THE QUADRIC ANSATZ FOR THE mn–DISPERSIONLESS KP EQUATION, AND SUPERSYMMETRIC EINSTEIN–WEYL SPACES

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ABSTRACT. We consider two multi–dimensional generalisations of the dispersionless Kadomtsev-Petviashvili (dKP) equation, both allowing for arbitrary dimensionality, and non–linearity. For one of these generalisations, we characterise all solutions which are constant on a central quadric. The quadric ansatz leads to a second order ODE which is equivalent to Painleve I or II for the dKP equation, but fails to pass the Painlevé test in higher dimensions. The second generalisation of the dKP equation leads to a class of Einstein–Weyl structures in an arbitrary dimension, which is characterised by the existence of a weighted parallel vector field, together with further holonomy reduction. We construct and characterise an explicit new family of Einstein–Weyl spaces belonging to this class, and depending on one arbitrary function of one variable.

1. INTRODUCTION

Let $u : U \longrightarrow \mathbb{R}$, where U is an open set in $\mathbb{R}^N$ with coordinates $x^a = (x, y_1, \dots, y_n, t)$. The mn–dispersionless Kadomtsev-Petviashvili (mn–dKP) equation [17, 20] is given by

$$u_{xt} - (u^m u_x)_x = \triangle u, \tag{1.1}$$

where $\triangle = \partial/\partial {y_1}^2 + \dots + \partial/\partial {y_n}^2$ and $u_x = \partial_x u$, etc. We refer the readers to [20] for a list of references and applications of equation (1.1) ranging from non-linear optics and acoustics to geometry for different values of integers (m, n).

The aim of this paper is to construct solutions to (1.1) which are constant on central quadrics $\mathcal{Q} \subset \mathbb{R}^N$, i. e. there exists a symmetric $N \times N$ matrix $\mathbf{M} = \mathbf{M}(u)$ such that

$$M_{ab}(u)x^a x^b = C, \quad \text{where} \quad C = \text{const}, \quad \text{and} \quad a, b = 1, \dots, N = n + 2. \tag{1.2}$$

If $m = n = 1$, then (1.1) becomes the standard dKP equation, and the ODE for the matrix $\mathbf{M}(u)$ resulting from the quadric ansatz (1.2) is either solvable by quadratures, or is equivalent to Painlevé I or Painlevé II (this last case being generic) [10]. In §2 we shall perform the analysis of the quadric ansatz for (1.1). The non–generic linearisable case leads to some explicit forms of $\mathbf{M}(u)$, which we shall present in §3.1. The other two cases lead to 2nd order ODEs generalising Painlevé I and II, which however fail to pass the Painlevé test (Theorem 3.3). Thus, in line with the integrability dogma [1], we conclude that (1.1) is only integrable if $m = n = 1$ (the dKP case), or $n = 0$ (the Riemann equation solvable by the method of characteristics), or $m = 0$ (the linear wave

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equation). It is known [9] that the dKP equation characterises a class of Einstein–Weyl metrics in $2+1$ dimensions which admit a parallel weighted vector field. While this correspondence does not extend to (1.1) for general (m, n), in §4 we shall demonstrate that a closely related equation

$$u_{xt} - (uu_x)_x + \frac{2(n-1)}{n} u_x^2 = \triangle u, \tag{1.3}$$

characterises a class of EW structures admitting a parallel null weighted vector field, and with a further assumption on the holonomy of the Weyl connection. Our main result in §4 is Theorem 4.1 characterising an explicit family of Einstein–Weyl spaces in arbitrary dimension.

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2. The quadric ansatz

The quadric ansatz (1.2) is applicable to equations of the form

$$\frac{\partial}{\partial x^a}\left(b^{ab}(u)\,\frac{\partial u}{\partial x^b}\right) = 0, \tag{2.1}$$

where $\mathbf{b}(u)$ is a symmetric $N \times N$ matrix. While the ansatz can be traced back to some works of Darboux [4], in the 'modern' times it has been applied to a class of dispersionless, as well as linear PDEs: the $SU(\infty)$–Toda equation [21], the dKP equation [10], the Laplace equation [8], as well as a general class of equations integrable by the method of hydrodynamic reductions [13]. The idea is to find and solve an ODE for $\mathbf{M}(u)$ (and keep in mind that this ODE is not a symmetry reduction of the underlying PDE, and the corresponding solutions in general do not admit any Lie–point, or generalised symmetries). The ODE arises as follows: Differentiating (1.2) implicitly w.r.t. x^a and substituting the resulting expression for $\partial u/\partial x^a$ into (2.1) yields a matrix ODE

$$g\mathbf{M}' = \mathbf{M}\mathbf{b}\mathbf{M}, \quad \text{where} \quad {}' = \frac{d}{du}$$

and the function $g = g(u)$ is defined by $2g' = \mathrm{Tr}(\mathbf{b}\mathbf{M})$. Setting $\mathbf{N} = -\mathbf{M}^{-1}$, this matrix ODE simplifies to

$$g\mathbf{N}' = \mathbf{b}, \tag{2.2}$$

and one finds that

$$g^2 \det \mathbf{N} = \zeta, \tag{2.3}$$

where ζ is an arbitrary constant.

