On the Hydrogen Atom in the Holographic Universe

S. Jalalzadeh¹, S. Abarghouei Nejad¹ and P.V. Moniz^{2,3}

¹Departamento de Física, Universidade Federal de Pernambuco, Recife, PE, 52171-900, Brazil

²Departmento de Física, Universidade da Beira Interior, 6200 Covilhã, Portugal

³Centro de Matemática e Aplicações (CMA-UBI), Covilhã, Portugal

E-mail: shahram.jalalzadeh@ufpe.br

E-mail: salmanabar@df.ufpe.br

E-mail: pmoniz@ubi.pt

Abstract. We investigate the holographic bound utilizing a homogeneous, isotropic, and non-relativistic neutral hydrogen gas present in the de Sitter space. Concretely, we propose to employ de Sitter holography intertwined with quantum deformation of the hydrogen atom using the framework of quantum groups. Particularly, the $\mathcal{U}_q(so(4))$ quantum algebra is used to construct a finite-dimensional Hilbert space of the hydrogen atom. As a consequence of the quantum deformation of the hydrogen atom, we demonstrate that the Rydberg constant is dependent on the de Sitter radius, L_{Λ} . This feature is then extended to obtain a finite-dimensional Hilbert space for the full set of all hydrogen atoms in the de Sitter universe. We then show that the dimension of the latter Hilbert space satisfies the holographic bound. We further show that the mass of a hydrogen atom m_{atom} , the total number of hydrogen atoms at the universe, N, and the retrieved dimension of the Hilbert space of neutral hydrogen gas, $\text{Dim}\mathcal{H}_{\text{bulk}}$, are related to the de Sitter entropy, S_{dS} , the Planck mass, m_{Planck} , the electron mass, m_{e} , and the proton mass m_{p} , by $m_{\text{atom}} \simeq m_{\text{Planck}} S_{\text{dS}}^{-\frac{1}{6}}$, $N \simeq S_{\text{dS}}^{\frac{2}{3}}$ and $\text{Dim}\mathcal{H}_{\text{bulk}} = 2^{\frac{m_{\text{e}}}{m_{\text{p}}}\alpha^2 S_{\text{dS}}}$, respectively.

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1. Introduction

The seminal result establishing a distinctive bound on the entropy of a spacelike region of spacetime was formulated initially by Bekenstein [1], and subsequently, it was gradually advanced upon fundamental research on the thermodynamic properties of massive black holes and other gravitational settings alike [2, 3, 4]. More recently, it was broadly extended into what is known as Bousso's covariant entropy conjecture [5], which conveys a well-defined holographic bound. In essence, it claims to be to be a feature of any physical theory in that $S \leq \frac{A}{4G}$, where G is the Newton's gravitational constant and A is the area bounding any region, which satisfies the spacelike projection theorem‡ [5]. These ideas were build from 't Hooft [8], Fischler [9] and Susskind [10] proposal of the holographic principle.

Let us be more concrete and start by summarizing the textbook thermodynamic Bekenstein bound for black holes plus its weak and strong forms, respectively, adopting from Smolin [11] as follows:

- The thermodynamic black hole entropy is $S_{bh} = A_{bh}/4G$, where A_{bh} is the area of the black hole horizon, constituting a crucial feature of the laws of black hole mechanics.
- The weak black hole entropy a measure of how much information can be gathered by an external observer about the interior of the black hole from measurements made outside the horizon. Besides the mass, angular momentum, and charge of the black hole, the aforementioned includes measurements of the radiation emitted by the black hole.
- The strong black hole entropy is a measure of how much information or number of degrees of freedom are encompassed in the interior region of the black hole.

These can be generalized for a bulk space $\mathcal{V}_{\text{bulk}}$ with a fixed boundary $\Sigma = \partial \mathcal{V}_{\text{bulk}}$ [11], namely:

(i) Weak holographic bound: Let \mathcal{H}_{Σ} be the Hilbert space of observables on the boundary Σ . Then

$$Dim \mathcal{H}_{\Sigma} \le e^{\frac{A_{\Sigma}}{4G}},\tag{1}$$

where A_{Σ} is the area of Σ and Dim \mathcal{H}_{Σ} is the smallest appropriate Hilbert space \mathcal{H}_{Σ} .

(ii) Holographic bound: Let $\mathcal{H}_{\text{bulk}}$ be the smallest Hilbert space of local observables measurable in the interior of a volume with boundary A in the bulk space. Then

Dim
$$\mathcal{H}_{\text{bulk}} \le \exp\left(\frac{A}{4G}\right)$$
, (2)

‡ It may be worthy of stressing the difference between the "holographic bound" and what is more appropriately known as "Bekenstein bound," i.e., S < ER, where E is the energy contained in a region of size R). Generally, S < ER < A/4G [6, 7].

where Dim $\mathcal{H}_{\text{bulk}}$ denotes the dimension of the Hilbert space. This latter pronouncement constitutes the core and essence of Bousso's covariant entropy conjecture [5], establishing a precise holographic bound for physical theories. It will be about this one that we will elaborate our paper.

