from Lemma 1. Further, we have

$$\frac{E_P(L_C)}{\max\{1, H(P)\}} \le b + 1 - bP(1) + bH(P) + bP(1)\log_2 P(1)$$

$$\le 2b + 1 - bP(1) + bP(1)\log_2 P(1).$$
(11)

Let $g_1(x) \triangleq 2b + 1 - bx + bx \log_2 x$. We only need to prove that $g_1(x) \leq b + 1$ over interval $[\frac{1}{2}, 1]$. We know that the curve of g_1 is U-shaped over interval $[\frac{1}{2}, 1]$ by its derivative. Thus, we have $g_1(x) \leq \max\{g_1(\frac{1}{2}), g_1(1)\} = b + 1$ over interval $[\frac{1}{2}, 1]$.

2) Case $H(P) \ge 1$ and $P(1) \ge 0.5$: In this case, we have

$$\frac{E_P(L_C)}{\max\{1, H(P)\}} \le \frac{b+1-bP(1)+bH(P)+bP(1)\log_2 P(1)}{H(P)}$$

= $b + \frac{b+1-bP(1)+bP(1)\log_2 P(1)}{H(P)}$
 $\le 2b+1-bP(1)+bP(1)\log_2 P(1)$
 $\stackrel{(a)}{\le} b+1.$ (12)

where (a) is due to $g_1(x) \le b+1$ over interval $[\frac{1}{2}, 1]$.

3) Case $H(P) \ge 1$ and P(1) < 0.5: In this case, we obtain

$$\frac{E_P(L_C)}{\max\{1, H(P)\}} \le b + \frac{b+1 - bP(1) + bP(1)\log_2 P(1)}{H(P)}$$

$$\stackrel{(a)}{\le} b + \frac{b+1 - bP(1) + bP(1)\log_2 P(1)}{-\log_2 P(1)}.$$
(13)

where (a) is due to Lemma 1. Let

$$g_2(x) \triangleq \frac{b+1-bx+bx\log_2 x}{-\log_2 x}$$

$$= \ln 2 \cdot \frac{bx-b-1}{\ln x} - bx.$$
(14)

We need to prove that $g_2(x) \leq 1$ over interval $(0, \frac{1}{2})$. We first prove that $g'_2(x) > 0$ over interval $(0, \frac{1}{2})$. Due to

$$g_2'(x) = \ln 2 \cdot \frac{b \ln x - b + \frac{b+1}{x}}{(\ln x)^2} - b,$$
(15)

then $g'_2(x) > 0$ over interval $(0, \frac{1}{2})$ is equivalent to $f(x) > \frac{1}{\ln 2}$ over interval $(0, \frac{1}{2})$, where $f(x) \triangleq \frac{\ln x - 1 + \frac{b+1}{bx}}{(\ln x)^2}.$ (16)

Finally, we obtain

$$f'(x) = \frac{-\ln x}{x^2(\ln x)^4} h(x)$$

$$\triangleq \frac{-\ln x}{x^2(\ln x)^4} \left(x \ln x + \frac{b+1}{b} \ln x - 2x + \frac{2b+2}{b} \right),$$

$$h'(x) = \ln x + \frac{b+1}{bx} - 1,$$

$$h''(x) = \frac{1}{x^2} \left(x - \frac{b+1}{b} \right).$$

(17)

Due to h''(x) < 0 over interval $(0, \frac{1}{2})$, we have

$$h'(x) > h'(\frac{1}{2})$$

= $\ln \frac{1}{2} + \frac{2b+2}{b} - 1$
> $-\ln 2 - 1 + 2$ (18)

over interval $(0, \frac{1}{2})$. Thus, h(x) strictly increases over interval $(0, \frac{1}{2})$. Due to

$$h(0.19) = 0.19 \ln 0.19 + \frac{b+1}{b} (2 + \ln 0.19) - 2 \times 0.19$$

$$\leq 0.19 \ln 0.19 + 2 \times (2 + \ln 0.19) - 0.38$$
(19)

$$< 0$$

and

$$h(0.24) = 0.24 \ln 0.24 + \frac{b+1}{b}(2 + \ln 0.24) - 2 \times 0.24$$

$$\geq 0.24 \ln 0.24 + \frac{13}{9} \times (2 + \ln 0.24) - 0.48$$

$$> 0,$$
(20)

there exists $x_0 \in (0.19, 0.24)$ such that $h(x_0) = 0$. Further, we have h(x) < 0 and f'(x) < 0over interval $(0, x_0)$, h(x) > 0 and f'(x) > 0 over interval $(x_0, \frac{1}{2})$. And hence, f(x) strictly decreases over interval $(0, x_0)$ and f(x) strictly increases over interval $(x_0, \frac{1}{2})$. Thus, we obtain

$$f(x) \ge f(x_0)$$

$$= \frac{1}{\ln x_0} - \frac{1}{(\ln x_0)^2} + \frac{b+1}{b} \cdot \frac{1}{x_0(\ln x_0)^2}$$

