

Flow Equations In Arbitrary Signature

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Abstract

We discuss general bosonic configurations of four-dimensional $N = 2$ supergravity coupled to vector multiplets in (t, s) space-time. The supergravity theories with Euclidean and neutral signature are described by the so-called para-special Kähler geometry. For extremal solutions, we derive in a unified fashion, using the equations of motion, the flow equations for all space-time signatures. Demanding that the solutions with neutral and Euclidean signatures admit unbroken supersymmetry, we derive the constraints, known as the stabilisation equations, on the para-covariantly holomorphic sections expressed in terms of the adapted coordinates. The stabilisation equations expressed in terms of the para-complex sections imply generalised flow equations in terms of para-complex central charge. For Euclidean and neutral signature, it is demonstrated that solutions for either signs of gauge kinetic terms are mapped into each other via field redefinitions.

1 Introduction

The study of supersymmetric and BPS gravitational backgrounds of supergravity theories in various space-time dimensions and signatures has been an active area of research in recent years. Such backgrounds are of importance for a better understanding of the non-perturbative structure of M theory, the stringy duality symmetries as well as quantum geometry. The first step in the systematic classification of solutions admitting supersymmetry in Einstein-Maxwell theory was taken in [1] building on the earlier work of [2]. A result obtained is that supersymmetric solutions with a time-like Killing vector are the well known IWP solutions [3]. Later, a great deal of progress has been made in the study of solutions of the more general Lorentzian $N = 2$ four-dimensional supergravity theories coupled with vector multiplets. Extensive details on $N = 2$ supergravity with vector multiplets and their associated special Kähler geometry can be found in [4].

Supersymmetric solutions of the general $N = 2$ supergravity theories were first explored in [5]. A main result of [5] is that the scalar fields of the supersymmetric solutions follow attractor equations with fixed points on the horizon which are independent of their values at infinity. The solutions at the near horizon are fully determined in terms of algebraic conditions, the so-called stabilisation equations,

$$i(\bar{Z}L^I - Z\bar{L}^I) = p^I, \quad i(\bar{Z}M_I - \bar{Z}\bar{M}_I) = q_I, \quad (1.1)$$

where Z is the central charge and p^I and q_I being the magnetic and electric charges.

Static solutions with non-constant scalar fields were later constructed in [6]. The analogue of the stationary IWP solutions in the general Lorentzian four-dimensional $N = 2$ supergravity were first analysed in [7]. The special Kähler geometry of the scalar fields proved to be an essential ingredient in the analysis and the study of solutions admitting supersymmetry. A rederivation of these solutions for either sign of the gauge fields kinetic terms was performed in [8] employing spinorial geometry method as outlined in [9]. The solutions with the standard sign of gauge kinetic terms are given by

$$ds^2 = -\frac{1}{|\beta|^2} (d\tau + \sigma)^2 + |\beta|^2 (dx^2 + dy^2 + dz^2), \quad (1.2)$$

with the conditions

$$i(\bar{\beta}L^I - \beta\bar{L}^I) = \tilde{H}^I, \quad i(\bar{\beta}M_I - \beta\bar{M}_I) = H_I, \quad (1.3)$$

$$d\sigma = * \left(H_I d\tilde{H}^I - \tilde{H}^I dH_I \right) \quad (1.4)$$

where \tilde{H}^I and H_I are harmonic functions on the flat coordinate space \mathbb{R}^3 and β is a complex function. Here $*$ is the Hodge star operator on flat \mathbb{R}^3 . Note that solutions with non-canonical sign of gauge fields kinetic term have a space-like Killing vector with the harmonic functions and the $*$ are on $\mathbb{R}^{1,2}$ [8]. Composite BPS configurations were considered in [10]. There, the correspondence between BPS states in type II string theory compactified on Calabi-Yau manifolds and BPS solutions of the four-dimensional $N = 2$ supergravity were investigated.

A thorough and detailed study of four-dimensional Euclidean supersymmetric theories has recently been conducted in [12–15]. In particular, Euclidean $N = 2$ supergravity theories were obtained as dimensional reductions of five-dimensional $N = 2$ supergravity theories [16] on a time-like circle. The couplings of the Euclidean theory were found to be described in terms of a para-special Kähler geometry. More on the construction and the study of $N = 2$ Euclidean supergravity theories can be found in [17, 18].

