

# Double Field Theory with matter and its cosmological application

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## Abstract

The phase space formulation of Double Field Theory (DFT) indicates that statistical matter can be included in terms of (T–)duality multiplets. We propose the inclusion of a perfect fluid in the geometry of DFT through a generalized energy–momentum tensor written in terms of a DFT pressure, energy density and velocity. The latter is an  $O(D, D)$  vector and satisfies two invariant constraints, in agreement with the on–shell constraints for the generalized momentum. We compute the conservation laws associated to the energy–momentum tensor considering general DFT backgrounds. Then we study cosmological backgrounds using a space–time split, and we find an expression for the DFT cosmological dynamics with the perfect fluid coupled. This proposal reproduces the equations of string cosmology upon parametrization of the DFT Einstein equations.

# Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
1.1	Main results . . . . .	4
1.2	Outline . . . . .	7
<b>2</b>	<b>Vacuum double field theory</b>	<b>7</b>
2.1	Basics . . . . .	7
2.2	Space–time split of double field theory . . . . .	10
<b>3</b>	<b>Incorporating matter into double field theory</b>	<b>12</b>
3.1	Double phase space and the point particle . . . . .	13
3.2	The perfect fluid in the double geometry . . . . .	15
<b>4</b>	<b>Cosmological ansatz</b>	<b>18</b>
4.1	Space–time split DFT with matter . . . . .	19
4.2	Parametrization . . . . .	20
4.3	From DFT to string cosmology . . . . .	21
<b>5</b>	<b>Conclusions and future directions</b>	<b>22</b>

# 1 Introduction

String Theory (ST) is a very good candidate to describe nature from first principles. One of the cornerstones of ST is duality, which is necessary to show the equivalence between their different formulations. Moreover, T-duality can be used as a guiding principle to construct a double geometry that describes strings in an  $O(D, D)$  invariant way,  $O(D, D)$  being an exact symmetry of ST. One possible framework to study the interplay between T-duality and the low energy limit of ST is Double Field Theory [1, 2, 3] (DFT)<sup>1</sup>. The main idea of DFT is to accomplish  $O(D, D)$  as a symmetry of the effective (or supergravity) action. The fundamental dimension of  $O(D, D)$  is  $2D$  and therefore the dimensions of the target space must be doubled as  $\hat{X}^M = (\tilde{x}_\mu, x^\mu)$  with  $\mu = 0, \dots, D - 1$ .

The minimal formulation of DFT includes an invariant group (non-dynamical) metric,  $\hat{\eta}_{\hat{M}\hat{N}}$ , and a field content given by a generalized metric  $\hat{\mathcal{H}}_{\hat{M}\hat{N}}(\hat{X})$  and a generalized dilaton  $\hat{d}(\hat{X})$  which are multiplets of the duality group. Standard diffeomorphisms and Abelian gauge transformations of the b-field are replaced by generalized diffeomorphisms in order to preserve the invariance of the group metric, i.e.,  $\delta_\xi \hat{\eta}_{\hat{M}\hat{N}} = 0$ . This formulation describes the universal NS-NS sector of the low energy limit of ST upon suitable parametrization of the fields and parameters and imposing the strong constraint, which ensures the closure of the generalized diffeomorphisms and removes the dependence on half of the coordinates.

The construction of DFT is inspired in toroidal compactifications where T-duality transformations appear as a symmetry and the fundamental fields can be cast in multiplets of the duality group. These compactifications, in turn, are compatible with string cosmology [6], string gas cosmology [7] and other scenarios with matter terms [8]. At the level of the worldsheet, this issue was addressed in [9] for cosmological vacuum solutions and in [10, 11] considering matter contributions and their relation with T-duality. Moreover in [8] a duality covariant formulation was obtained for string cosmology. With these results, it is expected that matter fields can be coupled to the standard formulation of DFT, as it was recently stressed in [12].

When coupling matter fields to the vacuum DFT at the level of the action, the equa-

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<sup>1</sup>For reviews see [4] and [5].

tions of motion derived from the variational principle can be recast into the form of a generalized Einstein equation [13]

$$\hat{\mathcal{G}}_{\hat{M}\hat{N}} = \hat{\mathcal{T}}_{\hat{M}\hat{N}}, \quad (1.1)$$

where  $\hat{\mathcal{G}}_{\hat{M}\hat{N}}$  is the symmetric DFT Einstein tensor and

$$\hat{\mathcal{T}}_{\hat{M}\hat{N}} = \hat{\mathcal{H}}_{\hat{M}\hat{N}} \left( \mathcal{L}_m - \frac{1}{2} \frac{\delta \mathcal{L}_m}{\delta \hat{d}} \right) - 2 \left[ \bar{P}_{\hat{M}\hat{K}} P_{\hat{N}\hat{L}} + \bar{P}_{\hat{N}\hat{K}} P_{\hat{M}\hat{L}} \right] \left( \frac{\delta \mathcal{L}_m}{\delta P_{\hat{K}\hat{L}}} - \frac{\delta \mathcal{L}_m}{\delta \bar{P}_{\hat{K}\hat{L}}} \right) \quad (1.2)$$

is the corresponding energy–momentum tensor with  $\mathcal{L}_m$  the matter Lagrangian coupled to the double geometry and  $P_{\hat{M}\hat{N}}$ ,  $\bar{P}_{\hat{M}\hat{N}}$  the DFT projectors<sup>2</sup>. Additionally the generalized Bianchi identities imply a vanishing divergence of the Einstein tensor,  $\nabla^{\hat{M}} \hat{\mathcal{G}}_{\hat{M}\hat{N}} = 0$ , and in turn the on–shell condition dictates a conservation law for matter, namely

$$\nabla^{\hat{M}} \hat{\mathcal{T}}_{\hat{M}\hat{N}} = 0. \quad (1.3)$$

The vacuum solutions for the DFT cosmological ansatz were studied in [14] and further solutions with matter were explored in [13, 15]. Moreover, higher–derivative terms were included in [12, 16, 17, 18] as well. Also in here we observe these works suggest that matter can be coupled to the standard construction of DFT. In turn having a well understood framework of matter coupled to DFT would allow to generalize the current computations of  $\alpha'$  corrections using the results of [19], where a systematic procedure to access to higher–derivative terms for vacuum DFT was introduced.

If one wants to describe matter from a statistical approach, as a gas or fluid in the double geometry, it is possible to proceed from a double kinetic perspective [20]<sup>3</sup>. Considering the phase space construction of DFT through the coordinates  $(\hat{X}^{\hat{M}}, \hat{\mathcal{P}}^{\hat{M}})$ , with  $\hat{\mathcal{P}}^{\hat{M}}$  being a generalized momentum vector, the analogous equation to (1.2) is given by

$$\hat{\mathcal{T}}^{\hat{M}\hat{N}}(\hat{X}) = \int d^{2D} \hat{\mathcal{P}} e^{-2\hat{d}} \hat{\mathcal{P}}^{\hat{M}} \hat{\mathcal{P}}^{\hat{N}} \hat{F}, \quad (1.4)$$

where  $\hat{F} = \hat{F}(\hat{X}, \hat{\mathcal{P}})$  is a generalized one–particle distribution function. In this double kinetic framework, the conservation law (1.3) for the energy–momentum tensor (1.4) can be obtained from the transfer equations related to the generalized Boltzmann equation for the evolution of  $\hat{F}$ .

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<sup>2</sup>We use the following conventions for the projectors:  $\bar{P}_{\hat{M}\hat{N}} = \frac{1}{2}(\hat{\eta}_{\hat{M}\hat{N}} + \hat{\mathcal{H}}_{\hat{M}\hat{N}})$  and  $P_{\hat{M}\hat{N}} = \frac{1}{2}(\hat{\eta}_{\hat{M}\hat{N}} - \hat{\mathcal{H}}_{\hat{M}\hat{N}})$ . We do not include hats in the  $O(D, D)$  projectors to present a light notation.

<sup>3</sup>See also [8, 15] where a different procedure to include matter was implemented.

## 1.1 Main results

While in General Relativity (GR) the explicit form of the energy–momentum tensor is well–known for several specific scenarios, e.g. the perfect fluid, this is not the case for DFT. The explicit form of the generalized distribution function is not known in any configuration since, in particular, the thermodynamics properties of the system and their equilibrium states are not fully understood. To avoid this last issue, in this work we introduce a proposal for the energy–momentum tensor of the perfect fluid in the geometry of DFT.

The ansatz is constructed taking a top–down perspective and proposing a generalization of the energy–momentum tensor for the perfect fluid in GR, adapting its structure in order to fulfill all the DFT constraints. Moreover, since DFT encodes the low energy limit of ST, this proposal must recover the fundamental equations of string cosmology. Although there exists a straightforward generalization of the definitions of the scalars (energy density and pressure), the construction of the generalized velocity of a point–particle in a particular orbit of the double phase space is more subtle. The generalized velocity for the particle in the double geometry is defined as

$$\hat{U}^{\hat{M}} = \frac{D\hat{X}^{\hat{M}}}{D\tau}, \quad (1.5)$$

where  $D\hat{X}^{\hat{M}}$  is the DFT–compatible 1–form differential which transforms as [21],

$$\delta_{\xi}(D\hat{X}^{\hat{M}}) = (\partial_{\hat{N}}\xi^{\hat{M}} - \partial^{\hat{M}}\xi_{\hat{N}})D\hat{X}^{\hat{N}}, \quad (1.6)$$

and  $\tau$  is an affine parameter in the double geometry, which reduces to the standard proper time upon parametrization.