$$> \frac{1}{\ln 0.24} - \frac{1}{(\ln 0.24)^2} + \frac{13}{9} \times \frac{1}{0.19(\ln 0.19)^2}$$

$$> \frac{1}{\ln 2}$$
(21)

over interval $(0, \frac{1}{2})$. Since $g'_2(x) > 0$ over interval $(0, \frac{1}{2})$, we obtain

$$g_{2}(x) < g_{2}(\frac{1}{2})$$

$$= \ln 2 \cdot \frac{\frac{b}{2} - b - 1}{-\ln 2} - \frac{b}{2}$$

$$= 1,$$
(22)

for all $x \in (0, \frac{1}{2})$.

The proof is completed.

Remark 1. In Theorem 2, the feasible range $1 \le b \le \frac{9}{4}$ is not tight. The upper bound of b is taken to be $\frac{9}{4}$ for the convenience of proving that $f(x) > \frac{1}{\ln 2}$ over interval $(0, \frac{1}{2})$.

When b = 1 in Theorem 2, the theoretical lower bound of $K_{\mathcal{C}}^*$ of the optimal UCI in [23] can be obtained. In fact, there is no such prefix code when $1 \le b < 1.5$.

Theorem 3. There is no prefix code C such that $L_{\mathcal{C}}(1) = 1$ and $L_{\mathcal{C}}(m) \leq b + 1 + b \lfloor \log_2 m \rfloor$ for all $2 \leq m \in \mathcal{N}$, where b is a constant less than $\frac{3}{2}$.

Proof. Suppose there is a prefix code C to meet the requirement.

1) For $m = 2, 3, L_{\mathcal{C}}(m) \le 2b + 1 < 4$. Thus, $L_{\mathcal{C}}(m) \le 3$.

2) For $m = 4, 5, 6, 7, L_{\mathcal{C}}(m) \le 3b + 1 < \frac{11}{2}$. Thus, $L_{\mathcal{C}}(m) \le 5$.

3) For $m = 8, 9, \dots, 15$, $L_{\mathcal{C}}(m) \le 4b + 1 < 7$. Thus, $L_{\mathcal{C}}(m) \le 6$.

Thus, we have

$$\sum_{m=1}^{\infty} \frac{1}{2^{L_{\mathcal{C}}(m)}} = \sum_{m=1}^{16} \frac{1}{2^{L_{\mathcal{C}}(m)}} + \sum_{m=17}^{\infty} \frac{1}{2^{L_{\mathcal{C}}(m)}}$$

$$\geq \frac{1}{2} + 2 \times \frac{1}{2^{3}} + 4 \times \frac{1}{2^{5}} + 8 \times \frac{1}{2^{6}} + \sum_{m=17}^{\infty} \frac{1}{2^{L_{\mathcal{C}}(m)}}$$

$$= 1 + \sum_{m=17}^{\infty} \frac{1}{2^{L_{\mathcal{C}}(m)}}$$

$$\geq 1.$$
(23)

This contradicts the Kraft's inequality [28]

$$\sum_{m=1}^{\infty} \frac{1}{2^{L_{\mathcal{C}}(m)}} \le 1,$$
(24)

so there is no such prefix code C.

IV. ι CODE TO ACHIEVE $K_{\mathcal{C}} = 2.5$

In this section, we provide a new UCI, termed ι code, to achieve $K_{\mathcal{C}} = 2.5$. First, we introduce some necessary notations. Let $\alpha(m)$ be m bits zeros followed by a single one, for all $m \in \mathcal{N}$. Let $\beta(m)$ be the binary representation of $m \in \mathcal{N}$. Let $[\beta(m)]$ be the binary string that removes the most significant bit one of $\beta(m)$. For example, $\alpha(3) = 0001$, $\beta(9) = 1001$ and $[\beta(9)] = 001$. Let $\{0, 1\}^*$ be a set containing all finite binary strings.

Next, the following defines an auxiliary code $\widetilde{\alpha} : \mathcal{N} \to \{0, 1\}^*$.

$$\widetilde{\alpha}(m) = \begin{cases} 1, & \text{if } m = 1, \\ \alpha(\frac{m}{2})0, & \text{if } m \ge 2 \text{ and } m \text{ is even,} \\ \alpha(\frac{m-1}{2})1, & \text{otherwise,} \end{cases}$$
(25)

for all $m \in \mathcal{N}$. Further, we define $\iota : \mathcal{N} \to \{0, 1\}^*$ below.

$$\iota(m) = \widetilde{\alpha}(|\beta(m)|)[\beta(m)], \tag{26}$$

for all $m \in \mathcal{N}$. To better understand both codes, Table II lists their first 16 codewords. From the definition, one can see that both code are prefix codes, and the decoding algorithm naturally corresponds.

