M5-branes Probing Flux Backgrounds

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ABSTRACT: We analyze the global symmetries and anomalies of $4d \mathcal{N} = 1$ field theories that arise from a stack of N M5-branes probing a class of flux backgrounds. These backgrounds consist of a resolved $\mathbb{C}^2/\mathbb{Z}_k$ singularity fibered over a smooth Riemann surface of genus $g \geq 2$, supported by a non-trivial G_4 -flux configuration labeled by a collection of 2(k-1) flux quanta, $\{N_i\}$. For k=2, this setup defines a non-trivial superconformal field theory (SCFT) in the IR, which is holographically dual to an explicit AdS_5 solution first described by Gauntlett, Martelli, Sparks, and Waldram. The generalization to $k \geq 3$ is hard to tackle directly within holography. Instead, in this paper we lay the groundwork for a systematic analysis of such a generalization by adopting anomaly inflow methods to identify continuous and discrete global symmetries of the 4d field theories. We also compute the 't Hooft anomalies for continuous symmetries at leading order in the limit of large N, N_i .

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1 Introduction and summary

Geometric and brane engineering provide a powerful framework for the construction of non-trivial quantum field theories (QFTs) and the analysis of their strongly coupled regimes. A prominent example is furnished by 4d superconformal field theories (SCFTs) realized as the low-energy limit of M5-brane configurations in M-theory. Large classes of strongly coupled 4d SCFTs can be realized by wrapping M5-branes on a Riemann surface with defects, preserving $\mathcal{N}=2$ [1, 2] or $\mathcal{N}=1$ [3–7] supersymmetry.

In this work we aim to investigate a class of constructions that remains largely unexplored: M5-branes probing flux backgrounds. As a case study, we consider M5-branes wrapped on a Riemann surface and probing a resolved $\mathbb{C}^2/\mathbb{Z}_k$ singularity. Such setups should be contrasted with M5-branes probing an unresolved $\mathbb{C}^2/\mathbb{Z}_k$ singularity, which yields

well studied 6d (1,0) SCFTs [8–13]. The latter may be further compactified to four dimensions on a Riemann surface, see e.g. [14–22].

Our analysis is motivated by a class of M-theory solutions, first discussed by Gauntlett, Martelli, Sparks, and Waldram (GMSW) [23], of the form $AdS_5 \times_w M_6$, a warped product of AdS_5 and an internal space M_6 . These solutions can be interpreted as the near-horizon geometry of a stack of M5-branes probing a resolved $\mathbb{C}^2/\mathbb{Z}_2$ singularity, fibered over a smooth genus-g Riemann surface Σ_g , and stabilized by a G_4 -flux configuration threading four-cycles constructed with Σ_g and the resolution two-cycle of $\mathbb{C}^2/\mathbb{Z}_2$ [24], the latter being a \mathbb{CP}^1 fibration over $\Sigma_g \times S^2$. Such an interpretation is supported by the features of an equivalent description of the internal space M_6 as a fibration of a 4d space M_4 over Σ_g , where M_4 is the resolution of the orbifold S^4/\mathbb{Z}_2 , obtained by the blow-up of the \mathbb{Z}_2 fixed points at the north and south poles of S^4 .

The setups described above admit a natural generalization, in which the group \mathbb{Z}_2 is replaced by \mathbb{Z}_k with $k \geq 3$. More precisely, we consider a different topology for M_6 : we still take M_6 to be a fibration of a 4d space M_4 over Σ_g , but now M_4 is the resolution of S^4/\mathbb{Z}_k , obtained by blowing up the fixed points of the \mathbb{Z}_k action at the poles of S^4 . The blow-up procedure generates a collection of k-1 two-cycles at each pole. A non-trivial G_4 -flux threads M_4 as well as the four-cycles obtained by combining these resolution cycles with Σ_g . In total, we have 2k-1 flux parameters: one flux quantum N is interpreted as the number of M5-branes in the stack, while two independent sets of k-1 flux quanta N_{N_i} , N_{S_i} $(i=1,\ldots,k-1)$ describe the resolution of the orbifold singularities at the north and south poles of S^4 .

The above discussion gives a concrete characterization of the topology of M_6 and the flux configuration threading it. The key physical question is whether this putative topology and its flux data can correspond to actual well-defined M-theory setups that yield non-trivial 4d field theories. A possible inroad into this problem is to search for explicit AdS_5 solutions in 11d supergravity in which the internal space has the topology of M_6 and is supported by a G_4 -flux configuration with the prescribed flux quanta. In other words, these solutions would be the $k \geq 3$ generalization of the k = 2 GMSW solutions. The BPS systems governing AdS_5 solutions preserving $\mathcal{N} = 1$ superconformal symmetry are well-understood [23, 25]. A direct search for AdS_5 solutions of the desired kind for $k \geq 3$, however, turns out to be prohibitively hard. Another possible strategy could be to try to construct a supergravity solution that describes the flux background probed by the M5-brane stack. Even in the case of k = 2, however, such a solution is not available, suggesting that it might be particularly challenging to push forward this approach for general k.

Faced with the difficulties outlined above, we turn to a different methodology for investigating if the topology and flux configuration under examination can yield interesting physics. Our approach is based on anomaly inflow. More precisely, the central working assumption of this paper is that the topology of M_6 and the associated flux configuration can be regarded as admissible boundary conditions for the 11d supergravity fields in the vicinity of a codimension-7 object, extended along four non-compact spacetime dimensions and furnishing a low-energy description of the wrapped M5-brane stack probing the flux background. The boundary conditions specified by M_6 , and the flux threading it, induce

an anomalous gauge variation of the low-energy M-theory effective action. Systematic methods have been developed [26]—building on [27, 28]—which take the topology and flux data on M_6 as input, and yield an inflow anomaly polynomial $I_6^{\rm inflow}$ that encodes the anomalous gauge variation induced by the boundary. According to the inflow paradigm, this variation cancels exactly against the 't Hooft anomalies of the 4d degrees of freedom that capture the IR dynamics of the wrapped M5-branes probing the flux background.

The main goal of this paper is the explicit implementation of this circle of ideas to the M_6 input data described before. This proves to be a non-trivial task. We uncover a rich pattern of symmetries and associated anomaly theories, demonstrating the power and flexibility of inflow methods.

Our analysis involves two classes of global symmetries in 4d. The first class consists of ordinary continuous symmetries associated with the isometries of M_6 . The second class is associated with cohomology classes in M_6 and consists both of higher-form symmetries [29] and ordinary symmetries (i.e. 0-form symmetries). Expansion of the M-theory 3-form onto non-trivial cohomology classes of M_6 yields a collection of external U(1) p-form gauge fields (with p ranging from 0 to 3) that enters the 5d low-energy effective action of M-theory reduced on M_6 . Following the general recipe of [30], we analyze the topological mass terms for these fields in the 5d low-energy effective action, thereby identifying which U(1) gauge symmetries in 5d are spontaneously broken to discrete, cyclic subgroups. Our findings are summarized in table 2. Depending on the choice of boundary conditions, these discrete 5d gauge symmetries correspond to different discrete global symmetries in four dimensions. 1

The main focus of this work is the study of continuous symmetries. As a result, we integrate out the U(1) p-form gauge fields that are topologically massive in 5d—and thus correspond to broken U(1)'s. The residual, unbroken symmetries include ordinary symmetries, as well as "(-1)-form" symmetries associated with the axionic fields in table 2, which are best understood in terms of anomalies in the space of coupling constants [32]. Notice that a careful bookkeeping of broken symmetries is essential in order to compute correctly the 't Hooft anomalies for unbroken symmetries, as discussed in [24] for the case of k=2.

The key ingredient in the computation of the 't Hooft anomalies for continuous symmetries is the construction of the class E_4 , which is the closed, equivariant completion of the cohomology class describing the background G_4 -flux configuration [26, 33]. Here, equivariance refers to the action of the continuous isometries of M_6 .² We are mainly interested in determining those terms in the inflow anomaly polynomial that are leading in the limit of large flux quanta N, N_{N_i} , N_{S_i} . To this end, it is sufficient to integrate E_4^3 along the internal M_6 directions. This procedure accounts for the effect of the two-derivative $C_3 \wedge G_4 \wedge G_4$ coupling in the low-energy M-theory effective action. In general, $I_6^{\rm inflow}$ also receives contributions from the higher-derivative coupling $C_3 \wedge X_8$ in the action—where X_8

¹For a related discussion for 3d ABJM-type models, see [31].

²In the process of carrying out this construction, we verify the absence of cohomological obstructions [34]. The latter would signal the spontaneous breaking of some of the continuous symmetries associated with isometries of M_6 . See [35, 36] for a realization of this mechanism in the context of wrapped M5-brane setups.

is a certain combination of Pontryagin classes constructed from the 11d metric, reported in (4.6)—but these terms are subleading in the limit of large N, N_{N_i} , N_{S_i} , and fall beyond the scope of this work. (They are studied in the special case k=2 in [24].)

The full result of the calculation outlined above (including the contribution of the axionic fields) is recorded in appendix C. Section 4.4 contains more compact expressions valid in some special cases of interest (we do not report the axionic terms). Equation (4.16) gives the full answer for k = 3, for arbitrary values of the flux parameters N, N_{N_i} , N_{S_i} . In (4.21) we record the result for generic k, in the special case in which all the resolution flux quanta N_{N_i} , N_{S_i} are equal.

The rest of this paper is organized as follows. In section 2 we describe the geometry and flux configuration of the internal space M_6 for generic $k \geq 2$, determining useful bases for the relevant (co)homology groups of M_6 . Section 3 is devoted to the analysis of the topological mass terms originating from reduction of 11d supergravity on M_6 . In section 4 we present the computation of the inflow anomaly polynomial for continuous symmetries, and record the results of this calculation. We conclude with a brief discussion in section 5. Several appendices collect useful technical material.

2 Geometric setup

We discuss in this section the salient topological features of the 6d internal space that is described by the fiber bundle

$$M_4 \hookrightarrow M_6 \to \Sigma_q$$
, (2.1)

with the fiber M_4 being the manifold obtained by resolving the fixed points of the orbifold S^4/\mathbb{Z}_k , i.e.

$$M_4 = [S^4/\mathbb{Z}_k]_{\text{resolved}} \tag{2.2}$$

for $k \geq 2$, and Σ_g is a higher-genus Riemann surface with $g \geq 2$.

2.1 Orbifold action and resolution of singularities

Under the orbifold action of \mathbb{Z}_k , the isometry group $SO(5) \supset SO(4) \cong SU(2)_L \times SU(2)_R$ of the four-sphere S^4 is reduced to a $U(1)_L \times SU(2)_R$ subgroup.³ With a slight abuse of notation, $SU(2)_R$ is identified as the R-symmetry of the theory, and $U(1)_L$ is identified as a flavor symmetry. For the purpose of illustrating the topology of the orbifold, we can write the metric of S^4/\mathbb{Z}_k as

$$ds^{2}(S^{4}/\mathbb{Z}_{k}) = d\eta^{2} + \sin^{2}\eta \left[\frac{1}{k^{2}} D\varphi^{2} + \frac{1}{4} ds^{2}(S_{\psi}^{2}) \right], \tag{2.3}$$

$$D\varphi = d\varphi + \frac{k}{2}\cos\theta \,d\psi \,, \quad ds^2(S_\psi^2) = d\theta^2 + \sin^2\theta \,d\psi^2 \,, \tag{2.4}$$

where $\eta, \theta \in [0, \pi]$, while the angular coordinates φ and ψ both have periodicities of 2π , thus allowing us to identify $U(1)_L = U(1)_{\varphi}$ and $SU(2)_R = SU(2)_{\psi}$. We make a few remarks

³Note that there is an enhanced symmetry for k=2 where we still have $SU(2)_L \times SU(2)_R$ as the isometry group [24].

regarding the metric shown above. Firstly, the angle ψ is the usual azimuthal angle of S_{ψ}^2 where the circle S_{ψ}^1 vanishes at $\theta=0,\pi$. Secondly, the latter term of the metric (2.3) is written, up to the factor of $1/k^2$ due to the orbifold action, as a Hopf fibration of S^3 over S_{ψ}^2 with fiber S_{φ}^1 . It should also be noted that there exist two orbifold fixed points at $\eta=0,\pi$ which are locally (charge-k) single-center Taub-NUT spaces.

We analyze in this work the scenario where the two orbifold singularities are resolved through blow-up. More details of this procedure are recorded in appendix A. The resultant resolved manifold, M_4 , is locally a multi-center Gibbons-Hawking space with k-1 aligned two-cycles separated by k (unit-charge) Kaluza-Klein monopoles at $\eta = 0, \pi$ respectively, hence reducing the SU(2) $_{\psi}$ isometry into its U(1) $_{\psi}$ subgroup. Accordingly, the fiber M_4 stands on its own as the fiber bundle

$$S_{\varphi}^1 \hookrightarrow M_4 \to S_{\psi}^1 \times M_2 \,, \tag{2.5}$$

where M_2 is the compact 2d space spanned by η and θ , whose boundary ∂M_2 is described by the four intervals with $\eta, \theta = 0, \pi$. An illustration of the topology of M_4 before and after the resolution of the orbifold singularities is provided in figure 1. Moreover, a metric on M_4 can be cast into the schematic form,

$$ds^{2}(M_{4}) = ds^{2}(M_{2}) + R_{\psi}^{2}(\eta, \theta)d\psi^{2} + R_{\varphi}^{2}(\eta, \theta)D\varphi^{2}, \qquad (2.6)$$

$$D\varphi = d\varphi - L(\eta, \theta)d\psi. \tag{2.7}$$

The function $L(\eta, \theta)$ encodes information about the fibration in M_4 whose significance will be discussed shortly. The functions $R_{\psi}(\eta, \theta)$ and $R_{\varphi}(\eta, \theta)$ parameterize respectively the radii of the circles S_{ψ}^1 and S_{φ}^1 with respect to the position on M_2 . In particular, $R_{\psi}(\eta, \theta)$ vanishes everywhere on the boundary of M_2 , whereas $R_{\varphi}(\eta, \theta)$ is nonvanishing everywhere on M_2 except at the positions of the 2k Kaluza-Klein monopoles on ∂M_2 .

It is instructive to use a single periodic parameter, $t \sim t + 1$, to parameterize the boundary ∂M_2 . The locations of the monopoles, t_i for $i = 1, \ldots, 2k$, further divides ∂M_2 into distinct intervals, so that

$$\partial M_2 = \bigcup_{i=1}^{2k} [t_i, t_{i+1}]. \tag{2.8}$$

Without loss of generality, we pick a convention that t_1 corresponds to $(\eta, \theta) = (0, 0)$ and t_{2k} corresponds to $(\eta, \theta) = (\pi, 0)$. As derived in appendix A, $L(\eta, \theta)$ is a piecewise constant function on ∂M_2 , described by

$$\ell_{i} \equiv L(t_{i} < t < t_{i+1}) = \begin{cases} i - \frac{k}{2} & \text{if } 1 \leq i \leq k, \\ \frac{3k}{2} - i & \text{if } k + 1 \leq i \leq 2k. \end{cases}$$
(2.9)

The difference $n_i = \ell_i - \ell_{i-1}$ measures the Kaluza-Klein charge of the *i*-th monopole. In particular, the charge of each monopole for $1 \le i \le k$ is +1, while for $k+1 \le i \le 2k$ the charge is -1, with the relative sign accounting for the opposite orientations relative to M_2 at $\eta = 0, \pi$. The sum of the charges along $\eta = 0$ and $\eta = \pi$ is equal to k and -k

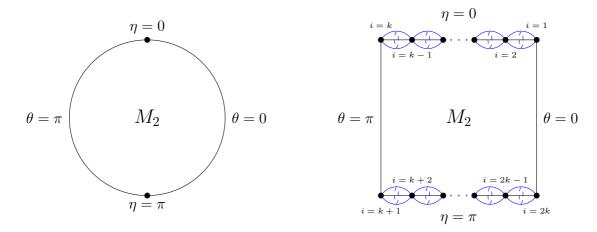


Figure 1: Illustration of the topology of M_4 , with the ψ and φ angles suppressed, before (left) and after (right) the resolution of the orbifold singularities at $\eta = 0, \pi$. We may roughly think of the coordinates η and θ as the 2d "latitude" and "longitude" respectively. The circle S^1_{ψ} vanishes along the entire boundary ∂M_2 , whereas S^1_{φ} vanishes only at the monopoles, which are labeled by the index $i = 1, \ldots, 2k$. The blue bubbles overlaid on ∂M_2 depict the resolution two-cycles connected with unit-charge Kaluza-Klein monopoles after the orbifold singularities are blown up.

respectively as expected. With each resolution two-cycle, there is an associated U(1) gauge symmetry, so there is an overall U(1)^{k-1} symmetry at $\eta = 0$ and similarly at $\eta = \pi$.

Following the same line of argument, it would be intriguing to generalize our results to partially resolved orbifold fixed points where there is a reduced number of monopoles but with generally non-unit charges, provided that (with the appropriate orientations) they sum up to k along each of the two boundary intervals at $\eta = 0, \pi$, i.e.

$$\sum_{i} n_{i} = \begin{cases} +k & \text{at } \eta = 0 \text{ with } 1 \leq n_{i} \leq k, \\ -k & \text{at } \eta = \pi \text{ with } -k \leq n_{i} \leq -1. \end{cases}$$

$$(2.10)$$

For each $|n_i| \geq 2$, a U(1)^{$|n_i|+1$} subgroup in the fully resolved setup enhances to an SU($|n_i|$) gauge symmetry [37]. Geometrically, this corresponds to collapsing a pair of adjacent resolution two-cycles into a charge- $|n_i|$ monopole. We show an example of such a partially resolved setup in figure 2 to illustrate the enhancement of gauge symmetries. A further analysis of these general scenarios is left to future work.

2.2 Topological twists

The twisting of M_4 over the Riemann surface introduces U(1) connections over Σ_g to the global angular forms $d\psi$ and $D\varphi$ originally defined on M_4 . Specifically, we preserve 4d $\mathcal{N}=1$ supersymmetry by performing the following topological twist [3, 4],

$$d\psi \to D\psi = d\psi - 2\pi\chi A_1^{\Sigma}, \qquad (2.11)$$

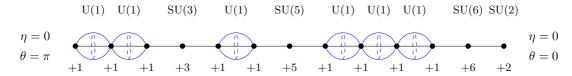


Figure 2: An example of a partially resolved orbifold fixed point for k = 25. Here we only show the interval corresponding to $\eta = 0$. (There is an analogous situation at $\eta = \pi$ with a generally different charge configuration.) Each number denotes the charge of the monopole above it, while each blue bubble represents a resolution two-cycle sandwiched between two unit-charge monopoles. Note that there is no two-cycle adjacent to any monopole whose charge is greater than one. There is a U(1) gauge symmetry associated with each two-cycle, and an enhanced $SU(|n_i|)$ gauge symmetry associated with each monopole with charge $1 \le |n_i| \le k$.

where $\chi = 2 - 2g$ is the Euler characteristic of Σ_g , and A_1^{Σ} is the local antiderivative of the normalized volume form V_2^{Σ} as defined by

$$\int_{\Sigma_g} V_2^{\Sigma} = 1. \tag{2.12}$$

Similarly, we may also promote

$$d\varphi \to D\varphi = d\varphi - LD\psi - 2\pi\zeta A_1^{\Sigma} \tag{2.13}$$

for some integer flavor twist parameter ζ . As we will soon see, the topological twist of the $U(1)_{\psi} \times U(1)_{\varphi}$ isometry group over the Riemann surface leads to non-trivial relations in the homology of M_6 .

2.3 Homology of M_6 and sum rules

The physics of the 5d supergravity theory obtained by reducing M-theory on the 6d internal space, which we will explore in more depth in the next section, depends crucially on the topology of M_6 . Therefore, it is important for us to understand the homology of M_6 .