From now, we shall regard the system (2.2, 2.3) as the reduction of (2.1) under the quadric ansatz (1.2). Solving this system for the components of $\mathbf{N}$ gives the matrix $\mathbf{M}$, and thus leads to an implicit solution to the nonlinear PDE (2.1) of the form (1.2), which is therefore constant on a central quadric.

The mn-dKP equation (1.1) is of the form (2.1), with

$$\mathbf{b}(u) = \left(\begin{array}{c|ccc|c} -u^m & 0 & \cdots & 0 & \frac{1}{2} \\ \hline 0 & & & & 0 \\ \vdots & & -\mathbf{1}_n & & \vdots \\ 0 & & & & 0 \\ \hline \frac{1}{2} & 0 & \cdots & 0 & 0 \end{array}\right), \tag{2.4}$$

where $\mathbf{1}_n$ denotes the $n \times n$ identity matrix. Equation (2.2) then implies that

$$\mathbf{N} = \left(\begin{array}{c|ccc|c} Y & \beta_1 & \cdots & \beta_n & Z \\ \hline \beta_1 & & & & \varepsilon_1 \\ \vdots & & \mathbf{X} & & \vdots \\ \beta_n & & & & \varepsilon_n \\ \hline Z & \varepsilon_1 & \cdots & \varepsilon_n & \phi \end{array}\right), \tag{2.5}$$

where β_i, ε_i, $i = 1, \ldots, n$, and ϕ are constants, Y and Z are functions of u, and $\mathbf{X}$ is an $n \times n$ symmetric matrix, whose diagonal components $X_1, X_2, \ldots, X_n$ are functions of u and off-diagonal components $\alpha_{ij}, i > j$ are all constants. Equation (2.2) also implies that

$$Y' = -g^{-1}u^m, \quad X_i{}' = -g^{-1}, \quad Z' = (2g)^{-1}.$$

The last equation gives g in terms of Z', so that

$$Y' + 2u^m Z' = 0, \tag{2.6}$$

and

$$\mathbf{X}(u) = -2Z(u)\mathbf{1}_n + \mathbf{X}_0 \tag{2.7}$$

where $\mathbf{X}_0$ is a constant matrix with diagonal components γ_i and off–diagonal components α_{ij}. The equation (2.3) becomes

$$\det \mathbf{N} \;=\; 4\,\zeta\,(Z')^2, \tag{2.8}$$

and, together with the form (2.5) gives an algebraic expression for Y in terms of Z, Z' and constants. Substituting this expression into (2.6) yields a second order scalar ODE for Z of the form

$$Z'' = A(Z)\,(Z')^2 + B(Z) + C(Z)u^m, \tag{2.9}$$

where A, B and C are rational functions of Z, of degrees respectively $n, 2n+1$ and n, which will be determined in §3.

If this equation can be solved, then reversing the steps above we can reconstruct the matrix $\mathbf{M}(u)$, and thus find an explicit solution to (1.1) constant on a central quadric. All solutions in the class (1.2) arise from this construction.

2.1. **Eliminating the constants.** To make further progress we exploit some symmetries of the mn–dKP equation (1.1) to eliminate some of the constants in (2.5). If $m = 1$, then the transformation

$$x^a \to \hat{x}^a = A^a{}_b x^b, \quad \text{where} \quad \mathbf{A} = \begin{pmatrix} 1 & c_1 & c_2 & \dots & c_n & k \\ & 1 & 0 & \dots & 0 & 2c_1 \\ & & 1 & \ddots & \vdots & 2c_2 \\ & & & \ddots & 0 & \vdots \\ & \mathbf{0} & & & 1 & 2c_n \\ & & & & & 1 \end{pmatrix} \tag{2.10}$$

together with

$$u \to \hat{u} = u + c_1^2 + c_2^2 + \dots + c_n^2 - k, \tag{2.11}$$

where c_i, $i = 1, \dots, n$, and k are constants is a symmetry of (1.1), which also preserves the quadric ansatz (1.2) with a replacement

$$\mathbf{M}(u) \to \hat{\mathbf{M}}(\hat{u}) = (\mathbf{A}^{-1})^T \mathbf{M}(u) \mathbf{A}^{-1}, \quad \text{or} \quad \hat{\mathbf{N}}(\hat{u}) = \mathbf{A}\, \mathbf{N}(u)\, \mathbf{A}^T. \tag{2.12}$$

Using (2.12) we can set some of the constants $(\varepsilon, \beta, \alpha, \gamma)$ to 0. There are three cases to consider depending on whether ϕ vanishes, or not.

