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# Constraining the generalized uncertainty principle with neutron interferometry

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**Abstract** – The non-zero minimal length arises in various theories of gravity, leading to the so-called generalized uncertainty principle (GUP). In this short paper we analyze the GUP effects on neutron interferometry, showing that the obtained phase shifts depend on the mass and velocity of the particle. New upper bounds on the dimensionless GUP parameter have been found that are in agreement with the literature.

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**Introduction.** – According to Heisenberg uncertainty principle (HUP), the minimum uncertainty on position can theoretically take the zero value which corresponds to a state with maximum (infinite) uncertainty on its momentum. This result is arguable when gravity is taken into account, even when we only consider its classical features [1]. Moreover, various quantum gravity scenarios suggest generalized versions of the HUP, leading to the so-called generalized uncertainty principle (GUP):

$$\Delta X \Delta P \geq \frac{\hbar}{2} [1 + \beta (\Delta P)^2 + \dots], \quad (1)$$

where  $\beta$  denotes the GUP parameter [2, 3]. The above expression imposes a non-zero lower bound on the minimum value of  $\Delta X$  which is of order of Planck length; this is also supported by gedanken experiments [4]. Such generalization has various implications for a wide range of physical systems [5]. Theoretically, the GUP parameter is usually assumed to be of the order of unity [6]; however, this choice renders quantum gravity effects too small to be measurable. On the other hand, if one does not impose the above condition a priori, current experiments predict large upper bounds on it, which are compatible with current observations and may signal the existence of a new length scale [7].

It is now several decades since the groundbreaking work

written by Werner and his co-workers showed that gravitational [8, 9] and rotational [10] effects were to be found in neutron interference experiments performed on the Earth's surface [11, 12]. The predicted and experimentally confirmed gravitational phase shift is the only expression in physics to feature both Newton's constant of gravitation  $G$  and Planck's quantum of action  $\hbar$ , which surely makes these experiments particularly noteworthy. The two experiments are referred to hereafter as the COW experiment and the neutron Sagnac effect. It is worth mentioning that the Sagnac effect has been deeply studied due to its importance in understanding fundamental physics (see for instance [13, 14]).

In the present work it will be shown how the non-zero minimal length affects the phase shifts obtained in different regimes; in particular, we will see that such corrections will depend on the mass and velocity of the particle. Then, starting from these corrections, upper bounds on the dimensionless parameter  $\beta$  will be found.

The paper is organized as follows. In Section II we briefly recall some important facts about the GUP and the possibility to study its effects on physical systems. Then, in Section III, we study the GUP-deformed COW effect. In Section IV we investigate the modification induced by the GUP on the neutron Sagnac phase shift, both in non-relativistic and in special relativistic regimes.

Furthermore, a general relativistic application involving the Kerr metric is considered and analyzed. Finally, in the last section, the GUP dimensionless parameter will be constrained, concluding the paper with some remarks.

**Deformed algebra.** – The investigations in string theory and quantum gravity (see, e.g., [15]) lead to the GUP:

$$\Delta X \geq \frac{\hbar}{2} \left( \frac{1}{\Delta P} + \beta \Delta P \right), \quad (2)$$

from which follows the existence of the fundamental minimal length  $\Delta X_{min} = \hbar\sqrt{\beta}$ , which is order of Planck's length  $l_p = \sqrt{\hbar G/c^3}$ . It was established that a minimal length can be obtained in the frame of small quadratic deformation of the Heisenberg algebra [2]

$$[X, P] = i\hbar(1 + \beta P^2). \quad (3)$$

If we now consider the classical limit (i.e.,  $\hbar \rightarrow 0$ ), the quantum-mechanical commutator for operators is replaced by the Poisson bracket for corresponding classical variables, that is

$$\frac{1}{\hbar}[X, P] \rightarrow \{X, P\}, \quad (4)$$

which in the deformed case (3) reads

$$\{X, P\} = (1 + \beta P^2). \quad (5)$$

Looking at Eq. (5), we see that Poisson brackets deform in a way similar to the quantum GUP commutator and it is clearly a violation of the equivalence principle (EP). That is, the equation of motion of a test particle in a gravitational field depends on the mass of the test particle itself (and on its speed) [16]. It should be mentioned here that an interesting discussion about the GUP modified Poisson equation can be found in Ref. [17]. This issue, namely if the GUP modifications on Sagnac effect can be disentangled from a (possible) violation of the EP is beyond the goals of this paper.