One-cycles Based on the earlier discussion, we observe that there is no one-cycle in M_4 , so the only one-cycles present in M_6 are those associated with the Riemann surface Σ_g . There are in total 2g one-cycles in M_6 that are formally the pullbacks (via the dual cohomology) of the standard \mathcal{A} and \mathcal{B} cycles of the Riemann surface, which we denote as

$$\mathcal{C}_1^{\Sigma,u} \in \left\{ \mathcal{C}_1^{\mathcal{A},p}, \mathcal{C}_1^{\mathcal{B},p} \right\} , \tag{2.14}$$

where u = 1, ..., 2g and p = 1, ..., g. The intersection pairing between the \mathcal{A} and \mathcal{B} cycles is as usual given by

$$\left\langle \mathcal{C}_{1}^{\mathcal{A},p}, \mathcal{C}_{1}^{\mathcal{A},q} \right\rangle = \left\langle \mathcal{C}_{1}^{\mathcal{B},p}, \mathcal{C}_{1}^{\mathcal{B},q} \right\rangle = 0, \quad \left\langle \mathcal{C}_{1}^{\mathcal{A},p}, \mathcal{C}_{1}^{\mathcal{B},q} \right\rangle = -\left\langle \mathcal{C}_{1}^{\mathcal{B},q}, \mathcal{C}_{1}^{\mathcal{A},p} \right\rangle = \delta^{pq}, \quad (2.15)$$

which can be compactly expressed by the following intersection matrices [30],

$$\mathcal{K}^{uv} = \begin{pmatrix} 0 & \delta^{pq} \\ -\delta^{pq} & 0 \end{pmatrix}, \quad \mathcal{K}_{uv} = \begin{pmatrix} 0 & \delta_{pq} \\ -\delta_{pq} & 0 \end{pmatrix}, \tag{2.16}$$

the latter being constructed such that $\mathcal{K}^{uv}\mathcal{K}_{vw} = -\delta_w^u$. To complete the discussion of one-cycles, we note that there exist dual harmonic one-forms $\lambda_{1,v}$ in de Rham cohomology that can be chosen to be orthonormal to the one-cycles $\mathcal{C}_1^{\Sigma,u}$, i.e.⁴

$$\int_{\mathcal{C}_1^{\Sigma,u}} \lambda_{1,v} = \delta_v^u \,. \tag{2.17}$$

Two-cycles The Riemann surface Σ_g is a two-cycle on its own, but when pulled back to M_6 it remains to be a bona fide two-cycle only at the positions of the monopoles where S^1_{ψ} and S^1_{φ} vanish simultaneously, thus yielding a set of 2k two-cycles in M_6 which we denote schematically as

$$C_2^{\Sigma,i} = \Sigma_q|_{t=t_i} \tag{2.18}$$

for $i=1,\ldots,2k$. We also have the resolution two-cycles resulting from the blow up of the orbifold singularities of S^4/\mathbb{Z}_k at $\eta=0,\pi$. Each of these resolution two-cycles extends between adjacent monopoles at $t=t_i,t_{i+1}$ on ∂M_2 , where S^1_{ψ} vanishes but S^1_{φ} is nonvanishing. By the same token, there exist two other two-cycles stretching between $\eta=0$ and $\eta=\pi$, or in other words, on the intervals $t_k < t < t_{k+1}$ and $t_{2k} < t < t_1$ respectively. In terms of figure 1, these two-cycles can be visualized as bubbles sitting on the $\theta=0,\pi$ intervals. Overall, this gives us another set of 2k two-cycles,⁵

$$C_2^i = [t_i, t_{i+1}] \times S_{\varphi}^1. \tag{2.19}$$

We hereafter refer to these collectively as the "resolution two-cycles."

The naïve counting of the two-cycles listed above tells us that the total number of two-cycles in M_6 is 4k. However, it turns out that the twisting of the $\mathrm{U}(1)_{\psi} \times \mathrm{U}(1)_{\varphi}$ bundle trivializes certain linear combinations of these two-cycles. As explained in detail in appendix B, this can be most easily seen by working with the dual de Rham cohomology group, $H^2(M_6) \cong H_2(M_6)$, and we find that the following homological relation is satisfied,

$$C_2^{\Sigma,i+1} - C_2^{\Sigma,i} = (\chi \ell_i - \zeta) C_2^i.$$
(2.20)

We see from (2.20) that only one of the $C_2^{\Sigma,i}$ is independent of the C_2^i . We end up with two overall homological sum rules relating the 2k cycles C_2^i ,

$$\sum_{i=1}^{2k} C_2^i = 0, \qquad \sum_{i=1}^{2k} \ell_i C_2^i = 0.$$
 (2.21)

Note that the first sum rule can be heuristically understood as the fact that the sum of all the two-cycles C_2^i forms the boundary of the 3d space composed of S_{φ}^1 and M_2 . The second Betti number can then be computed as $b_2(M_6) = 4k - (2k - 1) - 2 = 2k - 1$, where the former factor of 2k - 1 comes from the recurrence relation (2.20), while the

⁴In the remainder of this paper, any discussion invoking the use of cohomology groups is implicitly understood to be within the context of de Rham cohomology unless otherwise specified.

⁵For the sake of comprehension, we are ignoring the generally non-trivial bundle structures when writing such schematic expressions of cycles as direct products of subspaces.

factor of 2 comes from the two sum rules (2.21).⁶ For practical purposes, it is convenient to pick some complete basis of 2k-1 two-cycles, C_2^{α} , from among those listed in (2.18) and (2.19), and construct the corresponding dual cohomology class representatives, $\omega_{2,\beta}$, by orthonormalizing their inner products, i.e.

$$\int_{\mathcal{C}_2^{\alpha}} \omega_{2,\beta} = \delta_{\beta}^{\alpha} \,. \tag{2.22}$$

Three-cycles We can pair up one of the resolution two-cycles C_2^i and one of the Riemann surface one-cycles $C_1^{\Sigma,u}$ to form a three-cycle in M_6 ,

$$C_3^{i,u} = C_2^i \times C_1^{\Sigma,u} = [t_i, t_{i+1}] \times S_{\varphi}^1 \times C_1^{\Sigma,u}$$
 (2.23)

It is straightforward to check that these are all the non-trivial three-cycles residing in M_6 . Note in particular that there is no one-cycle in M_4 that can be paired with the Riemann surface. Given the structure of (2.23), it is unsurprising that the two sum rules for the two-cycles are directly inherited, i.e.

$$\sum_{i=1}^{2k} \mathcal{C}_3^{i,u} = 0, \qquad \sum_{i=1}^{2k} \ell_i \, \mathcal{C}_3^{i,u} = 0 \tag{2.24}$$

for each $u=1, \dots, 2g$. These sum rules can be worked out through an analysis resembling that for the two-cycles. The third Betti number is simply given by $b_3(M_6) = (2k)(2g) - 2g(2) = 4g(k-1)$. As in the previous cases, we can always choose a complete basis of three-cycles \mathcal{C}_3^x , with $x=1,\dots,b_3(M_6)$ being a collective index for (i,u), whose inner product with the dual cohomology class representatives, $\Lambda_{3,u}$, is orthonormal.

Four-cycles The obvious candidate for a four-cycle in M_6 is just

$$C_{4,C} = M_4$$
. (2.25)

We can also pair up a resolution two-cycle with the Riemann surface, i.e.

$$C_{4,i} = C_2^i \times \Sigma_q = [t_i, t_{i+1}] \times S_{\omega}^1 \times \Sigma_q. \tag{2.26}$$

Once again, we seek out relations between these four-cycles by working with the dual cohomology group, $H^4(M_6)$. We can repeat essentially the same exercise as before to arrive at the homological relations,

$$\sum_{i=1}^{2k} C_{4,i} = \chi C_{4,C}, \qquad \sum_{i=1}^{2k} \ell_i C_{4,i} = \zeta C_{4,C}.$$
 (2.27)

Unlike the cases we saw earlier, the right-hand sides of the relations above do not vanish; they are equal to M_4 up to multiplicative factors of the topological twists. These two sum rules are a manifestation of the non-trivial structure of the bundle $M_4 \hookrightarrow M_6 \to \Sigma_g$. For example, because of the twist associated with the U(1) $_{\psi}$ bundle, the sum of all $\mathcal{C}_{4,i}$ does

⁶Note that if we were to restrict to M_4 only, then we would have $b_2(M_4) = 2k - 2$ instead.

				b_3				
Σ_g	1	2g	1	-	-	-	-	2(1-g)
M_4	1	0	2k - 2	0	1	-	-	2k
M_6	1	2g	2k - 1	4g(k-1)	2k - 1	2g	1	2(1-g) $2k$ $4k(1-g)$

Table 1: Betti numbers and Euler characteristics of Σ_g , M_4 , M_6 .

not form the boundary of any manifold as one might intuitively expect, reflecting the fact that M_6 is not simply a product manifold. In fact, the two topological twists trivialize different linear combinations of four-cycles (or four-forms, in cohomology). The vanishing of the RHS of (2.21) and (2.24), on the other hand, can be attributed to the absence of any two-cycle or three-cycle in M_4 playing the role of $C_{4,C}$ in (2.27). As expected from Poincaré duality, the fourth Betti number is $b_4(M_6) = 2k - 2 + 1 = 2k - 1 = b_2(M_6)$. We again note that a set of basis four-cycles C_4^{α} and dual cohomology class representatives Ω_4^{β} can be suitably chosen such that they are orthonormal to one another.

Here we would like to make a detour and mention that one can define the flux quanta,

$$N = \int_{\mathcal{C}_{4,C}} \frac{G_4}{2\pi}, \qquad N_i = \int_{\mathcal{C}_{4,i}} \frac{G_4}{2\pi}, \qquad (2.28)$$

where G_4 is the M-theory four-form flux. They obey sum rules analogous to (2.27), i.e.

$$\sum_{i=1}^{2k} N_i = \chi N , \qquad \sum_{i=1}^{2k} \ell_i N_i = \zeta N , \qquad (2.29)$$

which implies that there are only 2k-1 independent flux quanta characterizing G_4 .

Five-cycles Similarly to the discussion of three-cycles, the only five-cycles in M_6 are

$$C_5^u = C_{4,C} \times C_1^{\Sigma,u}, \qquad (2.30)$$

where $u = 1, ..., b_5(M_6)$ with $b_5(M_6) = b_1(M_6) = 2g$ by virtue of Poincaré duality.

The Betti numbers and the Euler characteristics of Σ_g , M_4 , M_6 are tabulated in table 1.⁷ As a sanity check, all the Betti numbers are consistent with Poincaré duality. It is also explicitly verified that $\chi(M_6) = \chi(\Sigma_g) \chi(M_4)$ as expected from the Serre spectral sequence.⁸

Natural basis of (co)homology classes We will frequently employ a natural basis of homology classes, and hence cohomology classes, in the rest of this paper. In addition to the standard \mathcal{A} and \mathcal{B} cycles of the Riemann surface as the obvious choice of basis one-cycles $\mathcal{C}_1^{\Sigma,u}$, we choose the basis four-cycles $\mathcal{C}_{4,\alpha}$ to be

$$C_{4,1 \le \alpha \le k-1} = C_{4,1 \le i \le k-1}, \quad C_{4,\alpha=k} = C_{4,C}, \quad C_{4,k+1 \le \alpha \le 2k-1} = C_{4,k+1 \le i \le 2k-1}, \quad (2.31)$$

⁷Recall that the Euler characteristic of a manifold M can be computed using the alternating sum, $\chi(M) = \sum_{i=0}^{\dim M} (-1)^i b_i(M)$.

⁸Everywhere else we use the symbol χ to refer specifically to the Euler characteristic of the Riemann surface Σ_q .

which are respectively the k-1 four-cycles $C_{4,i}$, as in (2.26), in the "north" where $\eta=0$, the four-cycle $C_{4,C}=M_4$, and the k-1 four-cycles $C_{4,i}$ in the "south" where $\eta=\pi$. As an aside, it is convenient to adopt an intuitive naming convention,

$$C_{4,1 \le i \le k-1} \equiv C_{4,N_i}, \qquad C_{4,k+1 \le i \le 2k-1} \equiv C_{4,S_{2k-i}}.$$
 (2.32)

We select the basis two-cycles \mathcal{C}_2^{α} to be those which are Poincaré-dual to the basis four-cycles described above, so that their dual cohomology class representatives obey

$$\int_{M_6} \Omega_4^{\alpha} \wedge \omega_{2,\beta} = \delta_{\beta}^{\alpha} \,. \tag{2.33}$$

Last but not least, we pick the basis three-cycles C_3^x to be combinations of the basis two-cycles $C_2^{\alpha \neq k}$ and the basis one-cycles $C_1^{\Sigma,u}$.

3 Continuous and discrete flavor symmetries

In this section we analyze continuous zero-, one-, and two-form global symmetries for the 4d field theories of interest, and study their breaking to discrete subgroups. More precisely, we identify the global symmetries of the 4d QFTs from the gauge symmetries of 11d supergravity reduced on M_6 , taking care to account for spontaneous symmetry breaking from topological mass terms in the 5d low-energy effective action. Expansion of the three-form potential C_3 on the cohomology classes of M_6 provides part of the 5d spectrum of massless gauge fields. Additional gauge fields arise from the isometries of M_6 . In particular, we can couple the $\mathrm{U}(1)_{\psi}$, $\mathrm{U}(1)_{\varphi}$ isometries to a pair of massless, abelian gauge fields A_1^{ψ} , A_1^{φ} , respectively. However, these abelian gauge fields will not participate in the Stückelberg mechanism of primary interest in this section, so we will turn them off until section 4, and focus for the time being on the cohomology sector of the spectrum. The independence of gauge fields associated with isometries from those in the cohomology sector is given explicitly by (4.13).

The three-form potential C_3 gives rise to massless abelian p-form gauge fields in the reduction to 5d via the Kaluza-Klein mechanism. These are the fields in one-to-one correspondence with the cohomology classes of M_6 . We can identify them by expanding the variation of C_3 away from its background value as

$$\frac{\delta C_3}{2\pi} = \frac{a_0^x}{2\pi} \Lambda_{3,x} + \frac{A_1^{\alpha}}{2\pi} \wedge \omega_{2,\alpha} + \frac{B_2^u}{2\pi} \wedge \lambda_{1,u} + \frac{c_3}{2\pi} \,. \tag{3.1}$$

Recall that Ω_4^{α} , $\Lambda_{3,x}$, $\omega_{2,\alpha}$, and $\lambda_{1,u}$ are the de Rham cohomology class representatives of M_6 introduced in section 2. The zero-, one-, two-, and three-forms a_0^x , A_1^{α} , B_2^u , c_3 are dynamical, abelian 5d gauge potentials, with field strengths quantized in units of 2π . In terms of these field strengths $f_1^x = da_0^x$, $F_2^{\alpha} = dA_1^{\alpha}$, $H_3^u = dB_2^u$, and $\gamma_4 = dc_3$, we can express the four-form flux as

$$\frac{G_4}{2\pi} = N_\alpha \Omega_4^\alpha + \frac{f_1^x}{2\pi} \wedge \Lambda_{3,x} + \frac{F_2^\alpha}{2\pi} \wedge \omega_{2,\alpha} + \frac{H_3^u}{2\pi} \wedge \lambda_{1,u} + \frac{\gamma_4}{2\pi}.$$
 (3.2)

The fluxes N_{α} are also quantized,

$$\int_{\mathcal{C}_{4,\alpha}} \frac{G_4}{2\pi} = N_\alpha \in \mathbb{Z} \,, \tag{3.3}$$

in virtue of G_4 -flux quantization in M-theory [38].

At low energies, some of the continuous, abelian p-form gauge symmetries are spontaneously broken to discrete subgroups. In order to see this, we must consider the effects of topological terms in the 5d low-energy effective action. The relevant topological term in the 11d low-energy effective action is the Chern-Simons coupling,

$$S_{\rm CS} = -\frac{2\pi}{6} \int_{M_{11}} \frac{C_3}{2\pi} \frac{G_4}{2\pi} \frac{G_4}{2\pi} \,. \tag{3.4}$$

Note that here and in the equations to follow we have suppressed wedge products. Kinetic terms for the 5d gauge fields descend from 11d kinetic terms for G_4 via a standard Kaluza-Klein reduction, and likewise topological terms in the 5d low-energy effective action are obtained from reduction of S_{CS} . These topological terms in the 5d effective action can be expressed in terms of a six-form as

$$S_{\rm CS} = 2\pi \int_{\mathcal{M}_5} I_5^{(0)}, \qquad dI_5^{(0)} = I_6.$$
 (3.5)

The six-form I_6 can be compactly expressed in terms of intersection numbers,

$$\mathcal{K}_{\alpha\beta\gamma} \equiv \int_{M_6} \omega_{2,\alpha} \, \omega_{2,\beta} \, \omega_{2,\gamma} \,, \quad \mathcal{K}_{\alpha}^{\beta} \equiv \int_{M_6} \omega_{2,\alpha} \, \Omega_4^{\beta} \,, \quad \mathcal{K}_{xy} \equiv \int_{M_6} \Lambda_{3,x} \, \Lambda_{3,y} \,,$$

$$\mathcal{K}_{uv}^{\alpha} \equiv \int_{M_6} \lambda_{1,u} \, \lambda_{1,v} \, \Omega_4^{\alpha} \,, \quad \mathcal{K}_{u\alpha x} \equiv \int_{M_6} \lambda_{1,u} \, \omega_{2,\alpha} \, \Lambda_{3,x} \,.$$
(3.6)

With the field strengths F_2^I turned off, I_6 can be written as

$$I_{6} = -\frac{1}{6} \mathcal{K}_{\alpha\beta\gamma} \frac{F_{2}^{\alpha}}{2\pi} \frac{F_{2}^{\beta}}{2\pi} \frac{F_{2}^{\gamma}}{2\pi} - \mathcal{K}_{u\alpha x} \frac{f_{1}^{x}}{2\pi} \frac{F_{2}^{\alpha}}{2\pi} \frac{H_{3}^{u}}{2\pi} + \frac{1}{2} \mathcal{K}_{xy} \frac{f_{1}^{x}}{2\pi} \frac{f_{1}^{y}}{2\pi} \frac{\gamma_{4}}{2\pi} + \frac{1}{2} \mathcal{K}_{xy} \frac{f_{1}^{x}}{2\pi} \frac{f_{2}^{y}}{2\pi} \frac{\gamma_{4}}{2\pi} + \frac{1}{2} \mathcal{K}_{xy} \frac{f_{2}^{x}}{2\pi} \frac{f_{2}^{y}}{2\pi} \frac{f_{2}^{y}}{2\pi} \frac{\gamma_{4}}{2\pi} + \frac{1}{2} \mathcal{K}_{xy} \frac{f_{2}^{x}}{2\pi} \frac{f_{2}^{y}}{2\pi} \frac{f_$$

Note that the last two terms are quadratic in the external field strengths. These are the topological mass terms which spontaneously break one of the $b_2(M_6)$ zero-form symmetries and all $b_1(M_6)$ one-form symmetries to discrete subgroups. To see this more directly, we choose a basis of one-forms with intersection numbers as in (2.16), and a Poincaré-dual set of two- and four-forms, i.e. $\mathcal{K}_{\alpha}^{\beta} = \delta_{\alpha}^{\beta}$, so that

$$I_6 \supset -N \frac{\tilde{H}_{3,p}}{2\pi} \frac{H_3^p}{2\pi} - N_\alpha \frac{F_2^\alpha}{2\pi} \frac{\gamma_4}{2\pi},$$
 (3.8)

where we have split the index $u = 1, \ldots, 2g$ as

$$H_3^u = (H_3^p, \tilde{H}_{3,p}), \qquad p = 1, \dots, g.$$
 (3.9)

⁹Following the argument in [30, 39], we assume that there is no half-integral contribution to the flux quantization condition as far as the setup in this paper in concerned.