A large class of Lorentzian four-dimensional $N = 2$ supergravities are obtainable from five-dimensional $N = 2$ supergravity theory with $(1, 4)$ signature via a reduction on a space-like circle [16]. In turn, the five-dimensional theories are reductions of $(1, 10)$ supergravity on a Calabi-Yau threefold CY_3 [19]¹. Four-dimensional supergravity theories with various space-time signatures were obtained in [20] by reducing the eleven-dimensional supergravity theories [21, 22] on a CY_3 and a circle. A rigorous study of the four-dimensional theories in various space-time signatures and the classification of their four-dimensional $N = 2$ supersymmetry algebras have been performed in [23].

In the present work we are mainly interested in the extension of the results obtained for the solutions of the Lorentzian supergravity theories to all four-dimensional supergravities with vector multiplets in various space-time signatures. Our work is organized as follows. In section two we discuss general solutions for all signatures following on the analysis of [24]. We derive flow equations for extremal solutions through the analysis of the equations of motion. In section three, using spinorial geometry, we analyse solutions for a specific choice of spinor orbit and construct the corresponding IWP-like Euclidean and neutral signature solutions. We derive the stabilisation equations in terms of the so-called adapted coordinates and also in terms para-complex variables and derive generalised flow equations. We summarize and conclude in section four.

¹We use the notation (t, s) for space-time signature where t is the number of time dimensions and s is the number of spatial dimensions.

2 General Solutions and Flow equations

Ignoring the hypermultiplets, the $N = 2$ supergravity theories have the bosonic part of their Lagrangian given by

$$e^{-1}\mathcal{L} = R - 2g_{A\bar{B}}\partial_\mu z^A \partial^\mu \bar{z}^B - \frac{\kappa^2}{2}F^I \cdot \left(\text{Im}\mathcal{N}_{IJ}F^J + \text{Re}\mathcal{N}_{IJ}\tilde{F}^J \right). \quad (2.1)$$

For theories with Lorentzian signature, the fields z^A are n complex scalar fields, F^I and \tilde{F}^I ($I = 0, \dots, n$) are two-forms representing the gauge field-strengths and their duals and $\kappa^2 = \pm 1$. The Lorentzian theories are described in terms of special Kähler geometry which can be formulated in terms of the covariantly holomorphic sections

$$V = \begin{pmatrix} L^I \\ M_I \end{pmatrix}, \quad I = 0, \dots, n, \quad (2.2)$$

satisfying

$$i\langle V, \bar{V} \rangle = i(\bar{L}^I M_I - L^I \bar{M}_I) = 1, \quad (2.3)$$

$$D_{\bar{A}}V = \left(\partial_{\bar{A}} - \frac{1}{2}\partial_{\bar{A}}K \right) V = 0. \quad (2.4)$$

where K is the Kähler potential. Defining

$$U_A = D_A V = \left(\partial_A + \frac{1}{2}\partial_A K \right) V, \quad (2.5)$$

we have the useful relations

$$M_I = \mathcal{N}_{IJ}L^J, \quad D_A M_I = \bar{\mathcal{N}}_{IJ}D_A L^J \quad (2.6)$$

and

$$\begin{aligned} \langle D_A V, \bar{V} \rangle &= \langle D_A V, V \rangle = \langle D_A V, D_B V \rangle = 0, \\ g_{A\bar{B}} &= -i\langle D_A V, D_{\bar{B}} \bar{V} \rangle. \end{aligned} \quad (2.7)$$

The theories with Euclidean and neutral signature are formulated in terms of para-special Kähler geometry. Roughly speaking, the equations of para special Kähler geometry can be obtained from those of special Kähler geometry by replacing i with ι , with $\bar{\iota} = -\iota$ and $\iota^2 = 1$.

