

# Stability of motion and thermodynamics in charged black holes in $f(T)$ gravity

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We investigate the stability of motion and the thermodynamics in the case of spherically symmetric solutions in  $f(T)$  gravity using the perturbative approach. We consider small deviations from general relativity and we extract charged black hole solutions for two charge profiles, namely with or without a perturbative correction in the charge distribution. We examine their asymptotic behavior, we extract various torsional and curvature invariants, and we calculate the energy and the mass of the solutions. Furthermore, we study the stability of motion around the obtained solutions, by analyzing the geodesic deviation, and we extract the unstable regimes in the parameter space. We calculate the inner (Cauchy) and outer (event) horizons, showing that for larger deviations from general relativity or larger charges, the horizon disappears and the central singularity becomes a naked one. Additionally, we perform a detailed thermodynamic analysis examining the temperature, entropy, heat capacity and Gibb's free energy. Concerning the heat capacity we find that for larger deviations from general relativity it is always positive, and this shows that  $f(T)$  modifications improve the thermodynamic stability, which is not the case in other classes of modified gravity.

## I. INTRODUCTION

There are both theoretical and observational motivations for the construction of gravitational modifications, namely of extended theories of gravity that possess general relativity as a particular limit, but which in general exhibit a richer structure [1]. The first is based on the fact that since general relativity is non-renormalizable one could hope that more complicated extensions of it would improve the renormalizability properties [2, 3]. The second motivation is related to the observed features of the Universe, and in particular the need to describe its two accelerated phases, namely one at early times (inflation) and one at late times (dark energy era). The usual approach in the construction of gravitational modifications is to start from the Einstein-Hilbert action and extend it in various ways [4]. Nevertheless, one can start from the equivalent torsional formulation of gravity, and in particular from the Teleparallel Equivalent of General Relativity (TEGR) [5–8], and modify it accordingly, obtaining  $f(T)$  gravity [9–11],  $f(T, T_G)$  gravity [12],  $f(T, B)$  gravity [13–15], scalar-torsion theories [16–18], etc. Torsional gravity, can lead to interesting cosmological phenomenology and hence it has attracted a large amount of research [9, 19–45].

Additionally, torsional and  $f(T)$  gravity exhibit novel and interesting black hole and spherically symmetric solutions too [46–69]. In particular, spherically symmetric solution with a constant torsion scalar  $T$  have been studied in [70–72], while cylindrically charged black holes solutions using quadratic and cubic forms of  $f(T)$  have been derived [73–76]. Moreover, by using the Noether's symmetry approach, static spherically black hole solutions have been investigated in [77]. In similar lines, the research of static spherically symmetric solutions using  $f(T)$  corrections on TEGR was the focus of interest in many studies using the perturbative approach [78–82], while vacuum regular BTZ black hole solutions in Born-Infeld gravity have been extracted in [83, 84].

Although spherically symmetric solutions in  $f(T)$  gravity has been investigated in many works, the stability of motion around them has not been examined in detail. This issue is quite crucial, having in mind that modifications of general relativity are known to present various instabilities in various regimes of the parameter space. Hence, in this work we aim to derive charged spherically symmetric solution in  $f(T)$  gravity using the perturbative approach, and then examine the stability of motion and thermodynamic properties.

The arrangement of the manuscript is as follows: in Section II we extract the charged black-hole solutions for  $f(T)$  gravity, using the perturbative approach, for two charge profiles. In Section III we study the properties of the extracted

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perturbative solutions, and in particular their asymptotic forms, the invariants, and their energy. In Section IV we proceed to the investigation of the stability of motion around the solutions, by extracting and analyzing the geodesic deviation. Moreover, in Section V we study in detail the thermodynamic properties, focusing on the temperature, entropy, heat capacity and Gibbs free energy. The final Section VI is reserved for conclusions and discussion.

## II. CHARGED BLACK HOLE SOLUTIONS IN $f(T)$ GRAVITY

Let us extract charged black hole solutions following the perturbative approach. As usual, in torsional gravity as the dynamical field we use the orthonormal tetrad, whose components are  $h_a^\mu$ , with Latin indices (from 0 to 3) denoting the tangent space and Greek indices (from 0 to 3) marking the coordinates on the manifold. The relation between the tetrad and the manifold metric is  $g_{\alpha\beta} = \eta_{ij} h^i_\alpha h^j_\beta$ , with  $\eta_{ij}$  the Minkowski metric  $\eta_{ij} = \text{diag.}(-1, +1, +1, +1)$ . The torsion tensor is given as  $T^a_{\mu\nu} := \partial_\mu h^a_\nu - \partial_\nu h^a_\mu$ .

The action of  $f(T)$  gravity, alongside a minimally coupled electromagnetic sector, is [47, 48]

$$S_{f(T)} = \int d^4x |h| \left( \frac{1}{2\kappa^2} f(T) + \mathcal{F} \right), \quad (1)$$

with  $\kappa = 8\pi G$  the gravitational constant, and  $|h| = \det(h^a_\mu) = \sqrt{-g}$ . The torsion scalar  $T$  is written as  $T = T^a_{\mu\nu} S_a^{\mu\nu}$  in terms of the superpotential  $S_a^{\mu\nu} = \frac{1}{2}(K^{\mu\nu}_a - h_a^\mu T_\lambda^{\lambda\nu} + h_a^\nu T_\lambda^{\lambda\mu})$ , with the contortion tensor being  $K^{\mu\nu}_\alpha := -\frac{1}{2}(T^{\mu\nu}_\alpha - T^{\nu\mu}_\alpha - T_\alpha^{\mu\nu})$ . Additionally,  $\mathcal{F}$  is the gauge-invariant Lagrangian of electromagnetism given as  $\mathcal{F} = \frac{1}{4} \mathcal{F}_{\alpha\beta} \mathcal{F}^{\alpha\beta}$  [85]. Variation of action (1) with respect to the tetrad yields the field equations [86]:

$$\zeta_a^\mu \equiv \frac{1}{4} f(T) h_a^\mu + f_T \left[ T^b_{\nu a} S_b^{\mu\nu} + \frac{1}{h} \partial_\nu (h S_a^{\mu\nu}) \right] + f_{TT} S_a^{\mu\nu} \partial_\nu T - \frac{1}{2} \kappa^2 \Theta_a^\mu = 0, \quad (2)$$

with  $f_T \equiv \partial f / \partial T$  and  $f_{TT} \equiv \partial^2 f / \partial T^2$ , and where the electromagnetic stress-energy tensor is

$$\Theta_a^\mu = \mathcal{F}_{a\alpha} \mathcal{F}^{\mu\alpha} - \frac{1}{4} \delta_a^\mu \mathcal{F}_{\alpha\beta} \mathcal{F}^{\alpha\beta}. \quad (3)$$

Moreover, variation of action (1) with respect to the Maxwell field gives

$$\partial_\nu (\sqrt{-g} \mathcal{F}^{\mu\nu}) = 0. \quad (4)$$

We can rewrite equation (2) purely in terms of spacetime indices by contracting with  $g_{\mu\rho}$  and  $h^a_\sigma$ , resulting to

$$H_{\sigma\rho} = \frac{1}{2} \kappa^2 \Theta_{\sigma\rho}. \quad (5)$$

The symmetric part of (5) was sourced by the energy-momentum tensor (3), while their anti-symmetric part is a vacuum constraint for the considered matter models. The latter is equal to the variation of the action with respect to the flat spin-connection components [87, 88], namely

$$H_{(\sigma\rho)} = \frac{1}{2} \kappa^2 \Theta_{(\sigma\rho)}, \quad H_{[\sigma\rho]} = 0. \quad (6)$$

The explicit forms of these equations can be seen in Eqs. (26) and (30) of [17] by setting the scalar field  $\phi$  to zero, however we do not display them here since we will derive the spherically symmetric field equations directly from (2).

We proceed by focusing on spherically symmetric solutions. Employing the spherical coordinates  $(t, r, \theta, \phi)$  we write the suitable spherically symmetric tetrad space as:

$$b^a_\mu = \begin{pmatrix} \sqrt{a} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{b}} \cos(\phi) \sin(\theta) & r \cos(\phi) \cos(\theta) & -r \sin(\phi) \sin(\theta) \\ 0 & \frac{1}{\sqrt{b}} \sin(\phi) \sin(\theta) & r \sin(\phi) \cos(\theta) & r \cos(\phi) \sin(\theta) \\ 0 & \frac{1}{\sqrt{b}} \cos(\theta) & -r \sin(\theta) & 0 \end{pmatrix}, \quad (7)$$

where  $a \equiv a(r)$  and  $b \equiv b(r)$  are two positive  $r$ -dependent functions. The above tetrad corresponds to the usual metric

$$ds^2 = -a(r) dt^2 + \frac{dr^2}{b(r)} + r^2 d\Omega^2, \quad (8)$$

with  $d\Omega^2 = (d\theta^2 + \sin^2\theta d\phi^2)$ . Using (7) the torsion scalar becomes

$$T = \frac{2 \left[ 1 - \sqrt{b(r)} \right] \left[ ra'(r) - a(r)\sqrt{b(r)} + a(r) \right]}{r^2 a(r) b(r)}. \quad (9)$$

Note that  $T$  becomes zero in the case  $a = b \rightarrow 1$ .