3. Solutions

3.1. **Case I.** Assuming $\phi = \varepsilon_i = 0$ for $i = 1, \dots, n$ leads to the quadric ansatz equations which are solvable by quadrature,

$$u = 2 \int \left(\frac{-\zeta}{Z^2 \det \mathbf{X}} \right)^{\frac{1}{2}} dZ, \quad Y = -2 \int u^m \frac{dZ}{du}\, du. \tag{3.1}$$

To derive these formulae note that in this case $\det(\mathbf{N})$ is independent on Y. Therefore (3.1) arises from

$$4\zeta (Z')^2 = -Z^2 \det(\mathbf{X}) \tag{3.2}$$

which is (2.8), where $\det(\mathbf{X})$ is a polynomial of degree n in Z. If $n = 1$ or 2, then the corresponding expression for u is given by elementary functions. Two examples of such solutions are given below:

3.1.1. $(m, n) = (2, 1)$.

$$\frac{1}{\tan^2 u} \left[\frac{y^2}{4} \;+\; \Big(2 \sin^2 u \cos u \left(\cos u \ln(\cos u) + u \sin u \right) - u^2 \sin^2 u + \delta^2 \cos^4 u \Big) t^2 \right.$$
$$\left. -\; \delta (\cos^2 u) y t - (\sin^2 u) x t \right] \;=\; C, \quad \text{where} \quad \delta = \text{const}. \tag{3.3}$$

3.1.2. $(m, n) = (1, 2)$.

$$(4ae^{au}+1)\Big[4a^3(y_1^2+y_2^2)+4e^{-au}\big(a^2xt-at^2\ln(e^{-au}+4a)\big)-e^{-2au}t^2\big(au+\ln(e^{-au}+4a)\big)\Big] = C, \tag{3.4}$$

where $a > 0$ is a constant.

3.1.3. $m = 1$. There is also a simple class of solutions with $m = 1$, and arbitrary n. To find it suppose that all constants in the matrices $\mathbf{N}$ and $\mathbf{X}_0$ are zero. Then equation (3.2) with $\zeta = (-1)^{n+1}2^{n-2}$ becomes $(Z')^2 = Z^{n+2}$, which after integrating (3.1) gives

$$\begin{aligned} u\left(4u(\ln u)\,t^2 - 4xt + y_1^2 + y_2^2\right) &= C, \quad n = 2,\\ u^{2/n}\left(\frac{8}{n-2}u\,t^2 - 4xt + \sum_{i=1}^{n} y_i^2\right) &= C, \quad n \neq 2. \end{aligned}$$

3.2. **Case II.** We shall now assume that the constant ϕ in (2.5) is zero, but at least one of the ε_is is non–zero. If $m = 1$, the symmetry (2.10) can be used to eliminate all βs and γ_1. Then (2.8) with $R_n(Z) \equiv \det(\mathbf{X})$ gives

$$4\zeta\,(Z')^2 = YQ_{n-1}(Z) - Z^2R_n(Z), \quad \text{where} \quad Q_{n-1}(Z) \equiv \det\left(\begin{array}{ccc|c} & & & \varepsilon_1\\ & \mathbf{X} & & \vdots\\ & & & \varepsilon_n\\ \hline \varepsilon_1 & \cdots & \varepsilon_n & 0 \end{array}\right). \tag{3.5}$$

Differentiating (3.5) with respect to u and substituting $Y' = -2uZ'$ gives a second order ODE for Z of the form (2.9) where $m = 1$, and

$$A(Z) = \frac{1}{2}\frac{d}{dZ}(\ln Q_{n-1}), \; B(Z) = \frac{1}{8\zeta}\left(Z^2R_n\frac{d}{dZ}(\ln Q_{n-1}) - \frac{d}{dZ}(Z^2R_n)\right), \; C(Z) = -\frac{Q_{n-1}}{4\zeta}.$$

Proposition 3.1. *Let $\phi = 0$, and at least one of the ε_is is non–zero in (2.5). If $(m, n) = (1, 1)$, the equation (2.9) is equivalent to Painlevé I. For $m = 1$, and $n > 1$ equation (2.9) does not posses the Painlevé property.*

Proof. For an ODE to have the Painlevé property, its movable singularities can only be poles. Thus we follow the algorithm in [1] to determine whether the general solution of (2.9) admits a movable branch point. For convenience we first conduct the Painlevé test in the special case, where the constant matrix $\mathbf{X}_0$ is zero, so that $\mathbf{X} = -2Z\mathbf{1}_n$, and (with the definition $\varepsilon^2 \equiv {\varepsilon_1}^2 + \cdots + {\varepsilon_n}^2$)

$$2\zeta\,Z'' = \zeta\frac{(n-1)}{Z}(Z')^2 - 3(-2)^{n-2}Z^{n+1} - \varepsilon^2(-2)^{n-2}Z^{n-1}u. \tag{3.6}$$