n	$\widetilde{\alpha}$ code ι code	
1	1	1
2	01 0	010 0
3	01 1	010 1
4	001 0	011 00
5	001 1	011 01
6	0001 0	011 10
7	0001 1	011 11
8	00001 0	0010 000
9	00001 1	0010 001
10	000001 0	0010 010
11	000001 1	0010 011
12	0000001 0	0010 100
13	0000001 1	0010 101
14	00000001 0	0010 110
15	00000001 1	0010 111
16	00000001 0	0011 0000

TABLE II: The first 16 codewords of $\tilde{\alpha}$ code and ι code

Then, we analyze the K_{ι}^{*} of ι code. We obtain $L_{\iota}(1) = 1$ and

$$L_{\iota}(m) = |\widetilde{\alpha}(1 + \lfloor \log_2 m \rfloor)| + \lfloor \log_2 m \rfloor$$

= 2 + $\lfloor \frac{1 + \lfloor \log_2 m \rfloor}{2} \rfloor + \lfloor \log_2 m \rfloor$
 $\leq \frac{3}{2} \lfloor \log_2 m \rfloor + \frac{5}{2},$ (27)

for all $2 \le m \in \mathcal{N}$. Thus, we know that ι code is a UCI and $K_{\iota}^* \le 2.5$ due to Theorem 2. We consider the probability distribution $\overline{P} = (\frac{1}{2}, \frac{1}{2})$, and we obtain

$$\frac{E_{\overline{P}}(L_{\iota})}{\max\{1, H(\overline{P})\}} = 2.5.$$
(28)

Thus, $K_{\iota}^* \ge 2.5$. Further, we have $K_{\iota}^* = 2.5$. We find the frist UCI such that $K_{\mathcal{C}} = 2.5 < 2.75$. This means that the range of $K_{\mathcal{C}}^*$ of the optimal UCI is improved to $2 \le K_{\mathcal{C}}^* \le 2.5$.

Finally, we show that ι code is not asymptotically optimal. We obtain $L_{\iota}(1) = 1 + \frac{3}{2} \lfloor \log_2 1 \rfloor$ and

$$L_{\iota}(m) = 2 + \lfloor \frac{1 + \lfloor \log_2 m \rfloor}{2} \rfloor + \lfloor \log_2 m \rfloor$$

> $1 + \frac{3}{2} \lfloor \log_2 m \rfloor,$ (29)

for all $2 \le m \in \mathcal{N}$. Due to Theorem 1 and $L_{\iota}(m) \ge 1 + \frac{3}{2} \lfloor \log_2 m \rfloor$, for all $m \in \mathcal{N}$, ι code is not asymptotically optimal.

V. A FAMILY OF ASYMPTOTICALLY OPTIMAL UCIS

In this section, we introduce a family of asymptotically optimal UCIs. To better understand this family of asymptotically optimal UCIs, we first introduce a representative UCI in this family.

A. κ code to achieve $K_{\mathcal{C}} = \frac{8}{3}$

In this subsection, we present an asymptotically optimal UCI, termed κ code, to achieve $K_{\kappa} = \frac{8}{3} < 3.5$. Notably, κ code is a special case of a family of asymptotically optimal UCIs that will be introduced in the next subsection.

First, we define an auxiliary code $\widetilde{\gamma} : \mathcal{N} \to \{0, 1\}^*$ below.

$$\widetilde{\gamma}(m) = \begin{cases} \widetilde{\alpha}(m), & \text{if } m < 4, \\ \alpha(|\beta(m-2)|)[\beta(m-2)], & \text{otherwise,} \end{cases}$$
(30)

for all $m \in \mathcal{N}$. Further, we define $\kappa : \mathcal{N} \to \{0, 1\}^*$ below.

$$\kappa(m) = \tilde{\gamma}(|\beta(m)|)[\beta(m)], \tag{31}$$

for all $m \in \mathcal{N}$. Table III lists some codewords for $\tilde{\gamma}$ code and κ code. From definitions, we know that $\tilde{\gamma}$ code and κ code are prefix codes, and the decoding algorithm naturally corresponds. Due to the definition of $\tilde{\gamma}$ code and κ code, we obtain

$$L_{\tilde{\gamma}}(m) = \begin{cases} 1, & \text{if } m = 1, \\ 3, & \text{if } 2 \le m \le 3, \\ 2 + 2\lfloor \log_2(m-2) \rfloor, & \text{otherwise,} \end{cases}$$
(32)

and

$$L_{\kappa}(m) = \begin{cases} 1, & \text{if } m = 1, \\ 4, & \text{if } 2 \le m \le 3, \\ 5, & \text{if } 4 \le m \le 7, \\ 2 + \lfloor \log_2 m \rfloor + 2 \lfloor \log_2 (\lfloor \log_2 m \rfloor - 1) \rfloor, & \text{otherwise.} \end{cases}$$
(33)

Next, a lemma about the codeword length of κ code is given.

