Entanglement entropy of a superflow

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Abstract

We consider the theory of N free Dirac fermions with a uniformly winding mass, me^{iqx}, in two spacetime dimensions. This theory (which describes for instance a superconducting current in an N-channel wire) has been proposed to have a higher-spin gravity with scalar matter as the large-N dual. To order m^2, however, thermodynamic quantities in it can be computed using standard general relativity instead. Here, we consider the question if the same is true for the entanglement entropy (EE). By comparing results obtained on two sides of the duality, we find that general relativity indeed accounts correctly for the EE of an interval to order m^2 (and all orders in q).

1 Introduction

In semiclassical gravity, the Gibbons-Hawking (GH) formula [1] interprets the Euclidean effective action of a spacetime as the free energy (of gravity plus matter) and thus provides a method of computation of gravitational entropy. If the spacetime is asymptotically anti-de Sitter (AdS), the same effective action also determines the entropy of the dual conformal field theory (CFT) in the context of AdS/CFT duality [2]. Furthermore, if the CFT is deformed by a relevant operator, the correspondence [3, 4] between operators in the CFT and fields in AdS tells us the precise way in which the entropy of the CFT can be computed from the action of a deformed spacetime.

Replacing the thermal density matrix in the von Neumann entropy formula with one obtained by integrating out a subset of degrees of freedom defines entanglement entropy (EE). It has been shown [5] that the EE of a spherical region in a CFT vacuum is the same as the entropy of a thermal state in an auxiliary hyperbolic space and that application of the AdS/CFT correspondence to this thermal state, in the case when the gravitational dual is standard general relativity, reproduces the result of the minimal-surface formula of Ryu and Takayanagi [6]. Furthermore, it has been argued [7], in a similar context, that the EE can be related to the area of a minimal surface directly, without going through a thermal state as an intermediary; that argument applies to a region of any shape.

In studies of the AdS/CFT correspondence, of special interest are cases when the CFT is solvable, e.g., a free theory. One such case is the theory of N free Dirac fermions in two spacetime dimensions ($d = 2$), which has been conjectured [8] to be dual, in the large N limit, to a higher-spin gravity with scalar matter in AdS_3 [9]. Deformation of the CFT by a mass term corresponds to a nontrivial profile of a scalar. A scalar of amplitude A sources the higher-spin fields at the same (quadratic [9]) order in A as it sources the metric correction. To this order, however, neither those fields nor the metric correction contribute to variation of the action; the latter is given simply by the action of the scalar on the undeformed background. As a result, the $O(A^2)$ correction to the thermal entropy can be computed from the GH formula as if the dual theory were standard general relativity.

One may then wonder if to order A^2 the generalized GH formula [7] also works the same way as in general relativity, in particular, if the Ryu-Takayanagi (RT) formula for the EE still applies. In this paper, we address this question by making comparisons between results obtained on two sides of the duality.

The theory of N Dirac fermions in $d = 2$ is of interest and perhaps some practical importance in its own right. It can be used to describe a quantum wire with N transverse channels. If we identify the chiral charge of the fermion with the electric charge in the wire, the fermion mass corresponds to a superconducting pairing amplitude induced for instance by proximity to a larger superconductor. The mass winding along the wire as $\tilde{m}(x) = me^{iqx}$, where $m > 0$ and q are real constants, corresponds to a state with non-zero supercurrent. According to the preceding, we may hope to use standard general relativity to compute both the thermal and entanglement entropies to order m^2 and all orders in q.

Our main focus will be a non-vacuum, thermal state in the CFT, related holographically to the BTZ black hole [10]. In this case, there is potentially a small parameter m/T, where T is the temperature. On the gravity side, a small m/T corresponds to a small deformation of the BTZ background. In contrast, if m is larger than both T and $|q|$, the deformation cannot be small everywhere, and the only small dimensionless parameter proportional to m is ml, where l is the length of the entangling region. We briefly discuss this case towards the end of the paper.