Assume that the dominant behaviour of a solution near a movable singularity u_0 is of the form

$$Z \approx a(u-u_0)^p, \tag{3.7}$$

where a and p are constants. Then substitute (3.7) into (3.6) and balance the power of $u-u_0$ of two or more terms. If the balancing terms are dominant, i.e. their power of $u-u_0$ is most negative, then other terms can be ignored, and one can solve for a. For equation (3.6), it turns out that the only possible value of p is $p=-\frac{2}{n}$, which is not an integer for $n>2$ and suggests a movable algebraic branch point. Moreover, this result extends to the general case (2.9). This is because the assumption that leads to the special case (3.6) keeps only the highest degree terms in Q_{n-1} and R_n. The presence of the lower degree terms in the rational functions $A(Z)$, $B(Z)$ and $C(Z)$ will not change the possible dominant behaviour in the first step of the Painlevé test. Therefore we conclude that equation (2.9) does not posses the Painlevé property for $n>2$.

If $n=1$, then (3.6) and (2.9) are equivalent, after constant rescalings of dependent and independent variables, to the Painlevé I equation.

If $n=2$, with βs and γ_1 eliminated by the symmetry for $m=1$, then (2.9) becomes

$$4\zeta\, Z''=\frac{4\zeta}{2Z+\delta}\,(Z')^2+\frac{4Z^4-2\gamma Z^3-\alpha^2Z^2}{2Z+\delta}-8Z^3+3\gamma Z^2+\alpha^2 Z-\varepsilon^2(2Z+\delta)u, \quad (3.8)$$

where we let $\varepsilon^2\equiv\varepsilon_1^2+\varepsilon_2^2$, $\gamma\equiv\gamma_2$ and $\delta\equiv\frac{2\alpha\varepsilon_1\varepsilon_2-\gamma\varepsilon_1^2}{\varepsilon_1^2+\varepsilon_2^2}$. Here α is the off–diagonal component of the matrix $\mathbf{X}_0$. Substituting (3.7) in (3.8), the only possibility is $(p,a)=(-1,(-\zeta)^{1/2})$. Let $-\zeta=\kappa^2$, and take $a=\kappa$. The remaining steps in the algorithm will determine whether the general solution of (3.8) can be represented near a movable singular point u_0 by the Laurent series, with the leading term $\frac{\kappa}{(u-u_0)}$. It turns out that to satisfy (3.8) one needs to introduce a logarithmic term, which gives

$$Z(u)\approx\frac{\kappa}{(u-u_0)}+\frac{\gamma}{8}-\frac{32\varepsilon^2u_0-3\gamma^2-8\alpha^2}{192\kappa}(u-u_0)+\left(c+\frac{\varepsilon^2}{8\kappa}\ln(u-u_0)\right)(u-u_0)^2+O((u-u_0)^2),$$

where c is an arbitrary constant. The logarithmic term indicates a logarithmic branch point, and this shows that (3.8) does not posses the Painlevé property. Hence we conclude that equation (2.9) does not posses the Painlevé property for $m=1$, $n>1$.

□

3.3. **Case III.** This is the generic case, where we assume that $\phi\neq 0$ in (2.5). If $m=n=1$, then the ODE resulting from the quadric anzats reduces to Painlevé II [10]. For general n, and $m=1$ the symmetry (2.10) can be used to eliminate γ_1 and all εs. Equation (2.8) takes the form

$$4\,\zeta\,\left(Z'\right)^2\;=\;(\phi Y-Z^2)R_n(Z)+Q_{n-1}(Z), \quad (3.9)$$

where $R_n(Z)=\det\mathbf{X}$, and here $Q_{n-1}(Z)=-\phi\sum_{i=1}^n\beta_i\,\det\mathbf{B}_i$ with $\mathbf{B}_i$ denoting the matrices obtained from replacing the ith column of $\mathbf{X}$ (2.7) by the column vector $(\beta_1,\ldots,\beta_n)^T$. Solving this for Y in terms of Z and Z', and differentiating to eliminate Y' by $Y'=-2uZ'$

gives (2.9) where $m = 1$, and

$$A(Z) = \frac{1}{2}\frac{d}{dZ}(\ln R_n)\,,\; B(Z) = \frac{1}{8\zeta}\left(\frac{dQ_{n-1}}{dZ} - Q_{n-1}\frac{d}{dZ}(\ln R_n) - 2ZR_n\right),\; C(Z) = -\frac{\phi}{4\zeta}R_n.$$

Proposition 3.2. *Let $\phi \neq 0$ in (2.5). If $(m,n) = (1,1)$, the equation (2.9) is equivalent to Painlevé II. For $m = 1$ and $n > 1$ equation (2.9) does not posses the Painlevé property.*