Following the analysis of [24] for static solutions, we start by considering the following metrics

$$ds^2 = \epsilon_1 e^{2U} d\tau^2 + e^{-2U} (\epsilon_0 l^4 d\rho^2 + l^2 ds^2(\mathcal{M}_2)), \quad (2.8)$$

where U and l are functions of ρ only and \mathcal{M}_2 is a two-dimensional Einstein manifold with Ricci tensor

$$R_{ab} = kh_{ab}. \quad (2.9)$$

The two-dimensional space \mathcal{M}_2 can be represented by the metric

$$ds^2(\mathcal{M}_2) = h_{ab} dx^a dx^b = \epsilon_2 d\theta^2 + \epsilon_3 f^2 d\phi^2. \quad (2.10)$$

Here $\epsilon_i = \pm 1$, $i = 0, 1, 2, 3$. Euclidean and neutral signature solutions have $\epsilon_0 \epsilon_1 \epsilon_2 \epsilon_3 = \epsilon = 1$, while Lorentzian solutions have $\epsilon_0 \epsilon_1 \epsilon_2 \epsilon_3 = \epsilon = -1$. Moreover, the function f can take the values $\sin \theta$, $\sinh \theta$ or 1. The manifold \mathcal{M}_2 can be of spherical ($k = 1$), hyperbolic ($k = -1$) and flat topology ($k = 0$) depending on the choice of values of ϵ_2, ϵ_3 and f .

The Einstein equations of motion derived from the action (2.1) are given by

$$R_{\mu\nu} = 2g_{A\bar{B}} \partial_\mu z^A \partial_\nu \bar{z}^B + 2\kappa^2 \text{Im} \mathcal{N}_{IJ} \left(F_{\mu\lambda}^I F_{\nu}^{J\lambda} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta}^I F^{J\alpha\beta} \right). \quad (2.11)$$

The non-vanishing components of the Ricci-tensor of the metric (2.8) are

$$\begin{aligned} R_{\rho\rho} &= 2 (\partial_\rho \log l)^2 - 2\partial_\rho^2 \log l + \partial_\rho^2 U - 2 (\partial_\rho U)^2, \\ R_{\tau\tau} &= -\frac{\epsilon_0 \epsilon_1}{l^4} e^{4U} \partial_\rho^2 U, \\ R_{ab} &= \left(k - \frac{\epsilon_0}{l^2} (\partial_\rho^2 \log l - \partial_\rho^2 U) \right) h_{ab}. \end{aligned} \quad (2.12)$$

Thus one obtains from (2.11)

$$\begin{aligned} (\partial_\rho U)^2 + g_{A\bar{B}} \partial_\rho z^A \partial_\rho \bar{z}^B - e^{2U} V &= (\partial_\rho \log l)^2 - \partial_\rho^2 \log l, \\ \partial_\rho^2 U &= e^{2U} V, \\ \partial_\rho^2 \log l &= k\epsilon_0 l^2, \end{aligned} \quad (2.13)$$

where V is given by

$$V = -\frac{\epsilon_1 \kappa^2}{2} \text{Im} \mathcal{N}_{IJ} \left(e^{-4U} F_{\tau\rho}^I F_{\tau\rho}^J - \frac{\epsilon}{f^2} F_{\theta\phi}^I F_{\theta\phi}^J \right). \quad (2.14)$$

The equations (2.13) simplify further if we consider solutions with

$$(\partial_\rho \log l)^2 - \partial_\rho^2 \log l = c^2, \quad (2.15)$$

where the constant c can be regarded as a non-extremality parameter. Then (2.15) together with (2.13) imply the three possibilities $l = \frac{c}{\sinh c\rho}$, $l = \frac{c}{\cosh c\rho}$ and $l = e^{c\rho}$ for $\epsilon_0 k = 1$, $\epsilon_0 k = -1$ and $k = 0$, respectively. The Einstein equations of motion, for various topologies of \mathcal{M}_2 , reduce to

$$\begin{aligned} (\partial_\rho U)^2 + g_{A\bar{B}} \partial_\rho z^A \partial_\rho \bar{z}^B - e^{2U} V &= c^2, \\ \partial_\rho^2 U &= e^{2U} V. \end{aligned} \quad (2.16)$$

After solving for the gauge fields, the potential V takes the form

$$V = \frac{\epsilon_1 \kappa^2}{2} [\text{Im} \mathcal{N}^{MI} (\text{Re} \mathcal{N}_{MN} q_I p^N + \text{Re} \mathcal{N}_{IL} p^L (q_M - \text{Re} \mathcal{N}_{MN} p^N) - q_I q_M) + \epsilon \text{Im} \mathcal{N}_{IJ} p^I p^J] \quad (2.17)$$

where p and q represent the magnetic and electric charges. The potential V can be expressed as

$$V = -\epsilon \epsilon_1 \kappa^2 \left(|Z|^2 + g^{A\bar{B}} D_A Z D_{\bar{B}} \bar{Z} \right) \quad (2.18)$$