Both the generalized momentum  $\hat{\mathcal{P}}^{\hat{M}}$  corresponding to the trajectory of a point–particle and its generalized velocity  $\hat{U}^{\hat{M}}$  are vectors that indicate the same geometric direction in the tangent space and then they must be proportional. Choosing the proportionality constant is, in fact, defining an specific affine parameter  $\tau$  in (1.5). Indeed, we take the relation

$$\hat{\mathcal{P}}^{\hat{M}} = m\hat{U}^{\hat{M}} = mD\hat{X}^{\hat{M}}/D\tau, \quad (1.7)$$

where  $m$  is the invariant mass. Since the generalized momentum satisfies a mass–shell

condition and a strong constraint-like equation,

$$\hat{\mathcal{P}}^{\hat{M}} \hat{\mathcal{H}}_{\hat{M}\hat{N}} \hat{\mathcal{P}}^{\hat{N}} = -m^2, \quad (1.8)$$

$$\hat{\mathcal{P}}^{\hat{M}} \hat{\eta}_{\hat{M}\hat{N}} \hat{\mathcal{P}}^{\hat{N}} = 0, \quad (1.9)$$

the generalized velocity inherits the following relations

$$\hat{U}_{\hat{M}} \hat{\mathcal{H}}^{\hat{M}\hat{N}} \hat{U}_{\hat{N}} = -1 \quad (1.10)$$

$$\hat{U}_{\hat{M}} \hat{\eta}^{\hat{M}\hat{N}} \hat{U}_{\hat{N}} = 0. \quad (1.11)$$

The mass-shell condition for the particle in the double geometry (1.8) is related to the mass operator and level matching condition [4]. The constraints (1.9) and (1.11) take rid off the half of the components of the generalized velocity/momentum, in agreement with the number of degrees of freedom of these objects. Furthermore, (1.10) and (1.11) allow us to consider a rest frame upon a splitting of the DFT coordinates.

On the other hand, if we focus our attention on the structure of the energy-momentum tensor for the perfect fluid in GR, i.e.

$$T_{\mu\nu} = (p + e)u_\mu u_\nu + pg_{\mu\nu}, \quad (1.12)$$

we can straightforwardly generalize the second term in (1.12) by replacing  $g_{\mu\nu}$  with the generalized metric of DFT  $\hat{\mathcal{H}}_{\hat{M}\hat{N}}$ . However, the first term is ambiguous in terms of DFT projections, namely

$$b_1 U_{\hat{M}} U_{\hat{N}} + b_2 U_{\hat{M}} U_{\hat{N}} + b_3 U_{\hat{M}} U_{\hat{N}} + b_4 U_{\hat{M}} U_{\hat{N}}. \quad (1.13)$$

The simplest proposal is to consider only mixed components in (1.13) since this directly matches the structure of the generalized energy-momentum tensor coming from the variation of the DFT for the generalized scalar field [20]. We fix the remaining coefficients such that

$$\hat{\mathcal{T}}^{\hat{M}\hat{N}} = 2(\tilde{e} + \tilde{p}) \hat{U}^{\hat{M}} \hat{U}^{\hat{N}} + 2(\tilde{e} + \tilde{p}) \hat{U}^{\hat{M}} \hat{U}^{\hat{N}} + \tilde{p} \hat{\mathcal{H}}^{\hat{M}\hat{N}}, \quad (1.14)$$

where  $\tilde{e}(\hat{X}) = \tilde{e}$  and  $\tilde{p}(\hat{X}) = \tilde{p}$  are defined as the generalized energy density and pressure, and  $\hat{U}^{\hat{M}}$  is the generalized velocity which can be parametrized as

$$\hat{U}^{\hat{M}} = (\tilde{u}_\mu, u^\mu), \quad (1.15)$$

with  $\tilde{u}_\mu$  a dual velocity.

In addition to the construction of the generalization of the energy–momentum tensor for the perfect fluid (1.14) in the double geometry, we also explore the conservation equations derived from (1.3) and we elaborate on the DFT Einstein–type equations (1.1) sourced by (1.14) for a cosmological ansatz and their parametrization. As we have explained, this proposal must agree with the previous cosmological DFT constructions such as [13] and [14], upon parametrization. Considering the space–time split and a rest frame the non–trivial equations coming from the generalized Einstein equation are given by

$$\begin{aligned}
-\tilde{p} &= 2(\partial_0 d)^2 - 2\partial_{00}d - \frac{1}{16}\partial_0 A^{MN}\partial_0 A_{MN} + (\partial_0 \leftrightarrow \tilde{\partial}^0) \\
\frac{1}{2}(\tilde{e} + \tilde{p}) &= \partial_{00}d + \frac{1}{16}\partial_0 A^{KL}\partial_0 A_{KL} - (\partial_0 \leftrightarrow \tilde{\partial}^0) \\
0 &= -P_M{}^P \left[ \frac{1}{4}\partial_{00}A_{PQ} - \frac{1}{2}\partial_0 d \partial_0 A_{PQ} + (\partial_0 \leftrightarrow \tilde{\partial}^0) \right] \overline{P}^Q{}_N + (P \leftrightarrow \overline{P}).
\end{aligned} \tag{1.16}$$

These equations determine the set of DFT cosmologies whose matter source is (1.14) and their parametrization reads

$$\begin{aligned}
-2\tilde{p} &= 2(D-1)\dot{H} + D(D-1)H^2 - 4\ddot{\phi} + 4\dot{\phi}^2 - 4(D-1)H\dot{\phi} \\
&\quad + \frac{1}{4a^4}\delta^{ij}\delta^{kl}\dot{b}_{ik}\dot{b}_{jl} \\
-(\tilde{e} + \tilde{p}) &= (D-1)\left(\dot{H} + H^2\right) - 2\ddot{\phi} + \frac{1}{4a^4}\delta^{ij}\delta^{kl}\dot{b}_{ik}\dot{b}_{jl} \\
0 &= \left(\dot{H} + (D-1)H^2 - 2H\dot{\phi}\right)\delta^{ij} + \frac{1}{2a^4}\delta^{ik}\delta^{jl}\delta^{mn}\dot{b}_{km}\dot{b}_{ln} \\
0 &= \delta^{jl}\ddot{b}_{il} + \frac{1}{a^4}\delta^{jk}\delta^{lm}\delta^{no}\dot{b}_{ln}\dot{b}_{km}b_{io} + \left[(D-5)H - 2\dot{\phi}\right]\delta^{jk}\dot{b}_{ik} \\
&\quad - \left[2\dot{H} + 2(D-1)H^2 - 4H\dot{\phi}\right]\delta^{jk}b_{ik}.
\end{aligned} \tag{1.17}$$

Here we keep the usual coordinates and eliminate the dual ones. However, other T–dual solutions can be easily obtained from (1.16). These cosmologies reduce to the results of [14] when the matter and the  $b$ –field are neglected. Furthermore it is possible to describe string cosmologies scenarios [6] by implementing the following field redefinitions

$$\tilde{p} = e^{2\phi}p, \quad \tilde{e} = e^{2\phi}e. \tag{1.18}$$

The success of DFT with matter yields in the fact that it is possible to obtain string cosmologies from the double geometry, whose only stringy ingredient is given by the

duality group invariance. As we discuss in Section 4.3, it is worth noting that the proposal (1.14) is a first step in the description of the generalized perfect fluid in the double geometry, which reproduces string cosmologies with a fixed dilaton source  $\sigma$  (cf. (4.18)) upon a suitable parametrization and imposing the strong constraint.

## 1.2 Outline

This paper is organized as follows: In Section 2 we introduce the basics of DFT and we describe the space–time split decomposition. In Section 3 we incorporate matter into DFT: firstly we describe the construction of the DFT phase space from a kinetic theory point of view. Secondly we propose an explicit form for the generalized energy–momentum tensor related to a perfect fluid in the double geometry. Finally we study the conservation laws from the divergence equation. We analyze the cosmological ansatz of DFT with matter given by the generalized Einstein equation, in Section 4. We also show the agreement with string cosmology considering field redefinitions of the DFT energy density and pressure. Conclusions are given in Section 5.

# 2 Vacuum double field theory

In this section we introduce the basic aspects of the vacuum sector of DFT. We start by considering a formulation where the fundamental fields are the generalized metric and the generalized dilaton, which are in  $O(D, D)$  representations. Then, we describe the space–time split framework. In both approaches, the fundamental fields depend on both the ordinary coordinates and the dual ones.