Gauge fields	Multiplicity	5d gauge symmetries
<i>c</i> ₃	$b_0(M_6) = 1$	\mathbb{Z}_n two-form symmetry
$B_2^u = (B_2^i, \tilde{B}_{2,i})$	$b_1(M_6) = 2g$	$(\mathbb{Z}_N \times \mathbb{Z}_N)^g$ one-form symmetry
$A_1^{lpha}=(\mathcal{A}_1,\mathcal{A}_1^{\hat{lpha}})$	$b_2(M_6) = 2k - 1$	\mathbb{Z}_n and $\mathrm{U}(1)^{2k-2}$ zero-form symmetries
a_0^x	$b_3(M_6) = 4g(k-1)$	axionic

Table 2: Summary of the 5d p-form gauge fields and symmetry groups arising from cohomology classes of the internal space M_6 . Recall that $n \equiv \gcd(N_{\alpha})$.

With an appropriate basis rotation one can pick out the single one-form gauge field A_1 which couples to γ_4 [30],

$$-N_{\alpha}F_2^{\alpha}\gamma_4 = -n \, d\mathcal{A}_1\gamma_4 \,, \qquad n = \gcd(N_{\alpha}) \,. \tag{3.10}$$

We denote the remaining $b_2(M_6) - 1$ gauge fields which do not appear in any topological mass term by $\mathcal{A}_1^{\hat{\alpha}}$. Under this choice of basis, the topologically massive contributions to the 5d low-energy effective action can be written as

$$\frac{1}{2\pi} \int_{\mathcal{M}_5} \left(-nc_3 \, d\mathcal{A}_1 - N\tilde{B}_2^i \, dB_{2,i} \right) \,. \tag{3.11}$$

As discussed in [30], the gauge fields A_1 , c_3 , and B_2^i , $\tilde{B}_{2,i}$ are thus effectively continuum descriptions of discrete gauge fields, with gauge groups \mathbb{Z}_n , \mathbb{Z}_n , and $\mathbb{Z}_N \times \mathbb{Z}_N$, respectively. The spontaneous symmetry breaking of the original continuous gauge symmetries to these subgroups is governed by a Stückelberg mechanism. The resulting 5d gauge symmetry groups are summarized in table 2.

Note that in the limit where the 2k-2 northern and southern flux quanta $\{N_{N,i}, N_{S,i}\}$ (see our convention as defined around (2.32)) are taken to be zero in (3.10), the integer n simply becomes N, the only non-trivial flux parameter. In this case, we see that the self-dual \mathbb{Z}_N two-form symmetry of the 6d SCFT would yield a \mathbb{Z}_N zero-form symmetry and a \mathbb{Z}_N two-form symmetry upon reduction to 4d. By turning on additional flux parameters, these discrete symmetries of the 4d theory are broken to discrete \mathbb{Z}_n zero- and two-form symmetries [30].

The full holographic interpretation of these symmetry groups as global symmetries of a dual 4d SCFT would require a choice of boundary conditions. Different boundary conditions are generally associated with different boundary SCFTs. We refer the reader to [30] for a detailed discussion of possible scenarios.

4 Inflow anomaly polynomial

This section is devoted to the computation of the inflow anomaly polynomial for the continuous symmetries of the internal space M_6 we have described. The computation will

require us to first integrate out the topologically massive fields corresponding to discrete symmetries. For the sake of simplicity, we restrict attention to cases in which the flavor twist parameter ζ in (2.13) is fixed to zero.

4.1 Anomaly inflow methods for wrapped M5-branes

Consider a stack of N M5-branes with worldvolume W_6 in a background spacetime M_{11} . We are interested in setups in which four dimensions W_4 are left external and while the rest of the brane worldvolume directions are wrapped on a smooth, compact Riemann surface. The global symmetries of the field theory defined on the extended spacetime W_4 can admit 't Hooft anomalies. These anomalies are fully determined by the fibration

$$M_4 \hookrightarrow M_6 \to \Sigma_q$$
, (4.1)

and can be encoded in a six-form anomaly polynomial $I_6^{\rm QFT}$ using the descent formalism [27, 28]. As described in [26], this 't Hooft anomaly polynomial can be computed via anomaly inflow. Since the full M-theory is anomaly-free, the anomaly polynomial $I_6^{\rm 4dQFT}$ associated with the 4d theory must be exactly canceled by a combination of contributions from the classical anomalous variation of the effective 11d supergravity action and from decoupled modes,

$$I_6^{\rm inflow} + I_6^{\rm 4dQFT} + I_6^{\rm decoupled} = 0. \tag{4.2} \label{eq:4.2}$$

This mechanism allows us to access I_6^{4dQFT} directly via $-I_6^{\text{inflow}}$ in the large-N limit, where the contributions from decoupled modes are expected to be subleading.

To identify the inflow contribution $I_6^{\rm inflow}$ to the anomaly in 4d, we compute the fiber integral

$$I_6^{\text{inflow}} = \int_{M_6} \mathcal{I}_{12} \,,$$
 (4.3)

where \mathcal{I}_{12} is a characteristic class formally defined on a fiducial 12d space M_{12} such that $\partial M_{12} = M_{11}$, and given explicitly by

$$\mathcal{I}_{12} = -\frac{1}{6} E_4^3 - E_4 X_8 \,. \tag{4.4}$$

The N M5-branes act as a magnetic source for the four-form flux, resulting in classical anomalous variation of the 11d effective action related to \mathcal{I}_{12} via descent. This anomalous variation is encoded by the form E_4 , defined to be $G_4/2\pi$ evaluated near the brane stack under suitable boundary conditions [24, 26] with the normalization

$$\int_{M_4} E_4 = N. (4.5)$$

 E_4 is closed, globally defined, and invariant under the symmetries of the 4d theory. The eight-form X_8 is given in terms of the first and second Pontryagin classes of the tangent bundle TM_{11} by

$$X_8 = \frac{1}{192} \left[p_1^2(TM_{11}) - 4p_2(TM_{11}) \right] . \tag{4.6}$$

Note that the first term in (4.4) scales as N^3 , while the second term is linear in N. Since we are interested here in large-N perturbative anomalies, we will restrict attention to the E_4^3 term for the rest of this paper. We now turn to a more detailed discussion of the construction of the form E_4 .

4.2 Construction of E_4

In order to compute the d=4 inflow anomaly polynomial (4.3), we must first obtain the globally defined and closed four-form E_4 . On top of the p-form gauge fields introduced in the previous section, E_4 also depends on the abelian gauge fields A_1^{ψ} , A_1^{φ} associated with the isometries of M_6 , with field strengths given by

$$F_2^I = dA_1^I, I, J \in \{\psi, \varphi\}.$$
 (4.7)

The construction of E_4 follows from the expansion of $G_4/2\pi$ in (3.2),

$$E_4 = N_{\alpha} (\Omega_4^{\alpha})^{\text{eq}} + N \frac{f_1^x}{2\pi} \wedge (\Lambda_{3,x})^{\text{eq}} + N \frac{F_2^{\alpha}}{2\pi} \wedge (\omega_{2,\alpha})^{\text{eq}} + N \frac{H_3^u}{2\pi} \wedge (\lambda_{1,u})^{\text{eq}} + N \frac{\gamma_4}{2\pi}, \quad (4.8)$$

where all forms shown are defined on the fiducial space M_{12} . Here f_1^x , F_2^{α} , H_3^u , and γ_4 represent background field strengths, which we have re-scaled by factors of N so as to make the large-N scaling of the anomaly polynomial explicit. The forms $(\Omega_4^{\alpha})^{\text{eq}}$, $(\Lambda_{3,x})^{\text{eq}}$, $(\omega_{2,\alpha})^{\text{eq}}$, and $(\lambda_{1,u})^{\text{eq}}$ are extensions of the de Rham cohomology class representatives Ω_4^{α} , $\Lambda_{3,x}$, $\omega_{2,\alpha}$, and $\lambda_{1,u}$ on M_6 to the full M_{12} . These forms can be constructed by first gauging the representative of a given de Rham cohomology class with respect to the isometries, and then constructing a closed and globally defined completion. Suppressing indices labeling individual cohomology classes, these completions are of the forms

$$\lambda_{1}^{\text{eq}} = \lambda_{1}^{\text{g}},
\omega_{2}^{\text{eq}} = \omega_{2}^{\text{g}} + \frac{F_{2}^{I}}{2\pi} \omega_{0,I},
\Lambda_{3}^{\text{eq}} = \Lambda_{3}^{\text{g}} + \frac{F_{2}^{I}}{2\pi} \Lambda_{1,I}^{\text{g}},
\Omega_{4}^{\text{eq}} = \Omega_{4}^{\text{g}} + \frac{F_{2}^{I}}{2\pi} \Omega_{2,I}^{\text{g}} + \frac{F_{2}^{I}}{2\pi} \frac{F_{2}^{J}}{2\pi} \Omega_{0,IJ}.$$
(4.9)

where the label "eq" stands for "equivariant," while "g" stands for "gauged." The details about the parameterization of the auxiliary forms $\omega_{0,I}$, $\Lambda_{1,I}$, $\Omega_{2,I}$, and $\Omega_{0,IJ}$ are left to appendix B. Several ambiguities in the choice of such forms, and the corresponding field redefinitions that keep $I_6^{\rm inflow}$ invariant, are discussed in appendix D.

4.3 Integrating out massive fields

With (4.8) in hand, the E_4^3 term in (4.4) can be expanded as in (B.31). However, in order to study the perturbative anomalies for continuous global symmetries in 4d, we must first integrate out the topologically massive 5d fields. In particular, as discussed in section 3, the fields A_1 , c_3 , and B_2^i , $\tilde{B}_{2,i}$ correspond to topologically massive gauge fields in 5d, and thus cannot be interpreted as background fields for continuous symmetries in the 4d theory.

All topological mass terms quadratic in external field strengths can be eliminated from the polynomial (B.31) using the equations of motion for the 5d fields c_3 and B_2^u , respectively,

$$N_{\alpha}N_{\beta} \mathcal{J}_{I}^{\alpha\beta} \frac{F_{2}^{I}}{2\pi} + NN_{\beta} \mathcal{K}_{\alpha}^{\beta} \frac{F_{2}^{\alpha}}{2\pi} - \frac{1}{2} N^{2} \mathcal{K}_{xy} \frac{f_{1}^{x}}{2\pi} \frac{f_{1}^{y}}{2\pi} = 0, \qquad (4.10)$$

$$N_{\alpha} \mathcal{J}_{Iux}^{\alpha} \frac{f_1^x F_2^I}{2\pi} + \frac{1}{2} N_{\alpha} \mathcal{K}_{uv}^{\alpha} \frac{H_3^v}{2\pi} - N \mathcal{K}_{u\alpha x} \frac{f_1^x F_2^{\alpha}}{2\pi} = 0, \qquad (4.11)$$

where we have defined the integrals

$$\mathcal{J}_{I}^{\alpha\beta} \equiv \frac{1}{2} \int_{M_6} \left(\Omega_{2,I}^{\alpha} \, \Omega_4^{\beta} + \Omega_{2,I}^{\beta} \, \Omega_4^{\alpha} \right), \quad \mathcal{J}_{Iux}^{\alpha} \equiv \int_{M_6} \left(\Lambda_{1,xI} \, \lambda_{1,u} \, \Omega_4^{\alpha} - \lambda_{1,u} \, \Omega_{2,I}^{\alpha} \, \Lambda_{3,x} \right). \quad (4.12)$$

The forms $\Omega_{2,I}^{\alpha}$ and $\Lambda_{1,xI}$ introduced in constructing the closed completions $(\Omega_4^{\alpha})^{\text{eq}}$ and $(\Lambda_{3,x})^{\text{eq}}$ are defined only up to the addition of harmonic forms. As a result, the mixing between field strengths associated with isometries and those associated with the cohomology classes of M_6 implied by (4.10) and (4.11) can be removed by an appropriate choice of $\Omega_{2,I}^{\alpha}$ and $\Lambda_{1,xI}$ that fixes

$$N_{\alpha}N_{\beta}\mathcal{J}_{I}^{\alpha\beta} = 0, \qquad N_{\alpha}\mathcal{J}_{Iux}^{\alpha} = 0$$
 (4.13)

for all I and x. A detailed demonstration is provided in appendix B. Under this condition, all dependence on the field strengths F_2^I associated with the isometries of M_6 drops out of the equations of motion (4.10) and (4.11). Therefore we can safely restrict to the cohomology sector when integrating out the topologically massive fields. The Stückelberg mechanism governing the spontaneous symmetry breaking to the discrete subgroups described in section 3 is thus unaffected by the presence of gauge fields coupled to the isometries of M_6 .

4.4 Results for continuous symmetries

After integrating out topologically massive fields from (B.31), we can compute the inflow anomaly polynomial for continuous symmetries. In appendix C, we record the full anomaly polynomial $I_6^{\rm inflow}$ in the large-N limit for $\chi < 0$ and $\zeta = 0$, including all continuous higherform symmetries. We stress that $I_6^{\rm inflow}$ encodes the perturbative anomalies associated with continuous, but not discrete symmetries of the total worldvolume theory. A formal treatment of the latter will require an application of the technology developed in [30] using differential cohomology, which we defer to future work. Despite the appearance of multiple auxiliary functions in (C.1), the inflow anomaly polynomial is solely a function of the field strengths f_1^x , F_2^I , F_2^α , and the parameters k, χ , N_α . The precise functional dependence of $I_6^{\rm inflow}$ on these parameters, however, is contingent on the specific choice of the various forms appearing in (4.9). Nonetheless, the difference in $I_6^{\rm inflow}$ resulting from distinct choices can be compensated by redefinitions of F_2^α . We refer the reader to appendix D for a detailed discussion. Under appropriate field redefinitions, we verified that (C.1) reproduces the k=2 inflow anomaly polynomial in [24], which describes the 4d field theory dual of the GMSW solution.

To illustrate the generalized construction developed here, consider first the case when k=3, and let us restrict attention to the portion of I_6^{inflow} corresponding to zero-form symmetries by setting all $f_1^x=0$. In addition to the decoupling convention (4.13), we

adopt the convention that the bases of two- and four-cohomology classes are Poincaré-dual to one another,

$$\mathcal{K}^{\alpha}_{\beta} = \int_{M_6} \Omega_4^{\alpha} \wedge \omega_{2,\beta} = \delta^{\alpha}_{\beta} \,. \tag{4.14}$$

As a result, with the axionic field strengths f_1^x turned off, integrating out \mathcal{A}_1 using (4.10) is equivalent to imposing

$$\sum_{\alpha=1}^{2k-1} N_{\alpha} F_2^{\alpha} = 0 \tag{4.15}$$

as in [24]. In the four-cycle basis (2.31), with 2k-1 independent associated flux quanta N, N_{N_i} , N_{S_i} , the k=3 inflow anomaly polynomial can then be expressed as

$$I_6^{\text{inflow}} = \frac{I_6^{\text{N}}}{(2\pi)^3} + \left(N_{\text{N}_i} \leftrightarrow N_{\text{S}_i}, F_2^{\text{N}_i} \leftrightarrow -F_2^{\text{S}_i}\right),$$
 (4.16)

where we have defined

$$\begin{split} I_{6}^{N} &= \frac{3\chi N^{3}}{64} (F_{2}^{\psi})^{3} - \frac{N}{2\chi} (N_{N_{1}}^{2} + N_{N_{1}}N_{N_{2}} + N_{N_{2}}^{2}) (F_{2}^{\psi})^{3} \\ &+ \frac{1}{3\chi^{2}} (N_{N_{1}} + N_{N_{2}}) (2N_{N_{1}}^{2} + N_{N_{1}}N_{N_{2}} + 2N_{N_{2}}^{2}) (F_{2}^{\psi})^{3} \\ &- \frac{1}{9\chi^{3}N} (N_{N_{1}}^{2} + N_{N_{1}}N_{N_{2}} + N_{N_{2}}^{2} - N_{S_{2}}^{2} - N_{S_{2}}N_{S_{1}} - N_{S_{1}}^{2})^{2} (F_{2}^{\psi})^{3} \\ &+ \frac{1}{18\chi^{2}} (2N_{N_{1}}^{3} + 3N_{N_{1}}^{2}N_{N_{2}} - 3N_{N_{1}}N_{N_{2}}^{2} - 2N_{N_{2}}^{3}) (F_{2}^{\psi})^{2} F_{2}^{\varphi} - \frac{\chi N^{3}}{48} (F_{2}^{\varphi})^{2} F_{2}^{\psi} \\ &+ \frac{N}{9\chi} (N_{N_{1}}^{2} + N_{N_{1}}N_{N_{2}} + N_{N_{2}}^{2}) (F_{2}^{\varphi})^{2} F_{2}^{\psi} + \frac{9N^{2}}{8} (N_{N_{1}}F_{2}^{N_{1}} + N_{N_{2}}F_{2}^{N_{2}}) (F_{2}^{\psi})^{2} \\ &- \frac{3N}{2\chi} N_{N_{1}}N_{N_{2}} (F_{2}^{N_{1}} + F_{2}^{N_{2}}) (F_{2}^{\psi})^{2} - \frac{9N}{4\chi} (N_{N_{1}}^{2}F_{2}^{N_{1}} + N_{N_{2}}F_{2}^{N_{2}}) (F_{2}^{\psi})^{2} \\ &+ \frac{1}{\chi^{2}} (N_{N_{1}}^{2} + N_{N_{1}}N_{N_{2}} + N_{N_{2}}^{2}) (F_{2}^{\psi})^{2} - \frac{9N}{4\chi} (N_{N_{1}}F_{2}^{N_{1}} + N_{N_{2}}F_{2}^{N_{2}}) (F_{2}^{\psi})^{2} \\ &+ \frac{N^{2}}{6} (N_{N_{1}}F_{2}^{N_{1}} + N_{N_{2}}F_{2}^{N_{2}}) (F_{2}^{\psi})^{2} - \frac{9N}{4\chi} (N_{N_{1}}F_{2}^{N_{1}} + N_{N_{2}}F_{2}^{N_{2}}) (F_{2}^{\psi})^{2} \\ &- \frac{3\chi}{3\chi} \left[N_{N_{1}} (N_{N_{1}} + 2N_{N_{2}})F_{2}^{N_{1}} - N_{N_{2}} (N_{N_{2}} + 2N_{N_{1}})F_{2}^{N_{2}} \right] F_{2}^{\psi} + \frac{3N^{2}}{2} \left[N_{N_{1}} (F_{2}^{N_{1}})^{2} + N_{N_{2}} (F_{2}^{N_{2}})^{2} \right] F_{2}^{\psi} \\ &+ N^{2} (N_{N_{1}} + N_{N_{2}}) \left[(F_{2}^{N_{1}})^{2} - F_{2}^{N_{1}} F_{2}^{N_{2}} + (F_{2}^{N_{2}})^{2} \right] F_{2}^{\psi} + \frac{3N^{2}}{2} \left[N_{N_{1}} (F_{2}^{N_{1}} + N_{N_{2}} F_{2}^{N_{2}})^{2} \right] F_{2}^{\psi} \\ &- \frac{2N}{3\chi} (N_{N_{1}}^{2} + N_{N_{1}}N_{N_{2}} + N_{N_{2}}^{2} - N_{N_{1}}^{2} - F_{2}^{N_{1}} F_{2}^{N_{2}} + (F_{2}^{N_{2}})^{2} \right] F_{2}^{\psi} \\ &- \frac{2N}{3\chi} (N_{N_{1}}^{2} + N_{N_{1}}N_{N_{2}} + N_{N_{2}}^{2} - N_{N_{1}}^{2} - N_{N_{1}}^{2} N_{N_{2}} + N_{N_{2}}^{2} \right] \left[(F_{2}^{N_{1}})^{2} - F_{2}^{N_{1}} F_{2}^{N_{2}} + (F_{2}^{N_{2}})^{2} \right] F_{2}^{\psi} \\ &- \frac{2N}{3\chi} (N_{N_{1}}^$$

$$+\mathcal{O}(N, N_{N_1}, N_{N_2}, N_{S_1}, N_{S_2}).$$
 (4.17)