We start in Sec. 2 with establishing a relation between the scalar amplitude A in gravity and the fermion mass m in the CFT. The relation uses only the leading near-boundary asymptotics of the scalar and amounts to a computation of the scalar two-point function in the special, logarithmic case of the AdS/CFT correspondence. In Sec. 3, we consider the thermal entropy and in Sec. 4 the EE of an interval of length l in a thermal state of the CFT. In both cases, we find agreement between $O(m^2)$ results obtained on two sides of the duality. For the EE, our evidence in mostly numerical, but in the case of a short interval, which we consider in Sec. 5, some results can be extracted analytically. In particular, the coefficient of the leading $(ml)^2 \ln^2 l$ term in the $O(m^2)$ correction obtained from the RT formula is found to coincide with that obtained in Ref. [11] directly in field theory. We present a discussion of our results in Sec. 6.

2 Asymptotic analysis

We consider deformations of a Euclidean non-rotating BTZ black hole [10], due to a small-amplitude complex scalar field ϕ. The metric is of the form

$$ds^2 = r^2 d\tau^2 + F(r)dx^2 + G(r)dr^2 \,. \tag{1}$$

The asymptotic (near-boundary) region corresponds to large r, where $F(r) \to r^2$ and $G(r) \to 1/r^2$, after we set the AdS radius to unity. The coordinates x and τ are subject to the following peridocities:

$$x \sim x + L_x \,, \tag{2}$$

$$\tau \sim \tau + \beta \,. \tag{3}$$

The GH formula interprets $T = 1/\beta$ as the temperature of the spacetime. According to the AdS/CFT correspondence [2], it is also the temperature in the dual CFT, while L_x is the length of the spatial circle on which the CFT lives.

For the undeformed black hole

$$F(r) = F_0(r) \equiv r^2 + \alpha^2 \,, \qquad G(r) = G_0(r) \equiv [F_0(r)]^{-1} \,, \tag{4}$$

with

$$\alpha = 2\pi T \,. \tag{5}$$

The entropy of this space is $S_0 = \alpha L_x/(4G_N)$ [10], where G_N is Newton's constant. We now proceed to finding corrections to this result, due to the presence of the scalar.

The Gibbons-Hawking method of finding gravitational entropy requires computing the Euclidean action including, in our case, both gravity and the scalar. If we only want the action to the leading (A^2) order in the scalar amplitude A, we do not need to consider changes to the gravitational part of the action, since that part was extremal for $A = 0$. This leaves us with the bilinear action of the scalar,

$$I_E = \int d^3x \sqrt{g} \left(g^{mn} \partial_m \phi^* \partial_n \phi + M^2 \phi^* \phi \right) \,, \tag{6}$$

where g refers to the metric of the undeformed background. On the equations of motion, this reduces to the boundary term

$$I_E = \int d\tau dx \, \left(r g^{rr} \phi^* \partial_r \phi \right) \Big|_{r=r_m} \,, \tag{7}$$

where r_m is some large value of the radius.

Here, we focus on the logarithmic case $M^2 = -1$, corresponding, via the standard AdS/CFT dictionary [3, 4], to an operator of dimension $\Delta = 1$ in the CFT. We restrict attention to static x-dependent solutions of the form

$$\phi(x, r) = e^{iqx} \phi_{rad}(r) , \tag{8}$$

with a constant momentum q in the x direction. The leading and first subleading terms in the asymptotic of the scalar in this case can be combined into

$$\phi_{rad}(r) = \frac{A}{r} \ln(r\xi) + \ldots , \tag{9}$$

where A and ξ are constants, and the dots stand for terms suppressed by inverse powers of r. The constant ξ is a function of q and T,

$$\xi = \xi(q, T) , \tag{10}$$

but, in the present case, not of A itself. For $q = 0$, ξ can be considered as a holographic definition of the correlation length. We will occasionally use this terminology also for $q \neq 0$.

For a scalar described by a bilinear action on the undeformed BTZ background, the solution to the equations of motion is readily available, and one can read off it the full dependence of ξ on its arguments. We will make use of this solution in the next section. Here, let us observe that both Eqs. (7) and (9) depend only on the asymptotic form of the metric and so are a bit more general than the case at hand. We can use them to obtain a general relation of A to the mass parameter of the CFT. It will apply for instance also in the case of vacuum AdS, which has the same large-r asymptotic as the BTZ metric (4) but a different topology.