Inserting the above tetrad choice into the field equations (2) we acquire

$$\zeta_t^t = \frac{1}{4}f + \frac{b^{3/2}(ra' + 2a) + rab' - rba' - 2ba}{2r^2 ab^2} f_T + \frac{(\sqrt{b} - 1)}{rb} T' f_{TT} - \frac{\mathbb{Q}'^2}{2ab} = 0, \quad (10)$$

$$\zeta_r^r = \frac{\sqrt{b}(ra' + 2a) - 2(ra' + a)}{2r^2 ab} f_T + \frac{f}{4} - \frac{\mathbb{Q}'^2}{2ab} = 0, \quad (11)$$

$$\begin{aligned} \zeta_\theta^\theta = \zeta_\phi^\phi = & \frac{2a\sqrt{b} - (ra' + 2a)}{4rab} T' f_{TT} + \frac{f}{4} + \frac{\mathbb{Q}'^2}{2ab} \\ & + \frac{b(r^2 a'^2 - 6raa' - 2r^2 aa'' - 4a^2) + a[(2a + ra')(rb' + 4b^{3/2}) - 4ab^2]}{8r^2 a^2 b^2} f_T = 0, \end{aligned} \quad (12)$$

where primes denote derivatives with respect to  $r$ . In the above equations we have introduced the components of the electric field  $\mathbb{Q}_\mu = [\mathbb{Q}(r), 0, 0, 0]$ , where  $\mathcal{F}_{\mu\nu} = \mathbb{Q}_{\mu,\nu} - \mathbb{Q}_{\nu,\mu}$ . Hence, the non-vanishing components of the Maxwell field are

$$\frac{\mathbb{Q}'[a(rb' - 4b) + rba'] - 2rba\mathbb{Q}''}{2ra^2 b^2} = 0. \quad (13)$$

Note that equations (10)-(13) coincide with those of [89] when  $\mathbb{Q} = 0$ .

In the following we solve the above equations to first-order expansion around the Reissner-Nordström background, which allows us to extract analytical solutions (since in general the torsion scalar is not a constant, in which case one has the simple Reissner-Nordström solution). Hence, we assume the perturbative general vacuum charged solution as

$$a(r) = 1 - \frac{2M}{r} + \frac{s^2}{r^2} + \epsilon a_1(r), \quad (14)$$

$$b(r) = 1 - \frac{2M}{r} + \frac{s^2}{r^2} + \epsilon b_1(r), \quad (15)$$

$$\mathbb{Q}(r) = -\frac{s}{r} + \epsilon \mathbb{Q}_1(r). \quad (16)$$

Finally, concerning the  $f(T)$  function we will consider the power-law form

$$f(T) = T + \frac{1}{2}\alpha\epsilon T^2, \quad (17)$$

with  $\alpha$  the usual parameter of the  $T^2$  term and  $\epsilon \ll 1$  the small tracking parameter used to quantify the expansion in a consistent way [78, 89]. The above expression in the limit  $\epsilon \rightarrow 0$  recovers Teleparallel Equivalent of General Relativity, and it is known to be a good approximation for every realistic  $f(T)$  gravity [90–92], since the extra term quantifies the deviation from General Relativity.

Substituting (14)-(17) into (10)-(12), keeping  $\epsilon$  terms up to first order, we obtain:

$$\begin{aligned} \zeta_t^t = & \frac{\epsilon}{r^8 \varrho^2} \left\{ \alpha(\varrho - 1) [(10\varrho^2 + 5\varrho + 1)(\varrho - 1)^2 r^4 + 2s^2(8\varrho^2 + 4\varrho + 1)(\varrho - 1)r^2 + s^4(3\varrho + 1)] - r^4 s^2 a_1 - r^7 \varrho^6 b_1' \right. \\ & \left. + (\varrho^2 r^2 - 2r^2 + s^2)r^4 \varrho^4 b_1 + 2\varrho^2 r^6 s \mathbb{Q}_1' \right\} = 0, \\ \zeta_r^r = & \frac{\epsilon \left\{ \alpha(\varrho - 1)[(\varrho - 1)^2 r^2 + s^2(3\varrho - 1)][(\varrho - 1)r^2 + s^2] + r^7 \varrho^2 a_1'^2 + (\varrho^2 - 1)r^6 a_1 + r^6 \varrho^4 b_1 + 2\varrho^2 r^2 s \mathbb{Q}_1' \right\}}{r^8 \varrho^2} = 0, \\ \zeta_\theta^\theta = \zeta_\phi^\phi = & \frac{\epsilon}{4r^{10} \varrho^4} \left\{ 2\alpha\varrho [(5\varrho^2 + 4\varrho + 1)(\varrho - 1)^4 r^6 + s^2(\varrho - 1)^2(8\varrho^3 - 11\varrho^2 - 6\varrho - 3)r^4 + s^4 r^2(3 - 10\varrho^3 + 7\varrho^2) - s^6] \right. \\ & + 2\varrho^4 r^{10} a_1'' - 8\varrho^4 r^8 s \mathbb{Q}_1' + \varrho^2 r^7 [(1 - 3\varrho^2)r^2 + s^2] a_1' - \varrho^6 r^7 [(\varrho^2 + 1)r^2 - s^2] b_1' + a_1 [r^8(1 - \varrho^4) - 2r^6 s^2 + r^4 s^4] \\ & \left. + \varrho^4 b_1 [(\varrho^4 - 1)r^8 + 2r^6 s^2 - r^4 s^4] \right\} = 0, \end{aligned} \quad (18)$$

while the Maxwell field equation (13) becomes

$$2\rho^4 r^5 \mathbb{Q}'_1 + 4v\rho^4 r^4 \mathbb{Q}'_1 - s \{ \rho^2 r^3 a'_1 - \rho^6 r^3 b'_1 - [(\rho^2 - 1)r^2 + s^2](a_1 - \rho^2 b_1) \} = 0, \quad (19)$$

where  $\rho = \sqrt{1 - \frac{2M}{r} + \frac{s^2}{r^2}}$ . We solve the above equations separately in the cases where  $\mathbb{Q}_1(r) = 0$  and  $\mathbb{Q}_1(r) \neq 0$ , namely the cases with or without a perturbative correction in the charge profile.

• Case I:  $\mathbb{Q}_1(r) = 0$ :

In this case differential equations (18) and (19) admit the solution:

$$\begin{aligned} a_1(r) = & \frac{1}{s^6 r^6 \rho_{\Theta_1}} \left\{ 4sr^5 \alpha \rho \left\{ 5\rho_1^{3/2} r^2 \rho^2 \tan^{-1} \Theta + s [rM(5s^2 - 6M^2) + 2M^4 + s^2(M^2 - 2s^2)] \tan^{-1} \Theta_1 \right\} \right. \\ & + \rho_1 \left\{ 4\alpha s^2 r^5 \rho \ln \rho [(s^2 - 3M^2)r + M(M^2 + s^2)] \right. \\ & + \rho \left\{ 2r^6 \alpha \ln \rho_1 (2M^4 + M^2 s^2 - 2s^4) - s^2 \left\{ 8r^5 \alpha \ln(r) [(s^2 - 3M^2)r + M^3 + s^2 M] \right. \right. \\ & \left. \left. - s^2 [s^2 c_2 r^6 - (12\alpha M + s^2 c_1)r^5 + \alpha(4r^4 \rho_1^2 - 4/3s^4 r^2 + 3/5s^2)] \right\} \right\} \\ & \left. \left. - 20r^2 s^2 \rho \alpha \left[ r^3 (4/3s^2 - 2M^2) + 2/15Ms^4 + r^2 M(M^2 - 1/3s^2) + rs^2/3(M^2 - 1/5s^2) \right] \right\} \right\}, \quad (20) \end{aligned}$$