In addition, the above described holographic bound was widely applied on cosmological solutions and other gravitational collapsing systems. It has been studied employing several spacetimes, enlarging the diversity of its appraisal, by means of a significant set of contributions to the literature [12, 13, 14, 15, 16, 17], all confirming it. As remarked herewith, it was further advanced by Bousso in that it may be a universal law of nature [5, 18], within a background-independent formulation and bearing a holographic description of nature at its inception. Notwithstanding its allure, the conjecture presented in [5, 18], albeit quite successful, has not yet been proven. Approaches and surveys to find a route to do so, have so far involved general relativity or standard field theory [5, 18, 12, 13, 14, 15].

Within the context conveyed by the above paragraphs, the unpretentious purpose in our manuscript is to introduce a discussion on the holographic bound from another angle. We are not saying the conjecture cannot be proven strictly in the terms advocated throughout the herewith mentioned references [12, 13, 14, 15]; we are merely proposing to bring to the discussion a different (hopefully complementary) perspective, broadening the scope of discussion by means of another framework and tools.

Being more specific, our work will look at it but within a more basic and less complicated way. Concretely, employing a twofold research that constitutes the main contribution of this paper. On the one hand, it is argued that the existence of the de Sitter (DeS) horizon, which satisfies the spacelike projection theorem, suggests a holographic realization for the hydrogen atom gas. In particular, the infinitedimensional Hilbert space of the bound states of an atom is inconsistent with the holographic principle. This motivates us to introduce and explore the features of quantum deformation (within quantum groups) as a tool to bring more evidence supporting the universality of the holographic principle and the corresponding bound. A concrete quantum algebra is used to construct a finite dimensional Hilbert space of the hydrogen atom whose Rydberg constant is then shown to be dependent on the DeS radius. On the other hand, we subsequently take a (3+1)-dimensional spacetime filled with non-relativistic matter. A finite-dimensional Hilbert space for the set of hydrogen atoms in the DeS universe is estimated. We then show that the dimension of the corresponding Hilbert space satisfies the holographic bound plus that several quantities become consequently intertwined within the dimensions of the Hilbert space and the DeS entropy.

2. Quantum deformation of the hydrogen atom

The current observational paradigm presents our universe as accelerating and the cosmic event horizon increases monotonically, asymptotic to a specific radius. Hence,

it is reasonable to employ the working assumption where the late time universe is a DeS space, with a cosmic event horizon equal to the DeS radius $L_{\Lambda} := \sqrt{\frac{3}{\Lambda}} = 16.4 \pm 0.4$ Glyr = $(1.55 \pm 0.04) \times 10^{26}$ m. So, let us assume a non-relativistic hydrogen atom located at the origin of a DeS space with local coordinates (t, r, θ, ϕ) .

The cosmological constant is very small and that implies the observable universe to be large and nearly flat, so we will consider a usual non-relativistic Hamiltonian operator of the hydrogen atom in which the fine structure, all quantum field corrections are considered as a perturbation to it. In addition, we assume that the gravitational correction of the spacetime curvature to the Schrödinger equation [19, 20, 21]

$$V_{\rm g} = \frac{m_{\rm e}}{2} R_{oioj} x^i x^j, \tag{3}$$

constitutes a small perturbation potential, with R_{0i0j} being the Riemann tensor in Fermi normal coordinates (where the metric is rectangular and has vanishing first derivatives at each point of a curve), x^i is the position of the electron in the nucleus-centered and m_e is the electron mass. The subsequent spectrum of the hydrogen atom at such spacetime is

$$E_n = -\frac{\alpha^2}{2m_e n^2} + \frac{A_{n,j}}{4\alpha^2 m_e L_{\Lambda}^2} + \mathcal{O}\left(\frac{1}{n^3}\right),\tag{4}$$

where the second term in the above expression of the energy is the energy shift of the non-relativistic correction regarding the presence of the cosmological constant [19], A_{nj} are constants dependent to the quantum numbers of the state and $\mathcal{O}(1/n^3)$ represents the fine structure, the hyperfine structure and other corrections from quantum field theory such as the Lamb shift and the anomalous magnetic dipole moment of the electron. Hence, the wavelength of an emitted photon by a hydrogen atom, λ , is given by the modified Rydberg formula