Substituting Eq. (9) in (7), we find that the action is divergent in the limit $r_m \to \infty$ and so needs to be renormalized. This is achieved by adding a boundary counterterm [12]. We do that by first stepping away from the logarithmic case, i.e., considering M^2 somewhat above -1, so that

$$\lambda \equiv 1 + (1 + M^2)^{1/2} = 1 + s \tag{11}$$

with $0 < s < 1$, and then taking the limit $s \to 0$. The counterpart of Eq. (9) is

$$\phi_{rad}(r) = A_s r^{s-1} \left[1 - (\xi r)^{-2s} \right] + \ldots , \tag{12}$$

and the renormalized action, computed by the method of Ref. [12], is

$$I_{E,ren} = 2s \int d\tau dx |A_s|^2 \xi^{-2s} . \tag{13}$$

For Eq. (12) to reproduce (9) in the limit $s \to 0$, A_s must go infinity as $A/(2s)$ with a constant A. We then see that the logarithmic case requires an additional subtraction, of the $O(1/s)$ term in (13). After this subtraction, the renormalized action for $s = 0$ becomes

$$I_{E,ren} = -\int d\tau dx |A|^2 \ln(\mu\xi) \,, \tag{14}$$

where μ is a normalization momentum.

By the standard dictionary [3, 4], $I_{E,ren}$ generates the two-point function of an operator of dimension $\Delta = 1$ in the dual CFT. The result can be compared to that obtained directly in in the field theory of a N-component Dirac fermion in $d = 2$. The operator in question there is $O = \bar{\Psi}_R \Psi_L$, where $\Psi_{L,R}$ are the left- and right-moving components of the fermion field. The source of O in this theory is the position-dependent fermion mass me^{iqx}, where we can choose m to be real and positive. The two-point function, computed from a one-loop diagram is, to logarithmic accuracy, $(N/2\pi)\ln(\mu/Q)$ where Q is the largest of the three mass scales: $m, |q|$, and T. Comparing this to Eq. (14), we see that, if the results on two sides of the duality are to match, the amplitude A in the gravitational theory must be related to m as follows:

$$A = m\sqrt{\frac{N}{2\pi}} \,, \tag{15}$$

up to an inessential constant phase factor.

We conclude this section with a comment concerning the relative magnitude of q and m. In a superconductor, m represents the energy gap and q the superflow momentum per Cooper pair. It is well known that, in a conventional intrinsic superconductor, the q/m ratio cannot be arbitrarily large: at a sufficiently large q, the gap goes to zero signaling transition to the normal state [13]. Here, however, we consider q and m as independent parameters, as may be the case when superconductivity is induced by the proximity effect. Accordingly, we extend our analysis to arbitrarily large q (and indeed will not detect any obstacles to doing so).

3 Thermal entropy

The solution for the radial part of the scalar on the rigid BTZ background (4) is

$$\phi_{rad}(r) = \widetilde{A}(r^2+\alpha^2)^{-\lambda/2} {}_2F_1\left(a, b, 1; \frac{r^2}{r^2+\alpha^2}\right) \tag{16}$$

where $\widetilde{A}$ is a constant amplitude, λ is related to the mass squared of the scalar by Eq. (11), and ${}_2F_1$ is the hypergeometric function with

$$a = \frac{1}{2}\left(\lambda + \frac{iq}{\alpha}\right) \,, \qquad b = \frac{1}{2}\left(\lambda - \frac{iq}{\alpha}\right) \,. \tag{17}$$

As before, we focus on the case $\lambda = 1$ ($M^2 = -1$), when the asymptotic of ${}_2F_1$ is logarithmic [14]:

$$F(a, b, a+b; z) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \left[2\psi(1) - \psi(a) - \psi(b) - \ln(1-z)\right] + O[(1-z)\ln(1-z)] \,, \quad (18)$$

where $\Gamma(\cdot)$ and $\psi(\cdot)$ are the gamma and digamma functions, respectively. The asymptotic form of the solution then agrees with Eq. (9), with

$$A = \frac{2\widetilde{A}}{\pi} \cosh \frac{\pi q}{2\alpha} \,, \quad (19)$$

$$\ln \xi = -\ln \alpha + \psi(1) - \mathrm{Re}\, \psi\left(\frac{1}{2} + \frac{iq}{2\alpha}\right) . \quad (20)$$

The real part of the digamma function in (20) is a monotonically increasing function of q^2, approaching $\ln(q/\alpha)$ in the limit of large q/α. Note, then, that in this limit ξ becomes independent of the temperature, and to logarithmic accuracy $\ln \xi = -\ln q$.