with Z being the central charge given by

$$Z = L^I q_I - M_I p^I, \quad (2.19)$$

where L^I and M_I are para-complex for theories with neutral and Euclidean signature and complex for Lorentzian theories. For extremal spherically symmetric Lorentzian theories with $\kappa^2 = -1$, $\epsilon = \epsilon_1 = -1$, one reproduces the results of [24] where two gradient flow equations were obtained

$$\partial_\rho U = \pm e^U |Z|, \quad (2.20)$$

$$\partial_\rho z^A = \pm 2e^U g^{A\bar{B}} \partial_{\bar{B}} |Z|. \quad (2.21)$$

These first order differential equations were also obtained in [5] using supersymmetry. It can be shown that the flow equations (2.20) and (2.21) imply all the equations of motion. The plus sign possibility on the right hand side of (2.20) and (2.21) leads to physically unacceptable solutions [10]. The same analysis holds for extremal Lorentzian solutions with non-canonical sign of gauge kinetic terms where we have $\kappa^2 = 1$, $\epsilon = \epsilon_0 = -1$. For extremal Euclidean and neutral solutions, taking $\epsilon_i = -\kappa^2 = \epsilon = 1$, one also gets the relation

$$V = |Z|^2 + g^{A\bar{B}} D_A Z D_{\bar{B}} \bar{Z} \quad (2.22)$$

and as such we obtain the same flow equations expressed in terms of the para-complex central charge.

For $c^2 \neq 0$, one obtains non-extremal solutions. A general analysis for the study of non-supersymmetric solutions in arbitrary dimensions and metric signatures was recently given in [25]. We must note that the non-extremal metric ansatz (2.8) is related by a coordinate transformation to that used in [25] and [26].

3 Supersymmetric Euclidean and Neutral Solutions

In this section, we shall consider Euclidean as well as neutral signature solutions with either sign of the gauge kinetic term in a unified setting. A systematic analysis of Euclidean and neutral solutions for Einstein-Maxwell theory were recently considered in [27]. The Killing spinor equations for these theories are given by [20]:

$$\left(\nabla_\mu - \frac{1}{2} A_\mu \gamma_5 + \frac{\kappa}{4} \gamma \cdot F^I (\text{Im} L^J + \gamma_5 \text{Re} L^J) (\text{Im} \mathcal{N})_{IJ} \gamma_\mu \right) \varepsilon = 0, \quad (3.1)$$

$$\frac{\kappa}{2} (\text{Im} \mathcal{N})_{IJ} \gamma \cdot F^J \left[\text{Im} (D_{\bar{B}} \bar{L}^I g^{A\bar{B}}) + \gamma_5 \text{Re} (D_{\bar{B}} \bar{L}^I g^{A\bar{B}}) \right] \varepsilon + \gamma^\mu \partial_\mu (\text{Re} z^A - \gamma_5 \text{Im} z^A) \varepsilon = 0, \quad (3.2)$$

with $\kappa = i$ or $\kappa = -1$. Here A is the $U(1)$ Kähler connection given by

$$A = -\frac{i}{2} (\partial_A K dz^A - \partial_{\bar{A}} K d\bar{z}^A).$$

We start our analysis by considering metrics of the form

$$ds^2 = 2 \left(\mathbf{e}^1 \mathbf{e}^{\bar{1}} + \eta^2 \mathbf{e}^2 \mathbf{e}^{\bar{2}} \right), \quad (3.3)$$

where $\eta^2 = -1$ for solutions with $(2, 2)$ space-time signature and $\eta^2 = 1$ for Euclidean solutions. Using spinorial geometry, we shall analyze equations (3.1) and (3.2) for the spinor orbit $\varepsilon = \lambda 1 + \sigma e_1$, with real functions λ and σ . The action of the Dirac matrices on the Dirac spinors is given by

$$\gamma_2 = \sqrt{2} \eta i_{e^2}, \quad \gamma_{\bar{2}} = \sqrt{2} \eta e^2 \wedge, \quad \gamma_1 = \sqrt{2} i_{e^1}, \quad \gamma_{\bar{1}} = \sqrt{2} e^1 \wedge \quad (3.4)$$

where $\{1, e_1, e_2, e_{12} = e_1 \wedge e_2\}$ is the basis space of forms on \mathbb{R}^2 . Plugging $\varepsilon = \lambda 1 + \sigma e_1$ in (3.1) and using (3.4), we obtain the following geometric conditions on the spin connection