$$\frac{1}{\lambda} = \frac{m_{\rm e}\alpha^2}{4\pi} \left(\frac{1}{n_f^2} - \frac{1}{n_i^2} \right) + \frac{1}{4m_{\rm e}\alpha^2 L_{\Lambda}^2} (A_{n_i,j_i} - A_{n_f,j_f}) + \mathcal{O}(\frac{1}{n^3}), \tag{5}$$

where α is the fine structure constant and n_i, n_f are the principal quantum numbers of initial and final states involved in the transition, respectively. According to the Rydberg expression for the hydrogen atom spectra (5), the wavelength of an emitted photon between two successive states $n_i = n$ and $n_f = n - 1$, and $n \gg 1$ is given by $\lambda \simeq \frac{2\pi}{m_e\alpha^2}n^3$ for arbitrarily large values of n. Note that in obtaining this relation, by assuming $n \gg 1$ and $1/L_{\Lambda}^2 \ll 1$, we kept only the first term of the modified Rydberg formula.

Additionally, allow us to mention that an infinite-dimensional Hilbert space for an atom conveys an inconsistency with the holographic bound (2). This argument can be further elaborated as follows. In the presence of a cosmological constant Λ , any local observer eventually perceives the space as a box of size L_{Λ} . Therefore, in the presence of a cosmological horizon, as far as bound states are concerned, all adjacent transitions with $n_i \geq \left(\frac{\alpha^2 L_{\Lambda} m_e}{2\pi}\right)^{\frac{1}{3}}$ are forbidden. To clarify, let us consider a generic transition between two states n_f and n_i in which $n_i = n_f + k$ (in order to include adjacent (k = 1)

as well as others); the wavelength of the emitted photon, λ , must be less than the radius of the horizon L_{Λ} . Using the Rydberg formula, for large values of n_i and for a given k, n_i must satisfy the following inequality: $\frac{2\pi}{\alpha^2 m_e} \frac{n_i^3}{k} \leq L_{\Lambda}$. For example, with k=10 all transitions $n_i \geq \left(\frac{5\alpha^2 L_{\Lambda} m_e}{\pi}\right)^{\frac{1}{3}}$ are forbidden. Moreover, for transitions with larger k, subsequent larger values of n_i will be forbidden. Thus, there is a restriction on the maximal wavelength of the emitted photon, namely $\lambda \leq L_{\Lambda}$. Saturation of this inequality gives the highest allowed principal quantum number, namely

$$n_{\text{max}}^3 = \frac{\alpha^2 m_{\text{e}}}{2\pi} L_{\Lambda} \tag{6}$$

in DeS space. Hence, the existence of the DeS horizon, which satisfies the spacelike projection theorem, suggests a holographic realization for the hydrogen atom, whose Hilbert space ought to be finite-dimensional.

In the context conveyed throughout the previous paragraph, the purpose of the herein paper is to analyse whether a degree of holography can be suitably brought to discuss other features, namely the holographic entropy bound. Standard quantization methods, adopting (6), would merely constrain n_{max} . This obvious result, its interest notwithstanding, can be surpassed if we instead employ the features of quantum deformations (within quantum groups) as a tool.

There are a number of ways to construct a finite-dimensional Hilbert space. One method to retrieve the dimension of an Hilbert space into a finite value is through quantum deformation (by means of quantum groups) of the model, when the deformation parameter is a root of unity [22]. Deformed Hydrogen atom models are studied in different ways, such as using moyal-like noncommutative product as [23] and [24], or Kustaanheimo Stiefel transformation [25, 26]. Here, we study a q-deformatrion of dynamical symmetry of Hydrogen atom by using the quantum group $so_q(4)$. This is done (as in the case of a real deformation parameter, $q \in \mathbb{R}$, used in Refs. [26, 27, 28, 29, 30, 31]) by enlarging the corresponding symmetry group, using the Laplace-Runge-Lenz vector, and the separation of $so_q(4) = su_q(2) \otimes su_q(2)$.

Historically, quantum groups have emerged from studies on quantum integrable models, using quantum inverse scattering methods, which led to deformation of classical matrix groups and their corresponding Lie algebras [32, 33, 34]. Recently, quantum groups were found to play a major role in quantum integrable systems [35], conformal field theory [36], knot theory [37], solvable lattice models [38], topological quantum computations [39], molecular spectroscopy [40] and quantum gravity [41, 42, 43, 44, 45].