The small-amplitude condition for the scalar, under which the metric deformation is small and Eq. (16) is applicable, is

$$\kappa A \ll \xi^{-1} \,, \quad (21)$$

where $\kappa^2 = 8\pi G_N$, and G_N is Newton's constant. Written in terms of the fermion mass in the dual CFT, this becomes

$$m \ll \xi^{-1} \,, \quad (22)$$

where we have used the relation $G_N = 3/(2N)$ [15] and Eq. (15). Since ξ is determined by the larger of $\alpha = 2\pi T$ and q, the condition (22) is automatically satisfied if $m \ll 2\pi T$. This is analogous to the Ginzburg-Landau limit in a superconductor.

In accordance with the GH formula, the action (13) with the τ integral removed is interpreted as a correction to the free energy of the black hole:

$$\delta\Omega_{grav} = -L_x |A|^2 \ln(\mu\xi) \,, \quad (23)$$

where we now have an explicit expression for $\ln \xi$, Eq. (20). In view of Eq. (15), this can also be written as

$$\left.\frac{\partial \Omega_{grav}}{\partial m^2}\right|_{m=0} = \frac{N L_x}{2\pi} \ln(\mu\xi) \,. \quad (24)$$

The corresponding correction to the entropy then follows from the thermodynamics formula $S_{grav} = -\partial\Omega_{grav}/\partial T$. Note that dependence on the normalization point μ disappears upon taking the derivative with respect to T.

We now compare this result to the free energy of a multiplet of N free Dirac fermions in two spacetime dimensions. The mass Lagrangian density (in the Lorentzian signature) is

$$L_m = -m\left(e^{-iqx}\bar{\Psi}_L \Psi_R + e^{iqx}\bar{\Psi}_R \Psi_L\right) \,, \quad (25)$$

where $\Psi_{L,R}$ are the left- and right-moving component of the multiplets and m is a positive constant. A chiral transformation will remove the factor e^{iqx} in Eq. (25), at the price of an additional term appearing in the full Lagrangian:

$$L = L' - \frac{q}{2} \left(\Psi_R^\dagger \Psi_R + \Psi_L^\dagger \Psi_L \right) , \tag{26}$$

where L' is the Dirac Lagrangian density with a constant mass m. In a superconductor, the quantity in the brackets in (26) represents the electric current: recall that we identify the electric charge with the chiral charge of the Dirac fermion and set $v_F = 1$.

The latter form of the Lagrangian is convenient for finding the spectrum of elementary excitations. There are two branches, with energies

$$\epsilon_\pm(k) = \sqrt{k^2 + m^2} \pm \frac{1}{2} q \equiv \epsilon_0(k) \pm \frac{1}{2} q \, . \tag{27}$$

The free energy Ω_F of these fermions is that of the grand canonical ensemble with the chemical potential set to zero. Corrections to Ω_F due to the fermion mass are contained in

$$\frac{\partial \Omega_F}{\partial m^2} = N \sum_k \frac{1}{2\epsilon_0(k)} \left[n_F(\epsilon_+) + n_F(\epsilon_-) \right] , \tag{28}$$

where $n_F(\epsilon) = (e^{\beta\epsilon} + 1)^{-1}$ is the Fermi distribution. Our use of the grand canonical ensemble for an isolated system implies that we are working in the thermodynamic limit, when the sum over k can be replaced with an integral. A convenient way to compare the result to the one obtained on the gravity side is to expand both expressions in powers of q^2 and compare them term by term. The constant, power zero, term in (28) is obtained by replacing $\epsilon_\pm$ with ϵ_0 and is logarithmic:

$$\frac{N L_x}{2\pi} \int \frac{dk}{\epsilon_0(k)} n_F(\epsilon_0) = \frac{N L_x}{2\pi} \left(\ln \frac{T}{m} + \text{const} \right) . \tag{29}$$

This differs from the corresponding term in (24) by a T-independent constant. So, the results for the entropy match. Nonzero powers of q^2, obtained by expanding Eq. (28) in q, have finite limits at $m = 0$. These limits are seen to coincide, term by term, with powers of q^2 obtained by expanding $\ln \xi$ in Eq. (24).