$$\begin{aligned}
\omega_{1\bar{1}} &= \left(\partial_2 \log \frac{\lambda}{\sigma} - A_2 \right) \mathbf{e}^2 - \partial_1 \log \lambda \sigma \mathbf{e}^1 - \left(\partial_2 \log \frac{\lambda}{\sigma} - A_2 \right) \mathbf{e}^{\bar{2}} + \partial_1 \log \lambda \sigma \mathbf{e}^{\bar{1}}, \\
\omega_{2\bar{2}} &= \eta^2 \left(\left(\partial_{\bar{1}} \log \frac{\lambda}{\sigma} - A_{\bar{1}} \right) \mathbf{e}^{\bar{1}} - \left(\partial_1 \log \frac{\lambda}{\sigma} - A_1 \right) \mathbf{e}^1 + \partial_2 \log \lambda \sigma \mathbf{e}^2 - \partial_2 \log \lambda \sigma \mathbf{e}^{\bar{2}} \right), \\
\omega_{12} &= \kappa^2 (\partial_2 \log \lambda^2 - A_2) \mathbf{e}^1 + \eta^2 (\partial_1 \log \lambda^2 - A_1) \mathbf{e}^{\bar{2}}, \\
\omega_{\bar{1}\bar{2}} &= \kappa^2 (\partial_2 \log \sigma^2 + A_2) \mathbf{e}^{\bar{1}} + \eta^2 (\partial_{\bar{1}} \log \sigma^2 + A_{\bar{1}}) \mathbf{e}^{\bar{2}},
\end{aligned} \tag{3.5}$$

together with

$$\begin{aligned}
\text{Im} \mathcal{N}_{IJ} L_-^I (F_{1\bar{1}}^J - \eta^2 F_{2\bar{2}}^J) &= -\frac{\kappa \sqrt{2}}{\sigma \lambda} (\partial_1 + A_1) \sigma^2, \\
\text{Im} \mathcal{N}_{IJ} L_+^I (F_{1\bar{1}}^J + \eta^2 F_{2\bar{2}}^J) &= \frac{\sqrt{2} \bar{\kappa}}{\sigma \lambda} (\partial_1 - A_1) \lambda^2, \\
\text{Im} \mathcal{N}_{IJ} L_-^I F_{12}^J &= \frac{\kappa}{\sqrt{2} \lambda \sigma} (\partial_2 + A_2) \sigma^2, \\
\text{Im} \mathcal{N}_{IJ} L_+^I F_{12}^J &= -\frac{\kappa}{\sqrt{2} \lambda \sigma} (\partial_2 - A_2) \lambda^2.
\end{aligned} \tag{3.6}$$

where we have expressed our relations in terms of the so-called adapted coordinates (real light-cone coordinates) [15] defined as $X_{\pm} = \text{Re } X \pm \text{Im } X$. The analysis of (3.2) gives

$$\begin{aligned}
\partial_1 z_-^A &= \frac{\kappa \sigma}{\sqrt{2} \lambda} \text{Im} \mathcal{N}_{IJ} (D_{\bar{B}} \bar{L}^I g^{A\bar{B}})_- (F_{1\bar{1}}^J - \eta^2 F_{2\bar{2}}^J), \\
\partial_{\bar{1}} z_+^A &= \frac{\kappa \lambda}{\sqrt{2} \sigma} \text{Im} \mathcal{N}_{IJ} (D_{\bar{B}} \bar{L}^I g^{A\bar{B}})_+ (F_{1\bar{1}}^J + \eta^2 F_{2\bar{2}}^J), \\
\partial_2 z_+^A &= \frac{\sqrt{2} \kappa \lambda}{\sigma} \text{Im} \mathcal{N}_{IJ} (D_{\bar{B}} \bar{L}^I g^{A\bar{B}})_+ F_{12}^J, \\
\partial_2 z_-^A &= -\frac{\sqrt{2} \kappa \sigma}{\lambda} \text{Im} \mathcal{N}_{IJ} (D_{\bar{B}} \bar{L}^I g^{A\bar{B}})_- F_{12}^J.
\end{aligned} \tag{3.7}$$