The observation that the GUP can be obtained from the deformed Heisenberg algebra opens the possibility to study the influence of minimal length on properties of physical systems, both on the quantum level as well as on the classical one. In order to study the GUP effects on the physical systems considered here, let us introduce the following deformed algebra:

$$[X_i, P_i] = i\hbar\sqrt{1 + \beta P^2} (\delta_{ij} + \beta P_i P_j), \quad (6)$$

$$[X_i, X_j] = [P_i, P_j] = 0, \quad (7)$$

which can be obtained using the representation

$$\begin{aligned} \mathbf{X} &= \mathbf{x}, \\ \mathbf{P} &= \frac{\mathbf{p}}{\sqrt{1 - \beta \mathbf{p}^2}}, \end{aligned} \quad (8)$$

where  $\mathbf{x} = (x_1, x_2, x_3)$  and  $\mathbf{p} = (p_1, p_2, p_3)$  represent the coordinates and momentum in non-deformed space with canonical commutation relations

$$[x_i, p_i] = \hbar\delta_{ij}, \quad [x_i, x_j] = [p_i, p_j] = 0.$$

This specific representation is also considered in [3, 18].

**GUP-deformed COW effect.** – The milestone experiments of Colella, Overhauser, and Werner, commonly referred to as the COW experiments, provided the first link between general relativity and quantum mechanics. Indeed, the authors treated the gravitational field using Newtonian mechanics, but the formula for the phase shift contains both Newton's gravitational constant  $G$  and Planck's constant  $\hbar$ . The experimental setting is accurately described in Ref. [8]; it is based on the splitting of the neutron beam by Bragg diffraction from perfect crystals, as first implemented for  $X$  rays by Bonse and Hart (see [19] for further details). The interference involved is “topologically equivalent to a ring”, which we represent as a rectangle, of macroscopic dimensions (cm). The particles enter at the bottom left corner where the beam splits into two, such that they can travel at different heights in the gravitational field of the Earth, with different velocities. Then the beams recombine at the top right corner, where the interference takes place. Let us briefly describe a simple derivation of the COW effect. The spatial part of a plane matter wave (a neutron beam in this case) is given by  $e^{i\mathbf{k}\cdot\mathbf{r}}$ , where  $\mathbf{k}$  is the wave vector,  $k \equiv |\mathbf{k}| = 2\pi/\lambda$  is the wave number, with  $\lambda = \hbar/p$  ( $p$  is the particle's momentum). We can now easily compute the phase accumulated by the neutrons over a particular path:

$$\phi(\mathbf{r}) = \frac{1}{\hbar} \int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{p} \cdot d\mathbf{r}. \quad (9)$$

The phase difference is then

$$\Delta\phi = \frac{1}{\hbar} \int_{\mathbf{r}_0}^{\mathbf{r}} (\mathbf{p}_1 - \mathbf{p}_2) \cdot d\mathbf{r}, \quad (10)$$

where the subscript “1” refers to the lower route and “2” to the upper one. The only non-vanishing contributions to the phase shift are the ones coming from the horizontal parts of the routes; writing  $p_1 = mv$  and  $p_2 = mv'$ , where  $v$  and  $v'$  are the particle speeds, we get

$$\Delta\phi = \frac{1}{\hbar} m(v - v')L, \quad (11)$$

where  $L$  is the length of the interferometer. The principle of energy conservation gives

$$\frac{1}{2}mv'^2 = \frac{1}{2}mv^2 - mgH, \quad (12)$$

where, as usual,  $g$  denotes the gravitational acceleration and  $H$  the height of the interferometer. Since the product  $gH$  is much smaller than  $v^2$  (in the COW experiment they

considered thermal neutrons) we have that  $v_1 - v_2 \approx gH/v$ , and so we arrive at the final result:

$$\Delta\phi = \frac{mgA}{\hbar v}, \quad (13)$$

where  $A$  is the area of the interferometer.