## 2.1 Basics

DFT [1, 2, 3] is a proposal to rewrite the low energy limit of ST as a manifestly  $O(D, D)$  invariant theory. All the DFT fields and parameters are duality invariant objects. The double geometry consists in a  $2D$ –dimensional space with coordinates  $\hat{X}^{\hat{M}} = (\tilde{x}_\mu, x^\mu)$ ,  $\hat{M} = 1, \dots, 2D$ . Here  $x^\mu$  are coordinates on an embedded supergravity, and  $\tilde{x}_\mu$  are dual



coordinates. Considering a solution to the weak and strong constraints, namely

$$\begin{aligned}\partial_{\hat{M}}(\partial^{\hat{M}}\star) &= 0 \\ (\partial_{\hat{M}}\star)(\partial^{\hat{M}}\star) &= 0\end{aligned}\tag{2.1}$$

where  $\star$  is a generic field/parameter, half of the coordinates are taken away. For instance, simple solutions are  $\tilde{\partial}^\mu = 0$  or  $\partial_\mu = 0$  in which the fundamental fields depend only on  $x^\mu$  or  $\tilde{x}_\mu$ , respectively. In (2.1), contractions are given by the  $O(D, D)$  invariant metric,  $\hat{\eta}_{\hat{M}\hat{N}}$ .

The theory is invariant under a global  $O(D, D)$  symmetry which infinitesimally reads

$$\delta V_{\hat{M}} = V_{\hat{N}} \Omega^{\hat{N}}_{\hat{M}},\tag{2.2}$$

where  $V_{\hat{M}}$  is a generic  $O(D, D)$ -multiplet and  $\Omega \in O(D, D)$  is the generic parameter of the transformation. A generalized notion of diffeomorphisms can be defined in the double space. These are given by the generalized Lie derivative,

$$\delta_{\hat{\xi}} V^{\hat{M}} = \mathcal{L}_{\hat{\xi}} V^{\hat{M}} = \hat{\xi}^{\hat{N}} \partial_{\hat{N}} V^{\hat{M}} + (\partial^{\hat{M}} \hat{\xi}_{\hat{P}} - \partial_{\hat{P}} \hat{\xi}^{\hat{M}}) V^{\hat{P}} + \omega \partial_{\hat{N}} \hat{\xi}^{\hat{N}} V^{\hat{M}},\tag{2.3}$$

where  $\hat{\xi}^{\hat{M}}$  a generic parameter and  $\omega$  a density weight factor. The closure of these transformations is given by the C-bracket

$$\left[ \delta_{\hat{\xi}_1}, \delta_{\hat{\xi}_2} \right] V^{\hat{M}} = \delta_{\hat{\xi}_{21}} V^{\hat{M}}\tag{2.4}$$

where

$$\hat{\xi}_{12}^{\hat{M}} = \hat{\xi}_1^{\hat{P}} \frac{\partial \hat{\xi}_2^{\hat{M}}}{\partial \hat{X}^{\hat{P}}} - \frac{1}{2} \hat{\xi}_1^{\hat{P}} \frac{\partial \hat{\xi}_2^{\hat{P}}}{\partial \hat{X}^{\hat{M}}} - (1 \leftrightarrow 2).\tag{2.5}$$

The fundamental fields are the generalized dilaton  $\hat{d}$  and the generalized metric  $\hat{\mathcal{H}}_{\hat{M}\hat{N}}$ . In this work we consider that these fields integrate the vacuum sector of DFT. While the generalized metric transforms as a tensor under  $O(D, D)$  transformations and generalized diffeomorphisms ( $\omega = 0$ ), and it is an element of the duality group, i.e.

$$\hat{\mathcal{H}}_{\hat{M}\hat{P}} \hat{\eta}^{\hat{P}\hat{Q}} \hat{\mathcal{H}}_{\hat{Q}\hat{N}} = \hat{\eta}_{\hat{M}\hat{N}},\tag{2.6}$$

the generalized dilaton is an  $O(D, D)$  scalar and transforms non-covariantly under generalized diffeomorphisms

$$\delta_\xi \hat{d} = \xi^{\hat{N}} \partial_{\hat{N}} \hat{d} - \frac{1}{2} \partial_{\hat{M}} \xi^{\hat{M}}. \quad (2.7)$$

Both  $\hat{\mathcal{H}}_{MN}$  and  $\hat{\eta}_{MN}$  can be used to construct DFT projectors

$$P_{\hat{M}\hat{N}} = \frac{1}{2} \left( \hat{\eta}_{\hat{M}\hat{N}} - \hat{\mathcal{H}}_{\hat{M}\hat{N}} \right) \quad \text{and} \quad \bar{P}_{\hat{M}\hat{N}} = \frac{1}{2} \left( \hat{\eta}_{\hat{M}\hat{N}} + \hat{\mathcal{H}}_{\hat{M}\hat{N}} \right). \quad (2.8)$$

The action principle of DFT is given by

$$S_{DFT} = \frac{1}{2} \int d^{2D} \hat{X} e^{-2\hat{d}} \mathcal{L}_{DFT} \quad (2.9)$$

with the duality invariant Lagrangian

$$\begin{aligned} \mathcal{L}_{DFT} = & \frac{1}{8} \hat{\mathcal{H}}^{\hat{M}\hat{N}} \partial_{\hat{M}} \hat{\mathcal{H}}^{\hat{K}\hat{L}} \partial_{\hat{N}} \hat{\mathcal{H}}_{\hat{K}\hat{L}} - \frac{1}{2} \hat{\mathcal{H}}^{\hat{M}\hat{N}} \partial_{\hat{N}} \hat{\mathcal{H}}^{\hat{K}\hat{L}} \partial_{\hat{L}} \hat{\mathcal{H}}_{\hat{M}\hat{K}} \\ & + 4 \hat{\mathcal{H}}^{\hat{M}\hat{N}} \partial_{\hat{M}} \hat{d} \partial_{\hat{N}} \hat{d} - 2 \partial_{\hat{M}} \hat{\mathcal{H}}^{\hat{M}\hat{N}} \partial_{\hat{N}} \hat{d}. \end{aligned} \quad (2.10)$$

Since this Lagrangian transforms as a scalar under generalized diffeomorphisms, the action principle of DFT is invariant.

In DFT it is also possible to construct a generalized version of the Ricci scalar, i.e.,

$$\begin{aligned} \hat{\mathcal{R}} = & \frac{1}{8} \hat{\mathcal{H}}^{\hat{M}\hat{N}} \partial_{\hat{M}} \hat{\mathcal{H}}^{\hat{K}\hat{L}} \partial_{\hat{N}} \hat{\mathcal{H}}_{\hat{K}\hat{L}} - \frac{1}{2} \hat{\mathcal{H}}^{\hat{M}\hat{N}} \partial_{\hat{N}} \hat{\mathcal{H}}^{\hat{K}\hat{L}} \partial_{\hat{L}} \hat{\mathcal{H}}_{\hat{M}\hat{K}} + 4 \hat{\mathcal{H}}^{\hat{M}\hat{N}} \partial_{\hat{M}} \partial_{\hat{N}} \hat{d} \\ & + 4 \partial_{\hat{M}} \hat{\mathcal{H}}^{\hat{M}\hat{N}} \partial_{\hat{N}} \hat{d} - 4 \hat{\mathcal{H}}^{\hat{M}\hat{N}} \partial_{\hat{M}} \hat{d} \partial_{\hat{N}} \hat{d} - \partial_{\hat{M}} \partial_{\hat{N}} \hat{\mathcal{H}}^{\hat{M}\hat{N}}, \end{aligned} \quad (2.11)$$

and a generalized Ricci tensor

$$\hat{\mathcal{R}}_{\hat{M}\hat{N}} = P_{\hat{M}}^{\hat{P}} \hat{\mathcal{K}}_{\hat{P}\hat{Q}} \bar{P}^{\hat{Q}}_{\hat{N}} + \bar{P}_{\hat{M}}^{\hat{P}} \hat{\mathcal{K}}_{\hat{P}\hat{Q}} P^{\hat{Q}}_{\hat{N}}, \quad (2.12)$$

where

$$\begin{aligned} \hat{\mathcal{K}}_{\hat{M}\hat{N}} = & \frac{1}{8} \partial_{\hat{M}} \hat{\mathcal{H}}^{\hat{K}\hat{L}} \partial_{\hat{N}} \hat{\mathcal{H}}_{\hat{K}\hat{L}} - \frac{1}{4} \left( \partial_{\hat{L}} - 2 \partial_{\hat{L}} \hat{d} \right) \left( \hat{\mathcal{H}}^{\hat{L}\hat{K}} \partial_{\hat{K}} \hat{\mathcal{H}}_{\hat{M}\hat{N}} \right) + 2 \partial_{\hat{M}} \partial_{\hat{N}} \hat{d} \\ & - \frac{1}{2} \partial_{(\hat{M}} \hat{\mathcal{H}}^{\hat{K}\hat{L}} \partial_{\hat{L}} \hat{\mathcal{H}}_{\hat{N})\hat{K}} + \frac{1}{2} \left( \partial_{\hat{L}} - 2 \partial_{\hat{L}} \hat{d} \right) \left( \hat{\mathcal{H}}^{\hat{K}\hat{L}} \partial_{(\hat{M}} \hat{\mathcal{H}}_{\hat{N})\hat{K}} + \hat{\mathcal{H}}^{\hat{K}}_{(\hat{M}} \partial_{\hat{K}} \hat{\mathcal{H}}^{\hat{L}}_{\hat{N})} \right). \end{aligned}$$

It is straightforward to verify that these objects are fully covariant. Moreover, they can be constructed from a generalized Riemann tensor, but the latter cannot be entirely expressed in terms of the fundamental fields of DFT [22].