$$\begin{aligned} b_1(r) = & -\frac{4}{s^6 r^2 \rho^{7/2} \rho_1^2} \left\{ sr^4 \alpha \rho^{3/2} \rho_1^{5/2} \left\{ 5\rho_1^{3/2} [s^2(2M + r) - 3rM^2] \tan^{-1} \Theta \right. \right. \\ & \left. \left. + 2s [(2r + 5M)s^4 - M^2 s^2(6M + 11r) + 10rM^4] \tan^{-1} \Theta_1 \right\} \right. \\ & + \rho_1^2 \left\{ 2s^2 r^4 \alpha \rho^{3/2} [3M(M + r)s^2 - s^4 - 5rM^3] \ln \rho + r^4 \rho^{3/2} (2M^4 + 2M^2 s^2 - s^4) (2Mr - s^2) \alpha \ln \rho_1^2 \right. \\ & + s^2 \left\{ 4r^4 \rho^{3/2} \alpha \ln(r) [5M^3 r - 3Ms^2(M + r) + s^4] - 8/3\alpha r s^{10} + 1/10\alpha s^8 (23\rho^{3/2} + 40Mr^2 + 100r^3) \right. \\ & \left. + rs^6 [28\alpha r^4 - r^3/2(\rho^{3/2} c_2 + 168\alpha M) + 6\alpha r^2 M^2 - 8/3r\alpha \rho^{3/2} - 4M\alpha \rho^{3/2}] \right. \\ & \left. + r^3 s^4 \left\{ 46/3\alpha r^4 - 128Mr^3 \alpha + r^2 [(Mc_2 - c_1/2)\rho^{3/2} + 188M^2 \alpha] + 8\alpha r (65M^3/3 + 4\rho^{3/2}) + 16M\alpha \rho^{3/2} \right\} \right. \\ & \left. \left. - 2r^4 Ms^2 \alpha [2M^2 r^2 - 11Mr^3 + M\rho^{3/2} + 3r^4 + 16M^3 r] + 6r^6 M^3 \alpha (r - 2M)^2 \right\} \right\}, \quad (21) \end{aligned}$$

where  $\Theta = \frac{Mr - s^2}{s\rho}$ ,  $\Theta_1 = \frac{M-r}{\rho_1}$  and  $\rho_1 = \sqrt{M^2 - s^2}$ . Expressions (20),(21) are the solution of the field equations (2) and (4) up to  $\mathcal{O}(\epsilon)$ .

• Case II:  $\mathbb{Q}_1(r) \neq 0$ :

In this case the solution of (18) and (19) in the case  $\mathbb{Q}_1(r) \neq 0$  for the metric functions is

$$\begin{aligned} a_1(r) = & c_4 + \frac{1}{r^2 \alpha^{5/2}} \left\{ \int \frac{1}{r^6} \left\{ \alpha \left[ \rho^{1/2} r^2 (s^2 r^4 - r^6 - s^6 + r^2 s^2) + 2r\rho(3r^4 s^2 - 2r^6 - s^6) + r^2 \rho^{3/2} (5s^4 - 3r^4 + 6r^2 s^2) \right. \right. \right. \\ & \left. \left. + 2r\rho^2 (10s^4 + 8r^4 - 22r^2 s^2) + \rho^{5/2} (18s^4 + 15r^4 - 23r^2 s^2) + 2r\rho^3 (10r^2 - 9s^2) + 3r^2 \rho^{7/2} \right] \right. \\ & \left. \left. - 2r^7 s \rho^{5/2} \mathbb{Q}'_1 \right\} dr + c_3 \alpha^{5/2} \right\}, \quad (22) \end{aligned}$$

$$\begin{aligned} b_1(r) = & \frac{1}{r^4 \rho^{5/2}} \left\{ \alpha r^2 \rho^{5/2} + 4r\alpha \rho^2 (s^2 - r^2) + \rho^{3/2} [2r^6 s \mathbb{Q}'_1 + r^7 a'_1 + r^6 a_1 + 6\alpha r^4 - 10r^2 s^2 \alpha + 3\alpha s^4] \right. \\ & \left. - 4r\alpha \rho (r^2 - s^2)^2 - r^2 \rho^{1/2} [2r^2 \alpha s^2 + r^6 a_1 - \alpha s^4 - \alpha r^4] \right\}, \quad (23) \end{aligned}$$

while inserting these into (19) we finally acquire

$$\begin{aligned}
\mathbb{Q}_1(r) = & \frac{1}{15r^5 \varrho \varrho_1 s^6} \left\{ 30\alpha r^4 s \varrho \varrho_1 [4M^3 r - 2Mr s^2 - 3M^2 s^2 + s^4] \ln(r^2 \varrho^2) \right. \\
& + 30\alpha r^4 s \varrho \tanh^{-1} \left( \frac{M-r}{\varrho_1} \right) [(8M^4 - 8M^2 s^2 + s^4)r + Ms^2(5s^2 - 6M^2)] \\
& + 75\alpha r^4 \varrho \varrho_1^{5/2} \tanh^{-1} \left( \frac{Mr - s^2}{sr\varrho} \right) (5M^2 r - 4Ms^2 - s^2 r) + 300\alpha r^4 \varrho_1^2 \ln 2[(5M^2 - \epsilon^2)r - 4M\epsilon^2] \\
& - 15r^5 M\alpha s \varrho \varrho_1 \ln \varrho_1^2 (6M^2 - 5s^2) - 60r^4 \alpha s \varrho \varrho_1 \ln r [4M^3 r - 2Mr s^2 - 3M^2 s^2 + s^4] \\
& + s \varrho_1 \left\{ \varrho [15(rc_4 - c_3)r^4 s^5 + 2s^2 \alpha (3s^6 + 15r^4 s^2 - 10s^4 r^2 + 30Ms^2 r^3 - 60M^2 r^4)] \right. \\
& \quad + 5s^4 r \alpha (25r^4 - 125Mr^3 - 2s^4 + 23s^2 r^2 - 15r^2 M^2 - 6s^2 r M) \\
& \quad \left. + 25\alpha r^5 M^3 (29r - 30M) + 25Ms^2 r^4 \alpha (29M^2 - 13r^2 - 19Mr) \right\} \Bigg\}. \tag{24}
\end{aligned}$$

Hence, we have extracted the spherically symmetric solutions in the case of a quadratic deviation from Teleparallel Equivalent of General Relativity, which as we mentioned above is the first correction in every realistic  $f(T)$  gravity. We stress here that all the above expressions in the limit  $\epsilon \rightarrow 0$  recover the Schwarzschild results, hence our solutions provide the corrections on the latter brought about the  $f(T)$  gravity.

### III. PROPERTIES OF THE SOLUTIONS

In this section we examine the properties of the extracted perturbative solutions, and in particular their asymptotic forms, the invariants, and their energy.

#### A. Asymptotic forms

In the case  $\mathbb{Q}_1(r) = 0$ , the asymptotic form of the solutions (20),(21), up to  $\mathcal{O}(\epsilon)$  become

$$\begin{aligned}
a(r) \approx & 1 - \frac{2M}{r} + \frac{s^2}{r^2} \\
& + \epsilon \left\{ c_2 - \frac{c_1}{r} - \frac{\alpha}{r} \left[ \frac{68}{3M} + \frac{10M^2}{s^2 \varrho_1} - \frac{6}{\varrho_1} - \frac{76M}{3s^2} + \frac{20 \ln(2M-2s)}{s} - \frac{40M^2 \ln(2M-2s)}{s^3} \right] + \mathcal{O}\left(\frac{1}{r^5}\right) \right\}, \tag{25}
\end{aligned}$$

$$\begin{aligned}
b(r) \approx & 1 - \frac{2M}{r} + \frac{s^2}{r^2} + \epsilon \left\{ \frac{c_1}{r} - \frac{2Mc_2}{r} + \frac{\alpha}{r} \left[ \frac{68}{3M} + \frac{10M^2}{s^2 \varrho_1} - \frac{6}{\varrho_1} - \frac{76M}{3s^2} + \frac{20 \ln(2M-2s)}{s} - \frac{40M^2 \ln(2M-2s)}{s^3} \right] \right. \\
& \left. - \frac{4Mc_1 + (s^2 - 8M)s^2}{r^2} - \frac{8\alpha}{3s^3 r^2 \varrho_1} [30M(2M^2 - s^2)\varrho_1 \ln(2\varrho_1^2) - Ms(15M^2 - 9s^2) - 2s\varrho_1(17s^2 - 18M^2)] \right\}, \tag{26}
\end{aligned}$$

and thus the metric (8) becomes Minkowski for  $r \rightarrow \infty$ .