It is worth mentioning that the above results can also be obtained starting from the following Lagrangian:

$$L = \frac{p^2}{2m} + m\mathbf{g} \cdot \mathbf{r}, \quad (14)$$

with  $\mathbf{p} = m\dot{\mathbf{r}} = \frac{\partial L}{\partial \dot{\mathbf{r}}}$ .

In order to analyze the influence of minimal length on the COW effect, it is commonly supposed that the deformed Hamiltonian has the form of the non-deformed Hamiltonian where instead of canonical variables of non-deformed phase space are written variables of deformed phase space. Then, by using the  $P(p)$ -relation in (8), we get

$$\begin{aligned} H_{COW}^{GUP} &= \frac{P^2}{2m} + U(\mathbf{r}) \\ &= \frac{p^2}{2m(1 - \beta p^2)} + U(\mathbf{r}) \\ &= \frac{p^2}{2m} + \beta \frac{p^4}{2m} + U(\mathbf{r}) + \mathcal{O}(\beta^2) \\ &\simeq H_{COW} + \beta \frac{p^4}{2m}. \end{aligned} \quad (15)$$

The COW phase is then given by

$$\Delta\phi_{COW}^{GUP} = \frac{1}{\hbar} \oint \mathbf{p} \cdot d\mathbf{r} = \frac{mgA}{\hbar v} (1 - 6\beta m^2 v^2). \quad (16)$$

In the limit  $\beta \rightarrow 0$ , we immediately obtain the standard result. Therefore, the deviation induced by the GUP on the COW effect is

$$\left| \frac{\Delta\Phi_{COW}^{GUP} - \Delta\Phi_{COW}}{\Delta\Phi_{COW}} \right| = 6\beta m^2 v^2, \quad (17)$$

where  $\Delta\Phi_{COW} = \frac{mgA}{\hbar v}$  denotes the standard phase shift.

**GUP-deformed neutron Sagnac effect.** – In 1975, in a beautiful paper, Page observed that the rotation of the Earth could induce corrections to the phase shift elicited by the Earth’s Newtonian potential; these corrections are of the same order as the COW term derived in the previous section [20]. This phenomenon is commonly regarded as the counterpart of the Sagnac effect for matter waves. The actual experiment was then performed in 1979 by Werner, Staudenmann and Colella [10]. Starting from these considerations, the Hamiltonian governing the neutron’s motion will involve a third term in addition to the kinetic energy and the gravitational potential energy [21]. Moreover, the momentum becomes

$$\mathbf{p} = m\dot{\mathbf{r}} + m\boldsymbol{\omega} \times \mathbf{r}, \quad (18)$$

and the phase coming from the term in  $\boldsymbol{\omega}$  is

$$\Delta\alpha = \frac{1}{\hbar} \oint m[\boldsymbol{\omega} \times \mathbf{r}] \cdot d\mathbf{r} = \frac{2m\boldsymbol{\omega} \cdot \mathbf{A}}{\hbar}. \quad (19)$$

Let us now write down the Hamiltonian explicitly,

$$H_S = \frac{p^2}{2m} + m\mathbf{g} \cdot \mathbf{r} - \boldsymbol{\omega} \cdot \mathbf{L}, \quad (20)$$

where the subscript ‘‘S’’ stands for ‘‘Sagnac’’,  $\mathbf{p}$  is the momentum of the neutron,  $\mathbf{L} = \mathbf{r} \times \mathbf{p}$  is the angular momentum of the neutron’s motion about the center of Earth ( $\mathbf{r} = 0$ ), and  $m$  its mass; Eq. (20) can also be written as

$$H_S = \frac{p^2}{2m} + m\mathbf{g} \cdot \mathbf{r} - \mathbf{p} \cdot (\boldsymbol{\omega} \times \mathbf{r}). \quad (21)$$

By substituting canonical variables with variables of deformed space we have

$$\begin{aligned} H_S^{GUP} &= \frac{P^2}{2m} + m\mathbf{g} \cdot \mathbf{r} - \mathbf{P} \cdot (\boldsymbol{\omega} \times \mathbf{r}) \\ &= \frac{1}{2m} \frac{p^2}{1 - \beta p^2} + m\mathbf{g} \cdot \mathbf{r} - \frac{\mathbf{p} \cdot (\boldsymbol{\omega} \times \mathbf{r})}{\sqrt{1 - \beta p^2}}. \end{aligned} \quad (22)$$