Note that in addition to the symmetry of I_6^{inflow} under

$$N_{\mathcal{N}_i} \leftrightarrow N_{\mathcal{S}_i}, \qquad F_2^{\mathcal{N}_i} \leftrightarrow -F_2^{\mathcal{S}_i},$$
 (4.18)

of which we have made explicit use above, the inflow anomaly polynomial is invariant under the simultaneous exchanges

$$N_{\mathcal{N}_i} \leftrightarrow N_{\mathcal{N}_{k-i}}, \quad F_2^{\mathcal{N}_i} \leftrightarrow F_2^{\mathcal{N}_{k-i}}, \quad N_{\mathcal{S}_i} \leftrightarrow N_{\mathcal{S}_{k-i}}, \quad F_2^{\mathcal{S}_i} \leftrightarrow F_2^{\mathcal{S}_{k-i}}, \quad F_{\varphi} \to -F_{\varphi}, \quad (4.19)$$

Both symmetries are in fact present for any k. The symmetry under (4.18) can be exhibited for general k most easily in the case in which all resolution flux quanta are equal,

$$N_{\mathcal{N}_i}, N_{\mathcal{S}_i} = N_{\mathcal{N}} \tag{4.20}$$

for all i = 1, 2, ..., k-1. For these uniform flux configurations, the large-N inflow anomaly polynomial for continuous zero-form symmetries is given by

$$\begin{split} (2\pi)^3 I_6^{\text{inflow}} &= \frac{15k^2 \chi^3 N^3 - 60k^2 (k^2 - 1) \chi N N_N^2 + 16k (k^2 - 1) (3k^2 - 2) N_N^3}{1440 \chi^2} (F_2^{\psi})^3 \\ &- \frac{\chi^2 N^3 - 2(k^2 - 1) N N_N^2}{24 \chi} F_2^{\psi} (F_2^{\varphi})^2 \\ &+ \sum_{i=1}^{k-1} \frac{3k \chi N^2 N_N - 6(k^2 - 2ki + 2i^2) N N_N^2}{8 \chi} \left(F_2^{N_i} - F_2^{S_i} \right) (F_2^{\psi})^2 \\ &- \sum_{i=1}^{k-1} \frac{(k - 2i) N N_N^2}{\chi} \left(F_2^{N_i} - F_2^{S_i} \right) F_2^{\psi} F_2^{\varphi} - \sum_{i=1}^{k-1} \frac{N^2 N_N}{2k} \left(F_2^{N_i} - F_2^{S_i} \right) (F_2^{\varphi})^2 \\ &- \sum_{i=1}^{k-1} \frac{k \chi N^3 - 2(k^2 - 2ki + 2i^2 + 2) N^2 N_N}{4} \left[(F_2^{N_i})^2 + (F_2^{S_i})^2 \right] F_2^{\psi} \\ &+ \sum_{i=1}^{k-2} \frac{k \chi N^3 - 2(k^2 - 2ki + 2i^2 - k + 2i) N^2 N_N}{4} \left(F_2^{N_i} F_2^{N_{i+1}} + F_2^{S_i} F_2^{S_{i+1}} \right) F_2^{\psi} \\ &- \sum_{i=1}^{k-1} \sum_{j=1}^{k-1} \frac{N N_N^2}{\chi} \left(F_2^{N_i} + F_2^{S_i} \right) \left(F_2^{N_j} + F_2^{S_j} \right) F_2^{\varphi} \\ &+ \sum_{i=1}^{k-1} \frac{(k - 2i) N^2 N_N}{\chi} \left[(F_2^{N_i})^2 + (F_2^{S_i})^2 \right] F_2^{\varphi} \\ &- \sum_{i=1}^{k-2} \frac{(k - 2i - 1) N^2 N_N}{2} \left(F_2^{N_i} F_2^{N_{i+1}} + F_2^{S_i} F_2^{S_{i+1}} \right) F_2^{\varphi} \\ &- \sum_{i=1}^{k-1} \frac{2\chi N^3}{3} \left[(F_2^{N_i})^3 - (F_2^{S_i})^3 \right] \\ &- \sum_{i=1}^{k-2} \frac{(k - 2i - 2) \chi N^3}{4} \left[(F_2^{N_i})^2 F_2^{N_{i+1}} - (F_2^{S_i})^2 F_2^{S_{i+1}} \right] \end{split}$$

$$+\sum_{i=1}^{k-2} \frac{(k-2i)\chi N^{3}}{4} \left[F_{2}^{N_{i}} (F_{2}^{N_{i+1}})^{2} - F_{2}^{S_{i}} (F_{2}^{S_{i+1}})^{2} \right]$$

$$+\sum_{i=1}^{k-1} \sum_{j=1}^{k-1} N^{2} N_{N} \left[(F_{2}^{N_{i}})^{2} - (F_{2}^{S_{i}})^{2} \right] \left(F_{2}^{N_{j}} + F_{2}^{S_{j}} \right)$$

$$-\sum_{i=1}^{k-2} \sum_{j=1}^{k-1} N^{2} N_{N} \left(F_{2}^{N_{i}} F_{2}^{N_{i+1}} - F_{2}^{S_{i}} F_{2}^{S_{i+1}} \right) \left(F_{2}^{N_{j}} + F_{2}^{S_{j}} \right)$$

$$+ \mathcal{O}(N, N_{N}).$$

$$(4.21)$$

We observe the invariance of the expression under the exchange, $F_2^{N_i} \leftrightarrow -F_2^{S_i}$, as expected.

4.5 Parity symmetries

As noted above, the inflow anomaly polynomial for continuous, zero-form symmetries with general k is invariant under the same exchanges (4.18) and (4.19) as the k = 3 polynomial. In other words, I_6^{inflow} is even under the involutions,

$$\mathcal{P}_{\text{NS}}: N_{\text{N}_i} \leftrightarrow N_{\text{S}_i}, F_2^{\text{N}_i} \leftrightarrow -F_2^{\text{S}_i},$$
 (4.22)

$$\mathcal{P}_{\text{EW}_{\varphi}}: N_{N_i}, N_{S_i} \leftrightarrow N_{N_{k-i}}, N_{S_{k-i}}, F_2^{N_i}, F_2^{S_i} \leftrightarrow F_2^{N_{k-i}}, F_2^{S_{k-i}}, F_2^{\varphi} \to -F_2^{\varphi},$$
 (4.23)

and odd under the involution,

$$\mathcal{P}_{\text{EW}_{\psi}}: N_{N_{i}}, N_{S_{i}} \leftrightarrow N_{N_{k-i}}, N_{S_{k-i}}, F_{2}^{N_{i}}, F_{2}^{S_{i}} \leftrightarrow -F_{2}^{N_{k-i}}, -F_{2}^{S_{k-i}}, F_{2}^{\psi} \to -F_{2}^{\psi}.$$
(4.24)

We illustrate the effects of these involutions on the configuration of the "resolution fluxes," N_{N_i} , N_{S_i} , in figure 3. In addition, one can apply the actions of \mathcal{P}_{NS} and $\mathcal{P}_{EW_{\varphi}}$ (or $\mathcal{P}_{EW_{\psi}}$) successively to find that I_6^{inflow} is also even (or odd, in the case of $\mathcal{P}_{EW_{\psi}}$) under

$$\mathcal{P}_{\text{NS}}\mathcal{P}_{\text{EW}_{\varphi}}: N_{\text{N}_{i}} \leftrightarrow N_{\text{S}_{k-i}}, F_{2}^{\text{N}_{i}} \leftrightarrow -F_{2}^{\text{S}_{k-i}}, F_{2}^{\varphi} \rightarrow -F_{2}^{\varphi},$$
 (4.25)

$$\mathcal{P}_{NS}\mathcal{P}_{EW_{\psi}}: N_{N_{i}} \leftrightarrow N_{S_{k-i}}, F_{2}^{N_{i}} \leftrightarrow -F_{2}^{S_{k-i}}, F_{2}^{\psi} \to -F_{2}^{\psi}.$$
 (4.26)

Note that we can identify \mathcal{P}_{NS} and $\mathcal{P}_{EW_{\psi}}$ (or $\mathcal{P}_{EW_{\varphi}}$) with generators of the dihedral group D_2 acting on N_{N_i} , N_{S_i} , $F_2^{N_i}$, $F_2^{S_i}$.

Here we offer a geometric interpretation of the aforementioned symmetries in the anomaly polynomial in terms of discrete isometries of M_4 . With reference to (2.6), or (A.7) for a more explicit version, the metric of M_4 is invariant under each of the following discrete transformations,

$$\mathcal{P}_{\eta}: \quad \eta \to \pi - \eta \,, \tag{4.27}$$

$$\mathcal{P}_{\theta_{\varphi}}: \quad \theta \to \pi - \theta \,, \quad \varphi \to -\varphi \,,$$
 (4.28)

$$\mathcal{P}_{\theta, \mu}: \quad \theta \to \pi - \theta \,, \quad \psi \to -\psi \,.$$
 (4.29)

We interpret these as distinct parity symmetries of M_4 . As can be seen in figure 3, the exchange of flux quanta induced by \mathcal{P}_{η} is identical to that by \mathcal{P}_{NS} , leading us to identify

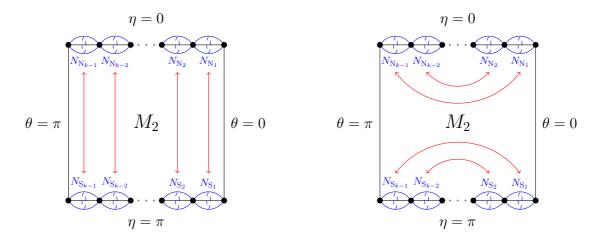


Figure 3: Illustration of the effects of the involutions \mathcal{P}_{NS} (left), and $\mathcal{P}_{EW_{\varphi}}$ or $\mathcal{P}_{EW_{\psi}}$ (right) on the flux quanta N_{N_i} , N_{S_i} . For example, under the action of \mathcal{P}_{NS} , the roles of, say, N_{N_1} and N_{S_1} , are interchanged, as indicated by one of the red double arrows in the diagram on the left. The two-form field strengths $F_2^{N_i}$, $F_2^{S_i}$ follow a similar exchange pattern.

the former as the geometric origin of the invariance of $I_6^{\rm inflow}$ under the latter. Similarly, we are led to the conclusion that $\mathcal{P}_{\theta_{\varphi}}$ and $\mathcal{P}_{\theta_{\psi}}$ are respectively the geometric origins of the even and odd parities of $I_6^{\rm inflow}$ under $\mathcal{P}_{\mathrm{EW}_{\varphi}}$ and $\mathcal{P}_{\mathrm{EW}_{\psi}}$. Note further that the global angular form $(D\varphi)^{\mathrm{g}} = D\varphi + A_1^{\varphi}$ has a definite (odd) parity only if $A_1^{\varphi} \to -A_1^{\varphi}$ under the action of $\mathcal{P}_{\theta_{\varphi}}$. Likewise, the global angular form $(D\psi)^{\mathrm{g}} = D\psi + A_1^{\psi}$ has a definite (odd) parity only if $A_1^{\psi} \to -A_1^{\psi}$ under the action of $\mathcal{P}_{\theta_{\psi}}$. These explain the sign changes of F_2^{φ} and F_2^{ψ} in (4.23) and (4.24) respectively.

We would like to remind the reader that the invariance of $I_6^{\rm inflow}$ under (4.22) and (4.23) is made manifest as a consequence of choosing the natural basis of (co)homology classes introduced in section 2. In another basis the transformation properties of $I_6^{\rm inflow}$ are generally obscured by nontrivial redefinitions of the flux quanta and field strengths.

5 Conclusion

In this work, we have started a systematic exploration of setups featuring wrapped M5-branes probing a family of flux backgrounds, consisting of a resolved S^4/\mathbb{Z}_k fibered over a higher-genus Riemann surface Σ_g . This class of setups for $k \geq 3$ is challenging to analyze directly in holography or in the probe picture. Anomaly inflow techniques, however, provide a powerful inroad into the investigation of these setups. Indeed, these methods hinge on robust topological and flux data, which can be inferred without solving explicitly the supersymmetry conditions and equations of motion.

Our findings exhibit a rich pattern of global symmetries and 't Hooft anomalies. This is already visible if we restrict our attention to continuous symmetries only, and focus on terms in the anomaly polynomial that are leading in the limit of large N, $N_{\rm N_i}$, $N_{\rm S_i}$. Our computations can be extended in two natural directions. Firstly, we could retain the topologically massive fields in our analysis in order to investigate 't Hooft anomalies

involving discrete symmetries. This task was addressed for k=2 in [30] (see in particular appendix E), and it would be interesting to consider the general case $k \geq 3$ using the results of this paper as a starting point. Secondly, we could access the subleading terms in N, N_{N_i} , N_{S_i} by taking into account the effect of the higher-derivative topological term $C_3 \wedge X_8$ in the M-theory effective action. This requires a determination of the class X_8 , computed in the presence of non-zero backgrounds for the connections A_1^{ψ} , A_1^{φ} associated with the isometries $\mathrm{U}(1)_{\psi}$, $\mathrm{U}(1)_{\varphi}$. A possible strategy to achieve this goal is to consider local expressions, given in terms of differential forms, to compute the class X_8 . This approach has been successful in the case k=2 [24]. The approach of that paper, however, relies on a special feature of the case k=2: the resolved $\mathbb{C}^2/\mathbb{Z}_2$ singularity admits a description both in terms of a Taub-NUT geometry and in terms of an Eguchi-Hanson geometry. The latter description greatly facilitates the computation of X_8 based on local expressions, but is unfortunately unavailable for $k \geq 3$. It would therefore be interesting to revisit the problem of the determination of the class X_8 for $k \geq 3$, possibly exploiting more refined computational techniques such as spectral sequences in equivariant cohomology.

According to the inflow paradigm, the output $I_6^{\rm inflow}$ of the inflow computation is to be identified with minus the anomaly polynomial of the 4d field theory realized by the wrapped M5-branes. The results of this work give us access to the cubic terms in N, $N_{\rm N_i}$, $N_{\rm S_i}$ inside the quantity $I_6^{\rm 4d~QFT} = -I_6^{\rm inflow}$. Armed with this knowledge, one can perform manipulations on $I_6^{\rm 4d~QFT}$ to learn about the 4d dynamics. In an upcoming paper [40], we adopt a-maximization [41] on $I_6^{\rm 4d~QFT}$ to explore whether the 4d IR dynamics can define a non-trivial SCFT, and to compare this putative SCFT to other theories originating from wrapped M5-branes.

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A Resolution of S^4/\mathbb{Z}_k orbifold singularities

Here we explicitly describe the blow-up of the orbifold singularities at the level of the metric. Under a coordinate transformation $\mu = \cos \eta$, the orbifold metric (2.3) becomes

$$ds^{2}(S^{4}/\mathbb{Z}_{k}) = \frac{d\mu^{2}}{1-\mu^{2}} + (1-\mu^{2}) \left[\frac{1}{k^{2}} D\varphi^{2} + \frac{1}{4} \left(d\theta^{2} + \sin^{2}\theta \, d\psi^{2} \right) \right], \tag{A.1}$$

Expanding around $\mu = \pm 1$, we obtain a single-center Taub-NUT space with charge k,

$$ds_{\pm}^{2}(S^{4}/\mathbb{Z}_{k}) = \frac{1}{V_{+}}D\varphi^{2} + V_{\pm}\left[dR_{\pm}^{2} + R_{\pm}^{2}\left(d\theta^{2} + \sin^{2}\theta \,d\psi^{2}\right)\right],\tag{A.2}$$

where $R_{\pm} = (1 \mp \mu)/k$ and $V_{\pm} = k/2R_{\pm}$. The singularities at $\mu = \pm 1$ are resolved by locally replacing the Taub-NUT space with a multi-center Gibbons-Hawking space,

$$ds_{\pm}^{2} = \frac{1}{V_{+}} \left(d\varphi + A_{\pm} \right)^{2} + V_{\pm} ds^{2}(\mathbb{R}^{3}), \qquad (A.3)$$

with the harmonic potential V_{\pm} having the general form,

$$V_{\pm}(\vec{R}) = v_{\pm} + \frac{1}{2} \sum_{i=1}^{i_{\pm}^{\max}} \frac{n_{\pm,i}}{R_{\pm,i}}, \tag{A.4}$$

where $R_{\pm,i}$ is the distance between the position \vec{R} in \mathbb{R}^3 and the *i*-th Kaluza-Klein monopole at $\mu = \pm 1$. The potential behaves asymptotically the same as that in (A.2) if and only if

$$v_{\pm} = 0, \qquad \sum_{i=1}^{N_{\pm}} n_{\pm,i} = k.$$
 (A.5)

The former condition states that the size of the S^1_{φ} circle goes to zero at each monopole. In this work, we focus on the case where both orbifold singularities are fully resolved by fixing $i_{\pm}^{\max} = k$ and $n_{\pm,i} = 1$ for all i, such that the space around each monopole is locally Euclidean, i.e. $\mathbb{R}^4/\mathbb{Z}_{n_{\pm,i}} = \mathbb{R}^4/\mathbb{Z} \cong \mathbb{R}^4$. Furthermore, the connection one-form A_{\pm} is related to the potential through $dV_{\pm} = \star_{\mathbb{R}^3} dA_{\pm}$, and has a straightforward solution,

$$A_{\pm} = \frac{1}{2} \sum_{i=1}^{k} \cos \theta_{\pm,i} \, d\psi_{\pm,i} \,, \tag{A.6}$$

where $\theta_{\pm,i}$ and $\psi_{\pm,i}$ are the standard polar and azimuthal angles in \mathbb{R}^3 with respect to the *i*-th monopole at $\mu = \pm 1$. We further impose that all the azimuthal angles are identified with the angle ψ of the unresolved orbifold, i.e. $\psi_{\pm,i} = \psi$, so as to preserve a U(1)_R subgroup of the original SU(2)_R isometry. Geometrically, this means that we have k-1 two-cycles, separated by k monopoles, aligned along the axis associated with the U(1)_{ψ} symmetry.

To summarize, the resultant resolved space, M_4 , has a $U(1)_{\psi} \times U(1)_{\varphi}$ isometry group, and, omitting the " \pm " notation for visual clarity, its metric near $\mu = \pm 1$ (or equivalently, $\eta = 0, \pi$) can be written as

$$ds^{2}(M_{4}) = \frac{2}{\sum_{i=1}^{k} R_{i}^{-1}} \left(d\varphi + \frac{1}{2} \sum_{i=1}^{k} \cos \theta_{i} \, d\psi \right)^{2} + \sum_{i=1}^{k} \frac{1}{2R_{i}} \left[dR_{i}^{2} + R_{i}^{2} \left(d\theta_{i}^{2} + \sin^{2} \theta_{i} \, d\psi^{2} \right) \right]. \tag{A.7}$$

By construction, θ_i takes a value of 0 or π at $\mu = \pm 1$, so the size of the circle S_{ψ}^1 , which is given by $\sum_{i=1}^k R_i \sin^2 \theta_i / 2$ at $\mu = \pm 1$ and $(1 - \mu^2) \sin^2 \theta / 4$ at $|\mu| < 1$, vanishes along the boundary ∂M_2 , where M_2 is defined as the 2d space spanned by (μ, θ) . It is useful to

This can be easily verified by performing the coordinate transformation $R_{\pm,i}=r_{\pm,i}^2/2n_{\pm,i}$, which leads to $ds_{\pm,i}^2=dr_{\pm,i}^2+r_{\pm,i}^2\left[(1/n_{\pm,i}^2)D\varphi^2+(1/4)\left(d\theta^2+\sin^2\theta\,d\psi^2\right)\right]$.