Proof. The result is obtained by first performing the Painlevé test on the special case case where $\mathbf{X} = -2Z\,\mathbf{1}_n$ (i.e. assuming $\mathbf{X}_0 = 0$ in (2.7)) and (2.9) is

$$2\zeta\, Z'' = \frac{n\zeta}{Z}(Z')^2 + (-2)^{n-1}\left(Z^{n+1} + \frac{\phi\beta^2}{4}Z^{n-2}\right) + (-2)^{n-1}\phi Z^n u, \tag{3.10}$$

where $\beta^2 \equiv \beta_1^2 + \cdots + \beta_n^2$. When $n = 1$, then, after a coordinate transformation [10] this family of ODEs (3.10) gives the Painlevé II equation. For $n > 2$, the ODE (3.10) fails the test at the first step of finding the dominant behaviour of the general solution, where it displays the dominant term of the form $a(u-u_0)^p$ with $p = -\frac{2}{n}$. For $n > 2$, this indicates an algebraic branch point of order $-\frac{2}{n}$, hence (3.10) does not have the Painlevé property. Then we argue that this result extends to the general form (2.9) as (2.9) differs from (3.10) only by the lower degree terms in the polynomials appearing in the rational functions $A(Z), B(Z)$ and $C(Z)$, and these will not affect the dominant behaviour analysis.

For $n = 2$, the form of (2.9) is still quite complicated by the presence of the constants α and $\gamma = \gamma_2$ in the matrix $\mathbf{X}$. After a translational change of variable $Z \to \hat{Z} = Z - \gamma/4$, and then dropping the hat, (2.9) becomes

$$\zeta\, Z'' = \frac{\zeta Z}{Z^2-\rho^2}(Z')^2 - (Z+\frac{\gamma}{4})(Z^2-\rho^2) - \frac{\phi\beta^2}{4}(2Z+\delta)\frac{Z}{Z^2-\rho^2} + \frac{\phi\beta^2}{4} - \phi(Z^2-\rho^2)u, \tag{3.11}$$

where $\beta^2 = \beta_1^2 + \beta_2^2$, $\delta = \frac{4\alpha\beta_1\beta_2 + \gamma(\beta_2^2-\beta_1^2)}{2\beta^2}$ and $\rho^2 = \dfrac{4\alpha^2+\gamma^2}{16}$.

The Painlevé test then shows that the general solution of (3.11) has a logarithmic branch point

$$Z(u) \approx \frac{\kappa}{(u-u_0)} - \frac{\phi u_0}{2} + \frac{\gamma}{8} + \left(c + \frac{\phi}{3}\ln(u-u_0)\right)(u-u_0) + O(u-u_0),$$

where $\kappa = \sqrt{-\zeta}$ and c and u_0 are arbitrary constants. Hence we conclude that (2.9) does not have the Painlevé property for $m = 1$, $n = 2$.

□

If $m > 1$, there is no obvious symmetry to eliminate the constants in $\mathbf{N}$. Nevertheless, the Painlevé analysis shows that the case $m = n = 1$ is the only case that the quadric ansatz reduction possesses the Painlevé property.

Theorem 3.3. *The quadric ansatz reduction of the mn-dKP equation does not posses the Painlevé property unless* $m = n = 1$.

Proof. The quadric ansatz reduction is of the form (2.9), where $A(Z)$ and $B(Z)$ are rational functions of degrees respectively n and $2n+1$, and $C(Z)$ is a polynomial of degree n. The non–zero constants in $\mathbf{N}$ for $m \geq 1$ only contribute to the lower degree terms in the polynomials appearing in $(A(Z), B(Z), C(Z))$ and thus will not change the dominant behaviour of a solution near a movable singularity u_0. Also, the term $C(Z)u^m$ is not leading for any m. Therefore, from the proofs of Propositions 3.1 and 3.2 we conclude that for $n > 2$, and any $m \geq 1$ the general solution has a movable algebraic branch point of order $-\frac{2}{n}$.

It remains to settle the case where $n = 1$ or 2, and $m > 1$. Performing the Painlevé test in these cases we find that the general solution exhibits a logarithmic branch point. In particular, we have the following form of the general solution:

Case II. $\phi = 0$ and at least one of the ε_is is non–zero in (2.5).