One also obtains

$$(\partial_{\bar{1}} + \kappa^2 \partial_1) \lambda = (\partial_{\bar{1}} + \kappa^2 \partial_1) \sigma = (\partial_1 + \kappa^2 \partial_{\bar{1}}) z_{\pm}^A = (A_{\bar{1}} + \kappa^2 A_1) = 0. \tag{3.8}$$

The torsion free condition implies that

$$\begin{aligned}
d(\mathbf{e}^1 - \kappa^2 \mathbf{e}^{\bar{1}}) &= -d \log \lambda \sigma \wedge (\mathbf{e}^1 - \kappa^2 \mathbf{e}^{\bar{1}}) \\
d\mathbf{e}^2 &= -d \log \lambda \sigma \wedge \mathbf{e}^2.
\end{aligned} \tag{3.9}$$

All the above conditions indicates that we can introduce the coordinates τ , x , y , and z , and write

$$\mathbf{e}^1 = -\frac{\kappa}{\sqrt{2}} \left(\lambda\sigma(d\tau + \phi) + \frac{1}{\lambda\sigma} idx \right), \quad \mathbf{e}^2 = \frac{1}{\sqrt{2}\lambda\sigma} (dy + idz). \quad (3.10)$$

The metric solution obtained from (3.3) is independent of the coordinate τ , and is given by

$$ds^2 = (\lambda\sigma)^2 (d\tau + \phi)^2 + \frac{1}{(\lambda\sigma)^2} (dx^2 + \eta^2 (dy^2 + dz^2)) \quad (3.11)$$

with

$$d\phi = \frac{2}{(\lambda\sigma)^2} *_3 \left(d \log \frac{\lambda}{\sigma} - A \right). \quad (3.12)$$

Here $*_3$ is the Hodge dual defined with respect to the metric $(dx^2 + \eta^2 (dy^2 + dz^2))$. Our orientation is such that $\epsilon_{1\bar{1}2\bar{2}} = \eta^2$. Using (3.6), (3.7), (3.10), (3.12) and the relations of the para-special Kähler geometry

$$g^{A\bar{B}} D_A L^I D_{\bar{B}} \bar{L}^J = \frac{1}{2} (\text{Im} \mathcal{N})^{IJ} - \bar{L}^I L^J, \quad (3.13)$$

$$dM_I = \bar{\mathcal{N}}_{IJ} dL^J + 2A \text{Im} \mathcal{N}_{IJ} L^J, \quad (3.14)$$

we obtain after some calculation

$$F^I = d [(\sigma^2 L_+^I - \kappa^2 \lambda^2 L_-^I) (d\tau + \phi)] + *d \left(\frac{\kappa^2 L_-^I}{\sigma^2} + \frac{L_+^I}{\lambda^2} \right), \quad (3.15)$$

$$G_I = d [(\sigma^2 M_{I+} - \kappa^2 \lambda^2 M_{I-}) (d\tau + \phi)] + *d \left(\kappa^2 \frac{M_{I-}}{\sigma^2} + \frac{M_{I+}}{\lambda^2} \right), \quad (3.16)$$

where $G_I = \text{Re} \mathcal{N}_{IJ} F^J + \text{Im} \mathcal{N}_{IJ} \tilde{F}^I$. The Bianchi identities together with Maxwell equations

$$dF^I = dG_I = 0, \quad (3.17)$$

give

$$\nabla^2 \left(\frac{L_+^I}{\lambda^2} + \frac{\kappa^2 L_-^I}{\sigma^2} \right) = 0, \quad \nabla^2 \left(\frac{M_{+I}}{\lambda^2} + \frac{\kappa^2 M_{-I}}{\sigma^2} \right) = 0, \quad (3.18)$$

where $\nabla^2 = \partial_x^2 + \eta^2 (\partial_y^2 + \partial_z^2)$, thus one obtains the stabilisation equations

$$\frac{L_+^I}{\lambda^2} + \frac{\kappa^2 L_-^I}{\sigma^2} = \tilde{H}^I, \quad \frac{M_{+I}}{\lambda^2} + \frac{\kappa^2 M_{-I}}{\sigma^2} = H_I. \quad (3.19)$$