## 2.2 Space–time split of double field theory

We start by splitting the DFT coordinates as

$$\hat{X}^{\hat{M}} = (\tilde{x}_0, x^0, X^M) \quad (2.13)$$

where  $M = 3, \dots, 2D$ ,  $\tilde{x}_0 = \tilde{t}$ ,  $x^0 = t$  and the partial derivatives are

$$\partial_{\hat{M}} = (\tilde{\partial}^0, \partial_0, \partial_M). \quad (2.14)$$

The weak constraint now implies

$$\partial_M \partial^M \star = -2\tilde{\partial}^0 \partial_0 \star \quad (2.15)$$

while the strong constraint is

$$\partial_M \square \partial^M \star = -\partial_0 \square \tilde{\partial}^0 \star - \partial_0 \star \tilde{\partial}^0 \square, \quad (2.16)$$

where  $\star$  and  $\square$  are arbitrary fields/parameters. The  $O(D, D)$  invariant metric decomposes as

$$\hat{\eta}_{\hat{M}\hat{N}} = \begin{pmatrix} \hat{\eta}^{00} & \hat{\eta}^0{}_0 & \hat{\eta}^0{}_N \\ \hat{\eta}_0{}^0 & \hat{\eta}_{00} & \hat{\eta}_{0N} \\ \hat{\eta}_M{}^0 & \hat{\eta}_{M0} & \hat{\eta}_{MN} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \eta_{MN} \end{pmatrix}, \quad (2.17)$$

while the generalized metric is given by

$$\hat{\mathcal{H}}_{\hat{M}\hat{N}} = \begin{pmatrix} -N^{-2} & \alpha & N^{-2}\mathcal{N}_N \\ \alpha & -\frac{1}{2}\alpha\mathcal{N}^K\mathcal{N}_K - N^2 + \mathcal{H}_{PK}\mathcal{N}^P\mathcal{N}^K & -\alpha\mathcal{N}_N + \mathcal{H}_{NK}\mathcal{N}^K \\ N^{-2}\mathcal{N}_M & -\alpha\mathcal{N}_M + \mathcal{H}_{MK}\mathcal{N}^K & \mathcal{H}_{MN} - N^{-2}\mathcal{N}_M\mathcal{N}_N \end{pmatrix} \quad (2.18)$$

where  $\alpha = \frac{1}{2}N^{-2}\mathcal{N}^M\mathcal{N}_M$  [23]. In (2.18)  $\mathcal{N}_M$  is a generalized shift vector and  $N$  is a generalized lapse function. The generalized dilaton,  $\hat{d}$ , can be redefined in terms of the generalized lapse function,

$$e^{-2\hat{d}} = Ne^{-2d}. \quad (2.19)$$

Since we are interested in cosmological ansatz, we consider  $\mathcal{N}_M = 0$  and  $N = 1$ . So far, the line element is given by

$$dS^2 = -d\tilde{t}^2 - dt^2 + \mathcal{H}_{MN}dX^M dX^N, \quad (2.20)$$

with  $\mathcal{H}_{MN} = \mathcal{H}_{MN}(\tilde{t}, t, X)$ . The dependence in all the dual coordinates is crucial to preserve the  $O(D, D)$  invariance. Since we are considering  $N = 1$ , the generalized dilaton is not redefined and we get  $\hat{d}(\hat{X}) = d(\tilde{t}, t, X)$ .

We can easily decompose both the DFT Lagrangian

$$\begin{aligned} \mathcal{L}_{DFT} = & \frac{1}{8}\mathcal{H}^{PQ}\partial_P\mathcal{H}^{MN}\partial_Q\mathcal{H}_{MN} - \frac{1}{8}\partial_0\mathcal{H}^{MN}\partial_0\mathcal{H}_{MN} - \frac{1}{8}\tilde{\partial}^0\mathcal{H}^{MN}\tilde{\partial}^0\mathcal{H}_{MN} \\ & - \frac{1}{2}\mathcal{H}^{PQ}\partial_Q\mathcal{H}^{MN}\partial_N\mathcal{H}_{PM} + 4\mathcal{H}^{MN}\partial_M d\partial_N d - 4\partial_0 d\partial_0 d \\ & - 4\tilde{\partial}^0 d\tilde{\partial}^0 d - 2\partial_M\mathcal{H}^{MN}\partial_N d, \end{aligned} \quad (2.21)$$

and the generalized curvatures

$$\begin{aligned} \hat{\mathcal{R}} = & \frac{1}{8}\mathcal{H}^{PQ}\partial_P\mathcal{H}^{MN}\partial_Q\mathcal{H}_{MN} - \frac{1}{8}\partial_0\mathcal{H}^{MN}\partial_0\mathcal{H}_{MN} - \frac{1}{8}\tilde{\partial}^0\mathcal{H}^{MN}\tilde{\partial}^0\mathcal{H}_{MN} \\ & - \frac{1}{2}\mathcal{H}^{PQ}\partial_Q\mathcal{H}^{MN}\partial_N\mathcal{H}_{PM} + 4\mathcal{H}^{MN}\partial_{MN}d - 4\partial_{00}d \\ & - 4\tilde{\partial}^{00}d + 4\partial_M\mathcal{H}^{MN}\partial_N d - 4\mathcal{H}^{MN}\partial_M d\partial_N d \\ & + 4\partial_0 d\partial_0 d + 4\tilde{\partial}^0 d\tilde{\partial}^0 d - \partial_{MN}\mathcal{H}^{MN} \end{aligned} \quad (2.22)$$

and

$$\begin{aligned} \hat{\mathcal{K}}_{\hat{M}\hat{N}} = & \frac{1}{8}\partial_{\hat{M}}\hat{\mathcal{H}}^{KL}\partial_{\hat{N}}\hat{\mathcal{H}}_{KL} - \frac{1}{4}\partial_L\hat{\mathcal{H}}^{KL}\partial_K\hat{\mathcal{H}}_{\hat{M}\hat{N}} + \frac{1}{4}\partial_{00}\hat{\mathcal{H}}_{\hat{M}\hat{N}} + \frac{1}{4}\tilde{\partial}^{00}\hat{\mathcal{H}}_{\hat{M}\hat{N}} \\ & - \frac{1}{4}\hat{\mathcal{H}}^{LK}\partial_{LK}\hat{\mathcal{H}}_{\hat{M}\hat{N}} - \frac{1}{2}\partial_0 d\partial_0\hat{\mathcal{H}}_{\hat{M}\hat{N}} - \frac{1}{2}\tilde{\partial}^0 d\tilde{\partial}^0\hat{\mathcal{H}}_{\hat{M}\hat{N}} + \frac{1}{2}\partial_L d\mathcal{H}^{LK}\partial_K\hat{\mathcal{H}}_{\hat{M}\hat{N}} \\ & + 2\partial_{\hat{M}\hat{N}}d - \frac{1}{2}\partial_{(\hat{M}}\mathcal{H}^{KL}\partial_L\hat{\mathcal{H}}_{\hat{N})K} + \frac{1}{2}\partial_L\mathcal{H}^{KL}\partial_{(\hat{M}}\hat{\mathcal{H}}_{\hat{N})K} + \frac{1}{2}\mathcal{H}^{KL}\partial_{L(\hat{M}}\hat{\mathcal{H}}_{\hat{N})K} \\ & - \partial_L d\mathcal{H}^{KL}\partial_{(\hat{M}}\hat{\mathcal{H}}_{\hat{N})K} + \frac{1}{2}\partial_L\hat{\mathcal{H}}^K{}_{(\hat{M}}\partial_K\hat{\mathcal{H}}^L{}_{\hat{N})} + \frac{1}{2}\hat{\mathcal{H}}_{0(\hat{M}}\partial_L\tilde{\partial}^0\hat{\mathcal{H}}^L{}_{\hat{N})} \\ & + \frac{1}{2}\hat{\mathcal{H}}^0{}_{(\hat{M}}\partial_L\partial_0\hat{\mathcal{H}}^L{}_{\hat{N})} + \frac{1}{2}\hat{\mathcal{H}}^K{}_{(\hat{M}}\partial_{LK}\hat{\mathcal{H}}^L{}_{\hat{N})} - \frac{1}{2}\partial_L d\hat{\mathcal{H}}^0{}_{(\hat{M}}\partial_0\hat{\mathcal{H}}^L{}_{\hat{N})} \\ & - \frac{1}{2}\partial_L d\hat{\mathcal{H}}_{0(\hat{M}}\tilde{\partial}^0\hat{\mathcal{H}}^L{}_{\hat{N})} - \frac{1}{2}\partial_L d\hat{\mathcal{H}}^K{}_{(\hat{M}}\partial_K\hat{\mathcal{H}}^L{}_{\hat{N})}. \end{aligned} \quad (2.23)$$

The previous expressions are useful to obtain closed expressions for rewriting cosmological solutions in a duality invariant way in the DFT with matter framework. Moreover, considering a cosmological principle [14], we can impose  $\mathcal{H}_{MN}(\tilde{t}, t, X) = A_{MN}(\tilde{t}, t)$  and  $d(\tilde{t}, t, X) = d(\tilde{t}, t)$  and the expressions for the curvatures drastically simplify. In this case, the remaining symmetry is  $O(1, 1) \times O(D - 1, D - 1)$ . We discuss about this point in Section 4.