4 Entanglement entropy

The Ryu-Takayanagi (RT) formula [6] for the EE has been argued [7] to follow from a generalized GH formula, combined with a holographic interpretation of the gravitational partition function. In general, one should not expect the RT formula to apply to higher-spin gravity; for the case without matter, alternative expressions have been proposed in

Refs. [16, 17, 18]. Given, however, that to the leading order in the mass parameter the higher-spin fields do not affect the action, one may wonder if to this order the RT formula remains applicable as well. Here, we compare results obtained from that formula to those from a direct lattice computation in field theory.

For a single interval, application of the RT formula amounts to computing the length of the geodesic, $x(r)$, connecting two given points $x = \pm x_m$ on the boundary. As before, the boundary is at a large $r = r_m$. The length of the interval is

$$l = 2x_m \,. \tag{30}$$

The geodesic has two symmetric branches: one, with $x < 0$, going from $r = r_m$ to the tip at $r = r_{tip}$, and the other, with $x > 0$, from r_{tip} back to r_m. The length of the geodesic in the metric (1) is given by

$$\mathcal{A}[x(r), F(r), G(r)] = 2\int_{r_{tip}}^{r_m} [G(r) + F(r)(x')^2]^{1/2} dr \,, \tag{31}$$

where $x' \equiv dx/dr$. The geodesic equation for $x(r)$ is obtained by extremizing this at fixed r_m and x_m, with r_{tip} obtained as a function of these parameters in the course of the procedure. The RT formula for the EE [6], $S_{ent} = \mathcal{A}/(4G_N)$, combined with the relation [15] $G_N = 3/(2N)$, then gives

$$S_{ent} = \frac{1}{6} N\mathcal{A} \,. \tag{32}$$

A simplification specific to $d = 2$ is that the integrand in Eq. (31) is independent of x, so the corresponding canonical "momentum" is conserved along the geodesic:

$$\frac{F(r)x'}{[G(r) + F(r)(x')^2]^{1/2}} = \text{const} = [F(r_{tip})]^{1/2} \,. \tag{33}$$

The value of the constant has been found by noting that $x' \to \infty$ at the tip. Moreover, the Hamilton-Jacobi theory applied to the functional (31) tells us that the same constant appears as the derivative of $\mathcal{A}$ with respect to the endpoint:

$$\frac{\partial \mathcal{A}}{\partial x_m} = 2[F(r_{tip})]^{1/2} \,. \tag{34}$$

With the help of Eqs. (30) and (32), this can be expressed as the derivative of the EE with respect to the length of the interval:

$$\frac{\partial S_{ent}}{\partial l} = \frac{N}{6}[F(r_{tip})]^{1/2} \,. \tag{35}$$

Expression (35) is curious in its own right, but is not the most convenient one if we are looking specifically at the case of small deformations (by a scalar of small amplitude A): it

r_{tip}/α	0.5	1	2
$l\alpha$	2.887	1.763	0.962

Table 1: The length l of the entangling interval (in units of the thermal wavelength α^{-1}, where $\alpha = 2\pi T$) as given by the undeformed geodesic (36) for cutoff radius $r_m/\alpha = 10^3$ and various values of r_{tip}.

requires us to consider variations of both $F(r)$ and r_{tip}, i.e., of the geodesic itself. In this case, we have found it more convenient to proceed directly from Eq. (31). Then, to the leading order in the scalar amplitude, we can use the geodesic that was the extremum of $\mathcal{A}$ for the undeformed BTZ background (4):

$$x_0(r) = \frac{1}{\alpha} \ln \frac{cr + \alpha(r^2 - r_{tip}^2)^{1/2}}{r_{tip}(r^2 + \alpha^2)^{1/2}}, \tag{36}$$

where $c \equiv (\alpha^2 + r_{tip}^2)^{1/2}$ is the value of the constant (33) for the undeformed case. Note that, here, r_{tip} is the value of r at the tip of the undeformed geodesic (36). As such, it is different (by terms of order A^2) from r_{tip} we would have to use in Eq. (35). On the other hand, in the method employed in what follows, to the required accuracy the two values are interchangeable, so we do not use separate notation for each.

For numerical work, it is convenient to choose r_{tip} at will and map it by Eq. (36) to a value of $l = 2x_m$. For future use, we present here a table of (r_{tip}, l) pairs obtained in this way (Table 1).