2.1. Standard tools

In our paper, we shall deal with the quantum deformation of the universal enveloping algebra, $so(4) \cong su(2) \otimes su(2)$, of the hydrogen atom. Since we are interested in $\mathcal{U}_q(su(2) \otimes su(2))$, let us review first some basic facts about the quantum group $\mathcal{U}_q(su(2))$. Explicitly, $\mathcal{U}_q(su(2))$ is a Hopf algebra over \mathbb{C} generated by set of operators

 $\{q^{J_0}, q^{-J_0}, J_{\pm}\}$ satisfying relations [46]

$$q^{J_0}J_{\pm}q^{-J_0} = q^{\pm}J_{\pm}, \quad [J_+, J_-] = [2J_0]_q$$
 (7)

where $q := \exp(\frac{2\pi i}{\mathfrak{D}})$, is the deformation parameter with $\mathfrak{D} := n_{\max} \in \mathbb{N}, \mathfrak{D} \geq 2$, and

$$[x]_q := \frac{q^x - q^{-x}}{q - q^{-1}} = \frac{\sin\left(\frac{2\pi x}{\mathfrak{D}}\right)}{\sin\left(\frac{2\pi}{\mathfrak{D}}\right)}.$$
 (8)

For $\mathfrak{D} \to \infty$ (or equivalently using the definition of $\mathfrak{D} = n_{\text{max}} \simeq L_{\Lambda}^{\frac{1}{3}} \sim \Lambda^{-\frac{1}{6}}, \Lambda \to 0$), $q \to 1$ and we recover the Lie algebra of SU(2). The Casimir operator is given by

$$J^{2} := \frac{1}{2}(J_{+}J_{-} + J_{-}J_{+}) + \frac{[2]_{q}}{2}[J_{0}]_{q}^{2}. \tag{9}$$

Let $V_j = \{|jm\rangle, m = -j(1)j\}$ be the Hilbert space of the representation theory of the $\mathcal{U}_q(su(2))$. Then

$$J_{\pm}|jm\rangle = \sqrt{[j\pm m+1]_q[j\mp m]_q}|j, m\pm 1\rangle,$$

$$J_0|jm\rangle = m|jm\rangle,$$

$$J^2|jm\rangle = [j]_q[j+1]_q|jm\rangle.$$
(10)

The invariants of $\mathcal{U}_q(su(2))$ at the root of unity are $\{J^2, J_{\pm}^{\mathfrak{D}'}, q^{\pm \mathfrak{D}'J_0}\}$ where

$$\mathfrak{D}' := \begin{cases} \mathfrak{D}/2, & \text{for even values of } \mathfrak{D}, \\ (\mathfrak{D} - 1)/2, & \text{for odd values of } \mathfrak{D}. \end{cases}$$
 (11)

At the nilpotent representation, which we are interested, $J_{+}^{\mathfrak{D}'}$ and $J_{-}^{\mathfrak{D}'}$ have the zero eigenvalue for all eigenvectors, $J_{\pm}^{\mathfrak{D}'}|jm\rangle=0$.

Let us return to the bound states of a hydrogen atom. It is well known that the Hamiltonian of the hydrogen atom

$$H = \frac{p^2}{2m_e} - \frac{\alpha}{r},\tag{12}$$

commutes with the orbital angular momentum \mathbf{L} and the Laplace-Runge-Lenz vector \mathbf{M} . Namely, $\mathbf{M} = \frac{1}{2m_{\rm e}}(\mathbf{P} \times \mathbf{L} - \mathbf{L} \times \mathbf{P}) - (\frac{\alpha \mathbf{r}}{r})$. Furthermore, H, \mathbf{L} and \mathbf{M} satisfy the following relations [47]

$$\mathbf{L}.\mathbf{M} = 0, \quad \mathbf{M}^2 = \alpha^2 + \frac{2}{m_e}(\mathbf{L}^2 + 1)H,$$
 (13)

including following algebra,

$$[L_{i}, H] = [M_{i}, H] = 0, [L_{i}, L_{j}] = i\epsilon_{ijk}L_{k}, [L_{i}, M_{j}] = i\epsilon_{ijk}M_{k}, [M_{i}, M_{j}] = -\frac{2i}{m_{e}}\epsilon_{ijk}L_{k}H.$$
 (14)

If we restrict ourselves to the bound states with energy E and replace H by E, then the vector operators \mathbf{L} and $\tilde{\mathbf{M}} := \mathbf{M}\sqrt{-\frac{m_e}{2E}}$ satisfy the so(4) commutation relations [47]. If we define the two vector operators

$$\mathbf{J}^{(1)} := \frac{1}{2}(\mathbf{L} + \tilde{\mathbf{M}}), \quad \mathbf{J}^{(2)} := \frac{1}{2}(\mathbf{L} - \tilde{\mathbf{M}}),$$
 (15)

then the components of $\mathbf{J}^{(1)}$ and $\mathbf{J}^{(2)}$ satisfy the commutation relations of two commuting sets of su(2) Lie algebras

$$[J_0^{(i)}, J_{\pm}^{(i)}] = \pm J_{\pm}^{(i)}, \quad [J_{+}^{(i)}, J_{-}^{(i)}] = 2J_0^{(i)}.$$
 (16)