On the other hand, in the case  $\mathbb{Q}_1(r) \neq 0$ , the asymptotic forms of the solutions (22),(23) up to  $\mathcal{O}(\epsilon)$  become

$$\begin{aligned}
a(r) = & 1 - \frac{2M}{r} + \frac{s^2}{r^2} \\
& + \epsilon \left\{ c_3 \left( 1 + \frac{2}{s} \right) + \frac{c_4}{r} - \frac{\alpha}{r} \left[ \frac{8M^3 \ln(r)}{s^4} - \frac{8M \ln(r)}{s^2} - \frac{136M^3}{3s^2 \varrho_1^2} + \frac{20M^5}{s^4 \varrho_1^2} + \frac{76M}{3\varrho_1^2} \right. \right. \\
& \quad \left. \left. + 2 \tanh^{-1} \left( \frac{M}{s} \right) \left( \frac{32M^2}{s^3} - \frac{20M^4}{s^5} - \frac{12}{s} \right) \right] + \frac{2sc_3}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right) \right\}, \tag{27}
\end{aligned}$$

$$\begin{aligned}
b(r) = & 1 - \frac{2M}{r} + \frac{s^2}{r^2} \\
& - 5\epsilon \left\{ \frac{12Mc_3 + 3c_4s}{rs} - \frac{\alpha}{r} \left[ \frac{60M^3}{s^4} + \frac{24M^3 \ln(r)}{s^4} - \frac{76M}{s^2} - 4 \tanh^{-1} \left( \frac{M}{s} \right) \left( \frac{24M^2}{s^3} + \frac{9}{s} + \frac{15M^4}{s^5} \right) \right] \right. \\
& \left. + \frac{\alpha}{3s^5 r^2} \left[ 240sM^4 - 208M^2s^3 - 120M^2s^3 \ln(r) + 36s^4 - 48 \tanh^{-1} \left( \frac{M}{s} \right) (3Ms^4 - 10M^3s^2 + 5M^5) \right] \right. \\
& \left. - \frac{(16M^2 - s^3)c_3 + 4c_4sM}{sr^2} + \mathcal{O} \left( \frac{1}{r^3} \right) \right\}, \tag{28}
\end{aligned}$$

which also become Minkowski in the limit  $r \rightarrow \infty$ .

## B. Invariants

Let us examine the behavior of various invariants in the obtained solutions. For the case  $\mathbb{Q}_1(r) = 0$ , and inserting the asymptotic forms (25),(26) into the tetrad (7) and metric (8), and then into the various tensor definitions we respectively acquire the following expressions for the torsion tensor square, the torsion vector square, the torsion scalar, the Kretschmann scalar, the Ricci tensor square, and the Ricci scalar:

$$\begin{aligned}
T^{\mu\nu\lambda}T_{\mu\nu\lambda} = & \frac{16M + 8\epsilon(c_1 - 2Mc_2)}{r^3} - \frac{16s^2 - 4M^2 + \epsilon[68M^2\epsilon c_2^2 - 16s^2c_2 - 4Mc_1 + 15c_1^2\epsilon + 8M^2c_2 - 64Mc_1c_2\epsilon]}{2r^4} \\
& + \mathcal{O} \left( \frac{1}{r^5} \right), \tag{29}
\end{aligned}$$

$$T^\mu T_\mu = \frac{4(2M + \epsilon[c_1 - 2Mc_2 + \epsilon c_1c_2 - 2M\epsilon c_2^2])}{r^3} + \mathcal{O} \left( \frac{1}{r^4} \right), \tag{30}$$

$$\begin{aligned}
T(r) = & \frac{4c_2\epsilon^2(2Mc_2 - c_1)}{r^3} + \frac{4M^2 - 8s^2 - \epsilon(8M^2c_2 - 8s^2c_2 - 4Mc_1) + \epsilon^2[9c_1^2 + 76M^2c_2^2 - 16s^2c_2^2 - 56Mc_1c_2]}{2r^4} \\
& + \mathcal{O} \left( \frac{1}{r^5} \right), \tag{31}
\end{aligned}$$

$$R^{\mu\nu\lambda\rho}R_{\mu\nu\lambda\rho} = \frac{48M^2 + 48\epsilon M(c_1 - 2c_2M) + 12\epsilon^2(c_1^2 - 6Mc_1c_2 + 16M^2c_2^2)}{r^6} + \mathcal{O} \left( \frac{1}{r^7} \right), \tag{32}$$

$$R^{\mu\nu}R_{\mu\nu} = \frac{4\epsilon^2c_2s^2[c_1 - 2Mc_2]}{r^7} + \frac{2s^2[2s^2 - 4s^2\epsilon c_2 - \epsilon^2(2c_1^2 - 13Mc_1c_2 - 8s^2c_2^2 + 18M^2c_2^2)]}{r^8} + \mathcal{O} \left( \frac{1}{r^9} \right), \tag{33}$$

$$\begin{aligned}
R = & - \frac{\epsilon^2(10M^2c_2^2 - 2s^2c_2^2 - 9Mc_1c_2 + 2c_1^2)}{r^4} - \frac{\epsilon^2(48M^3c_2^2 - 24s^2Mc_2^2 - 44M^2c_1c_2 + 10Mc_1^2 + 11s^2c_1c_2)}{r^5} \\
& - \frac{\epsilon^2(168M^4c_2^2 - 128s^2Mc_2^2 + 10s^2c_2^2 - 156M^3c_1c_2 + 81Ms^2c_1c_2 + 36M^2c_1^2 - 10s^2c_1^2)}{r^6} \\
& - \frac{\epsilon^2}{r^7} \left( 112M^3c_1^2 + 512M^5c_2^2 + 372M^2s^2c_1c_2 - 16M^3\alpha c_2 + 60Ms^2c_2^2 + 96Ms^4c_2^2 \right. \\
& \left. - 31s^4c_1c_2 - 520M^3q^2c_2^2 - 480M^4c_1c_2 \right) + \frac{16\epsilon\alpha M^3}{r^7} + \mathcal{O} \left( \frac{1}{r^8} \right). \tag{34}
\end{aligned}$$

Similarly, for the case  $\mathbb{Q}_1(r) \neq 0$  we obtain the same expressions, and the only difference is in the torsion tensor square, which now becomes

$$\begin{aligned}
T^{\mu\nu\lambda}T_{\mu\nu\lambda} = & \frac{32\epsilon\alpha(3s^4 - 8M^2s^2 + 5M^4) \tanh^{-1} \left( \frac{M}{s} \right)}{r^3s^5} \\
& + \frac{24s^4[\epsilon(2Mc_1 + c_2) - 2M] + 32\epsilon M(3c_1s^3 + 19\alpha s^2 - 15\alpha M^2)}{3r^3s^4} + \mathcal{O} \left( \frac{1}{r^4} \right). \tag{35}
\end{aligned}$$

The above invariants reveal the presence of the singularity at  $r = 0$  as expected, which is more mild than the case of simpleTEGR, a known feature of higher-order torsional theories [9, 65, 69, 89].

### C. Energy

One of the advantages of teleparallel formulation of gravity is the easy handling of the energy calculations, which is not the case in usual curvature formulation [9]. We start with the gravitational energy-momentum,  $P^a$ , which in integral form in four dimensions is [93]

$$P^a = - \int_V d^3x \partial_i \Pi^{ai} f_T, \quad (36)$$

where  $V$  is the three-dimensional volume and  $\Pi^{ai} = -4\pi S^{a0i}$  is expressed in terms of the superpotential components. In the TEGR limit, namely for  $f_T = 1$ , the above expression reduces to the form given in [94].

We start with the case  $\mathbb{Q}_1(r) = 0$ . Inserting the tetrad functions (25),(26) into the tetrad (7) we can calculate the involved superpotential component as

$$S^{001} = \frac{6M^2s^3(\epsilon c_2 - 1) + \epsilon s M(76M\alpha - 3c_1s^2 - 68\alpha s^2) + 60M\epsilon\alpha \ln(2(M-s)(2M^2-s^2))}{6Mr^2s^3(1+c_2\epsilon)} + \frac{6\epsilon\alpha(3s^2-5M^2)}{s^2(1+\epsilon c_2)\varrho_1}, \quad (37)$$

which substituted into (36) leads to

$$P^0 = E \approx M + \epsilon\alpha \left\{ M \frac{15M - 38\varrho_1}{3s^2\varrho_1} - 20 \frac{(2M^2 - s^2) \ln[2(M-s)]}{s^3} \right\} - \frac{M^2 + s^2}{2r} - 5\epsilon\alpha \left\{ M \frac{15M - 38\varrho_1}{3s^2\varrho_1 r} + 20 \frac{(2M^2 - s^2) \ln[2(M-s)]}{rs^3} \right\} = \mathcal{M} + O\left(\frac{1}{r}\right), \quad (38)$$

where  $\mathcal{M} \approx M - \epsilon\alpha \frac{38M}{3s^2}$  is the Arnowitt-Deser-Misner (ADM) mass that contains  $M$  and  $\epsilon$  up to first order.