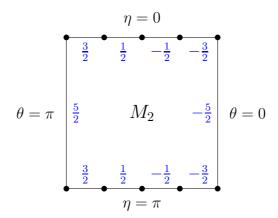


Figure 4: Value that the function L takes (written in blue) in each boundary interval of M_2 separated by monopoles for the case of k = 5.

define the function,

$$L(\mu, \theta) \equiv \begin{cases} -\frac{1}{2} \sum_{i=1}^{k} \cos \theta_i & \text{if } \mu = \pm 1, \\ -\frac{k}{2} \cos \theta & \text{if } |\mu| < 1, \end{cases}$$
(A.8)

such that the global angular form associated with S_{φ}^1 can be written as $D\varphi = d\varphi - L d\psi$ everywhere in M_4 . We observe that along ∂M_2 , the function L is piecewise constant and periodic. Specifically, if we trace the value of L anticlockwise along ∂M_2 , it starts with 1 - k/2 right next to $(\eta, \theta) = (0, 0)$, increases by 1 whenever we cross a monopole till reaching k/2 right after $(\eta, \theta) = (0, \pi)$, then it decreases by 1 whenever we cross a monopole till reaching -k/2 right after $(\eta, \theta) = (\pi, 0)$, and finally returns to 1 - k/2 after crossing the first monopole at $(\eta, \theta) = (0, 0)$. As an example, we illustrate the behavior of L along ∂M_2 for k = 5 in figure 4.

B Cohomology class representatives of M_6

The various homological relations in M_6 can be derived by working with its dual de Rham cohomology groups. We start with the relatively trivial case of constructing cohomology class representatives of $H^1(M_6)$. The most general one-form that is covariant under the action of the $U(1)_{\psi} \times U(1)_{\varphi}$ isometry group of M_6 can be written as

$$\lambda_1 = X_0^u \lambda_{1,u} + X_0^{\psi} \frac{D\psi}{2\pi} + X_0^{\varphi} \frac{D\varphi}{2\pi} + X_1^{\eta\theta},$$
 (B.1)

where X_0^u , X_0^{ψ} , X_0^{φ} are zero-forms and $X_1^{\eta\theta}$ is a one-form, all defined on M_2 , and the index u is summed over $1, \ldots, b_1(M_6)$. We want λ_1 to be closed and invariant under the isometries of M_6 . The former condition amounts to requiring

$$dX_0^u = 0\,, \quad \chi(X_0^\psi - LX_0^\varphi) + \zeta X_0^\varphi = 0\,, \quad d(X_0^\psi - LX_0^\varphi) = -L\,dX_0^\varphi = 0\,, \quad dX_1^{\eta\theta} = 0\,, \quad (\text{B.2})$$

which immediately tells us that X_0^u are constants. We can also combine the constraints above to deduce

$$d(X_0^{\psi} - LX_0^{\varphi}) = -\frac{\zeta}{\chi} dX_0^{\varphi} = -L dX_0^{\varphi} = 0,$$
 (B.3)

which together with (B.1) implies $X_0^{\psi} = X_0^{\varphi} = 0$, since L is a position-dependent function (with respect to M_2). This automatically makes λ_1 gauge-invariant. In addition, we conclude that $X_1^{\eta\theta}$ has to be exact because by construction,

$$\int_{\mathcal{C}_{1}^{\Sigma,u}} X_{1}^{\eta\theta} = 0 \tag{B.4}$$

for all u. Therefore, without loss of generality, we can simply pick $\lambda_{1,u}$ for $u=1,\ldots,2g$ to be the cohomology class representatives of $H^1(M_6)$.

Let us now write down the most general two-form that is covariant under the isometries of M_6 ,

$$\omega_2 = W_0^{\psi\varphi} \frac{D\psi}{2\pi} \wedge \frac{D\varphi}{2\pi} + W_1^{\psi} \wedge \frac{D\psi}{2\pi} + W_1^{\varphi} \wedge \frac{D\varphi}{2\pi} + W_0^{\Sigma} V_2^{\Sigma} + W_2^{\eta\theta}, \tag{B.5}$$

where $W_0^{\psi\varphi}, W_0^{\Sigma}$ are zero-forms, $W_1^{\psi}, W_1^{\varphi}$ are one-forms, and $W_2^{\eta\theta}$ is a two-form, all of which are defined on M_2 .¹¹ Imposing closure on ω_2 yields the following conditions,

$$W_0^{\psi\varphi} = 0$$
, $d(W_1^{\psi} - LW_1^{\varphi}) = 0$, $dW_0^{\Sigma} = -\chi(W_1^{\psi} - LW_1^{\varphi}) - \zeta W_1^{\varphi}$, $dW_2^{\eta\theta} = 0$. (B.6)

In addition, note that the interior products act on the connection forms as $\iota_{\psi}D\psi = 1$, $\iota_{\psi}D\varphi = -L$, $\iota_{\varphi}D\psi = 0$, $\iota_{\varphi}D\varphi = 1$. If we take the interior product of ω_2 with respect to the isometries of M_6 , we get

$$\iota_{\psi}\omega_2 = -(W_1^{\psi} - LW_1^{\varphi}), \qquad \iota_{\varphi}\omega_2 = -W_1^{\varphi}. \tag{B.7}$$

Both of these one-forms vanish when integrated over any of the one-cycles in M_6 , i.e. $\mathcal{C}_1^{\Sigma,u}$, because they do not contain any factor of $\lambda_{1,u}$ by construction, thus meaning they are exact,

$$W_1^{\psi} - LW_1^{\varphi} = dW_0^{\psi}, \qquad W_1^{\varphi} = dW_0^{\varphi},$$
 (B.8)

where W_0^{ψ} and W_0^{φ} are single-valued functions defined on M_2 . A similar argument implies $W_2^{\eta\theta}$ is also exact. Hence, a generic member of $H^2(M_6)$ can be parameterized as

$$\omega_2 = \left(dW_0^{\psi} + L dW_0^{\varphi}\right) \wedge \frac{D\psi}{2\pi} + dW_0^{\varphi} \wedge \frac{D\varphi}{2\pi} - \left(\chi W_0^{\psi} + \zeta W_0^{\varphi}\right) V_2^{\Sigma}. \tag{B.9}$$

We check that ω_2 as constructed above is invariant under the isometries of M_6 , ¹²

$$\mathcal{L}_{\psi}\omega_{2} = d\iota_{\psi}\omega_{2} + \iota_{\psi}d\omega_{2} = -d^{2}W_{0}^{\psi} = 0, \quad \mathcal{L}_{\varphi}\omega_{2} = d\iota_{\varphi}\omega_{2} + \iota_{\varphi}d\omega_{2} = -d^{2}W_{0}^{\varphi} = 0. \quad (B.10)$$

In Strictly speaking, we can also add terms linear in $\lambda_{1,u}$ to ω_2 , but they can only be trivial if we want ω_2 to be in $H^2(M_6)$.

¹²This statement can be equivalently phrased in terms of equivariant cohomology. The operator analogous to the ordinary exterior derivative, d, is the equivariant exterior derivative, $d_{\mathfrak{g}} = d - \iota_{I}$, where I labels the generator of the isometry group. Defining $\hat{\omega}_{2} = \left(dW_{0}^{\psi} + L \, dW_{0}^{\varphi}\right) \wedge \left(D\psi/2\pi\right) + dW_{0}^{\varphi} \wedge \left(D\varphi/2\pi\right)$, it can be shown that $\hat{\omega}_{2}$ is equivariantly closed, i.e. $d_{\mathfrak{g}}\hat{\omega}_{2} = 0$. A review of equivariant cohomology can be found, for example, in [42].

The inner products between a generic cohomology class representative (B.9) and the two-cycles (2.18), (2.19) are given by

$$\int_{\mathcal{C}_2^{\Sigma,i}} \omega_2 = -\chi W_0^{\psi}|_{t=t_i} - \zeta W_0^{\varphi}|_{t=t_i}, \quad \int_{\mathcal{C}_2^i} \omega_2 = W_0^{\varphi}|_{t_i}^{t_{i+1}}.$$
 (B.11)

Recall that ∂M_2 is the preimage of the zero section of the U(1) $_{\psi}$ bundle, so ω_2 can only be globally defined if terms explicitly containing $D\psi$ vanish on ∂M_2 . We therefore have to impose the following regularity constraint on ω_2 ,

$$[dW_0^{\psi} + L dW_0^{\varphi}]_{\partial M_2} = 0,$$
 (B.12)

and when combined with (B.11), it implies

$$\int_{\mathcal{C}_2^{\Sigma,i+1}} \omega_2 - \int_{\mathcal{C}_2^{\Sigma,i}} \omega_2 = (\chi \ell_i - \zeta) \int_{\mathcal{C}_2^i} \omega_2.$$
 (B.13)

Since ω_2 is an arbitrary element of $H^2(M_6)$, the relation above holds if and only if

$$C_2^{\Sigma,i+1} - C_2^{\Sigma,i} = (\chi \ell_i - \zeta) C_2^i. \tag{B.14}$$

Making use of the single-valuedness of W_0^{φ} , the sums of the second relation in (B.11) as well as (B.14) over $i = 1, \ldots, 2k$ respectively lead to two sum rules,

$$\sum_{i=1}^{2k} C_2^i = 0, \qquad \sum_{i=1}^{2k} \ell_i C_2^i = 0.$$
 (B.15)

If we follow the same lines of arguments for the three-forms, we find that a generic cohomology class representative in $H^3(M_6)$ can be parameterized as

$$\Lambda_3 = \left[\left(dS_0^{\psi,u} + L \, dS_0^{\varphi,u} \right) \wedge \frac{D\psi}{2\pi} + dS_0^{\varphi,u} \wedge \frac{D\varphi}{2\pi} \right] \wedge \lambda_{1,u} \,, \tag{B.16}$$

where $S_0^{\psi,u}$ and $S_0^{\varphi,u}$ are zero-forms defined on M_2 , while a generic cohomology class representative in $H^4(M_6)$ can be parameterized as

$$\Omega_4 = dT_1 \wedge \frac{D\psi}{2\pi} \wedge \frac{D\varphi}{2\pi} + \left(dU_0 + L\chi T_1 - \zeta T_1\right) \wedge \frac{D\psi}{2\pi} \wedge V_2^{\Sigma} + \chi T_1 \wedge \frac{D\varphi}{2\pi} \wedge V_2^{\Sigma}, \quad (B.17)$$

where U_0 and T_1 are respectively zero-forms and one-forms defined on M_2 . The two sum rules (2.24) for the three cycles can be readily derived based on the previous discussion for the two-cycles. For the four-cycles, on the other hand, we note that

$$\int_{\mathcal{C}_{4,C}} \Omega_4 = \int_{M_2} dT_1 = \int_{\partial M_2} T_1 \,, \quad \int_{\mathcal{C}_{4,i}} \Omega_4 = \chi \int_{t_i}^{t_{i+1}} T_1 \,. \tag{B.18}$$

Meanwhile, the regularity constraint analogous to (B.12) is

$$[dU_0 + L\chi T_1 - \zeta T_1]_{\partial M_2} = 0.$$
 (B.19)

It then follows that

$$\sum_{i=1}^{2k} C_{4,i} = \chi C_{4,C}, \qquad \sum_{i=1}^{2k} \ell_i C_{4,i} = \zeta C_{4,C}.$$
 (B.20)

The aforementioned parameterizations of λ_1 , ω_2 , Λ_3 , Ω_4 are only well-defined on M_6 . In order for the Kaluza-Klein expansion of G_4 to make sense in M_{11} , we need to construct the appropriate cohomology class representatives that are also invariant under gauge transformations in \mathcal{M}_5 . The first step to achieving this goal is to introduce U(1) connections over \mathcal{M}_5 to the global angular forms $D\psi$ and $D\varphi$ respectively by promoting

$$D\psi \to (D\psi)^{\mathrm{g}} = D\psi + A_1^{\psi}, \qquad D\varphi \to (D\varphi)^{\mathrm{g}} = D\varphi + A_1^{\varphi},$$
 (B.21)

where A_1^{ψ} and A_1^{φ} are 5d one-form gauge fields. For completeness, if the Riemann surface is a sphere, i.e. g = 0, which we do not consider in this paper unless otherwise specified, then there is an enhanced $SO(3)_{\Sigma}$ isometry. In this case, the volume form V_2^{Σ} has to be promoted to the (normalized) global angular form,

$$V_2^{\Sigma} \to \frac{e_2^{\Sigma}}{2} = \frac{1}{8\pi} \,\epsilon_{abc} (Dy^a \wedge Dy^b y^c - F_2^{ab} y^c) \,,$$
 (B.22)

where y^a with a=1,3 are coordinates of Σ_g embedded in \mathbb{R}^3 , and $Dy^a=dy^a-A_1^{ab}y_b$ with A_1^{ab} being an external $\mathrm{SO}(3)_\Sigma$ connection and F_2^{ab} is its field strength. We hereafter use the notation λ_1^{g} , ω_2^{g} , Λ_3^{g} , Ω_4^{g} to denote respectively the expressions (B.1), (B.9), (B.16), (B.17) with the replacements (B.21) implemented. Except for λ_1^{g} , which is trivial, these gauged forms are not suitable candidates for the cohomology class representatives because they are no longer closed and gauge-invariant. The proper candidates are of the following forms,

$$\lambda_{1}^{\text{eq}} = \lambda_{1}^{\text{g}},
\omega_{2}^{\text{eq}} = \omega_{2}^{\text{g}} + \frac{F_{2}^{I}}{2\pi} \omega_{0,I},
\Lambda_{3}^{\text{eq}} = \Lambda_{3}^{\text{g}} + \frac{F_{2}^{I}}{2\pi} \Lambda_{1,I}^{\text{g}},
\Omega_{4}^{\text{eq}} = \Omega_{4}^{\text{g}} + \frac{F_{2}^{I}}{2\pi} \Omega_{2,I}^{\text{g}} + \frac{F_{2}^{I}}{2\pi} \frac{F_{2}^{J}}{2\pi} \Omega_{0,IJ},$$
(B.23)

for a zero-forms $\omega_{0,I}$, $\Omega_{0,IJ}$, one-form $\Lambda_{1,I}$, two-form $\Omega_{2,I}$, where $\{I,J\} = \{\psi,\varphi\}$ label the isometries. If we impose closure and gauge-invariance on λ_1^{eq} , ω_2^{eq} , Λ_3^{eq} , Ω_4^{eq} , we find that they satisfy

$$2\pi \iota_{I} \lambda_{1} = 0,$$

$$2\pi \iota_{I} \omega_{2} + d\omega_{0,I} = 0,$$

$$2\pi \iota_{I} \Lambda_{3} + d\Lambda_{1,I} = 0, \quad 2\pi \iota_{(I} \Lambda_{1,J)} = 0,$$

$$2\pi \iota_{I} \Omega_{4} + d\Omega_{2,I} = 0, \quad 2\pi \iota_{(I} \Omega_{2,J)} + d\Omega_{0,IJ} = 0,$$
(B.24)

where the following identity,

$$d\alpha_p^{g} + A_1^{I}(\mathcal{L}_I \alpha_p)^{g} = (d\alpha_p)^{g} + F_2^{I}(\iota_I \alpha_p)^{g}$$
(B.25)

for some generic p-form α_p defined on M_6 , has been used [26, 30]. Note that $\omega_{0,I}$, $\Lambda_{1,I}$, $\Omega_{2,I}$, $\Omega_{0,IJ}$ are all defined only up to addition of closed forms.

We introduce below the explicit parameterizations for such forms in the specific context of this work. First of all, we have

$$d\Omega_{2,\psi} = -2\pi \iota_{\psi} \Omega_{4} = -L dT_{1} \wedge \frac{D\psi}{2\pi} - dT_{1} \wedge \frac{D\varphi}{2\pi} + dU_{0} \wedge V_{2}^{\Sigma},$$

$$d\Omega_{2,\varphi} = -2\pi \iota_{\varphi} \Omega_{4} = dT_{1} \wedge \frac{D\psi}{2\pi} + \chi T_{1} \wedge V_{2}^{\Sigma},$$
(B.26)

which admit the globally defined solutions (up to addition of closed two-forms),

$$\Omega_{2,\psi} = -\frac{1}{\chi} \left(dU_0 + L\chi T_1 \right) \wedge \frac{D\psi}{2\pi} - T_1 \wedge \frac{D\varphi}{2\pi} + 2U_0 V_2^{\Sigma} ,
\Omega_{2,\varphi} = \left(dY_{0,\psi} + L dY_{0,\varphi} + T_1 \right) \wedge \frac{D\psi}{2\pi} + dY_{0,\varphi} \wedge \frac{D\varphi}{2\pi} - \chi Y_{0,\psi} V_2^{\Sigma} ,$$
(B.27)

where $Y_{0,\psi}, Y_{0,\varphi}$ are zero-forms defined on M_2 . Note that the regularity of $\Omega_{2,\psi}$ and $\Omega_{2,\varphi}$ along ∂M_2 (where the circle S^1_{ψ} vanishes) requires $\left[dU_0 + L\chi T_1\right]_{\partial M_2} = \left[dY_{0,\psi} + L\,dY_{0,\varphi} + T_1\right]_{\partial M_2} = 0$. Consequently we can derive (up to additive constants)

$$\Omega_{0,\psi\psi} = -\frac{1}{\chi} U_0, \quad \Omega_{0,\psi\varphi} = \frac{1}{2} Y_{0,\psi}, \quad \Omega_{0,\varphi\varphi} = Y_{0,\varphi}.$$
(B.28)

The rest of the forms needed to complete the gauge-invariant cohomology class representatives can be similarly solved for to obtain (up to addition of closed one-forms)

$$\Lambda_{1,x\psi} = S_{0,x}^{\psi,u} \lambda_{1,u}, \qquad \Lambda_{1,x\varphi} = S_{0,x}^{\varphi,u} \lambda_{1,u},$$
(B.29)

and (up to additive constants)

$$\omega_{0,\psi} = W_0^{\psi}, \qquad \omega_{0,\varphi} = W_0^{\varphi}.$$
 (B.30)