Note that these two sets of generators are not independents and we have the following two identities between Casimirs of $(su(2))_{(1)}$ and $(su(2))_{(2)}$ [48]

$$C_1 := \mathbf{J}_{(1)}^2 - \mathbf{J}_{(2)}^2 = 0,$$

$$C_2 := \mathbf{J}_{(1)}^2 + \mathbf{J}_{(2)}^2 = -\frac{m_e \alpha^2}{4E} - \frac{1}{2}.$$
(17)

In fact, C_1 and C_2 are two independent Casimir operators of the original SO(4) Lie group, in which C_1 represents the orthogonality of the orbital angular momentum \mathbf{L} and the Laplace-Runge-Lenz vector \mathbf{M} , and C_2 is the Hamiltonian of the atom. If we let $|j_1m_1j_2m_2\rangle$ denote the basis vectors for the $(su(2))_{(1)} \otimes (su(2))_{(2)}$ the first Casimir in (17) implies $j_1 = j_2$ and the second Casimir gives us the Bohr formula

$$E_n = -\frac{m_e \alpha^2}{2n^2},\tag{18}$$

where we identify $2j_1 + 1 := n$ as the principal quantum number.

2.2. Quantum deformation and Hilbert space

One feasible way to define a q-deformed hydrogen atom is to quantum deform the Lie groups $(SU(2))_{(1)}$ and $(SU(2))_{(2)}$ each of them defined by (7). Also, the Pauli equations (17) have to be extended to the quantum algebra $\mathcal{U}_q(su(2))_{(1)} \otimes \mathcal{U}_q(su(2))_{(2)}$. Then, this deformation via Eqs.(9), (10) and (17) produces a quantum deformed hydrogen atom with modified Bohr formula given by

$$E_n = \frac{E_0}{4[j_1]_q[j_1+1]_q+1} = \frac{E_0}{1 + \frac{2}{\sin^2(\frac{2\pi}{\mathfrak{D}})} \left(\cos(\frac{2\pi}{\mathfrak{D}}) - \cos(\frac{2\pi n}{\mathfrak{D}})\right)},$$
 (19)

where $E_0 := -\frac{m_e \alpha^2}{2}$ is the energy of the ground state and as the undeformed case, $n := 2j_1 + 1$ is the principal quantum number. Furthermore, the discrete spectrum exhibits the same degeneracy as that of the hydrogen atom in flat space. It is clear that the q-deformed spectrum reduces to that of the ordinary hydrogen atom when Λ goes to zero. For large values of \mathfrak{D} and $n \ll \mathfrak{D}$, the expression of the q-deformed energy levels (19) allow us to compute the energy of emitted photons. The approximate value of the energy for $\mathfrak{D} \gg 1$ and $n \ll \mathfrak{D}$ in the spectrum (19) is

$$E_n \simeq R_E \left\{ -\frac{1}{n^2} - \left(\frac{1}{R_E L_\Lambda} \right)^{\frac{2}{3}} f_n \right\}, \quad f_n := 4\pi^{\frac{8}{3}} \left(\frac{1}{4n^4} - \frac{1}{3n^2} + \frac{1}{12} \right), (20)$$

where $R_{\rm E} = m_{\rm e}\alpha^2/2$ is the Rydberg energy. For values $\mathfrak{D} \gg 1$ we can use the non-deformed wave function of the non-perturbed hydrogen atom to calculate the energy shifts of the non-relativistic corrections regarding the presence of the cosmological constant and the corrections of the fine structure, the hyperfine structure and other

corrections from quantum field theory such as the Lamb shift and the anomalous magnetic dipole moment of the electron. The result is

$$E_n \simeq R_E \left\{ -\frac{1}{n^2} - f_n \left(\frac{1}{R_E L_\Lambda} \right)^{\frac{2}{3}} + \frac{A_{nj}}{8} \left(\frac{1}{R_E L_\Lambda} \right)^2 \right\} + \mathcal{O}\left(\frac{1}{n^3} \right). \tag{21}$$