Lemma 2. The codeword length of κ code

$$L_{\kappa}(m) \le \frac{8}{3} + \frac{5}{3} \lfloor \log_2 m \rfloor, \tag{34}$$

n	$\widetilde{\gamma}$ code κ code	
1	1	1
2	01 0	010 0
3	01 1	010 1
4	001 0	011 00
5	001 1	011 01
6	0001 00	011 10
7	0001 01	011 11
8	0001 10	0010 000
9	0001 11	0010 001
10	00001 000	0010 010
11	00001 001	0010 011
12	00001 010	0010 100
20	000001 0010	0011 0100
50	0000001 10000	000100 10010
100	00000001 100010	000101 100100

TABLE III: Some codewords of $\widetilde{\gamma}$ code and κ code

for all $2 \leq m \in \mathcal{N}$.

Proof. We first prove an auxiliary inequality as follows.

$$\lfloor \log_2(x-1) \rfloor \le \frac{1}{3} + \frac{1}{3}x,$$
(35)

for all $3 \le x \in \mathcal{N}$. When x = 3 or x = 4, we can verify directly. When x = 5, both sides of inequality (35) are 2. Hereafter, if the left side of inequality (35) is increased by 1, then x must be increased by at least 4. At the same time, the right side of inequality (35) is increased by at least $\frac{1}{3} \times 4 = \frac{4}{3} > 1$. Thus, inequality (35) holds. For inequality (34), when $m \le 7$, we can verify directly. When $m \ge 8$, we obtain

$$L_{\kappa}(m) = 2 + \lfloor \log_2 m \rfloor + 2 \lfloor \log_2(\lfloor \log_2 m \rfloor - 1) \rfloor$$

$$\leq 2 + \lfloor \log_2 m \rfloor + 2 \times (\frac{1}{3} + \frac{1}{3} \lfloor \log_2 m \rfloor)$$

$$= \frac{8}{3} + \frac{5}{3} \lfloor \log_2 m \rfloor.$$

$$\Box$$

Finally, we propose the main theorem in this subsection.

Theorem 4. (1) $2.5 \le K_{\kappa}^* \le \frac{8}{3}$;

- (2) κ code is asymptotically optimal.
- *Proof.* (1) Due to Theorem 2 and Lemma 2, we know that κ code is a UCI and $K_{\kappa}^* \leq \frac{8}{3}$. We consider $\overline{P} = (\frac{1}{2}, \frac{1}{2})$, and we obtain

$$\frac{E_{\overline{P}}(L_{\kappa})}{\max\{1, H(\overline{P})\}} = 2.5.$$
(37)

Thus, $K_{\kappa}^* \ge 2.5$. Further, we have $2.5 \le K_{\kappa}^* \le \frac{8}{3}$.

(2) The expected codeword length is

$$E_{P}(L_{\kappa}) = P(1) + 4(P(2) + P(3)) + 5\sum_{n=4}^{7} P(n) + \sum_{n=8}^{\infty} P(n)L_{\kappa}(n)$$

$$< 5 + \sum_{n=8}^{\infty} P(n)\log_{2} n + 2\sum_{n=8}^{\infty} P(n)\log_{2}(\log_{2} n)$$

$$\leq 5 + \sum_{n=2}^{\infty} P(n)\log_{2} n + 2\sum_{n=2}^{\infty} P(n)\log_{2}(\log_{2} n)$$

$$\stackrel{(a)}{\leq} 5 + H(P) + P(1)\log_{2} P(1) + 2\sum_{n=2}^{\infty} P(n)\log_{2}(\log_{2} n)$$

$$\leq 5 + H(P) + 2P(1)\log_{2} 1 + 2\sum_{n=2}^{\infty} P(n)\log_{2}(\log_{2} n)$$

$$\stackrel{(b)}{\leq} 5 + H(P) + 2\log_{2}\left(P(1) + \sum_{n=2}^{\infty} P(n)\log_{2} n\right)$$

$$\leq T_{\kappa}(H(P)) \triangleq 5 + H(P) + 2\log_{2}(1 + H(P)),$$
(38)

where (a) is due to inequality (9) and (b) is due to the convexity of the logarithm. Therefore, we have

$$\lim_{H(P)\to+\infty} R_{\kappa}(H(P)) = \lim_{H(P)\to+\infty} \frac{T_{\kappa}(H(P))}{H(P)} = 1.$$
(39)

And hence, κ code is asymptotically optimal.