### 3 Incorporating matter into double field theory

The original construction of DFT [2] is based on a rewriting of the low energy limit of ST where the invariance under  $O(D, D)$  is accomplished before compactification. In fact, the dynamics of the supergravity NS-NS vacuum sector regarding the metric  $g_{\mu\nu}$ , the  $b$ -field and the dilaton  $\phi$  can be completely rewritten within the framework of DFT and, particularly, in terms of  $O(D, D)$  multiplets. In this case a generalized vacuum Einstein equation can be constructed as

$$\hat{\mathcal{G}}_{\hat{M}\hat{N}} = 0, \quad (3.1)$$

where the LHS is given by the symmetric generalized Einstein tensor

$$\hat{\mathcal{G}}_{\hat{M}\hat{N}} = -\frac{1}{2}\hat{\mathcal{H}}_{\hat{M}\hat{N}}\hat{\mathcal{R}} - 2\hat{\mathcal{R}}_{\hat{M}\hat{N}}, \quad (3.2)$$

with  $\hat{\mathcal{R}}$  and  $\hat{\mathcal{R}}_{\hat{M}\hat{N}}$  given in equations (2.11) and (2.12) respectively. In order to get the full dynamics, equation (3.1) must be complemented with the equation of motion coming from the variation of the vacuum action (2.9) with respect to the dilaton, namely  $\hat{\mathcal{R}} = 0$ .

The inclusion of matter in the dynamics at the level of supergravity is also well understood both through a variational principle from an specific matter action or through the usual Einstein equation with a proper energy–momentum tensor. While the relation between supergravity, matter and T–duality was studied in several works, the generalized Einstein equation in terms of  $O(D, D)$  multiplets was constructed in [13] and takes the following form

$$\hat{\mathcal{G}}_{\hat{M}\hat{N}} = \hat{\mathcal{T}}_{\hat{M}\hat{N}}, \quad (3.3)$$

where  $\hat{\mathcal{T}}_{\hat{M}\hat{N}}$  is the generalized symmetric energy–momentum tensor which incorporates the effects of matter into the dynamics at the level of DFT.

The aim of this section is to find an explicit expression for  $\hat{\mathcal{T}}_{\hat{M}\hat{N}}$  written only in terms of manifestly  $O(D, D)$ -covariant quantities that recovers the well-known string cosmology upon suitable parametrization. So that we propose the inclusion of matter into the DFT formulation based on kinetic theory extending the results of [20], where the generalized

energy–momentum tensor was defined as the second moment of a generalized one–particle distribution function.

In the first part of this section we give a brief review of the phase space for the point particle in the double geometry. Then we construct the generalized energy–momentum tensor for a perfect fluid in the double geometry. This proposal is given in terms of a generalized velocity, which is consistently constrained. We present the conservation laws for the generalized energy–momentum tensor which are given by the generalized Euler equations and the generalized relativistic energy conservation equation.

### 3.1 Double phase space and the point particle

The phase space of DFT can be constructed considering an extension of the double geometry with coordinates,

$$\left\{ \hat{X}^{\hat{M}}, \hat{\mathcal{P}}^{\hat{M}} \right\}, \quad (3.4)$$

where  $\hat{\mathcal{P}}^{\hat{M}}$  is the generalized momentum. These coordinates are defined on a double tangent space, they transform as vectors with respect to  $O(D, D)$  and satisfy

$$\frac{\partial \hat{\mathcal{P}}^{\hat{M}}}{\partial \hat{X}^{\hat{N}}} = 0. \quad (3.5)$$

The strong constraint must also be extended as

$$\left( \frac{\partial}{\partial \hat{\mathcal{P}}^{\hat{M}}} \star \right) \left( \frac{\partial}{\partial \hat{\mathcal{P}}^{\hat{M}}} \star \right) = \frac{\partial}{\partial \hat{\mathcal{P}}^{\hat{M}}} \left( \frac{\partial}{\partial \hat{\mathcal{P}}^{\hat{M}}} \star \right) = 0, \quad (3.6)$$

$$\left( \frac{\partial}{\partial \hat{X}^{\hat{M}}} \star \right) \left( \frac{\partial}{\partial \hat{\mathcal{P}}^{\hat{M}}} \star \right) = \frac{\partial}{\partial \hat{X}^{\hat{M}}} \left( \frac{\partial}{\partial \hat{\mathcal{P}}^{\hat{M}}} \star \right) = 0. \quad (3.7)$$

Equations (3.6) and (3.7) guarantee that the generalized diffeomorphisms on the phase space,

$$\delta_{\hat{\xi}} V^{\hat{Q}}(\hat{X}, \hat{\mathcal{P}}) = \mathcal{L}_{\hat{\xi}} V^{\hat{Q}}(\hat{X}, \hat{\mathcal{P}}) + \hat{\mathcal{P}}^{\hat{N}} \frac{\partial \hat{\xi}^{\hat{M}}}{\partial \hat{X}^{\hat{N}}} \frac{\partial V^{\hat{Q}}(\hat{X}, \hat{\mathcal{P}})}{\partial \hat{\mathcal{P}}^{\hat{M}}} - \hat{\mathcal{P}}^{\hat{N}} \frac{\partial \hat{\xi}_{\hat{N}}}{\partial \hat{X}^{\hat{M}}} \frac{\partial V^{\hat{Q}}(\hat{X}, \hat{\mathcal{P}})}{\partial \hat{\mathcal{P}}^{\hat{M}}}, \quad (3.8)$$

satisfy a closure condition,

$$\left[ \delta_{\hat{\xi}_1}, \delta_{\hat{\xi}_2} \right] V^{\hat{M}}(\hat{X}, \hat{\mathcal{P}}) = -\delta_{\hat{\xi}_{12}} V^{\hat{M}}(\hat{X}, \hat{\mathcal{P}}), \quad (3.9)$$

where  $\hat{\xi}_{12}^M(\hat{X})$  is given by the C-bracket (eq. (2.5)). In (3.8) each diffeomorphism parameter depends only on the space-time coordinates,  $\hat{\xi}^{\hat{M}} = \hat{\xi}^{\hat{M}}(\hat{X})$ , and  $V^{\hat{Q}}(\hat{X}, \hat{\mathcal{P}})$  is a generic vector on the double phase space.

The generalized energy-momentum tensor is a tensor in the double space,

$$\hat{\mathcal{T}}^{\hat{M}\hat{N}}(\hat{X}) = \int \hat{\mathcal{P}}^{\hat{M}} \hat{\mathcal{P}}^{\hat{N}} \hat{F} e^{-2\hat{d}} d^{2D} \hat{\mathcal{P}}, \quad (3.10)$$

where  $\hat{F} = \hat{F}(\hat{X}, \hat{\mathcal{P}})$  is the one-particle generalized distribution function. This function transforms as a phase space scalar. Indeed the generalized energy-momentum tensor fulfills a conservation-like equation

$$\nabla_{\hat{M}} \hat{\mathcal{T}}^{\hat{M}\hat{N}} = 0, \quad (3.11)$$

as it is shown in [20] by computing the divergence of the second moment of the generalized Boltzmann equation for an equilibrium state.

On the other hand, the generalized momentum of a particle encodes the physics of an ordinary  $D$ -dimensional momentum  $p_\mu = g_{\mu\nu} p^\nu$  so we need to impose a strong-constraint like equation,

$$\hat{\mathcal{P}}^{\hat{M}} \hat{\eta}_{\hat{M}\hat{N}} \hat{\mathcal{P}}^{\hat{N}} = 0, \quad (3.12)$$

in order to get the correct number of degrees of freedom when parametrizing. In fact the equation (3.12) takes the same form as the level matching condition in toroidal compactifications [4]. Similarly, the mass-shell like condition is given by

$$-\hat{\mathcal{P}}^{\hat{M}} \hat{\mathcal{H}}_{\hat{M}\hat{N}} \hat{\mathcal{P}}^{\hat{N}} = m^2. \quad (3.13)$$

which takes the form of the mass-squared operator. When  $\mathcal{P}^{\hat{M}}$  arranges the  $D$ -dimensional momentum and winding modes for a closed string in a toroidal background, this object is an  $O(D, D, Z)$  multiplet [24] and the equation (3.13) reproduces the spectrum of the massless states of the string. Here the generalized momentum describes the momentum of a particle in the double geometry and, therefore, it is an  $O(D, D, R)$  multiplet. Furthermore, the parametrization of the phase-space diffeomorphism (3.8) requires  $\hat{\mathcal{P}}^{\hat{M}} = (\tilde{p}^\mu, p_\mu) = (0, p_\mu)$  which is a solution of (3.12) and in turn gives rise to the ordinary

mass–shell condition,

$$-p_\mu g^{\mu\nu} p_\nu = m^2, \quad (3.14)$$

through equation (3.13).