The expression of the q-deformed energy levels (21) allow us to compute the energy of emitted photons. Furthermore, we can write for the emitted photons a generic expression

$$\frac{1}{\lambda} = \frac{R_E}{2\pi} \left\{ \frac{1}{n_f^2} - \frac{1}{n_i^2} + \delta f_n \left(\frac{1}{R_E L_\Lambda} \right)^{\frac{2}{3}} - \frac{1}{8} \delta A_{n,j} \left(\frac{1}{R_E L_\Lambda} \right)^2 \right\} + \mathcal{O}\left(\frac{1}{n^3} \right). \tag{22}$$

where n_i, n_f are the principal quantum numbers of initial and final states involved in the transition, respectively, $\delta f_n := f_{n_f} - f_{n_i}$ and $\delta A_{n,j} := A_{n_f,j_f} - A_{n_i,f_i}$. Note that $R_E L_\Lambda \simeq 10^{34}$ and consequently, we can see that the effect of the quantum deformation, $\delta f_n \left(\frac{1}{R_E L_\Lambda}\right)^{\frac{2}{3}} \simeq \delta f_n \times 10^{-22}$, is larger than the effect of curvature, $\frac{\delta A_{nj}}{8} \left(\frac{1}{R_E L_\Lambda}\right)^2 \simeq \delta A_{nj} \times 10^{-66}$, but both of them are too small to be measurable with current spectroscopic methods. As we will see in the next section, the real impact of the cosmological constant is in the mass of the fundamental particles, number of particles in the universe, and finally in the holographic description of the possible bound states of the atoms.

If we neglect the third term in Eq.(22), which is not relevant to the q-deformation, and if we assume large n, then we can rewrite it as

$$\frac{1}{\lambda} = R_{\infty}' \left(\frac{1}{n_f^2} - \frac{1}{n_i^2} \right),\tag{23}$$

where

$$R'_{\infty} := \frac{m_{\rm e}\alpha^2 c}{4\pi\hbar} \left\{ 1 - \frac{4\pi^{\frac{8}{3}}}{3} \left(\frac{2\hbar}{L_{\Lambda}\alpha^2 m_{\rm e}c} \right)^{\frac{2}{3}} \right\} = R_{\infty} \left\{ 1 - \frac{2^{\frac{4}{3}}\pi^2}{3} \left(\frac{1}{R_{\infty}L_{\Lambda}} \right)^{\frac{2}{3}} \right\}. \quad (24)$$

is the q-deformed Rydberg constant in the SI units. It is pertinent to emphasize herein that Eqs.(24) and (28) (please see next paragraph) show that the q-deformed Rydberg constant is a function of the number of degrees of freedom of electron in the hydrogen atom. The Rydberg constant is one of the most precisely measured physical constants, with a relative standard uncertainty of under two parts in 10^{12} . The technological (spectroscopic) challenge [49] emerges from the smallness of Λ or equally from a very large value of the cosmic event horizon, L_{Λ} , within our currently observed ranges of reach.

In more realistic circumstances, the energy density of hydrogen gas could be a source of cosmological dynamics, and we should consider the time of the apparent cosmological horizon that is a boundary hypersurface of an anti-trapped region and has a topology of \mathbb{S}^2 . Then we should replace the DeS radius L_{Λ} in Eq. (24) with the Hubble radius c/H. Then, the Hubble parameter is a dynamical variable that satisfies the Friedmann equation

$$H^{2} = H_{0}^{2} \left\{ \Omega_{0m} (1+z)^{3} + \Omega_{\Lambda} \right\}, \tag{25}$$

where z, H_0 , $\Omega_{0m} = 8\pi G \rho_m(t_0)/3H_o^2$ and $\Omega_{\Lambda} = \Lambda c^2/3H_0^2$ are the redshift, the Hubble parameter, the density parameter of the cold matter (dark matter and the hydrogen gas) and the density parameter of the cosmological constant at the present epoch, respectively. In this case, the q-deformed Rydberg constant will be a function of the redshift

$$R'_{\infty} = R_{\infty} \left\{ 1 - \frac{2^{\frac{4}{3}} \pi^2}{3} \left(\frac{H_0}{cR_{\infty}} \right)^{\frac{2}{3}} \left(\Omega_{0m} (1+z)^3 + \Omega_{\Lambda} \right)^{\frac{1}{3}} \right\}.$$
 (26)

As we find, the order of the correction term of the Rydberg constant, $R_{\infty} = 10973731.568160(21)m^{-1}$ at the recombination time, z = 1089, is in order of $\mathcal{O}(10^{-9})$ which is out of the range of current measurements. On the other hand, in the radiation dominate area, where the contribution of radiation in the Friedmann equation is given by $\Omega_{0r}(1+z)^4$, the order of correction is $\mathcal{O}(10^{-5})$, which is in the range of current measurements. Hence, regarding the measurements' current scale, the effects of q-deformation are hidden behind the last scattering surface.