B. A family of asymptotically optimal UCIs

In this subsection, we propose a family of asymptotically Optimal UCIs, termed $\kappa[t]$ code, to further reduce the upper bound of $K_{\mathcal{C}}^*$. First, we provide the relevant definition. For any given positive integer t, we define a family of auxiliary codes $\tilde{\gamma}[t] : \mathcal{N} \to \{0, 1\}^*$ as follows:

$$\widetilde{\gamma}[t](m) = \begin{cases} \widetilde{\alpha}(m), & \text{if } m < 2t, \\ \alpha(|\beta(m+2-2t)|+t-2)[\beta(m+2-2t)], & \text{otherwise,} \end{cases}$$
(40)

for all $m \in \mathcal{N}$. Further, we define $\kappa[t] : \mathcal{N} \to \{0, 1\}^*$ as follows:

$$\kappa[t](m) = \widetilde{\gamma}[t](|\beta(m)|)[\beta(m)], \tag{41}$$

for all $m \in \mathcal{N}$. Two points need to be explained here. One is the prefix of $\tilde{\gamma}[t]$ code. The codeword of $\tilde{\gamma}[t]$ code starts with a series of consecutive zeros followed by a one. From the definition of $\tilde{\gamma}[t]$ code, we know that $\tilde{\gamma}[t](2t-1)$ starts with t-1 consecutive zeros followed by a one, and $\tilde{\gamma}[t](2t)$ starts with t consecutive zeros followed by a one. Thus, $\tilde{\gamma}[t]$ code a prefix code. The prefix of $\tilde{\gamma}[t]$ code guarantees the prefix of $\kappa[t]$ code. Their decoding algorithm naturally corresponds. The other is the special case of these two familys of codes. When t = 1, $\tilde{\gamma}[1]$ code is essentially Elias γ code and $\kappa[1]$ code is essentially Elias δ code. When t = 2, $\tilde{\gamma}[2]$ code is essentially $\tilde{\gamma}$ code and $\kappa[2]$ code is essentially κ code.

Due to the definition of $\tilde{\gamma}[t]$ code and $\kappa[t]$ code, we obtain

$$L_{\tilde{\gamma}[t]}(m) = \begin{cases} 1, & \text{if } m = 1, \\ 2 + \lfloor \frac{m}{2} \rfloor, & \text{if } 2 \le m < 2t, \\ t + 2 \lfloor \log_2(m + 2 - 2t) \rfloor, & \text{otherwise,} \end{cases}$$
(42)

and

$$L_{\kappa[t]}(m) = \begin{cases} 1, & \text{if } m = 1, \\ 2 + \lfloor \log_2 m \rfloor + \lfloor \frac{1 + \lfloor \log_2 m \rfloor}{2} \rfloor, & \text{if } 2 \le m < 2^{2t-1}, \\ t + \lfloor \log_2 m \rfloor + 2 \lfloor \log_2 (\lfloor \log_2 m \rfloor + 3 - 2t) \rfloor, & \text{otherwise.} \end{cases}$$
(43)

Next, a lemma about the codeword length of $\kappa[t]$ code is given.

Lemma 3. The codeword length of $\kappa[t]$ code

$$L_{\kappa[t]}(m) \le \frac{5}{2} + \frac{1}{2t+2} + \left(\frac{3}{2} + \frac{1}{2t+2}\right) \lfloor \log_2 m \rfloor, \tag{44}$$

for all $2 \leq m \in \mathcal{N}$.

Proof. We first prove an auxiliary inequality as follows:

$$t + 2\lfloor \log_2(x+3-2t) \rfloor \le \frac{5}{2} + \frac{1}{2t+2} + \left(\frac{1}{2} + \frac{1}{2t+2}\right)x,$$
(45)

for all $2t - 1 \le x \in \mathcal{N}$. When x = 2t - 1 or x = 2t, we can verify directly. When x = 2t + 1, both sides of inequality (45) are 4 + t. Hereafter, if the left side of inequality (45) is increased by 2, then x must be increased by at least 4. At the same time, the right side of inequality (45)

is increased by at least $\left(\frac{1}{2} + \frac{1}{2t+2}\right) \times 4 = 2 + \frac{2}{t+1} > 2$. Thus, inequality (45) holds. For inequality (44), when $2 \le m < 2^{2t-1}$, we have

$$L_{\kappa[t]}(m) = 2 + \lfloor \log_2 m \rfloor + \lfloor \frac{1 + \lfloor \log_2 m \rfloor}{2} \rfloor$$

$$\leq \frac{5}{2} + \frac{3}{2} \lfloor \log_2 m \rfloor$$

$$< \frac{5}{2} + \frac{1}{2t+2} + \left(\frac{3}{2} + \frac{1}{2t+2}\right) \lfloor \log_2 m \rfloor.$$
(46)

When $m \ge 2^{2t-1}$, we obtain

$$L_{\kappa[t]}(m) = t + 2\lfloor \log_2(\lfloor \log_2 m \rfloor + 3 - 2t) \rfloor + \lfloor \log_2 m \rfloor$$

$$\leq \frac{5}{2} + \frac{1}{2t+2} + \left(\frac{1}{2} + \frac{1}{2t+2}\right) \lfloor \log_2 m \rfloor + \lfloor \log_2 m \rfloor$$

$$= \frac{5}{2} + \frac{1}{2t+2} + \left(\frac{3}{2} + \frac{1}{2t+2}\right) \lfloor \log_2 m \rfloor.$$

(47)

Finally, we propose the main theorem in this subsection.