Additionally the generalized velocity for the particle in the double geometry is given by

$$\hat{U}^{\hat{M}} = \frac{D\hat{X}^{\hat{M}}}{D\tau}, \quad (3.15)$$

where  $DX^M$  is DFT compatible 1–form differential [21] and  $\tau$  is an affine parameter in the double geometry, which reduces to the standard proper time upon parametrization. Both the generalized momentum  $\hat{\mathcal{P}}^{\hat{M}}$ , evaluated on the trajectory of a point–particle, and the generalized velocity  $\hat{U}^{\hat{M}}$  are vectors that indicate the same geometric direction in the tangent space and then they must be proportional. Choosing the proportionality constant is, in fact, defining an specific affine parameter  $\tau$  in (3.15). Indeed, we take the relation

$$\hat{\mathcal{P}}^{\hat{M}} = m \hat{U}^{\hat{M}} = m D\hat{X}^{\hat{M}}/D\tau, \quad (3.16)$$

where  $m$  is the invariant mass in (3.13).

It is straightforward to rewrite the properties of the generalized momentum in terms of the generalized velocity as

$$\hat{U}_{\hat{M}} \hat{\mathcal{H}}^{\hat{M}\hat{N}} \hat{U}_{\hat{N}} = -1 \quad (3.17)$$

$$\hat{U}_{\hat{M}} \hat{\eta}^{\hat{M}\hat{N}} \hat{U}_{\hat{N}} = 0. \quad (3.18)$$

In the next part of this section we take advantage of the phase–space construction of DFT and we give a proposal for the generalized energy–momentum tensor of a perfect fluid in the double space. This is a simple proposal to effectively describe matter contributions in DFT such as fermionic [25] or Ramond–Ramond contributions [26], among others.

## 3.2 The perfect fluid in the double geometry

A relevant ingredient for including a perfect fluid in DFT is the generalized symmetric energy–momentum tensor  $\mathcal{T}_{\hat{M}\hat{N}}$ . In our case its index structure is given only by the



fundamental 2–index tensors of DFT,  $\hat{\mathcal{H}}_{\hat{M}\hat{N}}$  and  $\hat{\eta}_{\hat{M}\hat{N}}$ , and the generalized velocity vector of the fluid  $\hat{U}_{\hat{M}}$  which defines the temporal direction upon the split (2.13). Here we give a first proposal inspired by the structure of the ordinary energy–momentum tensor in the Riemannian geometries, i.e.,

$$T_{\mu\nu} = (p + e)u_{\mu}u_{\nu} + pg_{\mu\nu}. \quad (3.19)$$

While the second term of the previous expression can be easily generalized replacing  $g_{\mu\nu}$  by  $\hat{\mathcal{H}}_{\hat{M}\hat{N}}$ , the first term is ambiguous in terms of the DFT projections,

$$b_1 U_{\hat{M}} U_{\hat{N}} + b_2 U_{\hat{M}} U_{\hat{N}} + b_3 U_{\hat{M}} U_{\hat{N}} + b_4 U_{\hat{M}} U_{\hat{N}}. \quad (3.20)$$

The dynamics of the perfect fluid in Riemannian geometries is equivalent to the dynamics of a scalar field [27]. Since the dynamics for the generalized scalar field is given by a straightforward generalization of the Klein–Gordon equation it is expected that the minimal proposal for the generalized energy–momentum of the perfect fluid contains non-trivial  $b_2$  and  $b_3$ , which are present in the generalized energy–momentum tensor of the generalized scalar field (see eq. (4.24) in [20]). Therefore our proposal for the generalized energy–momentum tensor reads

$$\hat{\mathcal{T}}_{\hat{M}\hat{N}} = B (\hat{U}_{\hat{M}} \hat{U}_{\hat{N}} + \hat{U}_{\hat{M}} \hat{U}_{\hat{N}}) + C \hat{\mathcal{H}}_{\hat{M}\hat{N}}, \quad (3.21)$$

where we have neglected the  $b_1$  and  $b_4$  and possible dilatonic contributions. Each term of (3.21) transforms covariantly under T–duality through (2.2), so these transformations mix the components of the energy–momentum tensor as pointed out in [11]. The expression (3.21) hence capture all these (T–dual) tensors in a unified approach.

Using the space–time split formulation of DFT, the generalized velocity  $\hat{U}_{\hat{M}}$  splits according to

$$\hat{U}^{\hat{M}} = (\tilde{u}_0, u^0, U^M). \quad (3.22)$$

Furthermore, from the relations (3.17)–(3.18) we get

$$\hat{U}_{\hat{M}} \hat{U}^{\hat{M}} = -\frac{1}{2}, \quad \hat{U}_{\hat{M}} \hat{U}^{\hat{M}} = \frac{1}{2}. \quad (3.23)$$

Finally, we identify the quantities  $B$  and  $C$  in the generalized energy–momentum tensor (3.21) as

$$\begin{aligned} B &= 2[\tilde{e}(X) + \tilde{p}(X)] \\ C &= \tilde{p}(X). \end{aligned} \quad (3.24)$$

Here we define  $\tilde{e}(X)$  and  $\tilde{p}(X)$  as the generalized notions of the energy density and pressure from a formal analogy between (3.21) and the structure of the usual energy–momentum tensor of the relativistic perfect fluid (3.19). The tilded variables in these expressions may be related to the ordinary energy density and pressure through a field redefinition as we show in Section 4.3.

On the other hand, we extend the DFT projectors in the following way,

$$\bar{h}_{\hat{M}\hat{N}} = \bar{P}_{\hat{M}\hat{N}} + 2\bar{P}_{\hat{M}}^{\hat{P}}\bar{P}_{\hat{N}}^{\hat{Q}}\hat{U}_{\hat{P}}\hat{U}_{\hat{Q}}, \quad (3.25)$$

$$\underline{h}_{\hat{M}\hat{N}} = P_{\hat{M}\hat{N}} - 2P_{\hat{M}}^{\hat{P}}P_{\hat{N}}^{\hat{Q}}\hat{U}_{\hat{P}}\hat{U}_{\hat{Q}}, \quad (3.26)$$

and therefore  $\bar{h}_{\hat{M}\hat{P}}\bar{h}^{\hat{P}}_{\hat{N}} = \bar{h}_{\hat{M}\hat{N}}$  and  $\underline{h}_{\hat{M}\hat{P}}\underline{h}^{\hat{P}}_{\hat{N}} = \underline{h}_{\hat{M}\hat{N}}$ . These projectors also satisfy the orthogonality conditions

$$\bar{h}_{\hat{M}\hat{N}}\hat{U}^{\hat{M}} = 0, \quad (3.27)$$

$$\underline{h}_{\hat{M}\hat{N}}\hat{U}^{\hat{M}} = 0. \quad (3.28)$$

The conservation law for the generalized energy–momentum tensor was deduced in [20] and takes the form,

$$\nabla_{\hat{M}} \left( 2(\tilde{e} + \tilde{p})\hat{U}^{\hat{M}}\hat{U}^{\hat{N}} + 2(\tilde{e} + \tilde{p})\hat{U}^{\hat{M}}\hat{U}^{\hat{N}} + \tilde{p}\hat{\mathcal{H}}^{\hat{M}\hat{N}} \right) = 0. \quad (3.29)$$

We can extract the generalization of the energy conservation equation for a perfect fluid and the generalization of the relativistic Euler equation in the double space from (3.29).

Indeed the equation (3.29) may be projected using the generalized velocities,  $\hat{U}_{\hat{N}}$ ,  $\hat{U}_{\hat{N}}$  and the projectors  $\bar{h}_{\hat{N}\hat{P}}$ ,  $\underline{h}_{\hat{N}\hat{P}}$ . Thus we get four equations, one per each type of projection.