We close this section mentioning that, as a result of nilpotent realization [46], $(J_{\pm}^{(1,2)})^{\mathfrak{D}'}|j_1m_1j_2m_2\rangle = 0$, the Hilbert space of q-deformed hydrogen atom is finite-dimensional

$$\mathcal{H} = \bigoplus_{n=1}^{\mathfrak{D}'} \mathcal{H}_{n},$$

$$\mathcal{H}_{n} = \left\{ |j_{1}m_{1}j_{2}m_{2}\rangle; j_{1} = j_{2} = \frac{n-1}{2}; m_{i} = -j_{i}(1)j_{i} \right\},$$

$$\mathcal{H}_{\mathfrak{D}'} = \left\{ |j_{\max}m_{1}j_{\max}m_{2}\rangle; m_{i} = -j_{\max}(1)j_{\max} \right\},$$
(27)

where $j_{\text{max}} = \frac{\mathfrak{D}'-1}{2}$ is the azimuthal quantum number of the highest exited state. Since each j labels a distinct irreducible representation of $\mathcal{U}_q(su(2))$ and the number of m_i 's $(m_i = 2j_i + 1)$ is the dimensionality of the representation the dimension of Hilbert space for a q-deformed hydrogen atom is

Dim
$$\mathcal{H} = 2^{\sum_{i=1}^{\mathfrak{D}'} n_i^2} \simeq 2^{\frac{\mathfrak{D}'^3}{3}} = 2^{\frac{R_E L_{\Lambda}}{4\pi}} = 2^{\frac{m_e \alpha^2 L_{\Lambda}}{16\pi}}.$$
 (28)

2.3. The hydrogen gas in de Sitter space

To realize the relation of the dimension of that Hilbert space with the entropy of DeS space, let us now consider a dilute gas of N hydrogen atoms (as the baryonic matter in late time universe, where dark energy or the cosmological constant dominates) with homogeneous and isotropic distribution on DeS space. The radial position, x, and the radial velocity of an atom, v, then satisfy the Hubble law $v = \frac{1}{L_{\Lambda}}x$. This suggests that the fluctuations of position and velocity of the atom satisfy the same equation, $\Delta v = \frac{1}{L_{\Lambda}}\Delta x$. The Kinetic energy fluctuations then will $\Delta K = \frac{m_{\rm p}}{2}\Delta v^2 = \frac{m_{\rm p}}{2L_{\Lambda}}\Delta x\Delta v = \frac{1}{4L_{\Lambda}}$, where $m_{\rm p} \simeq m_{\rm atom}$ is the mass of proton and at the last equality we used the uncertainty principle $\Delta p\Delta x \geqslant 1/2$. In the thermodynamical limit $\Delta K/U \simeq 1/\sqrt{N}$, where U is the rest mass of the atom [50]. The above analysis gives

$$N \simeq (m_{\rm p} L_{\Lambda})^2 \simeq 5.4 \times 10^{83},$$
 (29)

where we assumed $m_{\rm atom} \simeq m_{\rm p}$.

Let us summarise some points:

• First of all, we know that the entropy of a non-relativistic gas of particles (or dust) is proportional to the total number of particles, so for hydrogen atom gas, by considering its components as point-like particles, we have [51]

$$S_{\rm gas} \simeq N.$$
 (30)

• Furthermore, like the Bekenstein-Hawking entropy of a black hole, the DeS entropy, S_{dS} , can be written [52]

$$S_{\rm dS} = \frac{\pi L_{\Lambda}^2}{G} = 2.88 \times 10^{122}.$$
 (31)

One can interpret this entropy as the weak holographic principle in which the total number of degrees of freedom living on the horizon is bounded by one-quarter of the area in Planck units [53, 54].

• These two entropies (30) and (31) are not distinct. The total entropy of dilute gas is interrelated to the entropy of DeS space via [51]

$$N \simeq S_{\rm gas} \simeq S_{\rm dS}^{\frac{2}{3}} = 2.02 \times 10^{81},$$
 (32)

where in the last equality we used the value of $S_{\rm dS}$ from (31). The result obtained in (32) is consistent [55] with the observed value $S_{\rm gas} = (9.5 \pm 4.5) \times 10^{80}$.

• Inserting relations (30) and (32) into (29) gives us

$$m_{\rm p} \simeq \left(\frac{1}{L_{\Lambda}G}\right)^{\frac{1}{3}} \simeq m_{\rm Planck} S_{\rm dS}^{-\frac{1}{6}} \simeq \left(\frac{\hbar^2 H_0}{Gc}\right)^{\frac{1}{3}},$$
 (33)

where $m_{\rm Planck}=1/\sqrt{G}$ is the Planck mass and $H_0\simeq c/L_\Lambda$ is the current observed value of the Hubble parameter. The expression in the far right of Eq.(33) is the Weinberg formula for the mass of the nucleon [56]. Weinberg's relation may then be understood, we speculate, as the operational requirement that the mass of the hydrogen atom (or an elementary particle) be such that is not determined solely by local microphysics, but in the part by the influence of the holographic screen. As a consequence of Eqs.(32) and (33), the total mass of hydrogen dust, $M_{\rm bulk}$, can be rewritten as

$$M_{\text{bulk}} \simeq m_{\text{atom}} N = m_{\text{Planck}} S_{\text{dS}}^{\frac{1}{2}},$$
 (34)

or equivalently

$$GM_{\rm bulk}^2 \simeq S_{\rm dS}.$$
 (35)

The left-hand side of this relation is the entropy of a black hole the size of the Universe. This shows that the Universe can have no more states than a black hole of the same size.