Now we are in a position to explicitly evaluate various intersection numbers associated with cycles of different degrees in M_6 , as well as miscellaneous integrals involved in the equations of motion (4.10) and (4.11). Before doing so, it is instructive to first record an expansion of E_4^3 as follows,

$$\begin{split} &\frac{1}{6}\,E_{4}^{3}\supset\\ &N_{\alpha}N_{\beta}N_{\gamma}\,\frac{F_{2}^{I}\,F_{2}^{J}\,F_{2}^{K}}{2\pi}\,\Omega_{0,IJ}^{\alpha}\big(\Omega_{2,K}^{\beta}\big)^{\mathrm{g}}\big(\Omega_{4}^{\gamma}\big)^{\mathrm{g}} + \frac{1}{6}\,N_{\alpha}N_{\beta}N_{\gamma}\,\frac{F_{2}^{I}\,F_{2}^{J}\,F_{2}^{K}}{2\pi}\,(\Omega_{2,I}^{\alpha}\big)^{\mathrm{g}}\big(\Omega_{2,J}^{\beta}\big)^{\mathrm{g}}\big(\Omega_{2,K}^{\gamma}\big)^{\mathrm{g}}\\ &+ NN_{\beta}N_{\gamma}\,\frac{F_{2}^{I}\,F_{2}^{J}\,F_{2}^{\alpha}}{2\pi}\,\omega_{0,\alpha I}\big(\Omega_{2,J}^{\beta}\big)^{\mathrm{g}}\big(\Omega_{4}^{\gamma}\big)^{\mathrm{g}} + NN_{\alpha}N_{\gamma}\,\frac{F_{2}^{I}\,F_{2}^{J}\,F_{2}^{\beta}}{2\pi}\,\frac{F_{2}^{\beta}}{2\pi}\,\Omega_{0,IJ}^{\alpha}(\omega_{2,\beta})^{\mathrm{g}}\big(\Omega_{4}^{\gamma}\big)^{\mathrm{g}}\\ &+ \frac{1}{2}\,NN_{\alpha}N_{\beta}\,\frac{F_{2}^{I}\,F_{2}^{J}\,F_{2}^{\gamma}}{2\pi}\,(\Omega_{2,I}^{\alpha}\big)^{\mathrm{g}}\big(\Omega_{2,J}^{\beta}\big)^{\mathrm{g}}\big(\omega_{2,\gamma}\big)^{\mathrm{g}} + N^{2}N_{\gamma}\,\frac{F_{2}^{I}\,F_{2}^{\alpha}\,F_{2}^{\beta}}{2\pi}\,\omega_{0,\alpha I}(\omega_{2,\beta})^{\mathrm{g}}\big(\Omega_{4}^{\gamma}\big)^{\mathrm{g}}\\ &+ \frac{1}{2}\,N^{2}N_{\alpha}\,\frac{F_{2}^{I}\,F_{2}^{\beta}\,F_{2}^{\gamma}}{2\pi}\,(\Omega_{2,I}^{\alpha}\big)^{\mathrm{g}}(\omega_{2,\beta})^{\mathrm{g}}(\omega_{2,\gamma})^{\mathrm{g}} + \frac{1}{6}\,N^{3}\,\frac{F_{2}^{\alpha}\,F_{2}^{\beta}\,F_{2}^{\gamma}}{2\pi}\,E_{2}^{\gamma}\,\omega_{0,\alpha I}(\omega_{2,\beta})^{\mathrm{g}}(\omega_{2,\gamma})^{\mathrm{g}}\\ &- \frac{1}{2}\,N^{2}N_{\alpha}\,\frac{f_{1}^{x}\,f_{1}^{y}\,f_{2}^{I}\,F_{2}^{J}\,F_{2}^{J}}{2\pi}\,\Omega_{0,IJ}^{\alpha}(\Lambda_{3,x})^{\mathrm{g}}(\Lambda_{3,y})^{\mathrm{g}} - N^{2}N_{\alpha}\,\frac{f_{1}^{x}\,f_{1}^{y}\,f_{2}^{I}\,F_{2}^{J}}{2\pi}\,\Lambda_{1,xI}\big(\Omega_{2,J}^{\alpha}\big)^{\mathrm{g}}(\Lambda_{3,y})^{\mathrm{g}} \end{split}$$

$$-\frac{1}{2}N^{2}N_{\alpha}\frac{f_{1}^{x}}{2\pi}\frac{f_{1}^{y}}{2\pi}\frac{F_{2}^{I}}{2\pi}\frac{F_{2}^{J}}{2\pi}\Lambda_{1,xI}\Lambda_{1,yJ}(\Omega_{4}^{\alpha})^{g} - \frac{1}{2}N^{3}\frac{f_{1}^{x}}{2\pi}\frac{f_{1}^{y}}{2\pi}\frac{F_{2}^{I}}{2\pi}\frac{F_{2}^{\alpha}}{2\pi}\omega_{0,\alpha I}(\Lambda_{3,x})^{g}(\Lambda_{3,y})^{g} \\ -N^{3}\frac{f_{1}^{x}}{2\pi}\frac{f_{1}^{y}}{2\pi}\frac{F_{2}^{I}}{2\pi}\frac{F_{2}^{\alpha}}{2\pi}\Lambda_{1,xI}(\omega_{2,\alpha})^{g}(\Lambda_{3,y})^{g} - N^{2}N_{\alpha}\frac{f_{1}^{x}}{2\pi}\frac{F_{2}^{I}}{2\pi}\frac{H_{3}^{u}}{2\pi}\Lambda_{1,xI}\lambda_{1,u}(\Omega_{4}^{\alpha})^{g} \\ +N^{2}N_{\alpha}\frac{f_{1}^{x}}{2\pi}\frac{F_{2}^{I}}{2\pi}\frac{H_{3}^{u}}{2\pi}\lambda_{1,u}(\Omega_{2,I}^{\alpha})^{g}(\Lambda_{3,x})^{g} + N^{3}\frac{f_{1}^{x}}{2\pi}\frac{F_{2}^{\alpha}}{2\pi}\frac{H_{3}^{u}}{2\pi}\lambda_{1,u}(\omega_{2,\alpha})^{g}(\Lambda_{3,x})^{g} \\ -\frac{1}{2}N^{3}\frac{f_{1}^{x}}{2\pi}\frac{f_{1}^{y}}{2\pi}\frac{\gamma_{4}}{2\pi}(\Lambda_{3,x})^{g}(\Lambda_{3,y})^{g} - \frac{1}{2}N^{2}N_{\alpha}\frac{H_{3}^{u}}{2\pi}\frac{H_{3}^{v}}{2\pi}\lambda_{1,u}\lambda_{1,v}(\Omega_{4}^{\alpha})^{g} \\ +NN_{\alpha}N_{\beta}\frac{F_{2}^{I}}{2\pi}\frac{\gamma_{4}}{2\pi}(\Omega_{2,I}^{\alpha})^{g}(\Omega_{4}^{\beta})^{g} + N^{2}N_{\beta}\frac{F_{2}^{\alpha}}{2\pi}\frac{\gamma_{4}}{2\pi}(\omega_{2,\alpha})^{g}(\Omega_{4}^{\beta})^{g},$$
 (B.31)

where we kept only terms with six internal legs because they survive under a fiber integration over M_6 .¹³ Let us also define the choice of basis of (co)homology classes via the expansions,

$$C_{4,C} = a_C^{\alpha} C_{4,\alpha}, \quad C_{4,i} = a_i^{\alpha} C_{4,\alpha}, \quad C_2^{\Sigma,i} = b_{\alpha}^{\Sigma,i} C_2^{\alpha}, \quad C_2^i = b_{\alpha}^i C_2^{\alpha}, \quad C_3^{i,u} = c_x^{i,u} C_3^x, \quad (B.32)$$

for some real coefficients $a_{\rm C}^{\alpha}$, a_i^{α} , $b_{\alpha}^{\Sigma,i}$, b_{α}^{i} , $c_x^{i,u}$, and the cycles $\mathcal{C}_{4,{\rm C}}$, $\mathcal{C}_{4,i}$, $\mathcal{C}_{2}^{\Sigma,i}$, \mathcal{C}_{2}^{i} , $\mathcal{C}_{3}^{i,u}$ are defined in section 2.¹⁴ In terms of the differential forms introduced earlier to parameterize the cohomology class representatives, the orthonormality between cycles and their dual cohomology class representatives can be rewritten as expressions for these expansion coefficients,

$$a_{\mathcal{C}}^{\alpha} = \int_{\partial M_2} T_1^{\alpha}, \qquad a_i^{\alpha} = \chi \int_{t_i}^{t_{i+1}} T_1^{\alpha},$$

$$b_{\alpha}^{\Sigma,i} = -\chi W_{0,\alpha}^{\psi}(t_i), \quad b_{\alpha}^{i} = W_{0,\alpha}^{\varphi}(t_{i+1}) - W_{0,\alpha}^{\varphi}(t_i), \quad c_x^{i,u} = S_{0,x}^{\varphi,u}(t_{i+1}) - S_{0,x}^{\varphi,u}(t_i).$$
(B.33)

The intersection numbers that are relevant in the computation of I_6^{inflow} are

$$\mathcal{K}_{uv} \equiv \int_{\Sigma_g} \lambda_{1,u} \wedge \lambda_{1,v} , \quad \mathcal{K}_{\alpha}^{\beta} \equiv \int_{M_6} \omega_{2,\alpha} \wedge \Omega_4^{\beta} , \quad \mathcal{K}_{xy} \equiv \int_{M_6} \Lambda_{3,x} \wedge \Lambda_{3,y} ,
\mathcal{K}_{uv}^{\alpha} \equiv \int_{M_6} \lambda_{1,u} \wedge \lambda_{1,v} \wedge \Omega_4^{\alpha} , \quad \mathcal{K}_{u\alpha x} \equiv \int_{M_6} \lambda_{1,u} \wedge \omega_{2,\alpha} \wedge \Lambda_{3,x} ,$$
(B.34)

with u, v = 1, ..., 2g and $\alpha, \beta = 1, ..., 2k - 1$ and x, y = 1, ..., 4g(k - 1). As discussed in section 2, by choosing $\lambda_{1,u}$ to be orthonormal to the standard \mathcal{A} and \mathcal{B} cycles of the Riemann surface Σ_g , the intersection number \mathcal{K}_{uv} can be compactly written in the following form,

$$\mathcal{K}_{uv} = \begin{pmatrix} 0 & \delta_{pq} \\ -\delta_{pq} & 0 \end{pmatrix} . \tag{B.35}$$

On the other hand, recall that the cohomology class representatives $\omega_{2,\alpha}$, $\Lambda_{3,x}$, Ω_4^{α} can be respectively parameterized as

$$\omega_{2,\alpha} = \left(dW_{0,\alpha}^{\psi} + L \, dW_{0,\alpha}^{\varphi} \right) \wedge \frac{D\psi}{2\pi} + dW_{0,\alpha}^{\varphi} \wedge \frac{D\varphi}{2\pi} - \chi W_{0,\alpha}^{\psi} V_2^{\Sigma} \,, \tag{B.36}$$

¹³If we include the g=0 case in our consideration, then we will have terms cubic in e_2^{Σ} which yield a nonvanishing integral as determined by the Bott-Cattaneo formula [43].

¹⁴The expansion of the one-cycles is trivial given that we use the standard \mathcal{A} and \mathcal{B} cycles of the Riemann surface as our basis one-cycles.

$$\Lambda_{3,x} = \left(dS_{0,x}^{\psi,u} + L \, dS_{0,x}^{\varphi,u} \right) \wedge \frac{D\psi}{2\pi} \wedge \lambda_{1,u} + dS_{0,x}^{\varphi,u} \wedge \frac{D\varphi}{2\pi} \wedge \lambda_{1,u} \,, \tag{B.37}$$

$$\Omega_4^{\alpha} = dT_1^{\alpha} \wedge \frac{D\psi}{2\pi} \wedge \frac{D\varphi}{2\pi} + \left(dU_0^{\alpha} + L\chi T_1^{\alpha}\right) \wedge \frac{D\psi}{2\pi} \wedge V_2^{\Sigma} + \chi T_1^{\alpha} \wedge \frac{D\varphi}{2\pi} \wedge V_2^{\Sigma}. \tag{B.38}$$

Let us first focus on (B.36), the regularity constraint (B.12) implies that within each open interval (t_i, t_{i+1}) on ∂M_2 , the functions $W_{0,\alpha}^{\psi}$ and $W_{0,\alpha}^{\varphi}$ are locally related by

$$W_{0,\alpha}^{\psi}(t_i < t < t_{i+1}) = w_{\alpha,i} - \ell_i W_{0,\alpha}^{\varphi}(t), \qquad (B.39)$$

with $w_{\alpha,i}$ being a real constant, which can be expressed in terms of the expansion coefficients in $\mathcal{C}_2^{\Sigma,i} = b_{\alpha}^{\Sigma,i} \, \mathcal{C}_2^{\alpha}$ and $\mathcal{C}_2^i = b_{\alpha}^i \, \mathcal{C}_2^{\alpha}$ for a generic basis of two-cycles \mathcal{C}_2^{α} as

$$w_{\alpha,i} = -\frac{1}{\chi} b_{\alpha}^{\Sigma,i} + \ell_i \left[W_{0,\alpha}^{\varphi}(t_1) + \sum_{i=1}^{i-1} b_{\alpha}^{j} \right],$$
 (B.40)

for some reference value $W_{0,\alpha}^{\varphi}(t_1)$. Here and elsewhere in this paper sums from j=1 to 0 are understood to be zero.

Following an analogous approach we can also derive

$$s_{x,i}^{u} = S_{0,x}^{\psi,u}(t_1) + \ell_i S_{0,x}^{\varphi,u}(t_1) + \sum_{j=1}^{i-1} (\ell_i - \ell_j) c_x^{j,u},$$
(B.41)

$$U_0^{\alpha}(t_i) = U_0^{\alpha}(t_1) - \sum_{j=1}^{i-1} \ell_j a_j^{\alpha}, \qquad (B.42)$$

for some reference values $S_{0,x}^{\psi,u}(t_1)$, $S_{0,x}^{\varphi,u}(t_1)$, $U_0^{\alpha}(t_1)$. As an aside, the exterior derivatives (on ∂M_2) of regularity constraints like (B.12) restrict

$$dW_{0,\alpha}^{\psi}\big|_{t=t_i} = dW_{0,\alpha}^{\varphi}\big|_{t=t_i} = dS_{0,x}^{\psi,u}\big|_{t=t_i} = dS_{0,x}^{\varphi,u}\big|_{t=t_i} = dU_0^{\alpha}\big|_{t=t_i} = T_1^{\alpha}\big|_{t=t_i} = 0 \qquad (B.43)$$

for all i = 1, ..., 2k.

To evaluate a given intersection number, one can convert the integral over M_6 into an integral over the boundary ∂M_2 via Stokes' theorem, which can be further broken into a sum of integrals over all the open intervals (t_i, t_{i+1}) . Note that (B.43) guarantees that such integrals do not receive any singular contribution potentially induced by discontinuities of the function L at the positions of the KK monopoles. For instance, we can deduce that

$$\mathcal{K}_{\alpha}^{\beta} = \int_{\partial M_2} \left(W_{0,\alpha}^{\varphi} \, dU_0^{\beta} - \chi W_{0,\alpha}^{\psi} \, T_1^{\beta} \right) = -\sum_{i=1}^{2k} w_{\alpha,i} a_i^{\beta} \,. \tag{B.44}$$

At first sight the expression above may naïvely seem to depend explicitly on the reference value $W_{0,\alpha}^{\varphi}(t_1)$. However, if we shift this reference value by $\delta W_{0,\alpha}^{\varphi}(t_1)$, then

$$\delta \mathcal{K}_{\alpha}^{\beta} = -\delta W_{0,\alpha}^{\varphi}(t_1) \sum_{i=1}^{2k} \ell_i a_i^{\beta} = 0, \qquad (B.45)$$

where we made use of the second sum rule in (2.27) (with $\zeta = 0$) in the second equality. This shows that the intersection number $\mathcal{K}^{\beta}_{\alpha}$ is in fact independent of the choice of the reference value, $W^{\varphi}_{0,\alpha}(t_1)$. Similarly, we find that

$$\mathcal{K}_{xy} = -\mathcal{K}_{uv} \sum_{i=1}^{2k} \left[c_x^{i,u} S_{0,y}^{\psi,v}(t_i) + c_y^{i,v} S_{0,x}^{\psi,u}(t_i) - \ell_i c_x^{i,u} c_y^{i,v} \right], \tag{B.46}$$

where the value of $S_{0,x}^{\psi,u}(t_i)$ is constrained by the regularity of $\Lambda_{3,x}$ to be

$$S_{0,x}^{\psi,u}(t_i) = S_{0,x}^{\psi,u}(t_1) - \sum_{j=1}^{i-1} \ell_j c_x^{j,u}.$$
 (B.47)

The intersection number \mathcal{K}_{xy} is invariant under shifts in the reference values $S_{0,x}^{\psi,u}(t_1)$ and $S_{0,y}^{\psi,v}(t_1)$, i.e.

$$\delta \mathcal{K}_{xy} = -\mathcal{K}_{uv} \left[\delta S_{0,y}^{\psi,v}(t_1) \sum_{i=1}^{2k} c_x^{i,u} + \delta S_{0,x}^{\psi,u}(t_1) \sum_{i=1}^{2k} c_y^{i,v} \right] = 0$$
 (B.48)

by virtue of the first sum rule in (2.24). In addition, $\mathcal{K}_{uv}^{\alpha}$ is trivially given by

$$\mathcal{K}_{uv}^{\alpha} = \mathcal{K}_{uv} a_{\mathrm{C}}^{\alpha} \,, \tag{B.49}$$

which obviously does not depend on the reference value of any auxiliary function, while the remaining intersection number,

$$\mathcal{K}_{u\alpha x} = -\mathcal{K}_{uv} \sum_{i=1}^{2k} \left[b_{\alpha}^{i} S_{0,x}^{\psi,v}(t_{i}) - \frac{1}{\chi} b_{\alpha}^{\Sigma,i} c_{x}^{i,u} - \ell_{i} b_{\alpha}^{i} c_{x}^{i,u} \right], \tag{B.50}$$

again does not change under a shift in the reference value, $S_{0,x}^{\psi,v}(t_1)$, i.e.

$$\delta \mathcal{K}_{u\alpha x} = -\mathcal{K}_{uv} \, \delta S_{0,x}^{\psi,v}(t_1) \sum_{i=1}^{2k} b_{\alpha}^i = 0,$$
 (B.51)

which follows from the first sum rule in (2.21).