In the case  $\mathbb{Q}_1(r) \neq 0$ , the above procedure leads to

$$S^{001} = -\frac{M}{r^2} + \epsilon \left[ \frac{12\alpha(5M^4 - 8M^2s^2 + 3s^4) \tanh^{-1}\left(\frac{M}{s}\right) + 12Mc_3s^4(s+2) + 4sM\alpha(17s^2 - 15M^2) + 3c_4s^5}{s^5r^2} \right], \quad (39)$$

and then to

$$P^0 = E \approx M + \epsilon \left[ \frac{12\alpha \tanh^{-1}\left(\frac{M}{s}\right) (5M^4 - 8M^2s^2 + 3s^4) + 3Mc_3s^4(s+2) + 4sM\alpha(17s^2 - 15M^2) + 3c_4s^5}{3s^5} \right] - \frac{M^2 + s^2}{2r} - \epsilon \left[ \frac{12M\alpha \tanh^{-1}\left(\frac{M}{s}\right) (10M^4 - 16M^2s^2 + 6s^4) + 3c_3s^4(3M^2s + 6M^2 - 2s^2 + s^3) + 8sM^2\alpha(19s^2 - 15M^2) + 6c_4Ms^5}{6rs^5} \right], \quad (40)$$

$$E \approx \mathcal{M}_1 + O\left(\frac{1}{r}\right), \quad (41)$$

where  $\mathcal{M}_1 = M + \epsilon \left[ \frac{68\alpha M + 3Mc_3s(s+2)}{3s^2} \right]$ . Note that for  $\epsilon = 0$ , i.e. in the TEGR limit, we recover the well-know energy expression of Reissner-Nordström spacetime [95].

### IV. GEODESIC DEVIATION AND STABILITY OF MOTION

In this section we proceed to the examination of the stability of motion around the obtained black hole solutions, investigating the geodesic deviation. The geodesic equations of a test particle in the gravitational field are given by

$$\frac{d^2x^\alpha}{d\lambda^2} + \left\{ \begin{matrix} \alpha \\ \beta\rho \end{matrix} \right\} \frac{dx^\beta}{d\lambda} \frac{dx^\rho}{d\lambda} = 0, \quad (42)$$

where  $s$  denotes the affine connection parameter and  $\left\{ \begin{matrix} \alpha \\ \beta\rho \end{matrix} \right\}$  the Levi-Civita connection. The geodesic deviation equations acquire the form [96]

$$\frac{d^2\psi^\sigma}{d\lambda^2} + 2 \left\{ \begin{matrix} \sigma \\ \mu\nu \end{matrix} \right\} \frac{dx^\mu}{d\lambda} \frac{d\psi^\nu}{d\lambda} + \left\{ \begin{matrix} \sigma \\ \mu\nu \end{matrix} \right\}_{,\rho} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \psi^\rho = 0, \quad (43)$$

where  $\psi^\rho$  is the 4-vector deviation.

In the case of the spherically symmetric ansatz (8) the above expressions give

$$\begin{aligned}\frac{d^2 t}{d\lambda^2} &= 0, \\ \frac{1}{2}a'(r)\left(\frac{dt}{d\lambda}\right)^2 - r\left(\frac{d\phi}{d\lambda}\right)^2 &= 0, \\ \frac{d^2 \theta}{d\lambda^2} &= 0, \\ \frac{d^2 \phi}{d\lambda^2} &= 0,\end{aligned}\tag{44}$$

and therefore for the geodesic deviation we finally obtain

$$\begin{aligned}\frac{d^2 \psi^1}{d\lambda^2} + b(r)a'(r)\frac{dt}{d\lambda}\frac{d\psi^0}{d\lambda} - 2rb(r)\frac{d\phi}{d\lambda}\frac{d\psi^3}{d\lambda} + \left\{\frac{1}{2}[a'(r)b'(r) + b(r)a''(r)]\left(\frac{dt}{d\lambda}\right)^2 - [b(r) + rb'(r)]\left(\frac{d\phi}{d\lambda}\right)^2\right\}\psi^1 &= 0, \\ \frac{d^2 \psi^0}{d\lambda^2} + \frac{b'(r)}{b(r)}\frac{dt}{d\lambda}\frac{d\psi^1}{d\lambda} &= 0, \\ \frac{d^2 \psi^2}{d\lambda^2} + \left(\frac{d\phi}{d\lambda}\right)^2 \psi^2 &= 0, \\ \frac{d^2 \psi^3}{d\lambda^2} + \frac{2}{r}\frac{d\phi}{d\lambda}\frac{d\psi^1}{d\lambda} &= 0.\end{aligned}\tag{45}$$

Using the circular orbit  $\theta = \frac{\pi}{2}$ ,  $\frac{d\theta}{d\lambda} = 0$ , and  $\frac{dr}{d\lambda} = 0$ , we acquire  $\left(\frac{d\phi}{d\lambda}\right)^2 = \frac{a'(r)}{r[2a(r) - ra'(r)]}$  and  $\left(\frac{dt}{d\lambda}\right)^2 = \frac{2}{2a(r) - ra'(r)}$ , and thus equations (45) can be rewritten as

$$\begin{aligned}\frac{d^2 \psi^1}{d\phi^2} + a(r)a'(r)\frac{dt}{d\phi}\frac{d\psi^0}{d\phi} - 2ra(r)\frac{d\psi^3}{d\phi} + \left\{\frac{1}{2}[a'^2(r) + a(r)a''(r)]\left(\frac{dt}{d\phi}\right)^2 - [a(r) + ra'(r)]\right\}\psi^1 &= 0, \\ \frac{d^2 \psi^2}{d\phi^2} + \psi^2 &= 0, \\ \frac{d^2 \psi^0}{d\phi^2} + \frac{a'(r)}{a(r)}\frac{dt}{d\phi}\frac{d\psi^1}{d\phi} &= 0, \\ \frac{d^2 \psi^3}{d\phi^2} + \frac{2}{r}\frac{d\psi^1}{d\phi} &= 0.\end{aligned}\tag{46}$$

The second equation of (46) corresponds to a simple harmonic motion, which indicates that the plane  $\theta = \pi/2$  is stable. Moreover, the other equations of (46) have solutions of the form:

$$\psi^0 = \zeta_1 e^{i\sigma\phi}, \quad \psi^1 = \zeta_2 e^{i\sigma\phi}, \quad \text{and} \quad \psi^3 = \zeta_3 e^{i\sigma\phi},\tag{47}$$

where  $\zeta_1, \zeta_2$  and  $\zeta_3$  are constants. Substituting (47) into (46), we extract the stability of motion condition as:

$$\frac{3abb' - \sigma^2 ab' - 2rb^{3/2}a'^{3/2} - rab'^2 + rab'a' + raba''}{ab'} > 0.\tag{48}$$

Equation (48) has the following solution in terms of the metric potentials

$$\sigma^2 = \frac{3abb' - 2rb^{3/2}a'^{3/2} - rab'^2 + rab'a' + raba''}{a^2 b'^2} > 0.\tag{49}$$

Hence, in order to conclude on the stability of motion around the obtained black-hole solutions, for the case  $\mathbb{Q}_1(r) = 0$  in the above expressions we insert  $a(r)$  and  $b(r)$  from (25), (26), while for the case  $\mathbb{Q}_1(r) \neq 0$  from (28).

In order to present the above results in a more transparent way, in Fig. 1 we depict the behavior of  $\sigma^2$  for various choices of the model parameters, for the two cases  $\mathbb{Q}_1(r) = 0$  and  $\mathbb{Q}_1(r) \neq 0$  separately. Note that for  $\mathbb{Q}_1(r) = 0$  we always obtain stability of motion as expected, while for  $\mathbb{Q}_1(r) \neq 0$  we find potentially unstable regions.