We now consider the linear approximation over the parameter of deformation  $\beta$ ; in this approximation the Hamiltonian reads

$$\begin{aligned} H_S^{GUP} &= \frac{p^2}{2m} + \frac{\beta}{2m} p^4 + m\mathbf{g} \cdot \mathbf{r} \\ &\quad - \mathbf{p} \cdot (\boldsymbol{\omega} \times \mathbf{r}) \left( 1 + \frac{1}{2} \beta p^2 \right) + \mathcal{O}(\beta^2). \end{aligned} \quad (23)$$

By recalling that  $\dot{\mathbf{r}} = \frac{\partial H}{\partial \mathbf{p}}$ , we obtain

$$\begin{aligned} \frac{\partial H_S^{GUP}}{\partial \mathbf{p}} &= \frac{\mathbf{p}}{m} + \frac{2\beta}{m} \mathbf{p}^3 - (\boldsymbol{\omega} \times \mathbf{r}) \\ &\quad - \frac{3}{2} \beta p^2 (\boldsymbol{\omega} \times \mathbf{r}) + \mathcal{O}(\beta^2) \\ &= \mathbf{v}. \end{aligned} \quad (24)$$

In first order over  $\beta$  we get

$$\mathbf{p} = m\mathbf{v} + m(\boldsymbol{\omega} \times \mathbf{r}) - 2\beta m^3 \mathbf{v}^3 + \frac{3}{2} \beta m^3 v^2 (\boldsymbol{\omega} \times \mathbf{r}). \quad (25)$$

The phase shift coming from the terms in  $\boldsymbol{\omega}$  is

$$\begin{aligned} \Delta\alpha_S^{GUP} &= \frac{1}{\hbar} \oint \mathbf{p} \cdot d\mathbf{r} = \frac{2m\boldsymbol{\omega} \cdot \mathbf{A}}{\hbar} \left( 1 + \frac{3}{2} \beta m^2 v^2 \right) \\ &= \Delta\alpha_S \left( 1 + \frac{3}{2} \beta m^2 v^2 \right). \end{aligned} \quad (26)$$

Thus, we have found a modification induced by the GUP on the neutron Sagnac effect; again, by considering the limit  $\beta \rightarrow 0$ , we immediately obtain the phase shift in non-deformed space. Therefore, a non-relativistic neutron

motion can generate an extra phase when GUP corrections are taken into account:

$$\left| \frac{\Delta\alpha_S^{GUP} - \Delta\alpha_S}{\Delta\alpha_S} \right| = \frac{3}{2}\beta m^2 v^2. \quad (27)$$

Let us now generalize the above discussion to the special relativistic case. In order to do that, let us consider the relativistic expression for kinetic energy and momentum, namely [22]

$$K = \gamma mc^2 - mc^2 \quad (28)$$

and

$$\mathbf{p} = \gamma m \mathbf{v}, \quad (29)$$

where  $\gamma$  denotes the Lorentz factor. By expanding the expression for the kinetic energy in powers of the small number ( $p/mc$ ), we get

$$\begin{aligned} K &= \sqrt{p^2 c^2 + m^2 c^4} - mc^2 \\ &= \frac{p^2}{2m} - \frac{p^4}{8m^3 c^2} + \mathcal{O}(1/c^4). \end{aligned} \quad (30)$$

Thus, the corresponding Hamiltonian for a particle in the relativistic limit can be written as

$$\begin{aligned} H_{R,S} &= \frac{p^2}{2m} - \frac{p^4}{8m^3 c^2} + m \mathbf{g} \cdot \mathbf{r} \\ &\quad - \gamma \mathbf{p} \cdot (\boldsymbol{\omega} \times \mathbf{r}) + \mathcal{O}(1/c^4), \end{aligned} \quad (31)$$

where ‘‘R,S’’ stands for ‘‘Relativistic, Sagnac’’ (we are considering relativistic effects up to first post-Minkowskian order).