Next we proceed to study the following integrals which appear in the equations of motion (4.10) and (4.11),

$$\mathcal{J}_{I}^{\alpha\beta} \equiv \frac{1}{2} \int_{M_{6}} \left(\Omega_{2,I}^{\alpha} \wedge \Omega_{4}^{\beta} + \Omega_{2,I}^{\beta} \wedge \Omega_{4}^{\alpha} \right),
\mathcal{J}_{Iux}^{\alpha} \equiv \int_{M_{6}} \left(\Lambda_{1,xI} \wedge \lambda_{1,u} \wedge \Omega_{4}^{\alpha} - \lambda_{1,u} \wedge \Omega_{2,I}^{\alpha} \wedge \Lambda_{3,x} \right),$$
(B.52)

with $I = \psi, \varphi$. We can follow essentially the same procedure as before to obtain

$$\mathcal{J}_{\psi}^{\alpha\beta} = \frac{1}{\chi} \sum_{i=1}^{2k} \left[a_i^{\alpha} U_0^{\beta}(t_i) + a_i^{\beta} U_0^{\alpha}(t_i) - \ell_i a_i^{\alpha} a_i^{\beta} \right], \tag{B.53}$$

$$\mathcal{J}_{\varphi}^{\alpha\beta} = \frac{\chi}{2} \sum_{i=1}^{2k} \left(\left[\frac{1}{\chi} a_{i}^{\alpha} + \ell_{i} (Y_{0,\varphi}^{\alpha}(t_{i+1}) - Y_{0,\varphi}^{\alpha}(t_{i})) \right] \left[\frac{1}{\chi} a_{i}^{\beta} + \ell_{i} (Y_{0,\varphi}^{\beta}(t_{i+1}) - Y_{0,\varphi}^{\beta}(t_{i})) \right] \right. \\
\left. - \left[\frac{1}{\chi} a_{i}^{\alpha} + \ell_{i} (Y_{0,\varphi}^{\alpha}(t_{i+1}) - Y_{0,\varphi}^{\alpha}(t_{i})) \right] Y_{0,\psi}^{\beta}(t_{i}) \\
\left. - \left[\frac{1}{\chi} a_{i}^{\beta} + \ell_{i} (Y_{0,\varphi}^{\beta}(t_{i+1}) - Y_{0,\varphi}^{\beta}(t_{i})) \right] Y_{0,\psi}^{\alpha}(t_{i}) \right. \\
\left. + \ell_{i} Y_{0,\psi}^{\alpha}(t_{i}) (Y_{0,\varphi}^{\beta}(t_{i+1}) - Y_{0,\varphi}^{\beta}(t_{i})) - \ell_{i} \left[\frac{1}{\chi} a_{i}^{\alpha} + \ell_{i} (Y_{0,\varphi}^{\alpha}(t_{i+1}) - Y_{0,\varphi}^{\alpha}(t_{i})) \right] Y_{0,\varphi}^{\beta}(t_{i+1}) \\
\left. + \ell_{i} Y_{0,\psi}^{\beta}(t_{i}) (Y_{0,\varphi}^{\alpha}(t_{i+1}) - Y_{0,\varphi}^{\alpha}(t_{i})) - \ell_{i} \left[\frac{1}{\chi} a_{i}^{\beta} + \ell_{i} (Y_{0,\varphi}^{\beta}(t_{i+1}) - Y_{0,\varphi}^{\beta}(t_{i})) \right] Y_{0,\varphi}^{\alpha}(t_{i+1}) \\
\left. + \ell_{i}^{2} (Y_{0,\varphi}^{\alpha}(t_{i+1}) Y_{0,\varphi}^{\beta}(t_{i+1}) - Y_{0,\varphi}^{\alpha}(t_{i}) Y_{0,\varphi}^{\beta}(t_{i})) \right), \tag{B.54}$$

$$\mathcal{J}^{\alpha}_{\psi ux} = -\frac{1}{\chi} \mathcal{K}_{uv} \sum_{i=1}^{2k} s^{v}_{x,i} a^{\alpha}_{i}, \qquad (B.55)$$

$$\mathcal{J}^{\alpha}_{\varphi ux} = \mathcal{K}_{uv} \sum_{i=1}^{2k} s^{v}_{x,i} \left(Y^{\alpha}_{0,\varphi}(t_{i+1}) - Y^{\alpha}_{0,\varphi}(t_{i}) \right). \tag{B.56}$$

As usual, the regularity constraint that has to be imposed on $\Omega_{2,\varphi}^{\alpha}$ is given by

$$\left[dY_{0,\psi}^{\alpha} + L \, dY_{0,\varphi}^{\alpha} + T_1^{\alpha} \right]_{\partial M_2} = 0.$$
 (B.57)

Integrating both sides of the relation above over ∂M_2 yields a continuity condition on $Y_{0,\psi}^{\alpha}(t)$,

$$a_{\mathcal{C}}^{\alpha} = -\sum_{i=1}^{2k} \ell_i \left(Y_{0,\varphi}^{\alpha}(t_{i+1}) - Y_{0,\varphi}^{\alpha}(t_i) \right) = \sum_{i=1}^k Y_{0,\varphi}^{\alpha}(t_i) - \sum_{i=k+1}^{2k} Y_{0,\varphi}^{\alpha}(t_i),$$
 (B.58)

which constrains the otherwise arbitrary values of $Y_{0,\varphi}^{\alpha}(t_i)$ for $i=1,\ldots,2k$. For example, one convenient choice of these values is

$$Y_{0,\varphi}^{\alpha}(t_i) = \begin{cases} \frac{a_{\mathcal{C}}^{\alpha}}{2k} & \text{if } 1 \le i \le k, \\ -\frac{a_{\mathcal{C}}^{\alpha}}{2k} & \text{if } k+1 \le i \le 2k. \end{cases}$$
(B.59)

Regardless of the choice of $Y_{0,\varphi}^{\alpha}(t_i)$, the regularity constraint (B.57) further determines

$$Y_{0,\psi}^{\alpha}(t_i) = Y_{0,\psi}^{\alpha}(t_1) - (1 - \delta_{i,1}) \sum_{j=1}^{i-1} \left[\frac{1}{\chi} a_j^{\alpha} + \ell_j \left(Y_{0,\varphi}^{\alpha}(t_{j+1}) - Y_{0,\varphi}^{\alpha}(t_j) \right) \right]$$
(B.60)

for some reference value $Y_{0,\psi}^{\alpha}(t_1)$.

Unlike the intersection numbers, the integrals $\mathcal{J}_{\psi}^{\alpha\beta}$, $\mathcal{J}_{\varphi}^{\alpha\beta}$, $\mathcal{J}_{\psi ux}^{\alpha}$, $\mathcal{J}_{\varphi ux}^{\alpha}$ are sensitive to the reference values $U_0^{\alpha}(t_1)$, $Y_{0,\psi}^{\alpha}(t_1)$, $S_{0,x}^{\psi,u}(t_1)$, $S_{0,x}^{\varphi,u}(t_1)$. Nevertheless, if we impose the convention

$$N_{\alpha}N_{\beta}\mathcal{J}_{I}^{\alpha\beta} = 0 \tag{B.61}$$

for each $I = \psi, \varphi$, then we can uniquely fix

$$N_{\alpha}U_{0}^{\alpha}(t_{1}) = \frac{1}{2\chi N} \left[\sum_{i=1}^{2k} \ell_{i} \left(N_{\beta} a_{i}^{\beta} \right)^{2} + 2 \sum_{i=2}^{2k} N_{\beta} a_{i}^{\beta} \sum_{j=1}^{i-1} \ell_{j} N_{\gamma} a_{j}^{\gamma} \right],$$

$$N_{\alpha}Y_{0,\psi}^{\alpha}(t_{1}) = \frac{1}{2\chi^{2}N} \left[\sum_{i=1}^{2k} \left[\left(N_{\beta} a_{i}^{\beta} \right)^{2} - 2\chi N_{\beta}U_{0}^{\beta}(t_{i+1}) N_{\gamma} \left(Y_{0,\varphi}^{\gamma}(t_{i+1}) - Y_{0,\varphi}^{\gamma}(t_{i}) \right) \right] + 2 \sum_{i=2}^{2k} N_{\beta} a_{i}^{\beta} \sum_{j=1}^{i-1} \left[N_{\gamma} a_{j}^{\gamma} + \chi \ell_{j} N_{\gamma} \left(Y_{0,\varphi}^{\gamma}(t_{j+1}) - Y_{0,\varphi}^{\gamma}(t_{j}) \right) \right] \right].$$
(B.63)

Similarly, if we impose the convention

$$N_{\alpha} \mathcal{J}_{Iux}^{\alpha} = 0 \tag{B.64}$$

for each $I = \psi, \varphi$, each $u = 1, \dots, 2g$, and each $x = 1, \dots, 4g(k-1)$, then we can uniquely fix

$$S_{0,x}^{\psi,u}(t_1) = -\frac{1}{\chi N} \sum_{i=2}^{2k} N_{\alpha} a_i^{\alpha} \sum_{j=1}^{i-1} (\ell_i - \ell_j) c_x^{j,u},$$
(B.65)

$$S_{0,x}^{\varphi,u}(t_1) = \frac{1}{N} \sum_{i=2}^{2k} N_{\alpha} \left[Y_{0,\varphi}^{\alpha}(t_{i+1}) - Y_{0,\varphi}^{\alpha}(t_i) \right] \sum_{i=1}^{i-1} (\ell_i - \ell_j) c_x^{j,u}.$$
 (B.66)

C The full inflow anomaly polynomial

With some algebra, it can be shown that in a given (co)homology basis parameterized by the expansion coefficients a_i^{α} , $b_{\alpha}^{\Sigma,i}$, b_{α}^i , $c_x^{i,u}$ as defined in (B.32), the full large-N inflow anomaly polynomial for arbitrary flux configurations is

$$\begin{split} I_{6}^{\text{inflow,large-}N} &= \\ \frac{2}{3\chi^{2}} \sum_{i=1}^{2k} \left[\ell_{i}^{2} N_{i}^{3} + 3N_{i} \, \tilde{U}_{0,i} \, \tilde{U}_{0,i+1} \right] \left(\frac{F_{2}^{\psi}}{2\pi} \right)^{3} \\ &- \frac{3}{2\chi} \sum_{i=1}^{2k} \left[\frac{2}{3\chi} \, \ell_{i} N_{i}^{3} + N_{i} \, \tilde{U}_{0,i} \, \tilde{Y}_{0,\psi,i+1} + N_{i} \, \tilde{Y}_{0,\psi,i} \, \tilde{U}_{0,i+1} \right. \\ & \qquad \qquad \left. + \left(\tilde{U}_{0,i+1}^{2} + \ell_{i} N_{i} \tilde{U}_{0,i} \right) \left(\tilde{Y}_{0,\varphi,i+1} - \tilde{Y}_{0,\varphi,i} \right) \right] \left(\frac{F_{2}^{\psi}}{2\pi} \right)^{2} \frac{F_{2}^{\varphi}}{2\pi} \\ &+ \sum_{i=1}^{2k} \left[\frac{1}{3\chi^{2}} \, N_{i}^{3} + N_{i} \, \tilde{Y}_{0,\psi,i} \, \tilde{Y}_{0,\psi,i+1} - \frac{1}{\chi} \, \ell_{i} N_{i}^{2} \, \tilde{Y}_{0,\varphi,i} - \frac{2}{\chi} \, N_{i} \tilde{U}_{0,i+1} \, \tilde{Y}_{0,\varphi,i+1} \right. \\ & \qquad \qquad \left. + \left(\frac{1}{\chi} \, N_{i} \, \tilde{U}_{0,i+1} + \tilde{U}_{0,i} \, \tilde{Y}_{0,\psi,i} + \tilde{U}_{0,i+1} \, \tilde{Y}_{0,\psi,i+1} \right) \left(\tilde{Y}_{0,\varphi,i+1} - \tilde{Y}_{0,\varphi,i} \right) \right] \frac{F_{2}^{\psi}}{2\pi} \left(\frac{F_{2}^{\varphi}}{2\pi} \right)^{2} \\ &+ \frac{1}{2} \sum_{i=1}^{2k} \left[\frac{1}{\chi} \, N_{i}^{2} \, \tilde{Y}_{0,\varphi,i+1} + 2N_{i} \, \tilde{Y}_{0,\psi,i+1} \, \tilde{Y}_{0,\varphi,i+1} - \left(N_{i} \, \tilde{Y}_{0,\psi,i} + \chi \, \tilde{Y}_{0,\psi,i} \, \tilde{Y}_{0,\psi,i+1} \right) \right] \\ &+ \frac{1}{2} \sum_{i=1}^{2k} \left[\frac{1}{\chi} \, N_{i}^{2} \, \tilde{Y}_{0,\varphi,i+1} + 2N_{i} \, \tilde{Y}_{0,\psi,i+1} \, \tilde{Y}_{0,\varphi,i+1} - \left(N_{i} \, \tilde{Y}_{0,\psi,i} + \chi \, \tilde{Y}_{0,\psi,i} \, \tilde{Y}_{0,\psi,i+1} \right) \right] \\ &+ \frac{1}{2} \sum_{i=1}^{2k} \left[\frac{1}{\chi} \, N_{i}^{2} \, \tilde{Y}_{0,\varphi,i+1} + 2N_{i} \, \tilde{Y}_{0,\psi,i+1} \, \tilde{Y}_{0,\varphi,i+1} - \left(N_{i} \, \tilde{Y}_{0,\psi,i} + \chi \, \tilde{Y}_{0,\psi,i} \, \tilde{Y}_{0,\psi,i+1} \right) \right] \\ &+ \frac{1}{2} \sum_{i=1}^{2k} \left[\frac{1}{\chi} \, N_{i}^{2} \, \tilde{Y}_{0,\varphi,i+1} + 2N_{i} \, \tilde{Y}_{0,\psi,i+1} \, \tilde{Y}_{0,\varphi,i+1} - \left(N_{i} \, \tilde{Y}_{0,\psi,i} + \chi \, \tilde{Y}_{0,\psi,i} \, \tilde{Y}_{0,\psi,i+1} \right) \right] \\ &+ \frac{1}{2} \sum_{i=1}^{2k} \left[\frac{1}{\chi} \, N_{i}^{2} \, \tilde{Y}_{0,\varphi,i+1} + 2N_{i} \, \tilde{Y}_{0,\psi,i+1} \, \tilde{Y}_{0,\psi,i+1} - \left(N_{i} \, \tilde{Y}_{0,\psi,i+1} + \chi \, \tilde{Y}_{0,\psi,i+1} \right) \right] \\ &+ \frac{1}{2} \sum_{i=1}^{2k} \left[\frac{1}{\chi} \, N_{i}^{2} \, \tilde{Y}_{0,\psi,i+1} + \frac{1}{\chi} \, N_{i}^{2} \, \tilde{Y}_{0,\psi,i+1} + \frac{1}{\chi} \, \tilde{Y}_{0,\psi,i+1} \right] \right] \\ &+ \frac{1}{\chi} \, \frac{1}{\chi} \, N_{i}^{2} \, \tilde{Y}_{0,\psi,i+1} + \frac{1}{\chi} \, N_{i}^{2} \, \tilde{Y}_{0,\psi,i+1} + \frac{1}{\chi} \, \tilde{Y}_{0,\psi,i+1} + \frac{1}$$

$$\begin{split} & - \tilde{U}_{0,i} \big(\check{Y}_{0,\varphi,i} + \check{Y}_{0,\varphi,i+1} \big) + \frac{\chi}{3} \, \ell_i^2 \big(\check{Y}_{0,\varphi,i+1} - \check{Y}_{0,\varphi,i} \big)^2 \bigg) \big(\check{Y}_{0,\varphi,i+1} - \check{Y}_{0,\varphi,i} \big) \bigg] \bigg(\frac{F_2^{\wp}}{2\pi} \bigg)^3 \\ & - \frac{3N}{N} \sum_{i=1}^{2k} \bigg[\frac{1}{2} \, w_{\alpha,i} \ell_i N_i^2 + w_{\alpha,i} N_i \, \check{U}_{0,i+1} \bigg] \bigg(\frac{F_2^{\wp}}{2\pi} \bigg)^2 \, \frac{F_2^{\wp}}{2\pi} \\ & + \frac{2N}{N} \sum_{i=1}^{2k} \bigg[\frac{1}{2} \, w_{\alpha,i} N_i^2 + \chi w_{\alpha,i} N_i \, \check{Y}_{0,\psi,i+1} + \chi w_{\alpha,i} \check{U}_{0,i} \big(\check{Y}_{0,\varphi,i+1} - \check{Y}_{0,\varphi,i} \big) \bigg] \frac{F_2^{\wp}}{2\pi} \, \frac{F_2^{\wp}}{2\pi} \, \frac{F_2^{\wp}}{2\pi} \\ & - \chi N \sum_{i=1}^{2k} \bigg[w_{\alpha,i} \check{Y}_{0,\psi,i+1} \check{Y}_{0,\varphi,i+1} - w_{\alpha,i} \check{Y}_{0,\psi,i} \check{Y}_{0,\varphi,i} + \frac{1}{2} \, w_{\alpha,i} \ell_i \big(\check{Y}_{0,\varphi,i+1}^2 - \check{Y}_{0,\varphi,i} \big) \bigg] \bigg(\frac{F_2^{\wp}}{2\pi} \bigg)^2 \, \frac{F_2^{\wp}}{2\pi} \\ & - \chi N \sum_{i=1}^{2k} \bigg[w_{\alpha,i} \check{w}_{\beta,i} N_i - (w_{\alpha,i+1} - w_{\alpha,i}) (w_{\beta,i+1} - w_{\beta,i}) (\ell_{i+1} - \ell_i) \, \check{V}_{0,\psi,i+1} - \check{Y}_{0,\varphi,i} \bigg] \bigg(\frac{F_2^{\wp}}{2\pi} \bigg)^2 \, \frac{F_2^{\wp}}{2\pi} \\ & + N^2 \sum_{i=1}^{2k} \bigg[(w_{\alpha,i+1} - w_{\alpha,i}) (w_{\beta,i+1} - w_{\beta,i}) (\ell_{i+1} - \ell_i) \, \check{Y}_{0,\psi,i+1} \bigg] \\ & - w_{\alpha,i} w_{\beta,i} \big(\check{Y}_{0,\varphi,i+1} - \check{Y}_{0,\varphi,i} \big) \bigg] \frac{F_2^{\wp}}{2\pi} \, \frac{F_2^{\wp}}{2\pi} \, \frac{F_2^{\wp}}{2\pi} \\ & + \frac{\chi N^3}{6} \sum_{i=1}^{2k} \bigg[(w_{\alpha,i+1} - w_{\alpha,i}) (w_{\beta,i+1} - w_{\beta,i}) (w_{\gamma,i+1} - w_{\gamma,i}) (\ell_i + \ell_{i+1}) \bigg] \\ & - 3 (w_{\alpha,i+1} - w_{\alpha,i}) (w_{\beta,i+1} - w_{\beta,i}) (w_{\gamma,i+1} - w_{\gamma,i}) (\ell_i + \ell_{i+1}) \\ & + 3 (w_{\alpha,i+1} - w_{\alpha,i}) (w_{\beta,i+1} w_{\gamma,i+1} - w_{\beta,i} w_{\gamma,i}) (\ell_{i+1} - \ell_i) \bigg] \frac{F_2^{\wp}}{2\pi} \, \frac{F_2^{\wp}}{2\pi} \, \frac{F_2^{\wp}}{2\pi} \\ & + \frac{N^2}{2\chi} \, \mathcal{K}_{uv} \sum_{i=1}^{2k} \bigg[s_{u,i}^{u} s_{u}^{v} s_{u}^{v} \tilde{U}_{0,i+1} S_{0,y,i}^{v,v} + c_{u}^{i} \tilde{U}_{0,i+1} S_{0,x,i}^{\psi,v} - s_{u}^{i} \tilde{V}_{0,\varphi,i} \bigg] \int_{0,x_i}^{F_2^{\wp}} \frac{F_2^{\wp}}{2\pi} \, \frac{F_2^{\wp}}{2\pi} \, \frac{F_2^{\wp}}{2\pi} \bigg)^2 \\ & + \frac{N^2}{2\chi} \, \mathcal{K}_{uv} \sum_{i=1}^{2k} \bigg[s_{u}^{i} c_{u}^{i} \tilde{V}_{0,i} + c_{u}^{i} \tilde{U}_{0,i+1} S_{0,y,i}^{\psi,v} + c_{u}^{i} \tilde{U}_{0,i+1} S_{0,x,i}^{\psi,v} - s_{u}^{i} \tilde{Y}_{0,\varphi,i} S_{0,y,i}^{\psi,v} \bigg) \\ & - s_{u}^{i} \ell_{u} \tilde{V}_{0,\psi,i} + c_{u}^{i} \tilde{U}_{0,i} + c_{u}^{i} \tilde{U}_{0,i} + c_{u}^{i} \tilde{V}_{0,\psi,i} \bigg) \int_{0,x_i}^{F_2^{\wp}} \frac{F_2^$$

$$-\ell_{i} \left(\tilde{Y}_{0,\varphi,i+1} S_{0,x,i+1}^{\varphi,u} S_{0,y,i+1}^{\varphi,v} - \tilde{Y}_{0,\varphi,i} S_{0,x,i}^{\varphi,u} S_{0,y,i}^{\varphi,v} \right) \left[\frac{f_{1}^{x}}{2\pi} \frac{f_{1}^{y}}{2\pi} \left(\frac{F_{2}^{\varphi}}{2\pi} \right)^{2} \right]$$

$$-\frac{N^{3}}{2\chi} \mathcal{K}_{uv} \sum_{i=1}^{2k} \left[\ell_{i} b_{\alpha}^{\Sigma,i+1} c_{x}^{i,u} c_{y}^{i,v} - b_{\alpha}^{\Sigma,i+1} c_{x}^{i,u} S_{0,y,i}^{\psi,v} - b_{\alpha}^{\Sigma,i+1} c_{y}^{i,v} S_{0,x,i}^{\psi,u} \right]$$

$$+ \chi b_{\alpha}^{i} S_{0,x,i}^{\psi,u} S_{0,y,i}^{\psi,v} + 2\chi \ell_{i}^{2} b_{\alpha}^{i} S_{0,x,i}^{\varphi,u} S_{0,y,i}^{\varphi,v} \right] \frac{f_{1}^{x}}{2\pi} \frac{f_{1}^{y}}{2\pi} \frac{F_{2}^{\psi}}{2\pi} \frac{F_{2}^{\alpha}}{2\pi}$$

$$-\frac{N^{3}}{2} \mathcal{K}_{uv} \sum_{i=1}^{2k} \left[w_{\alpha,i} c_{x}^{i,u} S_{0,y,i}^{\varphi,v} + w_{\alpha,i} c_{y}^{i,v} S_{0,x,i}^{\varphi,u} + w_{\alpha,i} c_{x}^{i,u} c_{y}^{i,v} \right] \frac{f_{1}^{x}}{2\pi} \frac{f_{1}^{y}}{2\pi} \frac{F_{2}^{\psi}}{2\pi} \frac{F_{2}^{\alpha}}{2\pi} \frac{F_{2}^{\alpha}}{2\pi} , \qquad (C.1)$$

where the shorthand notation,

$$N_i \equiv N_{\alpha} a_i^{\alpha}, \quad \tilde{U}_{0,i} \equiv N_{\alpha} U_0^{\alpha}(t_i), \quad \tilde{Y}_{0,\psi,i} \equiv N_{\alpha} Y_{0,\psi}^{\alpha}(t_i), \quad \tilde{Y}_{0,\varphi,i} \equiv N_{\alpha} Y_{0,\varphi}^{\alpha}(t_i), \quad (C.2)$$

is employed. We introduced the constants ℓ_i in (2.9), and the intersection matrix \mathcal{K}_{uv} is explicitly written in (B.35), whereas the definitions of the other auxiliary functions are recorded in appendix B.