The metric corrections are found from the xx and rr components of the Einstein equations,

$$\frac{G'}{G} + 2Gr = 2\kappa^2 r \left[-\frac{q^2 G}{F}\phi^2 + (\phi')^2 + M^2 G\phi^2 \right], \tag{37}$$

$$\frac{F'}{F} - 2Gr = 2\kappa^2 r \left[-\frac{q^2 G}{F}\phi^2 + (\phi')^2 - M^2 G\phi^2 \right]. \tag{38}$$

We now use ϕ to denote the radial dependence of the scalar and assume it real. The original scalar field is now $e^{iqx}\phi(r)$. Primes denote derivatives with respect to r. These equations have to be solved with the boundary conditions

$$G(0) = 1/\alpha^2, \qquad F(r_m) = r_m^2, \tag{39}$$

which correspond to varying the metric with the temperature and length of the x circle fixed. The boundary conditions for ϕ can be found from Eq. (8) (with $\lambda = 1$), which is applicable since we are working in the linearized theory. We have

$$\phi(0) = \widetilde{A}/\alpha, \qquad \phi'(0) = 0, \tag{40}$$

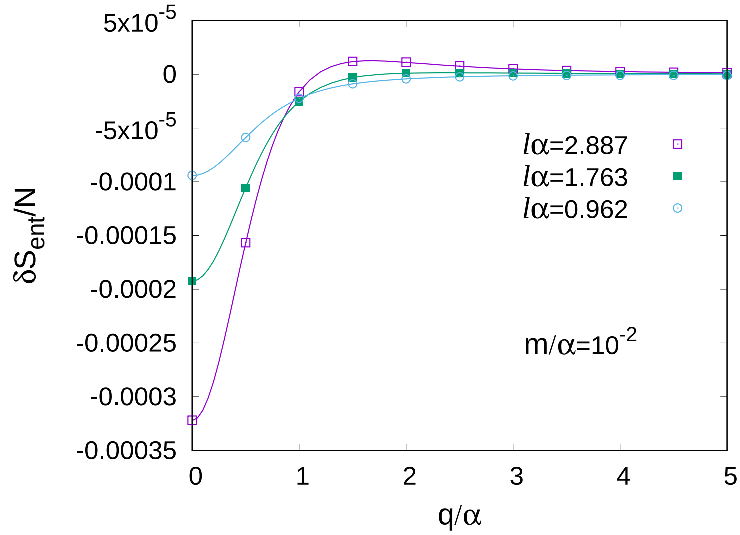

Figure 1: Entanglement entropy change per fermion due to a mass deformation for intervals of different lengths, as a function of the superflow momentum q (in units of $\alpha = 2\pi T$). Curves: results of a holographic calculation using general relativity as the dual. Points: results of a direct calculation in lattice field theory. The interval lengths are the same as in Table 1.

where $\tilde{A}$ is related to A by Eq. (19). The system consisting of Eqs. (37) and (38) and the equation for the scalar can now be solved numerically, and the solution can be used together with the geodesic (36) in Eq. (31).

In Fig. 1, we plot results obtained by this method for $m/\alpha = 10^{-2}$, $r_m/\alpha = 10^3$, and the values of r_{tip} shown in Table 1. The length of the geodesic (36) in the undeformed geometry is

$$\mathcal{A}_0 = 2 \ln \left(y + \sqrt{y^2 - 1} \right) , \tag{41}$$

where $y \equiv r_m / r_{tip}$, and we plot the quantity

$$\frac{\delta S_{ent}}{N} = \frac{1}{6} (\mathcal{A} - \mathcal{A}_0) , \tag{42}$$

which is the entropy change per fermion, due to a finite mass. This quantity goes to a finite limit at large r_m.

For comparison, in the same figure, we show results of a direct lattice computation of the EE for the corresponding values of the interval length l (Table 1). The method is described (for $q = 0$) in Ref. [11]. We use staggered fermions [19] with antiperiodic boundary conditions on a uniform lattice with an even number N of lattice sites. The Hamiltonian corresponding to the additional Lagrangian of Eq. (26) is discretized as

$$H_{add} = \frac{q}{2} \sum_{n=0}^{N-1} \Psi_n^\dagger \Psi_n \,, \tag{43}$$

where Ψ_n is a one-component Fermi operator. The results in Fig. 1 are for $N = 5 \times 10^4$. The good agreement seen in the figure confirms applicability of general relativity to computation of $O(m^2)$ terms in the EE.