$n = 1$:

$$Z(u) \approx \frac{8\zeta}{(u-u_0)^2} + \frac{\gamma}{6} - \frac{12\varepsilon(u_0^m\varepsilon + \beta) - \gamma^2}{480\zeta}(u-u_0)^2 - \frac{m u_0^{m-1}\varepsilon^2}{24\zeta}(u-u_0)^3 + \left(c + \frac{m(m-1)u_0^{m-2}\varepsilon^2}{56\zeta}\ln(u-u_0)\right)(u-u_0)^4 + O((u-u_0)^4)$$

$n = 2$:

$$Z(u) \approx \frac{\kappa}{(u-u_0)} + \frac{\gamma_1+\gamma_2}{8} - \frac{32(u_0^m(\varepsilon_1^2+\varepsilon_2^2) + \beta_1\varepsilon_1 + \beta_2\varepsilon_2) - 8\alpha^2 - 3(\gamma_1^2+\gamma_2^2) + 2\gamma_1\gamma_2}{192\kappa}(u-u_0) + \left(c + \frac{m u_0^{m-1}(\varepsilon_1^2+\varepsilon_2^2)}{8\kappa}\ln(u-u_0)\right)(u-u_0)^2 + O((u-u_0)^2)$$

Case III. $\phi \neq 0$ in (2.5).

$n = 1$:

$$Z(u) \approx \frac{8\zeta}{(u-u_0)^2} - \frac{2}{3}u_0^m\phi + \frac{\gamma}{6} - m u_0^{m-1}\phi(u-u_0) + \left(c + \frac{2m(m-1)u_0^{m-2}\phi}{5}\ln(u-u_0)\right)(u-u_0)^2 + O((u-u_0)^2)$$

$n = 2$:

$$Z(u) \approx \frac{\kappa}{(u-u_0)} - \frac{u_0^m\phi}{2} + \frac{\gamma_1+\gamma_2}{8} + \left(c + \frac{m u_0^{m-1}\phi}{3}\ln(u-u_0)\right)(u-u_0) + O(u-u_0),$$

Here, $\kappa = \sqrt{-\zeta}$, c is an arbitrary constant, and $\gamma \equiv \gamma_1$ for $n = 1$.

□

4. Einstein–Weyl geometry

There is a, by now well established, link between 2+1 dimensional Einstein–Weyl geometry, and dispersionless integrable systems [22, 9, 7, 2, 11, 14]. In particular, the Manakov–Santini equation [16] is known to be the general local normal form of the Einstein–Weyl equations [12].

In this section we construct a Weyl structure in an arbitrary dimension $N = n + 2$, such that the Einstein–Weyl condition reduces to a single dispersionless PDE. In the case when $n = 1$ this PDE is the dKP equation, and the Einstein–Weyl structure is that of [9], and for $n > 1$ the PDE is (1.3). The main result (Theorem 4.1) is an explicit class of Einstein–Weyl spaces depending on one arbitrary function of one variable.

Recall [19] that a Weyl structure on a manifold U (which is really just an open set in $\mathbb{R}^N$, as our considerations are local) consists of a conformal structure $[h]$ represented by a metric h, and a torsion–free connection D which is compatible with $[h]$ in the sense that $Dh = \nu \otimes h$ for some one–form ν. This compatibility is invariant under the conformal change of metric:

$$h \to \Omega^2 h, \quad \nu \to \nu + 2d(\ln \Omega), \tag{4.1}$$

where $\Omega : U \to \mathbb{R}^+$. A Weyl structure is said to be non–closed if $d\nu \neq 0$, or equivalently if D is not a Levi–Civita connection of any metric in the class $[h]$. The Einstein–Weyl (EW) equations hold if the symmetrised Ricci tensor of D is proportional to some metric $h \in [h]$. The EW equations can be regarded as a system of PDEs for the representative metric h, and the associated one–form ν:

$$\chi_{ab} \equiv R_{ab} + \frac{N-2}{2}\nabla_{(a}\nu_{b)} + \frac{N-2}{4}\nu_a\nu_b - \frac{1}{N}h_{ab}\left(R + \frac{N-2}{2}\nabla_c\nu^c + \frac{N-2}{4}\nu_c\nu^c\right) = 0, \tag{4.2}$$

where ∇, R_{ab} and R are respectively the Levi–Civita connection, the Ricci tensor, and the Ricci scalar of h. A tensor V is said to be of weight k, if $V \to \Omega^k V$ under the conformal rescaling (4.1). If V is a vector field of weight k, then the weighted covariant derivative of V with respect to the Weyl connection is given by

$$\widetilde{D}_a V^b = \nabla_a V^b - \frac{1}{2}\nu_c V^c {\delta_a}^b - \frac{1}{2}(k+1)\nu_a V^b + \frac{1}{2}V_a\nu^b. \tag{4.3}$$