Again, by substituting canonical variables with variables of deformed space and by using the  $P(p)$  relation, we can write the system Hamiltonian as a function of canonical coordinates  $p$  and  $r$ :

$$\begin{aligned} H_{R,S}^{GUP} &= \frac{1}{2m} \frac{p^2}{1 - \beta p^2} - \frac{1}{8m^3 c^2} \frac{p^4}{(1 - \beta p^2)^2} \\ &\quad + m \mathbf{g} \cdot \mathbf{r} - \gamma \frac{\mathbf{p}}{\sqrt{1 - \beta p^2}} \cdot (\boldsymbol{\omega} \times \mathbf{r}), \end{aligned} \quad (32)$$

where we omitted  $\mathcal{O}(1/c^4)$  for the sake of simplicity. In first order of  $\beta$ , we obtain

$$\begin{aligned} H_{R,S}^{GUP} &= \frac{p^2}{2m} + \frac{\beta}{2m} p^4 - \frac{p^4}{8m^3 c^2} (1 + 2\beta p^2) + m \mathbf{g} \cdot \mathbf{r} \\ &\quad - \gamma \mathbf{p} \cdot (\boldsymbol{\omega} \times \mathbf{r}) \left( 1 + \frac{1}{2} \beta p^2 \right), \end{aligned} \quad (33)$$

where we also omitted  $\mathcal{O}(\beta^2)$ . By considering terms up to  $p^4$ , and computing the derivative of the GUP-corrected Hamiltonian with respect to  $\mathbf{p}$ , we get

$$\begin{aligned} \frac{\partial H_{R,S}^{GUP}}{\partial \mathbf{p}} &= \frac{\mathbf{p}}{m} + \frac{2}{m} \mathbf{p}^3 \left( \beta - \frac{1}{4m^2 c^2} \right) \\ &\quad - \gamma (\boldsymbol{\omega} \times \mathbf{r}) - \frac{3}{2} \gamma \beta p^2 (\boldsymbol{\omega} \times \mathbf{r}) \\ &= \mathbf{v}, \end{aligned} \quad (34)$$

obtaining, in first order of  $\beta$ ,

$$\begin{aligned} \mathbf{p} &= m \mathbf{v} - 2\gamma^3 m^3 \mathbf{v}^3 \left( \beta - \frac{1}{4m^2 c^2} \right) \\ &\quad + m \gamma (\boldsymbol{\omega} \times \mathbf{r}) + \frac{3}{2} \beta \gamma^3 m^3 v^2 (\boldsymbol{\omega} \times \mathbf{r}). \end{aligned} \quad (35)$$

Then, the phase shift coming from the terms in  $\boldsymbol{\omega}$  reads

$$\begin{aligned} \Delta\alpha_{R,S}^{GUP} &= \frac{1}{\hbar} \oint \mathbf{p} \cdot d\mathbf{r} \\ &= \gamma \Delta\alpha_S \left( 1 + \frac{3}{2} \beta m^2 v^2 + \frac{3}{2} \beta \frac{m^2 v^4}{c^2} \right). \end{aligned} \quad (36)$$

In the semiclassical limit, namely for  $c \rightarrow \infty$  (or, equivalently, for  $\gamma \rightarrow 1$ ), Eq. (36) reduces to Eq. (26), as expected. Therefore, the discrepancy in the relativistic case is given by

$$\left| \frac{\Delta\alpha_{R,S}^{GUP} - \gamma \Delta\alpha_S}{\gamma \Delta\alpha_S} \right| = \frac{3}{2} \beta m^2 v^2 \left( 1 + \frac{v^2}{c^2} \right). \quad (37)$$

Some comments are in order here. The GUP and the related minimum length are normally considered in non-relativistic quantum mechanics. Extending it to relativistic theories is important for having a Lorentz invariant minimum length. Indeed, in Ref. [23], a covariant generalization of the standard GUP is formulated. By starting from the GUP proposed in the above mentioned paper, one can follow our approach to find the modified Sagnac phase. We have found that the order of  $\beta$  would not change significantly. However, a possible violation of Lorentz invariance is still object of debate.