Let us comment on short-distance cutoffs required in both calculations. On the gravity side, the cutoff is represented by the maximum radius r_m [20]; on the lattice, by the lattice spacing h. Because the difference (42) is finite in the limit $r_m \to \infty$, the precise relation between r_m and h does not matter, as long as h remains much smaller than all the physical length scales. For definiteness, we have set $h = 1/r_m$. Then, for instance, for $r_m/\alpha = 10^3$ (the value used for compiling Table 1), $1/(h\alpha) = 10^3$, so l in units of the lattice spacing is obtained by multiplying an entry for $l\alpha$ in Table 1 by 1000.

5 Limit of a short interval

In the limit of a short interval, $l \ll \xi$, the leading terms in δS_{ent} can be computed analytically. Indeed, in this case, the entire geodesic lies near the boundary, and we can use the asymptotic expression (9) for ϕ and linearized Einstein equations to find the leading logarithms in the metric functions there. The computation is similar to those done in Refs. [21, 22, 23] for the vacuum AdS deformed by scalars of different masses. In our case, we need to keep track of deformations of both G and F, since both contribute to the geodesic length (31). Define the deformations G_1 and F_1 by

$$G(r) = G_0(r) + G_1(r)\,, \qquad F(r) = F_0(r) + F_1(r)\,. \tag{44}$$

In the asymptotic region, linearized Eqs. (37) and (38) become

$$\frac{G_1'}{G_0} + 4rG_1 = \frac{2\kappa^2 A^2}{r^3}\left[-2\ln(r\xi) + 1 + \ldots\right], \tag{45}$$

$$\frac{F_1'}{F_0} - \frac{2rF_1}{F_0^2} = 2rG_1 + \frac{2\kappa^2 A^2}{r^3}\left[2\ln^2(r\xi) - 2\ln(r\xi) + 1 + \ldots\right]. \tag{46}$$

Here and in the next equation dots denote terms suppressed by inverse powers of r. The solution to the first of these is

$$G_1(r) = \frac{2\kappa^2 A^2}{r^4}\left[-\ln^2(r\xi) + \ln r + \text{const} + \ldots\right] \tag{47}$$

Substituting this into the equation for F_1, we find that logarithmic terms on the right-hand side all cancel, and the leading behavior of F_1 at large r is a constant.

To the linear order in F_1 and G_1, the change in the length (31) is

$$\delta\mathcal{A}[x(r),F(r),G(r)] = \int_{r_{tip}}^{r_m} \frac{G_1(r)+F_1(r)(x')^2}{[G_0(r)+F_0(r)(x')^2]^{1/2}} dr\,. \tag{48}$$

As before, we can use here the undeformed geodesic (36) since it was extremal to the zeroth order. The term with F_1 does not produce any logarithms of r_{tip}, so as far as those are concerned

$$\delta\mathcal{A}[x(r),F(r),G(r)] \approx \int_{r_{tip}}^{r_m} (r^2 - r_{tip}^2)^{1/2} G_1(r) dr\,. \tag{49}$$

The integral is convergent in the limit $r_m \to \infty$, and we assume this limit in what follows. Substituting Eq. (47) for G_1 and computing the integrals, we obtain

$$\delta\mathcal{A}[x(r),F(r),G(r)] = \frac{2\kappa^2 A^2}{3r_{tip}^2}\left[-\ln^2(r_{tip}\xi) + \left(2\ln 2 - \frac{5}{3}\right)\ln r_{tip} + O(1)\right]\,, \tag{50}$$

where $O(1)$ refers to counting of powers of $\ln r_{tip}$. For a near-boundary geodesic, Eq. (36) can be approximated by

$$x(r) = \left(\frac{1}{r_{tip}^2} - \frac{1}{r^2}\right)^{1/2}\,. \tag{51}$$

In the limit $r_m \to \infty$, this gives $l = 2x_m = 2/r_{tip}$. Finally, using using Eq. (15) to relate A to the fermion mass, we obtain

$$\delta S_{ent} = \frac{1}{6} N m^2 l^2 \left[-\ln^2(l/\xi) + \frac{5}{3}\ln l + O(1)\right]\,, \tag{52}$$

where we now count powers of $\ln l$. The leading dependence of Eq. (42) on q is due to $\ln\xi$ in

$$\ln^2(l/\xi) = \ln^2 l - 2\ln l \ln\xi + O(1) \tag{53}$$

with $\ln\xi$ given by Eq. (20). Note that, in the limit $q \gg T$, expression (53) becomes independent of the temperature.