Now, we define the total number of discrete states of all hydrogen atoms in the Universe by

$$Dim \mathcal{H}_{bulk} := (Dim \mathcal{H})^N, \tag{36}$$

which leads to

$$Dim \mathcal{H}_{\text{bulk}} = 2^{\frac{m_e}{m_p} \alpha^2 S_{\text{dS}}}.$$
 (37)

Given the value $\alpha^2 \frac{m_e}{m_p} \simeq 2.9 \times 10^{-8}$, it is clear that (37) satisfies the holographic bound (2). As it is shown in [53], the horizon of DeS is a 2-dimensional lattice where the number of cells is equal to the DeS entropy (31). Hence, Eq.(37) shows that the number of degrees of freedom of all hydrogen atoms in the universe is proportional to the number of cells on the DeS boundary. This is congruent with the holographic principle and then the holographic entropy bound, which asserts that all natural phenomena within the bulk of a region of space is fully realised by the finite set of degrees of freedom which reside on the boundary, and that this number should not be larger than one binary degree of freedom per Planck area [5, 57].

3. Conclusions and outlook

We conclude by presenting a summary, plus adding a discussion and a brief outlook.

The context that guided our herewith research was that of the holographic entropy bound, a broad conjecture to apply to all physical systems. In particular, it was proposed [12] that the total observable entropy in the Universe would be bounded by the inverse of the cosmological constant, including the case of cosmologies dominated by ordinary matter. Such assertion would constitute a universal law of nature [5, 18]: universes with a positive cosmological constant would be described by a fundamental theory with only a finite number of degrees of freedom. This is yet to be fully proved and, so far, it has been broadly tested on cosmological solutions and suitable gravitational collapsing systems, within geometrical setups, for states which have energy eigenvalue below a threshold and are localized at space region of specific width. All mentioned reports have confirmed, albeit in restricted configurations, as remarked within a significant set of contributions, namely [12, 13, 14, 15].

In more detail, the purpose of our paper was to introduce a discussion on the holographic bound but from another angle. Concretely, we proposed to employ de Sitter holography intertwined with a specific quantization of the hydrogen atom using the framework of quantum groups.

Specifically, a concrete quantum algebra (namely, $\mathcal{U}_q(so(4))$) was used to construct a Hilbert space, whose retrieved dimension is proportional to $2^{\frac{m_e\alpha^2L_{\Lambda}}{16\pi}}$. We then established that a consequence of the quantum deformation of the hydrogen atom was that the Rydberg constant becomes dependent on the de Sitter radius, L_{Λ} . We obtained a finite-dimensional Hilbert space for the full set of all hydrogen atoms in the de Sitter universe. We then showed that the dimension of the latter Hilbert space is $2^{\frac{m_e}{m_p}\alpha^2 S_{dS}}$

and it satisfies the holographic entropy bound. It is well-known that to formulate quantum electrodynamics, we just need two dimensionless constants: the first one is the fine structure constant, α , and the second one is the ratio of the electron mass to the proton mass $\beta = \frac{m_e}{m_p}$ [58]. Apart from numerical factors like the atomic number, Z, or integral quantum numbers, the whole physical properties of atoms, molecules, and solids can be determined as functions of α and β [59]. Equation (37) shows that these two parameters also play a crucial role in the holographic bound of the hydrogen atom gas. Furthermore, we also expressed that the mass of a hydrogen atom m_{atom} and the total number of atoms inside the cosmic event horizon, N, are related (through simple expressions that the holography bound conjecture endorses) to the de Sitter entropy, S_{dS} and the Planck mass, m_{Planck} , by $m_{\text{atom}} \simeq m_{\text{Planck}} S_{\text{dS}}^{-\frac{1}{6}}$, and $N \simeq S_{\text{dS}}^{\frac{2}{3}}$.

Although we used a simple model, we are confident it can be extended to the case of radiation or even more elaborated, a spin-1 field theory description within the framework we used, even if restricted to a de Sitter space. Perhaps bolder, gravitational degrees of freedom could eventually be considered within quantum groups and constructing the Hilbert space, hopefully finite. Thus, we trust the features of quantum deformation (within quantum groups) may be considered as reliable complementary tool to explore holography, herein brought in an interesting intertwined manner.

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