The mn–dKP equation (1.1) can be written in the form $d \star du = 0$, where the Hodge endomorphism $\star : \Lambda^1 \to \Lambda^{n+1}$ corresponds to the metric (note that the inverse metric corresponds to the matrix $\mathbf{b}(u)$ in (2.1))

$$h = dy_1^2 + \cdots + dy_n^2 - 4dxdt - 4u^m dt^2. \tag{4.4}$$

In the case $m = n = 1$ there exists a one–form $\nu = -4u_x dt$ such that the Einstein–Weyl condition reduces to the dKP equation [9]. It turns out that there is no one–form which, together with the metric (4.4) gives the mn–dKP equation if $n > 1$. We shall instead take the metric (4.4) with $m = 1$ (which can always be achieved by re-defining the function u) as a starting point. It can then be verified by an explicit computation of (4.2) that for the Weyl structure represented by

$$h = dy_1^2 + \cdots + dy_n^2 - 4dxdt - 4udt^2, \quad \nu = -\frac{4}{n}u_x dt \tag{4.5}$$

with $x^0 := t$, $x^i := y_i$, $i = 1, \dots, n$, and $x^{n+1} := x$, all components of χ_{ab} except χ_{00} vanish identically. The resulting Einstein–Weyl equation $\chi_{00} = 0$ is a scalar PDE (1.3)

$$u_{xt} - (uu_x)_x + \frac{2(n-1)}{n} u_x^2 = \triangle u.$$

Moreover, the vector field $V = \partial/\partial x$ is null, and covariantly constant with weight $-\frac{n}{2}$ with respect to D. Equation (1.3) is the dKP equation if $n = 1$, or its generalisation [18, 5] if $n > 1$.

This class of Einstein–Weyl structures falls into a larger class of solutions which admit a parallel weighted spinor [5]. The particular case (4.5) corresponds to Example 4 in this reference. To understand a coordinate invariant characterisation of (4.5), assume that an $(n+2)$–dimensional Weyl space represented by a pair (h, ν) admits a covariantly constant null vector field with weight $-n/2$. Then the one–form $\mathbf{V} = h(V, \cdot)$ dual to V satisfies

$$d\mathbf{V} = \frac{4-n}{4} \nu \wedge \mathbf{V}. \tag{4.6}$$

The Frobenius theorem implies that there exist functions (v, t) on U so that $\mathbf{V} = vdt$. We shall use t as one of the local coordinates on U. The existence of canonical (up to some freedom) remaining $(n+1)$ coordinates $(y_1, \dots, y_n, x)$ is also guaranteed by the Frobenius theorem: the distribution $\mathcal{V}$ of null curves is spanned by $V = \partial/\partial x$, and its integrable orthogonal complement $\mathcal{V}^\perp$ is spanned by $\{\partial/\partial x, \partial/\partial y_1, \dots, \partial/\partial y_n\}$.

If $n \neq 4$ (so that the $\dim(U) \neq 6$), the we can rescale the metric so that $\mathbf{V} = -2dt$ and

$$h = f^{ij}\, dy_i dy_j - 4dxdt + 2A^i dy_i dt - 4udt^2, \quad \nu = bdt,$$

where the functions f^{ij}, A^i, b and u at this stage depend on all coordinates. Going back to (4.3) with $V = \partial/\partial x$ and $k = -n/2$, and considering its symmetrised part shows that f^{ij} and A^i do not depend on x. Moreover a coordinate transformation $y_i \to \hat{y}_i(y_j, t)$ together with a redefinition of $f^{ij}(y, t)$ and $u(x, y, t)$ can be used to set $A^i = 0$. The parallel weighted condition on V also imples that $b = -(4/n)u_x$.

The final step reducing the functions f^{ij} to the identity $n \times n$ matrix is achieved in [5] by considering the connection induced by D on the screen bundle (see [15])

$$\mathcal{S} \equiv \mathcal{V}^\perp / \mathcal{V} \subset TU, \tag{4.7}$$

and restricting its holonomy to $\mathbb{R} \otimes \mathrm{Id}$ (equivalently the $\mathfrak{so}(n)$ projection of the holonomy algebra of this connection is zero). Now the metric and the one–form are given by (4.5), and the Einstein–Weyl equations reduce to[1] (1.3).

To this end we shall construct an explicit subclass of examples of (4.5) and (1.3) under the additional assumption that the screen bundle distribution $\mathcal{V}^\perp/\mathcal{V}$ generates an

[1] In the special case $n = 4$, we start–off with the metric (4.5), and the general one–form ν and impose the weighted parallel conditon on $V = \partial/\partial x$ to reduce the one–form ν to (4.5).

isometric action of $\mathbb{R}^n$ or T^n on the Einstein–Weyl space. This will be done by linearising (1.3) by a contact transformation.