First we project (3.29) using the different projections of the generalized velocity and we get

$$\hat{U}_{\hat{N}}\nabla_{\hat{M}} \left( 2(\tilde{e} + \tilde{p})\hat{U}^{\hat{M}}\hat{U}^{\hat{N}} + 2(\tilde{e} + \tilde{p})\hat{U}^{\hat{M}}\hat{U}^{\hat{N}} + \tilde{p}\hat{\mathcal{H}}^{\hat{M}\hat{N}} \right) = 0, \quad (3.30)$$

$$\hat{U}_{\hat{N}}\nabla_{\hat{M}} \left( 2(\tilde{e} + \tilde{p})\hat{U}^{\hat{M}}\hat{U}^{\hat{N}} + 2(\tilde{e} + \tilde{p})\hat{U}^{\hat{M}}\hat{U}^{\hat{N}} + \tilde{p}\hat{\mathcal{H}}^{\hat{M}\hat{N}} \right) = 0. \quad (3.31)$$

Therefore, the generalized relativistic energy conservation equation reads

$$\begin{aligned} -(\tilde{e} + \tilde{p})\nabla_{\hat{M}}\hat{U}^{\hat{M}} - U^{\hat{M}}\nabla_{\hat{M}}(\tilde{e} + \tilde{p}) + 2(\tilde{e} + \tilde{p})\hat{U}_{\hat{N}}(\hat{U}^{\hat{M}}\nabla_{\hat{M}}U^{\hat{N}} + \hat{U}^{\hat{M}}\nabla_{\hat{M}}U^{\hat{N}}) &= -\hat{U}^{\hat{M}}\nabla_{\hat{M}}\tilde{p}, \\ -(\tilde{e} + \tilde{p})\nabla_{\hat{M}}\hat{U}^{\hat{M}} - U^{\hat{M}}\nabla_{\hat{M}}(\tilde{e} + \tilde{p}) - 2(\tilde{e} + \tilde{p})\hat{U}_{\hat{N}}(\hat{U}^{\hat{M}}\nabla_{\hat{M}}U^{\hat{N}} + \hat{U}^{\hat{M}}\nabla_{\hat{M}}U^{\hat{N}}) &= -\hat{U}^{\hat{M}}\nabla_{\hat{M}}\tilde{p}. \end{aligned}$$

On the other hand the projection (3.29) with  $\bar{h}_{\hat{N}\hat{P}}$  is

$$\bar{h}_{\hat{N}\hat{P}}\nabla_{\hat{M}}\left(2(\tilde{e} + \tilde{p})\hat{U}^{\hat{M}}\hat{U}^{\hat{N}} + 2(\tilde{e} + \tilde{p})\hat{U}^{\hat{M}}\hat{U}^{\hat{N}} + \tilde{p}\hat{\mathcal{H}}^{\hat{M}\hat{N}}\right) = 0. \quad (3.32)$$

Consequently, the generalized Euler equation projected onto  $\bar{h}_{\hat{N}\hat{P}}$ , written in a duality invariant way, is

$$2(\tilde{e} + \tilde{p})\bar{h}_{\hat{N}\hat{P}}\left(\hat{U}^{\hat{M}}\nabla_{\hat{M}}\hat{U}^{\hat{N}} + \hat{U}^{\hat{M}}\nabla_{\hat{M}}\hat{U}^{\hat{N}}\right) = -\bar{h}_{\hat{N}\hat{P}}\nabla^{\hat{N}}\tilde{p}. \quad (3.33)$$

Analogously we obtain the complementary generalized Euler equation projected with  $\underline{h}_{\hat{N}\hat{P}}$ ,

$$2(\tilde{e} + \tilde{p})\underline{h}_{\hat{N}\hat{P}}\left(\hat{U}^{\hat{M}}\nabla_{\hat{M}}\hat{U}^{\hat{N}} + \hat{U}^{\hat{M}}\nabla_{\hat{M}}\hat{U}^{\hat{N}}\right) = \underline{h}_{\hat{N}\hat{P}}\nabla^{\hat{N}}\tilde{p}. \quad (3.34)$$

So far we have described how to incorporate matter into the standard formulation of DFT through the generalized energy–momentum tensor for a perfect fluid. In the next section we use this framework to explore the cosmological ansatz with matter.

## 4 Cosmological ansatz

DFT is a powerful framework to rewrite the low energy limit of ST in a T-duality covariant way and, particularly, it is possible to make contact with the vacuum solutions of string cosmology using a suitable parametrization of the DFT fields and parameters. The pioneer work [14] introduced the cosmological equations for the dilaton and the metric tensor that came from the generalized Einstein tensor, while in [13] a generalized energy–momentum tensor for the matter coupled to the double geometry was included. Our main goal in this work is to achieve the manifestly covariant system of equations that couples the vacuum fields of the double geometry to matter using a cosmological ansatz at the DFT level. Indeed, in this section we elaborate on a common cosmological DFT framework considering the matter source as a perfect fluid through (3.21).

We firstly perform a space–time split of all the multiplets and then we derive the full dynamics regarding the matter and vacuum DFT fields (4.11). Secondly we parametrized the vacuum fields as usual and write down the parametrized cosmological equations including  $g_{\mu\nu}$ ,  $\phi$  and  $b$ –field contributions (4.16). In addition we find a parametrization of the DFT matter description, (4.22) and (4.23), which is compatible with the actual pressure and energy density in the string cosmology limit [6].

#### 4.1 Space–time split DFT with matter

Considering the space–time split (2.18) with  $\mathcal{N}_M = 0$ ,  $N = 1$  and a cosmological principle for the fundamental fields, namely  $\mathcal{H}_{MN} = A_{MN}(\tilde{t}, t)$  and  $d = d(\tilde{t}, t)$ , it is straightforward to decompose the generalized Ricci scalar

$$\begin{aligned}\hat{\mathcal{R}} &= -\frac{1}{8}\partial_0 A^{MN}\partial_0 A_{MN} - \frac{1}{8}\tilde{\partial}^0 A^{MN}\tilde{\partial}^0 A_{MN} - 4\partial_{00}d \\ &\quad - 4\tilde{\partial}^{00}d + 4\partial_0 d\partial_0 d + 4\tilde{\partial}^0 d\tilde{\partial}^0 d,\end{aligned}\tag{4.1}$$

the non–vanishing components of the generalized Ricci tensor

$$\begin{aligned}\hat{\mathcal{R}}_{00} &= -\hat{\mathcal{R}}^{00} = -\frac{1}{2}\hat{\mathcal{K}}^{00} + \frac{1}{2}\hat{\mathcal{K}}_{00} \\ \hat{\mathcal{R}}_{MN} &= P_M{}^P\hat{\mathcal{K}}_{PQ}\bar{P}^Q{}_N + \bar{P}_M{}^P\hat{\mathcal{K}}_{PQ}P^Q{}_N,\end{aligned}\tag{4.2}$$

and the non–vanishing components of  $\mathcal{K}_{\hat{M}\hat{N}}$

$$\hat{\mathcal{K}}_{00} = \frac{1}{8}\partial_0 A^{KL}\partial_0 A_{KL} + 2\partial_{00}d\tag{4.3}$$

$$\hat{\mathcal{K}}^{00} = \frac{1}{8}\tilde{\partial}^0 A^{KL}\tilde{\partial}^0 A_{KL} + 2\tilde{\partial}^{00}d\tag{4.4}$$

$$\hat{\mathcal{K}}_0{}^0 = \hat{\mathcal{K}}^0{}_0 = \frac{1}{8}\partial_0 A^{KL}\tilde{\partial}^0 A_{KL}\tag{4.5}$$

$$\hat{\mathcal{K}}_{MN} = \frac{1}{4}\partial_{00}A_{MN} + \frac{1}{4}\tilde{\partial}^{00}A_{MN} - \frac{1}{2}\partial_0 d\partial_0 A_{MN} - \frac{1}{2}\tilde{\partial}^0 d\tilde{\partial}^0 A_{MN},\tag{4.6}$$

with (2.15) already imposed.

On the other hand, we consider that the matter is given by a perfect fluid in the double space, i.e.,

$$\hat{\mathcal{T}}^{\hat{M}\hat{N}} = 2(\tilde{\epsilon} + \tilde{p})\hat{U}^{\hat{M}}\hat{U}^{\hat{N}} + 2(\tilde{\epsilon} + \tilde{p})\hat{U}^{\hat{M}}\hat{U}^{\hat{N}} + \tilde{p}\hat{\mathcal{H}}^{\hat{M}\hat{N}}.\tag{4.7}$$

We choose a rest frame such that  $\hat{U}^{\hat{M}} = (0, 1, 0)$  and, therefore, the non-vanishing components of the generalized energy-momentum tensor are

$$\hat{\mathcal{T}}^{00} = (\tilde{e} + \tilde{p}) - \tilde{p} \quad (4.8)$$

$$\hat{\mathcal{T}}_{00} = -(\tilde{e} + \tilde{p}) - \tilde{p} \quad (4.9)$$

$$\hat{\mathcal{T}}_{MN} = \tilde{p} A_{MN}. \quad (4.10)$$

From (3.3), we get

$$\begin{aligned} -\tilde{p} &= 2(\partial_0 d)^2 - 2\partial_{00}d - \frac{1}{16}\partial_0 A^{MN}\partial_0 A_{MN} + (\partial_0 \leftrightarrow \tilde{\partial}^0) \\ \frac{1}{2}(\tilde{e} + \tilde{p}) &= \partial_{00}d + \frac{1}{16}\partial_0 A^{KL}\partial_0 A_{KL} - (\partial_0 \leftrightarrow \tilde{\partial}^0) \\ 0 &= -P_M{}^P \left[ \frac{1}{4}\partial_{00}A_{PQ} - \frac{1}{2}\partial_0 d \partial_0 A_{PQ} + (\partial_0 \leftrightarrow \tilde{\partial}^0) \right] \bar{P}^Q{}_N + (P \leftrightarrow \bar{P}). \end{aligned} \quad (4.11)$$

These are the space-time split DFT equations which can be used to determine cosmological solutions with matter. Strictly speaking, the expressions (4.11) are combinations of the different components of the generalized Einstein equation.