For concreteness, let us we focus on the natural basis of homology classes that we described towards the end of section 2. It is straightforward to deduce that the choice of basis four-cycles,

$$C_{4,1 \le \alpha \le k-1} = C_{4,1 \le i \le k-1}, \quad C_{4,\alpha=k} = C_{4,C}, \quad C_{4,k+1 \le \alpha \le 2k-1} = C_{4,k+1 \le i \le 2k-1}, \quad (C.3)$$

can be equivalently written in terms of the coefficients,

$$a_{\mathcal{C}}^{\alpha} = \delta_{k}^{\alpha} , \qquad a_{i \neq k, 2k}^{\alpha} = \delta_{i}^{\alpha} ,$$

$$a_{i=k}^{1 \leq \alpha \leq k-1} = -\frac{\alpha}{k} , \quad a_{i=k}^{\alpha = k} = \frac{\chi}{2} , \quad a_{i=k}^{k+1 \leq \alpha \leq 2k-1} = -\frac{2k - \alpha}{k} ,$$

$$a_{i=2k}^{1 \leq \alpha \leq k-1} = -\frac{k - \alpha}{k} , \quad a_{i=2k}^{\alpha = k} = \frac{\chi}{2} , \quad a_{i=2k}^{k+1 \leq \alpha \leq 2k-1} = -\frac{\alpha - k}{k} .$$
(C.4)

Note that we used the two four-cycle sum rules (2.27) to determine the coefficients in the latter two lines. The two-cycles that are Poincaré-dual to the basis four-cycles satisfy

$$\mathcal{K}^{\alpha}_{\beta} = \int_{M_6} \Omega_4^{\alpha} \wedge \omega_{2,\beta} = -\sum_{i=1}^{2k} a_i^{\alpha} w_{\beta,i} = \delta^{\alpha}_{\beta}. \tag{C.5}$$

Applying the condition above to the "natural" four-cycles yields the constraints,

$$-\delta_{\beta}^{1 \leq \alpha \leq k-1} = w_{\beta,\alpha} - \frac{\alpha}{k} w_{\beta,k} - \frac{k-\alpha}{k} w_{\beta,2k},$$

$$-\delta_{\beta}^{\alpha=k} = \frac{\chi}{2} (w_{\beta,k} + w_{\beta,2k}),$$

$$-\delta_{\beta}^{k+1 \leq \alpha \leq 2k-1} = w_{\beta,\alpha} - \frac{2k-\alpha}{k} w_{\beta,k} - \frac{\alpha-k}{k} w_{\beta,2k},$$
(C.6)

which have the solution,

$$w_{\beta,i\neq k,2k} = -\delta_{\beta}^{i} - \frac{\delta_{\beta}^{k}}{\chi}, \qquad w_{\beta,k} = w_{\beta,2k} = -\frac{\delta_{\beta}^{k}}{\chi}. \tag{C.7}$$

The $b_{\beta}^{\Sigma,i}$ and b_{β}^{i} coefficients are related to the constants $w_{\beta,i}$ through

$$b_{\beta}^{\Sigma,i} = \chi \frac{\ell_{i-1} w_{\beta,i} - \ell_{i} w_{\beta,i-1}}{\ell_{i} - \ell_{i-1}}, \quad b_{\beta}^{i} = \frac{w_{\beta,i+1} - w_{\beta,i}}{\ell_{i+1} - \ell_{i}} - \frac{w_{\beta,i} - w_{\beta,i-1}}{\ell_{i} - \ell_{i-1}}, \quad (C.8)$$

as can be derived utilizing (B.39) and the single-valuedness of $W_{0,\beta}^{\varphi}(t)$. We also defined the "natural" three-cycles in section 2 to be $C_3^{x=(\beta\neq k,v)} = C_2^{\alpha\neq k} \times C_1^{\Sigma,v}$, with the index xparameterized here as a tuple $(\beta \neq k, v)$. It follows from (B.32) that

$$c_{x=(\beta\neq k,v)}^{i,u} = b_{\beta}^i \, \delta_v^u \,. \tag{C.9}$$

The collection of expansion coefficients presented above fully specifies our basis of homology classes that is relevant to the anomaly polynomial.

As explained in detail in appendix B, the reference values of the auxiliary functions used in (C.1) are uniquely fixed under the convention (4.13), apart from $Y_{0,\varphi}^{\alpha}$ for which we choose to parameterize as in (B.59). To summarize, we have

$$\tilde{U}_{0,i} = \frac{1}{2\chi N} \left(\sum_{j=1}^{2k} \ell_j N_j^2 + 2 \sum_{j=2}^{2k} N_j \sum_{m=1}^{j-1} \ell_m N_m \right) - \sum_{j=1}^{i-1} \ell_j N_j , \qquad (C.10)$$

$$\tilde{Y}_{0,\psi,i} = \frac{1}{2\chi^2 N} \left(\sum_{j=1}^{2k} N_j^2 - \frac{2\chi N}{k} \sum_{j=1}^k \ell_j N_j + 2 \sum_{j=2}^{2k} N_j \sum_{m=1}^{j-1} \left[N_m + \frac{\chi N}{k} \ell_m (\delta_{m,2k} - \delta_{m,k}) \right] \right) - \sum_{j=1}^{i-1} \left[\frac{1}{\chi} N_j + \frac{N}{k} \ell_j (\delta_{j,2k} - \delta_{j,k}) \right],$$
(C.11)

$$\sum_{j=1}^{\infty} \left[\chi - \frac{1}{k} - \frac{1}{k} \right]^{N}$$

$$\left\{ \frac{N}{2k} - \text{if } 1 \le i \le k, \right\}$$
(C.13)

$$\tilde{Y}_{0,\varphi,i} = \begin{cases}
\frac{N}{2k} & \text{if } 1 \le i \le k, \\
-\frac{N}{2k} & \text{if } k+1 \le i \le 2k,
\end{cases}$$
(C.12)

$$S_{0,x,i}^{\psi,u} = -\frac{1}{\chi N} \sum_{j=2}^{2k} N_j \sum_{m=1}^{j-1} (\ell_j - \ell_m) c_x^{m,u} - \sum_{j=1}^{i-1} \ell_j c_x^{j,u}, \qquad (C.13)$$

$$S_{0,x,i}^{\varphi,u} = -\frac{1}{k} \sum_{j=1}^{2k-1} \left(\frac{k}{2} + \ell_j\right) c_x^{j,u} - \frac{1}{k} \sum_{j=1}^{k-1} \left(\frac{k}{2} - \ell_j\right) c_x^{j,u} + \sum_{j=1}^{i-1} c_x^{j,u}, \tag{C.14}$$

$$s_{x,i}^{u} = -\frac{1}{\chi N} \sum_{j=2}^{2k} N_{j} \sum_{m=1}^{j-1} (\ell_{j} - \ell_{m}) c_{x}^{m,u} - \frac{1}{k} \ell_{i} \sum_{j=1}^{2k-1} \left(\frac{k}{2} + \ell_{j}\right) c_{x}^{j,u} - \frac{1}{k} \ell_{i} \sum_{j=1}^{k-1} \left(\frac{k}{2} - \ell_{j}\right) c_{x}^{j,u} + \sum_{i=1}^{i-1} (\ell_{i} - \ell_{j}) c_{x}^{j,u},$$
(C.15)

where the sums over j from 1 to i-1 are to be understood to not contribute if i=1. Lastly, it is worth emphasizing again that all the auxiliary functions listed above are (rational) functions of only χ , N, N_{α} , a_i^{α} , $b_{\alpha}^{\Sigma,i}$, b_{α}^i , $c_x^{i,u}$, so there is no extra data required to compute

¹⁵Recall that the quantity $n_i = \ell_i - \ell_{i-1}$ is the charge of the *i*-th Kaluza-Klein monopole along ∂M_2 . For a fully resolved setup, we have $n_i = +1$ for $1 \le i \le k$ and $n_i = -1$ for $k+1 \le i \le 2k$.

D Basis-(in)dependence of the anomaly polynomial

The explicit expression of $I_6^{\mathrm{inflow,large-}N}$ depends on the specific choices of

- 1. the basis of cohomology classes, i.e. Ω_4^{α} , $\Lambda_{3,x}$, $\omega_{2,\alpha}$, $\lambda_{1,u}$;
- 2. the non-closed forms associated with isometries, i.e. $\Omega_{2,I}^{\alpha}$, $\Lambda_{1,xI}$, $\omega_{0,\alpha I}$;

used in constructing E_4 , both of which can be further shifted with exact forms. We argue that the invariance of $I_6^{\text{inflow},\text{large-}N}$ can be restored with appropriate redefinitions of either or both the flux parameters and the external field strengths. ¹⁶

Let us first consider a generic change of basis of cohomology classes (related to the basis of homology classes via orthonormality), plus a shift in the choice of representative within each cohomology class,

$$\left(\Omega_4^{\alpha}\right)' = (\mathcal{R}_4)^{\alpha}_{\beta} \Omega_4^{\beta} + d\Omega_3^{\alpha}, \quad \Lambda'_{3,x} = (\mathcal{R}_3)^y_x \Lambda_{3,y} + d\Lambda_{2,x},
\omega'_{2,\alpha} = (\mathcal{R}_2)^{\beta}_{\alpha} \omega_{2,\beta} + d\omega_{1,\alpha}, \quad \lambda'_{1,u} = (\mathcal{R}_1)^v_u \lambda_{1,v} + d\lambda_{0,u},$$
(D.1)

for some constant matrices $\mathcal{R}_p \in \mathrm{GL}(b_p(M_6), \mathbb{R})$ and globally defined, gauge-invariant forms Ω_3^{α} , $\Lambda_{2,x}$, $\omega_{1,\alpha}$, $\lambda_{0,u}$. It results in the shifts below by solving the closure constraints on E_4' ,

We can check that under the redefinitions, ¹⁷

$$N_{\alpha}' = (\mathcal{R}_{4}^{-1})_{\alpha}^{\beta} N_{\beta} , \quad (f_{1}^{x})' = (\mathcal{R}_{3}^{-1})_{y}^{x} f_{1}^{y} , \quad (F_{2}^{\alpha})' = (\mathcal{R}_{2}^{-1})_{\beta}^{\alpha} F_{2}^{\beta} , \quad (H_{3}^{u})' = (\mathcal{R}_{1}^{-1})_{v}^{u} H_{3}^{v} ,$$
(D.3)

the four-form flux E_4 merely acquires an additional globally defined exact piece, ¹⁸

$$E_4' = E_4 + d \left[N_\alpha' \left(\Omega_3^\alpha \right)^g + N \frac{(f_1^x)'}{2\pi} (\Lambda_{2,x})^g + N \frac{(F_2^\alpha)'}{2\pi} (\omega_{1,\alpha})^g + N \frac{(H_3^u)'}{2\pi} (\lambda_{0,u})^g \right], \quad (D.4)$$

implying that $I_6^{\text{inflow},\text{large-}N}$ is invariant. It guarantees as well that the equations of motion (4.10) and (4.11) are automatically preserved.

Alternatively, we may consider shifts of the non-closed forms in E_4 associated with isometries of M_6 , by linear combinations of harmonic forms plus exact forms,

$$(\Omega_{2,I}^{\alpha})' = \Omega_{2,I}^{\alpha} + (\mathcal{T}_{2,I})^{\alpha\beta} \,\omega_{2,\beta} + d\Omega_{1,I}^{\alpha} \,, \quad \Lambda'_{1,xI} = \Lambda_{1,xI} + (\mathcal{T}_{1,I})_x^u \,\lambda_{1,u} + d\Lambda_{0,xI} \,, \omega'_{0,\alpha I} = \omega_{0,\alpha I} + (\mathcal{T}_{0,I})_{\alpha} \,,$$
(D.5)

¹⁶We also expect the $\mathcal{O}(N)$ contribution to I_6^{inflow} from $-E_4X_8 \subset \mathcal{I}_{12}$ to be invariant, but we refrain from discussing it in detail given that we do not explicitly construct X_8 in this paper.

¹⁷Strictly speaking, if we were to preserve the integral quantization condition (3.3), then we should limit $\mathcal{R}_4 \in GL(2k-1,\mathbb{Z})$.

¹⁸Here we implicitly used the identity $d(\omega_p)^g + A^I (\mathcal{L}_I \omega_p)^g = (d\omega_p)^g + F^I (\iota_I)^g$ with $\mathcal{L}_I \omega_p = 0$ for some gauge-invariant p-form ω_p [26].

for some constant real matrices $\mathcal{T}_{2,I}$, $\mathcal{T}_{1,I}$, constant real vectors $\mathcal{T}_{0,I}$, and some globally defined, gauge-invariant forms $\Omega_{1,I}^{\alpha}$ and $\Lambda_{0,xI}$. The matrices $\mathcal{T}_{2,I}$ and $\mathcal{T}_{1,I}$ are not totally unconstrained as we will soon see. Demanding closure of E'_4 requires that

$$\left(\Omega_{0,IJ}^{\alpha}\right)' = \Omega_{0,IJ}^{\alpha} + (\mathcal{T}_2)_{(I}^{\alpha\beta}\omega_{0,\beta|J)} + 2\pi\iota_{(I}\Omega_{1,J)}^{\alpha} + (\mathcal{T}_{0,IJ})^{\alpha}, \tag{D.6}$$

for some constant real vectors $\mathcal{T}_{0,IJ}$. It is again straightforward to check that under the redefinitions, ¹⁹

$$\frac{(F_2^{\alpha})'}{2\pi} = \frac{F_2^{\alpha}}{2\pi} - \frac{N_{\beta}}{N} (\mathcal{T}_{2,I})^{\alpha\beta} \frac{F_2^I}{2\pi}, \quad \frac{(H_3^u)'}{2\pi} = \frac{H_3^u}{2\pi} - (\mathcal{T}_{1,I})_x^u \frac{f_1^x}{2\pi} \frac{F_2^I}{2\pi},
\frac{\gamma_4'}{2\pi} = \frac{\gamma_4}{2\pi} - \frac{N_{\alpha}}{N} (\mathcal{T}_{0,IJ})^{\alpha} \frac{F_2^I}{2\pi} \frac{F_2^J}{2\pi} - (\mathcal{T}_{0,I})_{\alpha} \frac{F_2^{\alpha}}{2\pi} \frac{F_2^I}{2\pi},$$
(D.7)

the four-form flux E_4 is shifted with an exact piece,

$$E_4' = E_4 + d \left[N_\alpha \frac{F_2^I}{2\pi} (\Omega_{1,I}^\alpha)^g + N \frac{f_1^x}{2\pi} \frac{F_2^I}{2\pi} (\Lambda_{0,xI})^g \right].$$
 (D.8)

Note that, however, this is not sufficient to keep $I_6^{\text{inflow},\text{large-}N}$ invariant.²⁰ We also need to enforce the equivalence between the shifted equations of motion,

$$N_{\alpha}N_{\beta} \left\{ \mathcal{J}_{I}^{\alpha\beta} + \mathcal{K}_{\gamma}^{\alpha} \left[(\mathcal{T}_{2,I})^{\beta\gamma} - (\mathcal{T}_{2,I})^{\gamma\beta} \right] \right\} \frac{F_{2}^{I}}{2\pi} + NN_{\beta} \, \mathcal{K}_{\alpha}^{\beta} \, \frac{F_{2}^{\alpha}}{2\pi} - \frac{1}{2} \, N^{2} \, \mathcal{K}_{xy} \, \frac{f_{1}^{x}}{2\pi} \frac{f_{1}^{y}}{2\pi} = 0 \,, \quad (D.9)$$

$$N_{\alpha} \left\{ \mathcal{J}_{Iux}^{\alpha} - \frac{3}{2} \mathcal{K}_{uv}^{\alpha} (\mathcal{T}_{1,I})_{x}^{v} - \mathcal{K}_{u\beta x} \left[(\mathcal{T}_{2,I})^{\alpha\beta} - (\mathcal{T}_{2,I})^{\beta\alpha} \right] \right\} \frac{f_{1}^{x}}{2\pi} \frac{F_{2}^{I}}{2\pi} + \frac{1}{2} N_{\alpha} \mathcal{K}_{uv}^{\alpha} \frac{H_{3}^{v}}{2\pi} - N \mathcal{K}_{u\alpha x} \frac{f_{1}^{x}}{2\pi} \frac{F_{2}^{\alpha}}{2\pi} = 0 , \quad (D.10)$$

and (4.10) and (4.11) respectively, which amounts to imposing the following constraints on $\mathcal{T}_{2,I}$ and $\mathcal{T}_{1,I}$,

$$N_{\alpha}N_{\beta}\mathcal{K}_{\gamma}^{\alpha}\left[\left(\mathcal{T}_{2,I}\right)^{\beta\gamma} - \left(\mathcal{T}_{2,I}\right)^{\gamma\beta}\right] = 0, \quad \frac{3}{2}N_{\alpha}\mathcal{K}_{uv}^{\alpha}\left(\mathcal{T}_{1,I}\right)_{x}^{v} + N_{\alpha}\mathcal{K}_{u\beta x}\left[\left(\mathcal{T}_{2,I}\right)^{\alpha\beta} - \left(\mathcal{T}_{2,I}\right)^{\beta\alpha}\right] = 0,$$
(D.11)

so as to maintain the invariance of $I_6^{\text{inflow,large-}N}$. We verified as a consistency check that after the aforementioned redefinitions of the field strengths, the expression we obtain using (C.1) for $I_6^{\text{inflow,large-}N}$ at k=2 correctly reproduces the independently derived result of [24].

¹⁹If we were to impose that these redefined field strengths are also appropriately quantized, then it would require $N_{\beta}(\mathcal{T}_{2,I})^{\alpha\beta}/N$, $(\mathcal{T}_{1,I})^{u}_{x}$, $N_{\alpha}(\mathcal{T}_{0,IJ})^{\alpha}/N$, $(\mathcal{T}_{0,I})_{\alpha} \in \mathbb{Z}$.

²⁰The paradox can be resolved by understanding the fact that f_1^x , F_2^I , F_2^α , H_3^u are not mutually independent, so (4.8) can be interpreted as an expansion in an overcomplete basis of external field strengths.

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