*General relativistic corrections.* We now assume that the external gravitational field of Earth is described by the Kerr metric. The weak field approximation up to the first post-Newtonian order, i.e., up to  $\mathcal{O}(1/c^2)$ , gives [24]

$$\begin{aligned} ds^2 &\simeq \left( c^2 + 2\phi + \frac{2\phi^2}{c^2} + \frac{\phi \mathbf{b}^2}{c^2 r^4} (x'^2 + y'^2) \right) dt^2 \\ &\quad - \frac{\phi}{c^2} \frac{4\mathbf{b}}{r^2} (x' dy' - y' dx') dt \\ &\quad - \left( 1 - \frac{2\phi}{c^2} \right) (dx'^2 + dy'^2 + dz'^2), \end{aligned} \quad (38)$$

where  $\phi$  is the Newtonian gravitational potential of the Earth,  $\phi = -GM/r$  with  $r = \sqrt{x'^2 + y'^2 + z'^2}$ , and  $\mathbf{b}$  is its angular momentum per unit mass,  $\mathbf{b} = \mathbf{J}/M = \frac{2}{5} R^2 \boldsymbol{\omega}$ . In order to express physical quantities observed on the laboratory coordinate ( $xyz$ -frame), we replace  $(x', y', z')$  with  $(x, y, z)$  defined by

$$\begin{aligned} t' &= t, \\ x' &= x \cos(\omega t) - y \sin(\omega t), \\ y' &= x \sin(\omega t) + y \cos(\omega t), \\ z' &= z'. \end{aligned} \quad (39)$$

Following [25], we use the 3+1 formalism for representing the spacetime, where the four-dimensional metric tensor

Table 1: Upper bounds on the GUP parameter considering different sources.

Sources	$v$ (m/s)	Eq.	$\beta$ ( $\text{GeV}^{-2}$ )	$\beta_0$
Ultracold Neutron	5	27	$\leq 2.6 \times 10^{16}$	$\leq 2.6 \times 10^{54}$
Cold Neutron	800	27	$\leq 1.04 \times 10^{12}$	$\leq 1.04 \times 10^{50}$
Thermal Neutron	2200	27	$\leq 1.3 \times 10^{11}$	$\leq 1.3 \times 10^{49}$
Fission Neutron	$10^7$	37	$\leq 6 \times 10^3$	$\leq 6 \times 10^{44}$

$g_{\mu\nu}$  is split as follows:

$$\begin{aligned} g_{00} &= N^2 - \gamma_{ij} N^i N^j, \\ g_{0i} &= -\gamma_{ij} N^j, \\ g_{ij} &= -\gamma_{ij}, \end{aligned} \quad (40)$$

where  $N$  is the lapse function,  $N^i$  is the shift vector and  $\gamma_{ij}$  is the spatial metric on the  $t$ -constant hypersurface. The slowly rotating and weak gravitational field of Earth in the observer's rest frame is expressed by the quantities

$$\begin{aligned} N &= c \left( 1 + \frac{\phi}{c^2} + \frac{\phi^2}{2c^4} \right), \\ N^x &= - \left[ 1 + \frac{4\phi R^2}{5c^2 r^2} \right] \omega y, \\ N^y &= \left[ 1 + \frac{4\phi R^2}{5c^2 r^2} \right] \omega x, \\ N^z &= 0, \\ \gamma_{ij} &= \left( 1 - \frac{2\phi}{c^2} \right) \delta_{ij}. \end{aligned} \quad (41)$$

In order to derive the Hamiltonian of a free neutron moving in the curved spacetime produced by the gravitational field of the Earth, we introduce the relativistic Lagrangian for a particle with mass  $m$  [26]:

$$\begin{aligned} L_{GR} &= -mc \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} \\ &= -mc \sqrt{N^2 - \gamma_{ij} (N^i + \dot{x}^i) (N^j + \dot{x}^j)}. \end{aligned} \quad (42)$$