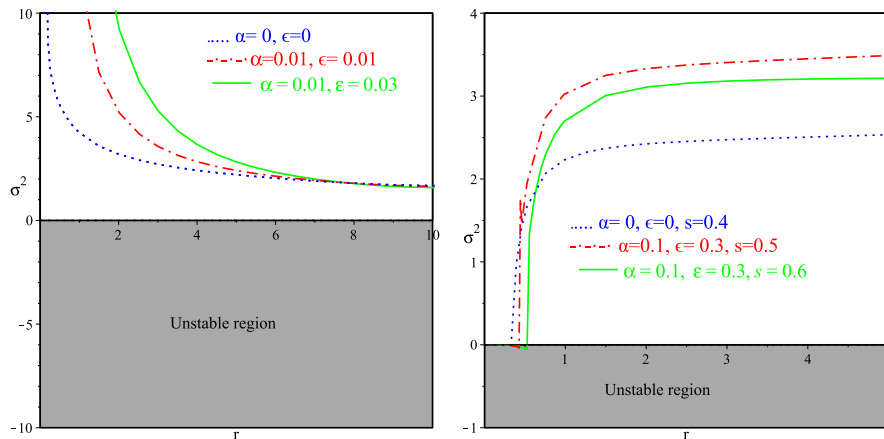


FIG. 1: The black-hole stability of motion parameter  $\sigma^2$  versus  $r$ . Left graph:  $\mathbb{Q}_1(r) = 0$  with  $M = 9$ ,  $c_1 = c_2 = s = 1$  and various choices of the model parameters  $\alpha$  and  $\epsilon$  in Planck mass units. Right graph:  $\mathbb{Q}_1(r) \neq 0$  with  $M = 0.01$ ,  $c_1 = c_2 = 1$  and various choices of the model parameters  $\alpha$ ,  $\epsilon$  and  $s$  in Planck mass units.

## V. THERMODYNAMICS

In this section we perform an analysis of the thermodynamic properties of the obtained black-hole solutions. Since the nature of the solutions and especially their thermodynamic features change for  $\mathbb{Q}_1(r) = 0$  and  $\mathbb{Q}_1(r) \neq 0$ , in the following we examine the two cases separately.

### A. Thermodynamics of the black hole solution with $\mathbb{Q}_1(r) = 0$

We start by investigating the black-hole solution of the case  $\mathbb{Q}_1(r) = 0$  given in (25),(26). In the left graph of Fig. 2 we display the metric potentials  $a(r)$  and  $b(r)$ . As we can see,  $a(r)$  may exhibit two horizons while  $b(r)$  does not. In the right graph of Fig. 2 we focus on  $a(r)$ , in order to make more transparent the behavior of its possible two horizons, acquired by solving  $a(r) = 0$ , namely  $r_-$  which denotes the inner Cauchy horizon of the black hole and  $r_+$  which is the outer event horizon. In particular, for small  $\alpha$  values, namely small deviations from general relativity, we obtain two horizons, however as  $\alpha$  increases there is a specific value in which the two horizons become degenerate ( $r_- = r_+ = r_d$ ), while for larger values the horizon disappears and the central singularity becomes a naked one. This is a known feature of torsional gravity, namely for some regions of the parameter space naked singularities appear [47, 48, 74]. Finally, let us calculate the total mass contained within the event horizon  $r_+$ . We find the mass-radius expression as

$$\mathcal{M}_+ \equiv \mathcal{M}(\nabla_+) \approx \frac{r_+}{2}, \quad (50)$$

where  $\mathcal{M}_+$  is given by Eq. (38) for  $r_+$  in place of  $r$ .

We proceed by examining the temperature. The Hawking black-hole temperature is defined as [97–100]

$$T_+ \equiv T(r_+) = \frac{a'(r_+)}{4\pi}, \quad (51)$$

with  $r = r_+$  the event horizon, which satisfies  $a'(r_+) \neq 0$ . Additionally, in the framework of  $f(T)$  gravity, the black-hole entropy is given by [101–103]

$$S_+ \equiv S(r_+) = \frac{A}{4f_T(r_+)}, \quad (52)$$

where  $A$  is the area. Inserting the  $f(T)$  form (17) and the solution (25),(26) into the above definitions we find

$$T_+ \simeq \frac{3r_+^2 s^3 + \epsilon\alpha[30s^2 r_+ + 11s^3 + 30r_+^3 \ln(2/r_+) + 45sr_+^2] + 3\epsilon s^3 r_+^2 c_2}{12\pi r_+^3}, \quad (53)$$

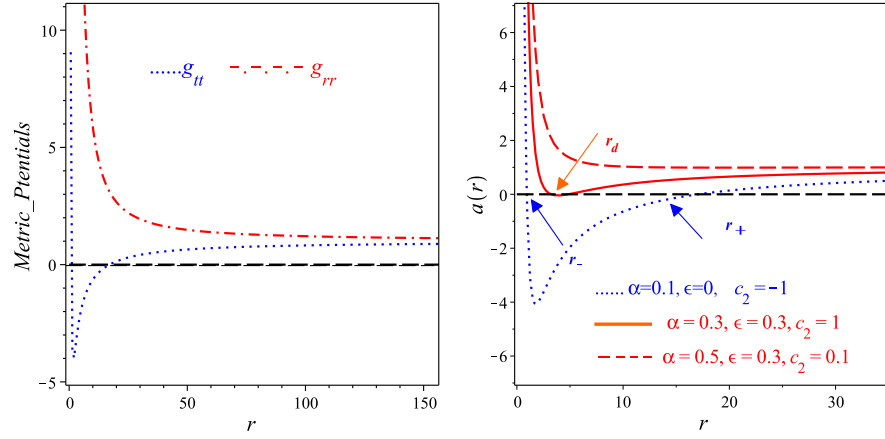


FIG. 2: *Left graph:* The two metric potentials  $g_{tt} \equiv a(r)$  and  $g_{rr} \equiv b(r)$  given in (25), (26) versus  $r$ , for the black hole solution with  $\mathbb{Q}_1(r) = 0$ , for  $M = 9$ ,  $c_1 = c_2 = 1$ ,  $\alpha = 0.1$ ,  $\epsilon = 0.1$ , and  $s = 4$ , in Planck mass units. *Right graph:* The metric potential  $a(r)$  versus  $r$ , for  $M = 9$ ,  $c_1 = 1$ ,  $s = 4$  and various values of the model parameters  $c_2$ ,  $\alpha$  and  $\epsilon$ , in Planck mass units.  $r_-$  and  $r_+$  are the inner and outer horizons respectively, while  $r_d$  is the degenerate horizon in which the above two coincide.

and

$$S_+ \simeq \pi r_+^2 \left[ 1 + 2\epsilon\alpha \left( \frac{4}{r_+^2} + \frac{\mathcal{M}_+^\epsilon - \epsilon f^\epsilon}{r_+^4} \right) \right]. \quad (54)$$

These expressions indicate that for  $\epsilon = 0$  we recover the standard general-relativity temperature and entropy. In Fig. 3 we depict the temperature and entropy versus the horizon, for various values of the model parameters. As we can see the entropy is always positive and exhibits a quadratic behavior, while the temperature is always positive when  $\epsilon > 0$  but for vanishing  $\epsilon$  it is positive only for  $r_d > r_+$ .

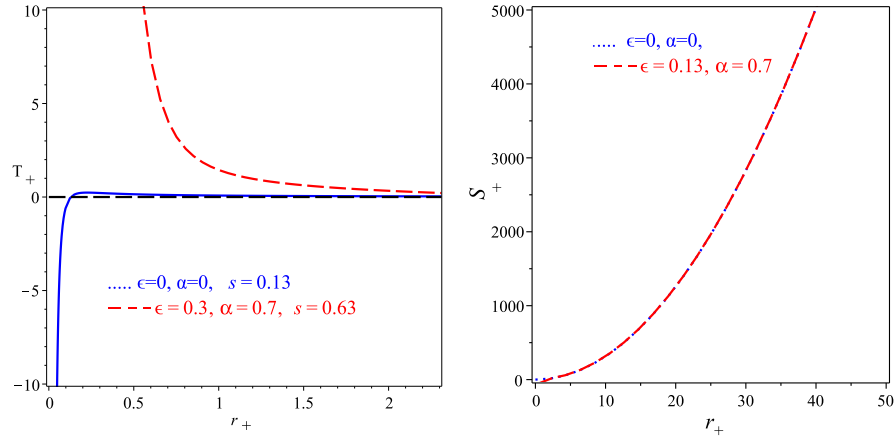


FIG. 3: *The temperature (left graph) and entropy (right graph) versus the horizon*, for the black hole solution with  $\mathbb{Q}_1(r) = 0$ , for  $c_1 = 1$ ,  $c_2 = 5$  and  $s = 4$  and for various values of the model parameters  $\alpha$  and  $\epsilon$ , in Planck mass units.