Here \tilde{H}^I and H_I are harmonic functions in the three-dimensional space with metric $dx^2 + \eta^2 (dy^2 + dz^2)$. Furthermore, (3.19) imply the relations

$$A = -\frac{\kappa^2}{2} (\lambda\sigma)^2 \left(H_I d\tilde{H}^I - \tilde{H}^I dH_I \right) + d \log \frac{\lambda}{\sigma}, \quad (3.20)$$

$$d\phi = \kappa^2 *_3 \left(H_I d\tilde{H}^I - \tilde{H}^I dH_I \right). \quad (3.21)$$

In what follows, we recast the solutions in terms of para-complex functions. For definiteness, we consider the case with $\kappa^2 = -1$ and introduce the para-complex functions

$$\mathbb{Y}^I = \bar{\beta}L^I, \quad \mathbb{F}_I = \bar{\beta}M_I, \quad (3.22)$$

where L^I , M_I and β are all para-complex function. The metric (3.11) then takes the form

$$ds^2 = \frac{1}{|\beta|^2}(d\tau + \omega)^2 + |\beta|^2(dx^2 + \eta^2(dy^2 + dz^2)).$$

The function β can be related to λ and σ as

$$\beta = \frac{1}{2\lambda^2\sigma^2} [(\lambda^2 + \sigma^2) + \iota(\lambda^2 - \sigma^2)]. \quad (3.23)$$

The symplectic constraint (2.3) implies

$$e^{-2U} = |\beta|^2 = \iota(\bar{\mathbb{Y}}^I\mathbb{F}_I - \mathbb{Y}^I\bar{\mathbb{F}}_I) \quad (3.24)$$

and the stabilisation equations (3.19) take the familiar form

$$\iota(\mathbb{Y}^I - \bar{\mathbb{Y}}^I) = \tilde{H}^I, \quad \iota(\mathbb{F}_I - \bar{\mathbb{F}}_I) = H_I. \quad (3.25)$$

which can be written in a more compact form

$$2\text{Im}(\bar{\beta}V) = H, \quad H = \begin{pmatrix} \tilde{H}^I \\ H_I \end{pmatrix}. \quad (3.26)$$

Using the relations (2.7) (for para-complex variables) together with

$$dV = D_A V dz^A + \iota AV,$$

we obtain from (3.26) the relations

$$Zd\rho = (d - \iota A)\beta, \quad (3.27)$$

$$\partial_\rho z^A = -e^U g^{A\bar{B}} e^{\iota\gamma} D_{\bar{B}} \bar{Z} \quad (3.28)$$

where

$$\beta = e^{-U} e^{\iota\gamma} = e^{-U} (\cosh \gamma + \iota \sinh \gamma). \quad (3.29)$$

Note that the relation (3.27) implies

$$\partial_\rho U = -\text{Re}\left(\frac{Z}{\beta}\right), \quad (3.30)$$

$$(d\gamma - A) = \text{Im}\left(\frac{Z}{\beta}\right) d\rho. \quad (3.31)$$

The flow equations discussed in the previous section are obtained by setting

$$A = d\gamma. \quad (3.32)$$

Inspecting the stabilisation equations as expressed in terms of the adapted coordinates in (3.19) and following on the arguments presented in [18], it can be seen that the solution corresponding to $\kappa^2 = -1$ can be mapped into the solution with $\kappa^2 = 1$, via the symmetry

$$M'_{-I} = -L'_-, \quad M'_{+I} = L'_+, \quad (3.33)$$

$$L'^I_- = -M'_{-I}, \quad L'^I_+ = M'_{+I}, \quad (3.34)$$

together with interchanging magnetic with electric charges. Here the prime coordinates represent the fields in $\kappa^2 = 1$ theory. Note that in terms of para-complex sections, this transformation reads

$$\begin{pmatrix} L'^I \\ M'_I \end{pmatrix} = \iota \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} L^I \\ M_I \end{pmatrix} \quad (3.35)$$

which together with electric-magnetic duality imply

$$Z' = -\iota Z. \quad (3.36)$$

Note also that

$$\iota (\bar{L}'^I M'_I - L'^I \bar{M}'_I) = \iota (\bar{L}^I M_I - L^I \bar{M}_I) = 1. \quad (3.37)$$

It is also evident from (3.19) that solutions for $\kappa^2 = 1$ (with primed variables) can be obtained from the solutions corresponding to $\kappa^2 = -1$, via the field redefinitions

$$L'^I_+ = L^I_+, \quad L'^I_- = -L^I_-, \quad M'_{+I} = M_{+I}, \quad M'_{-I} = -M_{-I}. \quad (3.38)$$