Using the canonical momentum, we obtain the following classical Hamiltonian for the particle:

$$\begin{aligned} H_{GR} &\equiv p_i \dot{x}^i - L_{GR} - mc^2 \\ &= N \sqrt{m^2 c^2 + \gamma^{ij} p_i p_j} - N^i p_i - mc^2, \end{aligned} \quad (43)$$

where the rest mass energy is subtracted from the conventional definition of the Hamiltonian for later convenience. As before, by substituting the canonical variables with variables of deformed space and by using the  $P(p)$  relation, we can write

$$H_{GR}^{GUP} = N \sqrt{m^2 c^2 + \gamma^{ij} P_i P_j} - N^i P_i - mc^2, \quad (44)$$

where the momentum in the non-deformed space is defined as

$$P_i = \frac{p_i}{\sqrt{1 - \beta \gamma_{ij} p^i p^j}}. \quad (45)$$

By defining  $p^2 \equiv \gamma_{ij} p^i p^j$ , the Hamiltonian (44) becomes (in first order over  $\beta$ )

$$\begin{aligned} H_{GR}^{GUP} &= N \sqrt{m^2 c^2 + p^2 (1 + \beta p^2)} \\ &\quad - N^i p_i \left( 1 + \frac{1}{2} \beta p^2 \right) - mc^2, \end{aligned} \quad (46)$$

where we omitted  $\mathcal{O}(\beta^2)$  for the sake of simplicity. By assuming  $\gamma_{ij} p^i p^j \ll m^2 c^2$ , we get

$$\begin{aligned} H_{GR}^{GUP} &= \left( \frac{N}{c} - 1 \right) mc^2 + \frac{N}{c} \frac{p^2}{2m} (1 + \beta p^2) \\ &\quad - N^i p_i \left( 1 + \frac{1}{2} \beta p^2 \right) + \mathcal{O}(1/c^2). \end{aligned} \quad (47)$$

The above Hamiltonian can be rewritten in the following way. First, let us notice that the quantity  $N^i p_i$  can be written as

$$N^i p_i = \left[ 1 + \frac{4\phi R^2}{5c^2 r^2} \right] \boldsymbol{\omega} \cdot \mathbf{L}, \quad (48)$$

where  $\mathbf{L} = \mathbf{r} \times \mathbf{p}$  with  $\mathbf{r} = (x, y, z)$  and  $\mathbf{p} = (p_x, p_y, p_z)$ . The first term on the right-hand side of Eq. (47) can be written as

$$\left( \frac{N}{c} - 1 \right) mc^2 = m\phi + \frac{m}{2c^2} \phi^2. \quad (49)$$

Finally,

$$\begin{aligned} \frac{N}{c} \frac{p^2}{2m} (1 + \beta p^2) &= \frac{p^2}{2m} + \beta \frac{p^4}{2m} \\ &\quad + \frac{\phi}{c^2} \frac{p^2}{2m} (1 + \beta p^2) + \mathcal{O}(1/c^4). \end{aligned} \quad (50)$$

Therefore, the Hamiltonian becomes

$$\begin{aligned} H_{GR,S}^{GUP} &= \frac{p^2}{2m} + \beta \frac{p^4}{2m} + m\phi - \boldsymbol{\omega} \cdot \mathbf{L} \left( 1 + \frac{\beta}{2} p^2 \right) \\ &\quad + \frac{\phi}{c^2} \frac{p^2}{2m} (1 + \beta p^2) + \frac{m\phi^2}{2c^2} \\ &\quad - \frac{4\phi R^2}{5c^2 r^2} \left( 1 + \frac{\beta}{2} p^2 \right) \boldsymbol{\omega} \cdot \mathbf{L}. \end{aligned} \quad (51)$$

By defining

$$\gamma_G \equiv \frac{4\phi R^2}{5c^2 r^2}, \quad (52)$$

the derivative with respect to  $\mathbf{p}$  of the modified GR Hamiltonian can be written as

$$\begin{aligned} \frac{\partial H_{GR,S}^{GUP}}{\partial \mathbf{p}} &= \frac{\mathbf{p}}{m} \left(1 + \frac{\phi}{c^2}\right) + \frac{2\beta \mathbf{p}^3}{m} \left(1 + \frac{\phi}{c^2}\right) \\ &\quad - (\boldsymbol{\omega} \times \mathbf{r}) \left(1 + \frac{3}{2}\beta p^2\right) (1 + \gamma_G) \\ &= \mathbf{v}, \end{aligned} \quad (53)$$