Theorem 5. (1) $2.5 \le K_{\kappa[t]}^* \le 2.5 + \frac{1}{2t+2}$; (2) $\kappa[t]$ code is a family of asymptotically optimal UCIs.

Proof. (1) Due to Theorem 2 and Lemma 3, we know that $\kappa[t]$ code is a UCI and $K_{\kappa[t]}^* \leq \frac{5}{2} + \frac{1}{2t+2}$. We consider $\overline{P} = (\frac{1}{2}, \frac{1}{2})$, and we obtain

$$\frac{E_{\overline{P}}(L_{\kappa[t]})}{\max\{1, H(\overline{P})\}} = 2.5.$$
(48)

Thus, $K_{\kappa[t]}^* \ge 2.5$. Further, we have $2.5 \le K_{\kappa[t]}^* \le 2.5 + \frac{1}{2t+2}$.

(2) When t = 1, Elias [15] has proven it. When $t \ge 2$, we obtain the following inequality derivation similar to (38).

$$E_{P}(L_{\kappa[t]}) = \sum_{n=1}^{\infty} P(n)L_{\kappa[t]}(n)$$

$$< L_{\kappa[t]}(2^{2t-1}-1) + \sum_{n=2^{2t-1}}^{\infty} P(n)\log_{2}n + 2\sum_{n=2^{2t-1}}^{\infty} P(n)\log_{2}(\log_{2}n)$$

$$\leq 3t - 1 + \sum_{n=2}^{\infty} P(n)\log_{2}n + 2\sum_{n=2}^{\infty} P(n)\log_{2}(\log_{2}n)$$

$$\stackrel{(a)}{\leq} T_{\kappa[t]}(H(P)) \triangleq 3t - 1 + H(P) + 2\log_{2}(1 + H(P)),$$
(49)

where (a) is due to (38). Therefore, we have

$$\lim_{H(P) \to +\infty} R_{\kappa[t]}(H(P)) = \lim_{H(P) \to +\infty} \frac{T_{\kappa[t]}(H(P))}{H(P)} = 1.$$
(50)

Thus, $\kappa[t]$ code is a family of asymptotically optimal UCIs.

When t tends to infinity, the value of $K_{\kappa[t]} = \frac{5}{2} + \frac{1}{2t+2}$ can be infinitely close to 2.5. An interesting thing needs to be explained here. When t is no longer a fixed value and tends to infinity, we can essentially regard $\lim_{t \to +\infty} \kappa[t]$ code as ι code. But at this time, $\lim_{t \to +\infty} \kappa[t]$ code is not asymptotically optimal.

VI. $K_{\mathcal{C}}^*$ of the Classic UCIs

In this section, we provide a more precise range of $K_{\mathcal{C}}^*$ of the classic UCIs by Theorem 2. The main results of this section are summarized as follows.

Theorem 6. (1) δ code is asymptotically optimal UCI and $2.5 \leq K_{\delta}^* \leq 2.75$;

- (2) ω code is asymptotically optimal UCI and $2.1 < K_{\omega}^* \leq 3$;
- (3) η code is UCI and $2.5 \le K_{\eta}^* \le \frac{8}{3}$;
- (4) θ code is asymptotically optimal UCI and $2.5 \le K_{\theta}^* \le 2.8$.

From Table I, we only need to prove that $K_{\delta}^* \leq 2.75$, $K_{\omega}^* \leq 3$, $K_{\eta}^* \leq \frac{8}{3}$ and $K_{\theta}^* \leq 2.8$. We first prove the following lemma.

Lemma 4. For all $2 \leq m \in \mathcal{N}$, we obtain

- (1) $L_{\delta}(m) \leq 2.75 + 1.75 \lfloor \log_2 m \rfloor;$
- (2) $L_{\omega}(m) \leq 3 + 2\lfloor \log_2 m \rfloor;$
- (3) $L_{\eta}(m) \leq \frac{8}{3} + \frac{5}{3} \lfloor \log_2 m \rfloor;$
- (4) $L_{\theta}(m) \le 2.8 + 1.8 \lfloor \log_2 m \rfloor.$

Proof. (1) We prove the following inequality

$$\left|\log_2(1+x)\right| \le 0.875 + 0.375x,\tag{51}$$

for all $x \in \mathcal{N}$. When $x \leq 2$, we can verify directly. When x = 3, both sides of inequality (51) are 2. Hereafter, if the left side of inequality (51) is increased by 1, then x must be

increased by at least 4. At the same time, the right side of inequality (51) is increased by at least $0.375 \times 4 = 1.5 > 1$. Thus, inequality (51) holds. Further, we obtain

$$L_{\delta}(m) = 1 + \lfloor \log_2 m \rfloor + 2 \lfloor \log_2 (1 + \lfloor \log_2 m \rfloor) \rfloor$$

$$\stackrel{(a)}{\leq} 1 + \lfloor \log_2 m \rfloor + 2(0.875 + 0.375 \lfloor \log_2 m \rfloor)$$

$$= 2.75 + 1.75 \lfloor \log_2 m \rfloor,$$
(52)

for all $2 \leq m \in \mathcal{N}$, where (a) is due to inequality (51).