This is the transformation discussed in [23]. In terms of the para-complex variables this is given by

$$\begin{pmatrix} L'^I \\ M'_I \end{pmatrix} = \iota \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} L^I \\ M_I \end{pmatrix} \quad (3.39)$$

and it implies that

$$Z' = \iota Z. \quad (3.40)$$

In this case we have

$$\iota (\bar{L}'^I M'_I - L'^I \bar{M}'_I) = -\iota (\bar{L}^I M - L^I \bar{M}_I) = -1. \quad (3.41)$$

The transformation (3.39) is local and connects the Lagrangians of theories with either sign of gauge kinetic terms [23]. As explained in [23], the transformation is induced by an isomorphism between the two $N = 2$ superalgebras arising from the dimensional reduction of the five-dimensional supersymmetry algebras.

As examples, we construct some explicit solutions corresponding to the minimal models defined by

$$M_I = Q_{IJ} L^J \quad (3.42)$$

with Q_{IJ} being the elements of a constant (para)-complex symmetric matrix. The stabilisation equations (3.25) lead to the solutions

$$\mathbb{Y}^K = \bar{\beta} L^K = -\frac{1}{2} \text{Im} Q^{IK} \left(\bar{Q}_{IJ} \tilde{H}^J - H_I \right),$$

We then obtain from (3.24)

$$e^{-2U} = \frac{1}{2} H^t X H \quad (3.43)$$

where

$$X = \begin{pmatrix} \text{Re} Q (\text{Im} Q)^{-1} \text{Re} Q - \text{Im} Q & -\text{Re} Q (\text{Im} Q)^{-1} \\ -(\text{Im} Q)^{-1} \text{Re} Q & (\text{Im} Q)^{-1} \end{pmatrix}. \quad (3.44)$$

Next we can construct supersymmetric solutions for theories with cubic prepotential which can be obtained from eleven dimensions via a compactification on a CY_3 and a circle. These theories can be defined by the relations

$$M_0 = -C_{IJK} \frac{L^I L^J L^K}{(L^0)^2}, \quad M_I = 3C_{IJK} \frac{L^J L^K}{L^0}. \quad (3.45)$$

The stabilization equations, for the specific choice $\tilde{H}^0 = H_I = 0$, can be solved by

$$\mathbb{Y}^I = \frac{1}{2} \iota \tilde{H}^I, \quad \mathbb{Y}^0 = \frac{1}{2} \sqrt{-\frac{C_{IJK} \tilde{H}^I \tilde{H}^J \tilde{H}^K}{H_0}} \quad (3.46)$$

and we obtain

$$e^{-2U} = 2 \sqrt{-C_{IJK} H_0 H^I H^J H^K}. \quad (3.47)$$

4 Final Remarks

In this paper we have considered solutions to four-dimensional supergravity theories in arbitrary space-time signature. The first order differential flow equations, in terms of the

(para)-complex central charge, for the metric and scalar fields were derived via the analysis of the equations of motion. The spinorial geometry methods were employed in the analysis of the Killing spinor equations and a class of supersymmetric Euclidean and neutral IWP-like solutions were derived. The solutions and in particular the stabilisation equations characterizing the evolution of the scalar fields were expressed in terms of adapted coordinates (real light-cone coordinates). The stabilisation conditions expressed in terms of para-complex variables were used to find the generalised flow equations characterizing our supersymmetric solutions. For different signs of the gauge kinetic terms, the solutions were related to each other via a field redefinition in line with the discussions in [18, 23].

BPS states in type II theory compactified on CY_3 can be described in terms of branes wrapped on various supersymmetric cycles. To match the two spectrums, multi-centered composite solutions, corresponding to multi-centered harmonic functions, must be taken into consideration [10]. Some explicit solutions of such composite solutions were also given in [11]. It would be of interest to perform similar analysis for Euclidean and neutral solutions in relation to the various CY_3 compactifications of type II string theories obtained as circle reductions of M-theory with space-time signatures $(1, 10)$, $(2, 9)$ and $(5, 6)$ and their mirrors [21]. It is also of interest to generalise the results of [27] to include vector multiplets and thus have a complete classification of supersymmetric solutions. Another important direction is the construction of supergravity theories with higher derivative terms in Euclidean and neutral signatures and the study of their BPS solutions. We hope to report on some of these research directions in future publications.

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