leading to

$$\begin{aligned} \mathbf{p} &= m\mathbf{v} \left(1 - \frac{\phi}{c^2}\right) - 2\beta m^3 \mathbf{v}^3 \\ &\quad + m(\boldsymbol{\omega} \times \mathbf{r}) \left(1 + \frac{3}{2}\beta m^2 v^2\right) \end{aligned} \quad (54)$$

$$\times (1 + \gamma_G) \left(1 - \frac{\phi}{c^2}\right). \quad (55)$$

Now we can compute the phase shift, obtaining

$$\begin{aligned} \Delta\alpha_{GR,S}^{GUP} &= \Delta\alpha_S \left[ \left(1 + \frac{3}{2}\beta m^2 v^2\right) \right. \\ &\quad \left. \times \left(1 - \frac{\phi}{c^2}\right) (1 + \gamma_G) \right]. \end{aligned} \quad (56)$$

If  $c \rightarrow \infty$  (Galilean limit), Eq. (56) approaches Eq. (26), as expected.

$$\begin{aligned} \left| \frac{\Delta\alpha_{GR,S}^{GUP} - \Delta\alpha_S}{\Delta\alpha_S} \right| &= -\frac{\phi}{c^2} + \frac{3}{2}\beta m^2 v^2 \left(1 - \frac{\phi}{c^2}\right) \\ &\quad + \gamma_G \left(1 + \frac{3}{2}\beta m^2 v^2\right), \end{aligned} \quad (57)$$

up to second order in  $\phi$ .

**Bounds on  $\beta$  and conclusion remarks.** – To the best of our knowledge, the estimated error on the measurements of the phase shift in the COW experiment is of the order of 1%, confirmed by Michelson–Morley and Michelson–Gale experiments [27]; as the minimal length is not detectable in the current experimental setups, the GUP distributions should be smaller than the accuracy of Sagnac interferometers. This means that the deviations that we have obtained in the different regimes should satisfy the following condition:

$$\left| \frac{\Delta\alpha_S^{GUP} - \Delta\alpha}{\Delta\alpha} \right| \leq 10^{-2}. \quad (58)$$

For example, let us consider a neutron ( $m \approx 10^{-27}$  kg) having speed  $v \approx 10^3$  m/s, which experiences a modified COW phase due to the presence of GUP corrections. By using Eq. (17), we obtain  $6\beta m^2 v^2 \leq 10^{-2}$ , and so  $\beta \leq 6.6 \times 10^{11}$  GeV $^{-2}$ . Defining the dimensionless GUP parameter as  $\beta_0 = \beta M_p c^2$ , we get  $\beta_0 \leq 6.6 \times 10^{49}$ ; this result is in agreement with previous bounds. Let us now focus on the neutron Sagnac experiment; in this case, an

error of about 30% is considered [28]. Since the phase shifts we have found depend on the mass and velocity of the particle, we collect the upper bounds on  $\beta$  in table 1 for different sources (see [29]). All the bounds we have found are in agreement with previous ones (see for example Refs. [30–32]). It should be mentioned that, in order to find the constraints, we assumed  $\gamma \simeq 1.005$ , and the factors  $1 \pm \frac{\phi}{c^2}, \frac{R}{r}$  approximately equal to 1.

In summary, among the numerous quantum gravity effects, one which is of particular importance is the so-called GUP. It is well known that HUP lies at the heart of quantum mechanics; according to this principle, upon the loss of information on the momentum, the length can be arbitrarily precisely measured. On the other hand, various quantum gravity models predict the existence of a minimum measurable length. So, to be consistent with the still unknown theory of quantum gravity, the HUP should be consistently modified. Its effects on a wide range of physical systems have been recently investigated. Moreover, in recent years, rapid technological progress in matter wave interference experiments has been made, which provides us with a new tool with which to study the GUP. We have argued that there are some extra distributions induced by the GUP effects on the neutron Sagnac effect which depend on the particle velocity and its mass. Finally, we have used such results to constrain the dimensionless parameter  $\beta$ .

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