(2) Our objective is to prove that

$$L_{\omega}(m) = 1 + \sum_{n=1}^{s} (\lambda^{n}(m) + 1) \le 3 + 2\lfloor \log_{2} m \rfloor,$$
(53)

for all $2 \le m \in \mathcal{N}$. Let $a_1 \triangleq 2$ and $a_{m+1} \triangleq 2^{a_m}$ for all $m \in \mathcal{N}$. When $s \le 2$; that is, $a_1 = 2 \le m < 16 = a_3$, we can verify directly. When s = 3; that is, $a_3 = 16 \le m < 65536 = a_4$, since

$$\lfloor \log_2 x \rfloor \le \frac{1}{2}x,\tag{54}$$

for all $x \in \mathcal{N}$ and

$$\lfloor \log_2 \lfloor \log_2 x \rfloor \rfloor + 1 \le \frac{1}{2}x,\tag{55}$$

for all $2 \leq x \in \mathcal{N}$, we obtain

$$L_{\omega}(m) = 3 + \lfloor \log_2 m \rfloor + \lfloor \log_2 \lfloor \log_2 m \rfloor \rfloor + (\lfloor \log_2 \lfloor \log_2 \lfloor \log_2 m \rfloor \rfloor \rfloor + 1)$$

$$\leq 3 + \lfloor \log_2 m \rfloor + \frac{1}{2} \lfloor \log_2 m \rfloor + \frac{1}{2} \lfloor \log_2 m \rfloor$$

$$= 3 + 2 \lfloor \log_2 m \rfloor,$$
(56)

for all $a_3 \leq m < a_4$. When $s \geq 4$; that is, $m \geq a_4$, we consider the following three inequalities.

2.1) We have

$$\lambda^{2}(m) + 1 = \lfloor \log_{2} \lfloor \log_{2} m \rfloor \rfloor + 1$$

$$\stackrel{(a)}{\leq} \frac{1}{2} \lfloor \log_{2} m \rfloor + 1$$
(57)

for all $2 \leq m \in \mathcal{N}$, where (a) is due to inequality (54).

2.2) We prove the following inequality

$$\lambda^{3}(m) + 1 \le \frac{1}{4} \lfloor \log_{2} m \rfloor, \tag{58}$$

for all $a_4 \leq m \in \mathcal{N}$. When $m = a_4$, we obtain

$$3 = \lambda^3(a_4) + 1 < \frac{1}{4} \lfloor \log_2 a_4 \rfloor = 4.$$
(59)

Hereafter, if the left side of inequality (58) is increased by 1, then m must be increased by at least $2^{2^{2^3}} - 2^{2^{2^2}} = 2^{256} - 2^{16}$. At the same time, the right side of inequality (58) is increased by at least $\frac{1}{4}(2^{256} - 2^{16}) > 1$. Thus, inequality (58) holds.

2.3) We prove the following inequality

$$\lambda^{t}(m) + 1 \le \frac{1}{2^{t-1}} \lfloor \log_2 m \rfloor, \tag{60}$$

for all $a_t \leq m \in \mathcal{N}$, where t is any given integer greater than or equal to 4. When $m = a_t$, due to

$$\lambda^{t}(a_{t}) = \lambda^{t-1}(a_{t-1}) = \dots = \lambda(a_{1}) = 1,$$
(61)

we obtain

$$2 = \lambda^{t}(a_{t}) + 1 = \frac{1}{2^{3}}a_{3} \le \frac{1}{2^{t-1}}a_{m-1} = \frac{1}{2^{t-1}}\lfloor \log_{2} a_{t} \rfloor.$$
 (62)

Hereafter, if the left side of inequality (60) is increased by 1, then m must be increased by at least $a_{t+1} - a_t$. At the same time, the right side of inequality (60) is increased by at least

$$\frac{1}{2^{t-1}} (\lfloor \log_2 a_{t+1} \rfloor - \lfloor \log_2 a_t \rfloor)$$

= $\frac{1}{2^{t-1}} (a_t - a_{t-1}) \ge \frac{1}{2^3} (a_4 - a_3) > 1.$ (63)

Thus, inequality (60) holds.