Theorem 4.1. *Let (h, ν) be an $(n+2)$–dimensional Einstein–Weyl structure U which admits a parallel weighted null vector field V with weight $-\frac{n}{2}$, and such that*

- *The connection on the screen bundle $\mathcal{S}$ defined by (4.7) induced by D has holonomy $\mathbb{R} \otimes Id$.*
- *The sections of the screen bundle $\mathcal{S}$ generate the isometric action of the group of translations $\mathbb{R}^n$ on U.*

Then there exits local coordinates (t, y_i, s) such that the one–form $\mathbf{V} \equiv h(V, \cdot) = -2dt$, the isometric action is generated by $\{\partial/\partial y_1, \dots, \partial/\partial y_n\}$ and the Einstein–Weyl structure is given by

$$\begin{aligned} h &= dy_1{}^2 + \cdots + dy_n{}^2 + \frac{4G(s)}{(t-s)^{\frac{n}{n-2}}} ds dt, \quad \nu = -\frac{4}{(n-2)(t-s)} dt \quad \text{if } n \neq 2 \\ h &= dy_1{}^2 + dy_2{}^2 + 4G(s) e^{-st} ds dt, \quad \nu = -2s dt \quad \text{if } n = 2, \end{aligned} \tag{4.8}$$

where $G = G(s)$ is an arbitrary function of one variable.

Proof. The existence of the parallel weighed null vector field, and the holonomy reduction in Theorem 4.1 implies - as explained above - that the metric, and the one–form take the local normal form (4.5). The additional symmetry assumption in the Theorem then implies that $u = u(x, t)$, and the Einstein–Weyl condition (1.3) becomes

$$u_{xt} - u u_{xx} + \kappa {u_x}^2 = 0, \quad \text{where} \quad \kappa = \frac{n-2}{n}. \tag{4.9}$$

Rewrite (4.9) as a differential ideal

$$\begin{aligned} \omega_1 &\equiv du - u_x dx - u_t dt = 0, \\ \omega_2 &\equiv du_x \wedge dx + u du_x \wedge dt + \kappa {u_x}^2 dt \wedge dx = 0 \end{aligned} \tag{4.10}$$

and set

$$H = u - x u_x, \quad p = u_x$$

Rewrtting ω_1 (4.10) as

$$\begin{aligned} dH &= u_t dt - x du_x \\ &= H_t dt + H_p dp \end{aligned}$$

gives

$$x = -H_p, \quad u_t = H_t, \quad u = H - pH_p \tag{4.11}$$

where now $H = H(p, t)$. Substituting (4.11) into ω_2 in (4.10) gives

$$H_{pt} + pH_p - H - \kappa p^2 H_{pp} = 0. \tag{4.12}$$

The corresponding Einstein–Weyl structure (4.5) takes the form

$$h = {dy_1}^2 + \cdots + {dy_n}^2 + 4F(dpdt + \kappa p^2 dt^2), \quad \nu = -\frac{4}{n} pdt, \quad \text{where} \quad F \equiv H_{pp}. \tag{4.13}$$

This only depends on the second derivatives of the function H, so the function F is constrained by one PDE obtained from differentiating (4.12) with respect to p:

$$F_t + (1-2k)pF - \kappa p^2 F_p = 0. \tag{4.14}$$

This PDE can be solved explicitly, and the form of the general solution depends on $n = 2/(1-k)$:

$$F = \begin{cases} p^{-\frac{n-4}{n-2}} G\Big(t - \frac{n}{(n-2)p}\Big) & \text{if } n \neq 2, \\ G(p)e^{-pt}, & \text{if } n = 2, \end{cases} \tag{4.15}$$

where in both cases G is an arbitrary function of one variable. Introducing a new variable s by

$$s = \begin{cases} t - \frac{n}{(n-2)p} & \text{if } n \neq 2, \\ p & \text{if } n = 2, \end{cases}$$

absorbing the overall constant into the arbitrary function G, and adopting (y_i, t, s) as local coordinates on U yields (4.8).

□

5. Conclusions

We have demonstrated that solutions to the mn–dKP equation (1.1) constant on central quadrics are characterised by solutions to a 2nd order scalar ODE. In the generic case this ODE is of Painlevé type if $m = n = 1$, but does not posses the Painlevé property if $m \cdot n > 1$. This rules out the integrability of (1.1) for these values of m, n. There are other approaches to dispersionless integrability of (1.1) discussed in [20], and in particular Boris Kruglikov informed us that the approach taken in references [14, 3] could also be used to rule out non–integrable cases, and perhaps narrow them down to $m = n = 1$.

Equation (1.1) with $n = 1$ has been studied numerically in [6], where an asymptotic description of a gradient catastrophe in generalised KP equation was conjectured, and related to special solutions of Painlevé I. In [20] the analytical approach to this shock formation has been presented. It would be interesting to understand whether our explicit solutions shed more light on these shock formations.

In §4 we have related another multi–dimensional generalisation of the dKP equation (1.3) to a class of Lorentzian Einstein–Weyl structures. This class admits a parallel weighted null vector, and thus a parallel weighted spinor which makes it interesting in both physics (supersymmetric solutions to Einstein–Weyl equations [18]), and geometry, where the existence of such spinor corresponds to a holonomy reduction of the Weyl connection [5].

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