## 4.2 Parametrization

In this section we parametrize all the fundamental quantities in order to analyze the dynamics of the usual supergravity fields. Indeed the parametrization of the  $O(D-1, D-1)$  generalized metric reads

$$A_{MN} = \frac{1}{a^2} \begin{pmatrix} \delta^{ij} & -\delta^{ik}b_{kj} \\ b_{ik}\delta^{kj} & a^4\delta_{ij} - b_{ik}\delta^{kl}b_{lj} \end{pmatrix}, \quad (4.12)$$

where the spatial indices are  $i, j = 1, \dots, D-1$ . Every component only depends on the ordinary time  $t$  due to the solution to the strong constraint  $\tilde{\partial}^0 = 0$ . Naturally, it is possible to inspect the dual solution imposing  $\partial_0 = 0$ , or any other T-dual combination.

The parametrization of the generalized dilaton is given by

$$e^{-2d} = a^{D-1}e^{-2\phi}, \quad (4.13)$$

and, consequently,  $d = \phi - \frac{(D-1)}{2}\ln(a)$ . The spatial DFT projectors reads

$$P_{MN} = \frac{1}{2}(\eta_{MN} - A_{MN}), \quad \bar{P}_{MN} = \frac{1}{2}(\eta_{MN} + A_{MN}) \quad (4.14)$$

while

$$\eta_{MN} = \begin{pmatrix} 0 & \delta_j^i \\ \delta_i^j & 0 \end{pmatrix}. \quad (4.15)$$

Applying the expressions (4.12)–(4.15) to the system (4.11) is a straightforward computation, but breaking the duality group in terms of  $Gl(D)$  representations generates large expressions which are suitable to manage with a software for symbolic algebraic tensor manipulation [28]. Using this code, the parametrization of the dynamical DFT cosmological equations is given by

$$\begin{aligned} -2\tilde{p} &= 2(D-1)\dot{H} + D(D-1)H^2 - 4\ddot{\phi} + 4\dot{\phi}^2 - 4(D-1)H\dot{\phi} \\ &\quad + \frac{1}{4a^4}\delta^{ij}\delta^{kl}\dot{b}_{ik}\dot{b}_{jl} \\ -(\tilde{e} + \tilde{p}) &= (D-1)\left(\dot{H} + H^2\right) - 2\ddot{\phi} + \frac{1}{4a^4}\delta^{ij}\delta^{kl}\dot{b}_{ik}\dot{b}_{jl} \\ 0 &= \left(\dot{H} + (D-1)H^2 - 2H\dot{\phi}\right)\delta^{ij} + \frac{1}{2a^4}\delta^{ik}\delta^{jl}\delta^{mn}\dot{b}_{km}\dot{b}_{ln} \\ 0 &= \delta^{jl}\ddot{b}_{il} + \frac{1}{a^4}\delta^{jk}\delta^{lm}\delta^{no}\dot{b}_{ln}\dot{b}_{km}b_{io} + \left[(D-5)H - 2\dot{\phi}\right]\delta^{jk}\dot{b}_{ik} \\ &\quad - \left[2\dot{H} + 2(D-1)H^2 - 4H\dot{\phi}\right]\delta^{jk}b_{ik} \end{aligned} \quad (4.16)$$

where  $H = \dot{a}/a$ , and we keep the same notation for the pressure and the energy density at this supergravity level. Equations (4.16) describe the family of cosmologies that can be rewritten in a DFT framework using the generalized Einstein equation (3.3) with a generalized perfect fluid as matter. When  $\tilde{e} = 0$ ,  $\tilde{p} = 0$  and  $b_{ij} = 0$ , the vacuum solutions exactly agree with [14].

### 4.3 From DFT to string cosmology

Here we analyze the relation between the DFT cosmology with matter and string cosmologies. One possible string cosmology scenario that incorporates matter is based on the following action

$$S_{\text{SC}} = \int d^D x \sqrt{-|g|} e^{-2\phi} \left[ R + 4(\nabla\phi)^2 \right] + S_M, \quad (4.17)$$

whose fundamental gravitational fields are the metric tensor  $g_{\mu\nu}$  and the dilaton  $\phi$ , while  $b_{ij} = 0$ . Moreover  $R$  is the usual Ricci scalar and  $S_M$  is the action term of matter. The

equations of motion for the gravitational sector come from the variation of the action with respect to the metric tensor  $\delta_g S_{\text{SC}} = 0$  and to the dilaton  $\delta_\phi S_{\text{SC}} = 0$ . In both variations the matter term acts as a source, indeed the source  $\delta_g S_M$  is related to the energy–momentum tensor  $T_{\mu\nu}$  and, analogously, a dilaton charge (or a dilaton source)  $\sigma$  is given by

$$\sigma = \frac{-1}{\sqrt{-g}} \frac{\delta S_M}{\delta \varphi} = -e^{-2\varphi} \left[ \frac{\delta L_{\text{mat}}}{\delta \varphi} - 2L_{\text{mat}} \right]. \quad (4.18)$$

Regarding the usual cosmological ansatz, the equations of motion in this string cosmology framework become [6]

$$-e^{2\phi} \sigma = 2(D-1)\dot{H} + D(D-1)H^2 - 4\ddot{\phi} + 4\dot{\phi}^2 - 4(D-1)H\dot{\phi} \quad (4.19)$$

$$-e^{2\phi} \left( e + \frac{\sigma}{2} \right) = (D-1) \left( \dot{H} + H^2 \right) - 2\ddot{\phi} \quad (4.20)$$

$$e^{2\phi} \left( p - \frac{\sigma}{2} \right) = \dot{H} + (D-1)H^2 - 2H\dot{\phi}. \quad (4.21)$$

It turns out that these equations are compatible with DFT when

$$\tilde{p} = e^{2\phi} p, \quad (4.22)$$

$$\tilde{e} = e^{2\phi} e \quad (4.23)$$

and  $b_{ij} = 0$ . In this case the consistency between both approaches forces  $p = \sigma/2$ . This means that there exists a subfamily of string cosmologies that can be written in a duality covariant way. On the other hand, the inclusion of the remaining cosmologies into a DFT framework may require a Lagrangian–fluid correspondence, as in [27], since in that case both  $\sigma$  (eq. (4.18)) and the generalized energy–momentum tensor could be explicitly computed from first principles.

## 5 Conclusions and future directions

In this work we have extended the standard vacuum DFT construction in order to include matter through a generalized energy–momentum tensor, which mimics the dynamics of a perfect fluid in the double geometry. We propose an ansatz for this tensor (3.21) in terms of a generalized notion of pressure, energy density and velocity, the latter satisfying

a strong constraint-like equation and a mass-shell-like condition (3.17)–(3.18). We explicitly derive the conservation laws for this tensor (3.32) and (3.33)–(3.34), which enable the possibility to explore aspects of hydrodynamics at the DFT level.

We perform a space-time split where the dependence of the fundamental fields over the dual coordinates is preserved. We impose that the generalized shift vector and the generalized lapse function satisfy  $\mathcal{N}_M = 0$  and  $N = 1$  respectively, while the generalized metric and the generalized dilaton satisfy a cosmological ansatz according to  $\mathcal{H}_{MN} = A_{MN}(\tilde{t}, t)$  and  $d = d(\tilde{t}, t)$ . This framework preserves the duality group  $O(1, 1) \times O(D - 1, D - 1)$  and allows us to rewrite string cosmology equations in a duality invariant language as shown in (4.11). Interestingly enough, the matter contributions do not affect all the components of the generalized Einstein equation, and the DFT cosmologies can be directly related to string cosmologies upon field redefinitions of  $\tilde{p}$  and  $\tilde{e}$  according to (4.22) and (4.23) when  $b_{ij} = 0$ . In a more general case, the present construction (4.11) allows one to obtain a family of duality invariant b-field contributions to string cosmology as in (4.16). The results of this work pave the way for the understanding of the inclusion of statistical matter in DFT with applications to ST.

We finish this section with a list of possible follow-up projects:

- **Duality invariant distribution function:** Making use of the construction given in [20] and considering a generalization of the Maxwell-Juttner distribution function, the generalized energy-momentum tensor for the perfect fluid might be computed from a kinetic approach. This is a broad line of investigation since the duality invariant distribution function would allow the inclusion of thermodynamic concepts in the double geometry such as the generalized entropy current or the study of equilibrium states.
- **Beyond perfect fluids in DFT:** The inclusion of a generalized entropy current from the phase space formulation should be the next step to interpret duality transformations on the generalized energy-momentum tensor as imperfect terms. At the level of the low energy limit of string theory, this issue was studied in [11].
- **$\alpha'$ -corrections:** A systematic way to introduce these corrections in a DFT back-



ground was given in [19]. The corrections are related to higher-derivative terms which deform the generalized Einstein tensor when matter terms are neglected. It would be interesting to test this procedure when matter deformations are taken into account in the double geometry. This might bring new perspectives for string cosmology.

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