Due to inequality (57), (58) and (60), we obtain

$$L_{\omega}(m) = 2 + \lfloor \log_2 m \rfloor + \sum_{n=2}^{s} (\lambda^n(m) + 1)$$

$$\leq 2 + \lfloor \log_2 m \rfloor + 1 + \sum_{n=1}^{s-1} \frac{\lfloor \log_2 m \rfloor}{2^n}$$

$$= 3 + (2 - \frac{1}{2^{s-1}}) \lfloor \log_2 m \rfloor$$

$$< 3 + 2 \lfloor \log_2 m \rfloor.$$
(64)

$$L_{\eta}(m) = 3 + \lfloor \log_{2}(m-1) \rfloor + \lfloor \frac{\lfloor \log_{2}(m-1) \rfloor}{2} \rfloor$$

$$\leq \frac{8}{3} + \frac{1}{6} \times 2 + \frac{3}{2} \lfloor \log_{2} m \rfloor$$

$$\leq \frac{8}{3} + \frac{1}{6} \lfloor \log_{2} m \rfloor + \frac{3}{2} \lfloor \log_{2} m \rfloor$$

$$= \frac{8}{3} + \frac{5}{3} \lfloor \log_{2} m \rfloor.$$
(65)

(4) We prove the following inequality

$$0.2 + 1.5 \lfloor \log_2 x \rfloor \le 0.8x,\tag{66}$$

for all $3 \le x \in \mathcal{N}$. When x = 3, we can verify directly. When x = 4, both sides of inequality (66) are 3.2. Hereafter, if the left side of inequality (66) is increased by 1.5, then x must be increased by at least 4. At the same time, the right side of inequality (66) is increased by at least $0.8 \times 4 = 3.2 > 1.5$. Thus, inequality (66) holds. For $L_{\theta}(m) \le 2.8 + 1.8 \lfloor \log_2 m \rfloor$, when $m \le 7$, we can verify directly. When $m \ge 8$, we obtain

$$L_{\theta}(m) = 3 + \lfloor \log_2 m \rfloor + \lfloor \log_2 \lfloor \log_2 m \rfloor \rfloor + \lfloor \frac{\lfloor \log_2 \lfloor \log_2 m \rfloor \rfloor}{2} \rfloor$$

$$\leq 3 + \lfloor \log_2 m \rfloor + 1.5 \lfloor \log_2 \lfloor \log_2 m \rfloor \rfloor$$

$$= 2.8 + \lfloor \log_2 m \rfloor + (0.2 + 1.5 \lfloor \log_2 \lfloor \log_2 m \rfloor \rfloor)$$

$$\leq 2.8 + 1.8 \lfloor \log_2 m \rfloor.$$
(67)

Due to Lemma 4 and Theorem 2, we have $K_{\delta}^* \leq 2.75$, $K_{\omega}^* \leq 3$, $K_{\eta}^* \leq \frac{8}{3}$ and $K_{\theta}^* \leq 2.8$. Furthermore, Theorem 6 is proved.

From Theorem 6, $K_{\iota}^* = 2.5$ and $2.5 \leq K_{\kappa[t]}^* \leq 2.5 + \frac{1}{2t+2}$, we obtain Table IV to compare the expansion factor between our ι code, $\kappa[t]$ code and the classic UCIs previously proposed. Currently, only ι code can achieve $K_{\iota} = 2.5$. For asymptotically optimal UCIs, the current best result is that $\kappa[t]$ code can achieve $K_{\kappa[t]} = 2.5 + \frac{1}{2t+2}$, for all $t \in \mathcal{N}$.

VII. CONCLUSIONS

In this paper, we study the expansion factor of UCI further, and Table IV summarizes the work of this paper. From Table IV, the proposed ι code improves the expansion factor of optimal UCI to $K_c = 2.5$, and the proposed $\kappa[t]$ code improves the expansion factor of asymptotically

Code	The range of $K_{\mathcal{C}}^*$	Asymptotically optimal
γ code	$K_{\gamma}^* = 3$	No
η code	$2.5 \le K_\eta^* \le \frac{8}{3}$	No
ι code	$K_{\iota}^{*} = 2.5$	No
δ code	$2.5 \le K_{\delta}^* \le 2.75$	Yes
ω code	$2.1 < K_{\omega}^* \le 3$	Yes
θ code	$2.5 \le K_{\theta}^* \le 2.8$	Yes
κ code	$2.5 \le K_{\kappa}^* \le \frac{8}{3}$	Yes
$\kappa[t]$ code	$2.5 \le K_{\kappa[t]}^* \le 2.5 + \frac{1}{2t+2}$	Yes

TABLE IV: The latest research results for $K_{\mathcal{C}}^*$ of some UCIs

optimal UCIs to $K_c \Rightarrow 2.5$. This work further reduces the range of the expansion factor to $2 \le K_c^* \le 2.5$. There are several unresolved issues, as listed below.

- 1) one can see that the explicit value of $K_{\mathcal{C}}^*$ of the optimal UCI is still unknown.
- 2) ω code is the only UCI whose lower bound of K_c^* is less than 2.5. Can ω code achieve $K_{\omega} < 2.5$?

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