We now focus on the heat capacity, which is a crucial quantity concerning the thermodynamic stability [104–106], since our perturbative approach to the black-hole solution allows for an easy calculation. The heat capacity at the event horizon is defined as [107–109]:

$$C_+ \equiv C(r_+) \simeq \frac{\partial \mathcal{M}_+}{\partial T_+} = \frac{\partial \mathcal{M}_+}{\partial r_+} \left( \frac{\partial T_+}{\partial r_+} \right)^{-1}, \quad (55)$$

and positive heat capacity implies thermodynamic stability. Substituting (50) and (53) into (55) we obtain the heat

capacity as

$$C_+ \simeq \frac{2\pi r_+^2}{3s^3} \left[ \epsilon \alpha (30r_+ + 7s - 60r \ln 2) - 3s^3 \right]. \quad (56)$$

Expression (56) implies that  $C_+$  does not diverge and thus we do not have a second-order phase transition. In the left graph of Fig. 4 we depict  $C_+$  as a function of the horizon. As we can see, in the  $\epsilon = 0$  case we have  $C_+ < 0$  due to the negative derivative of the temperature, as expected for the the Reissner Nordström black hole. Nevertheless, for  $\epsilon > 0$  we obtain positive heat capacity. This is one of the main results of the present work, namely that  $f(T)$  modifications improve the thermodynamic stability. Note that this is not the case in other gravitational modifications, since for instance in  $f(R)$  gravity the heat capacity is positive only conditionally [110–112].

We close this subsection by the examination of the Gibb's free energy. In terms of the the mass, temperature and entropy at the event horizon this is defined as [103, 113]:

$$G(r_+) = \mathcal{M}(r_+) - T(r_+)S(r_+). \quad (57)$$

Inserting (50), (53) and (54) into (57), we obtain

$$G_+ \equiv G(r_+) = \frac{3s^3(3s^2 + r_+^2) + \epsilon \{ \alpha [30r_+^3 \ln(2r_+) + 31sr_+^2 + 3s^3r_+^2 - 30s^2r_+ - 231s^3] \} + 3\epsilon s^3 r_+ (r_+ c_2 - c_1)}{12r_+^3}. \quad (58)$$

In the right graph of Fig. 4 we depict the behavior of Gibb's free energy. As we observe it is always positive, for both  $\epsilon = 0$  and  $\epsilon > 0$ .

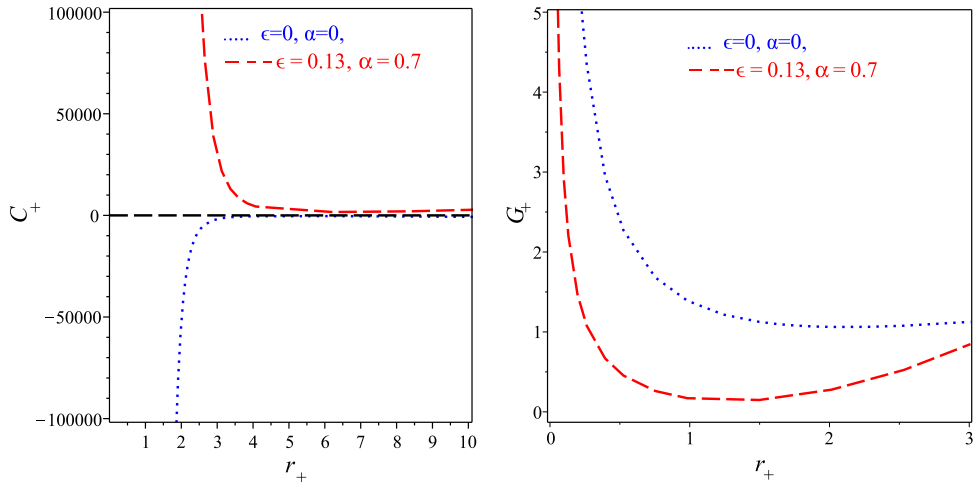


FIG. 4: The heat capacity (left graph) and the Gibb's energy (right graph) versus the horizon, for the black hole solution with  $\mathbb{Q}_1(r) = 0$ , for  $c_1 = 1$ ,  $c_2 = 5$  and  $s = 4$  and for various values of the model parameters  $\alpha$  and  $\epsilon$ , in Planck mass units.

### B. Thermodynamics of the black hole solution with $\mathbb{Q}_1(r) \neq 0$

In this subsection we repeat the above thermodynamic analysis in the case of the black hole solution for  $\mathbb{Q}_1(r) \neq 0$  given in (27),(28). In the left graph of Fig. 5 we depict the metric potentials  $a(r)$  and  $b(r)$ , and as we observe  $a(r)$  may exhibit two horizons while  $b(r)$  does not. In the right graph of Fig. 5 we present  $a(r)$ . Similarly to the previous subsection, we see that for small  $\alpha$  values, namely small deviations from general relativity, we obtain two horizons, however as  $\alpha$  increases there is a specific value in which the two horizons become degenerate ( $r_- = r_+ = r_d$ ), while for larger values the horizon disappears and the central singularity becomes a naked one. However, the interesting feature is that for the same  $\alpha$  value, the parameter  $s$  that quantifies the charge profile also affects the horizon structure, and in particular larger  $s$  leads to the appearance of the naked singularity.

The mass-radius relation takes the form

$$M_+ = \frac{s^2 + r_+^2}{2r_+} + \epsilon \left[ \frac{3sc_3(2s^2 + 2r_+^2 + sr_+^2) + 3s^2c_4r_+ + 56\alpha(s^2 + r_+^2)}{s^2r_+} \right], \quad (59)$$

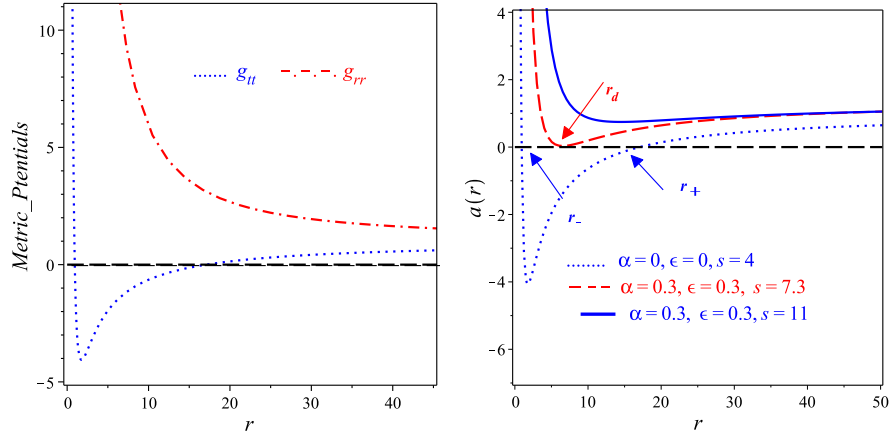


FIG. 5: *Left graph:*The two metric potentials  $g_{tt} \equiv a(r)$  and  $g_{rr} \equiv b(r)$  given in (27),(28) versus  $r$ , for the black hole solution with  $\mathbb{Q}_1(r) \neq 0$ , for  $M = 9$ ,  $c_1 = c_2 = 1$ ,  $\alpha = 0.1$ ,  $\epsilon = 0.1$ , and  $s = 4$ , in Planck mass units. *Right graph:* The metric potential  $a(r)$  versus  $r$ , for  $M = 9$ ,  $c_3 = 1$ ,  $c_4 = 1$  and various values of the model parameters  $\alpha$ ,  $\epsilon$  and  $s$ , in Planck mass units.  $r_-$  and  $r_+$  are the inner and outer horizons respectively, while  $r_d$  is the degenerate horizon in which the above two coincide.

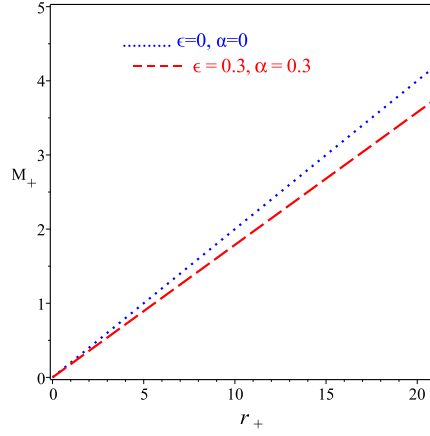


FIG. 6: The mass-radius relation (59) for the black hole solution with  $\mathbb{Q}_1(r) \neq 0$ , for  $c_3 = 1$ ,  $c_4 = 0$ , and  $s = 4$ , and various values of the model parameters  $\alpha$ ,  $\epsilon$  in Planck mass units.

and it is plotted in Fig. 6, where we can verify that  $M_+$  is always positive.