Our results so far have been for the case $m \ll 1/\xi$, where ξ is determined by the larger of T and q. Let us comment on the counterpart of Eq. (52) for the opposite case, when m is the largest mass scale in the problem. For example, let us consider deformations of the global AdS_3 by a scalar of the form (8) subject to the condition $|q| \ll m$. This is appropriate for the system at zero temperature and a low superflow speed. The metric is

$$ds^2_{T=0} = F(r)d\tau^2 + r^2 dx^2 + G(r)dr^2\,. \tag{54}$$

The undeformed AdS of unit radius corresponds to $F(r) = [G(r)]^{-1} = 1 + r^2$. The dual CFT now lives on a unit circle: $L_x = 2\pi$. As before, we focus on a scalar of mass $M^2 = -1$ corresponding to the operator $O = \bar{\Psi}_R \Psi_L$ in the free-fermion CFT. The amplitude A is related to the mass m of the fermion by the same Eq. (15). The key difference from the preceding case is that the correlation length in the CFT is now determined by the mass, and the role of the thermodynamic limit is taken over by the condition

$$m \gg 2\pi/L_x = 1\,. \tag{55}$$

For this condition to apply, the amplitude of the scalar must be relatively large, and as result the metric deformation cannot be small everywhere. Indeed, we can define the correlation length holographically as the radius r of the region where the metric correction is comparable to the unperturbed metric.

As the scalar decreases at large r, eventually, at $r \gg 1/\xi$, the metric deformation becomes small. We are back in the domain of linearized theory, where the asymptotic formula (9) for the scalar applies. The correlation length ξ in it, however, is now determined primarily by the mass, and the linearized theory provides no information on it beyond the estimate $\xi \sim m^{-1}$. Working in parallel with the computation above, we obtain the same Eq. (52) for short intervals; however, uncertainty in the estimate $\xi \sim m^{-1}$ prevents one from determining the coefficient of the $\ln l$ term. The result is

$$\delta S_{ent} = \frac{1}{6} N m^2 l^2 \left[-\ln^2(ml) + O(\ln l) \right] \,, \tag{56}$$

for $ml \ll 1$. We note that the leading correction in this case coincides with that obtained in Ref. [11] by a direct calculation in field theory. Agreement up to a numerical factor, between the $\ln^2 l$ terms in the holographic and field-theory calculations for $\Delta = 1$, has been noted in Ref. [23]. Here, we show that the coefficients match as well.

6 Discussion

In this paper, we have aimed to understand conditions under which the entropies of the $d = 2$ free-fermion CFT deformed by an x-dependent mass term me^{iqx} can be computed holographically using standard general relativity as the dual. We have considered both the thermal entropy (of the entire space) and the entanglement entropy (EE) of an interval. In both instances, we have found that one can use general relativity to compute the entropy to order $m^2\xi^2$, where ξ is the correlation length set by the larger of the temperature and the momentum q. For the thermal entropy, this can be seen as a consequence of the proposed duality [8] between this CFT and the higher-spin theory of Ref. [9]: to order m^2, the effective actions in that theory and in general relativity are the same.

For the EE, the reasoning is less direct since in that case, before applying duality, one makes transformations of the CFT coordinates and metric [5, 7]. One could argue, however, that, while these transformations can make the coordinate dependence of the fermion mass complicated, they do not affect counting of the powers of m, so to compute the entropy to order m^2 one may still be able to replace the higher-spin gravity dual with the standard one. Our results lend support to this argument.

Finally, we remark that holographic formulas for the EE in higher-spin AdS_3 gravity without matter have been proposed (within the Chern-Simons formulation) in Refs. [16, 17, 18]. One may expect that, in the presence of a scalar necessary to describe a mass deformation, these formulas will have to be modified. It would be interesting to find out how.

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