For the temperature (51) we obtain

$$T_+ \simeq \frac{3s^2(r_+^2 - s^2) + \epsilon[r_+^2(56\alpha + 3s^2c_3 + 6c_3s) + 6s^2(4\alpha - sc_3)]}{12\pi s^2 r_+^3}. \quad (60)$$

Moreover, for the entropy (52) we find

$$S_+ \simeq \frac{\pi}{r_+^3} [r_+^5 - 2\epsilon\alpha M(r_+ M - 2s^2)], \quad (61)$$

which again for  $\epsilon = 0$  recovers the general relativity result. In Fig. 7 we depict the temperature and entropy versus the horizon, for various values of the model parameters. We mention that in this case both temperature and entropy may acquire negative values, however the entropy, which is always quadratically increasing, is positive when  $r_+ > r_d$ .

For the heat capacity  $C_+ = \frac{\partial M_+}{\partial r_+} \left( \frac{\partial T_+}{\partial r_+} \right)^{-1}$ , using (59) and (60), we acquire

$$C_+ \simeq -2\pi(2s^2 + r_+^2 - 2s^2\epsilon c_3 - 8 - \epsilon\alpha). \quad (62)$$

Expression (62) implies that  $C_+$  does not diverge and therefore we do not have a second-order phase transition. In the left graph of Fig. 8 we present  $C_+$  as a function of the horizon. As we can see, in the  $\epsilon = 0$  case we have  $C_+ < 0$  due

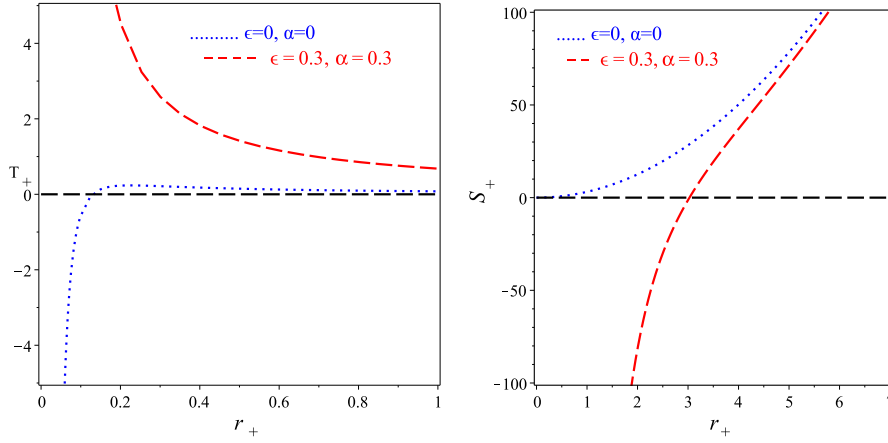


FIG. 7: The temperature (left graph) and entropy (right graph) versus the horizon, for the black hole solution with  $Q_1(r) \neq 0$ , for  $M = 9$ ,  $c_3 = c_4 = 1$  and  $s = 4$ , and various values of the model parameters  $\alpha$ ,  $\epsilon$  in Planck mass units.

to the negative derivative of the temperature, as expected for the the Reissner Nordström black hole. Nevertheless, for  $\epsilon > 0$ , namely in the case where the  $f(T)$  correction is switched on, we may obtain positive heat capacity. Finally, for the Gibb's free energy  $G(r_+) = M(r_+) - T(r_+)S(r_+)$ , using (59), (60) and (61), we find

$$G_+ \simeq \frac{3s^2(r_+^2 + s^2) + \epsilon[3c_3s(2r_+^2 + sr_+^2 + 6s^2) + 6s^2r_+c_4 + 8\alpha(7r_+^2 + 11s^2)]}{12s^2r_+}. \quad (63)$$

In the right graph of Fig. 8 we present Gibb's free energy as a function of the horizon. As we can see for both  $\epsilon = 0$  and  $\epsilon > 0$  it is always positive.

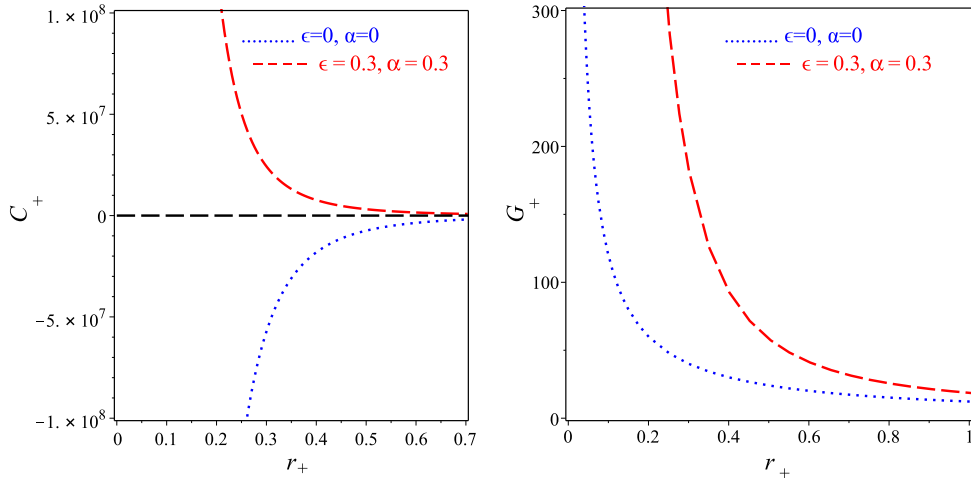


FIG. 8: The heat capacity (left graph) and the Gibb's energy (right graph) versus the horizon, for the black hole solution with  $Q_1(r) \neq 0$ , for  $M = 9$ ,  $c_3 = c_4 = 1$  and  $s = 4$ , and various values of the model parameters  $\alpha$ ,  $\epsilon$  in Planck mass units.

## VI. CONCLUSIONS AND DISCUSSION

We investigated the stability of motion and thermodynamics in the case of spherically symmetric solutions in  $f(T)$  gravity using the perturbative approach. In particular, we considered small deviations from teleparallel equivalent of general relativity and we extracted charged black hole solutions for two charge profiles, namely with or without a perturbative correction in the charge distribution. Firstly, we examined their asymptotic behavior showing that for large distances they become Minkowski. Then we extracted various torsional and curvature invariants, which revealed

the presence of the central singularity as expected. Moreover, we calculated the energy and the mass of the solutions. As we showed, all results recover the general relativity ones in the case where the  $f(T)$  deviation goes to zero.

As a next step we investigated the stability of motion around the obtained black hole solutions, by extracting and studying the geodesic deviation of a test particle in their gravitational field. Assuming a secular orbit, we extracted the corresponding stability of motion condition in terms of the metric potentials. As we saw, in the case where the perturbative correction to the charge profile is absent the solution is always stable, however in the case where it is present we obtained unstable regimes in the parameter space.

Additionally, we performed a detailed analysis of the thermodynamic properties of the black hole solutions. In particular, we extracted the inner (Cauchy) and outer (event) horizons, the mass profile, the temperature, the entropy, the heat capacity and the Gibb's free energy. As we showed, for small  $\alpha$  values, namely small deviations from general relativity, we obtain the two horizons, however as  $\alpha$  increases there is a specific value in which the two horizons become degenerate, and for larger values the horizon disappears and the central singularity becomes a naked one, a known feature of torsional gravity. Furthermore, we saw that for the same  $\alpha$  value, the parameter  $s$  that quantifies the charge profile also affects the horizon structure, and in particular larger  $s$  leads to the appearance of the naked singularity.

Concerning the temperature and entropy, we showed that although there are regimes in which they become negative, for  $r_+ > r_d$  they are always positive definite. Concerning the heat capacity we saw that it does not diverge and thus we do not have a second-order phase transition. However, the most interesting result is that it becomes positive for larger deviations from general relativity, which shows that  $f(T)$  modifications improve the thermodynamic stability, which is not the case in other gravitational modifications. Finally, for the Gibb's free energy, we showed that it is always positive, for all torsional additions and for both charge-profile cases.

In summary, the present work indicates that torsional modification of gravity may have an advantage comparing to other gravitational modification classes, when stability issues are raised, which may serve as an additional motivation for the corresponding investigations. One particular interesting issue is to investigate in detail whether torsional modified gravity leads to smoother (weaker) central singularities comparing to general relativity or curvature modified gravity. This issue will be the focus of interest of a separate project.

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