CONWAY INVARIANT JACOBI FORMS ON THE LEECH LATTICE

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ABSTRACT. In this paper we study Jacobi forms associated with the Leech lattice Λ which are invariant under the Conway group Co₀. We determine and construct generators of modules of both weak and holomorphic Jacobi forms of integral weight and fixed index $t \leq 3$. As applications, (1) we find the modular linear differential equations satisfied by the holomorphic generators; (2) we determine the decompositions of many products of orbits of Leech vectors; (3) we calculate the intersections between orbits and Leech vectors; (4) we derive some conjugate relations among orbits modulo $t\Lambda$

1. Introduction

In 1985 Eichler and Zagier introduced the theory of Jacobi forms in their monograph [14]. These forms are holomorphic functions in two variables $(\tau, z) \in \mathbb{H} \times \mathbb{C}$ which are modular in τ with respective to $\mathrm{SL}_2(\mathbb{Z})$ and quasi-periodic in z. Later, Gritsenko [15] defined Jacobi forms of lattice index by replacing z with many variables associated with an integral positive-definite lattice. The Jacobi form creates an elegant bridge between different types of modular forms. For example, Jacobi forms can be identified as modular forms for the Weil representation of $\mathrm{SL}_2(\mathbb{Z})$ through the theta decomposition, and are connected to modular forms on symmetric domains of type IV by the Fourier–Jacobi expansion (see [17]). Jacobi forms also have many applications in mathematical physics, such as the elliptic genera of some manifolds including K3 surfaces, and the topological string partition functions on various Calabi–Yau threefolds. It is a natural question to determine the structure of the space of Jacobi forms. This question was solved by Wirthmüller [27] for Jacobi forms associated with root systems not of E_8 type, and by [24, 25, 22] for the exceptional root system E_8 . The ring of Jacobi forms indexed by lattices of rank two was recently described in [26]. Little is known about spaces of Jacobi forms associated with other lattices, especially irreducible lattices of large rank.

In this paper we investigate Jacobi forms on the Leech lattice which are invariant under the Conway group Co_0 , and give many applications to computational aspects of the Leech lattice. The Leech lattice is the unique even unimodular positive-definite lattice of rank 24 which has no roots. It was discovered by Leech in 1967 [18], and its uniqueness was proved by Conway in 1969 [9]. This lattice has many remarkable properties. For example, it plays a role in constructing the fake monster Lie algebra [2] and proving the monstrous moonshine conjecture [3], and it achieves the densest sphere packing in \mathbb{R}^{24} [8]. The group Co_0 is the automorphism group of the Leech lattice, whose structure was first described by Conway [10]. The quotient of Co_0 by its center gives a sporadic simple group of order 4, 157, 776, 806, 543, 360, 000. Therefore, the Leech lattice is highly symmetrical, and we expect that the space of Conway invariant Jacobi forms will not be too large. Due to the importance of the Leech lattice and the Conway group, we also expect that Conway invariant Jacobi forms will have some applications in mathematics and physics. These motivate us to study such Jacobi forms.

Date: August 17, 2022.

²⁰²⁰ Mathematics Subject Classification. 11F50.

Key words and phrases. Jacobi forms, Leech lattice, Conway group.

We introduce the definition of Conway invariant Jacobi forms and state the main results. Let Λ denote the Leech lattice equipped with bilinear form (-, -).

Definition 1.1. Let $k \in \mathbb{Z}$ be an integer and $t \in \mathbb{N}$ be a non-negative integer. If a holomorphic function $\varphi : \mathbb{H} \times (\Lambda \otimes \mathbb{C}) \to \mathbb{C}$ satisfies the conditions

(i) Conway invariance:

$$\varphi(\tau, \sigma(\mathfrak{z})) = \varphi(\tau, \mathfrak{z}), \quad \sigma \in \mathrm{Co}_0$$

(ii) Quasi-periodicity:

$$\varphi(\tau, \mathfrak{z} + x\tau + y) = \exp(-t\pi i(x, x)\tau - 2t\pi i(x, \mathfrak{z}))\varphi(\tau, \mathfrak{z}), \quad x, y \in \Lambda,$$

(iii) Modularity:

$$\varphi\left(\frac{a\tau+b}{c\tau+d}, \frac{\mathfrak{z}}{c\tau+d}\right) = (c\tau+d)^k \exp\left(t\pi i \frac{c(\mathfrak{z},\mathfrak{z})}{c\tau+d}\right) \varphi(\tau,\mathfrak{z}), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

and the Fourier expansion of φ takes the form

$$\varphi(\tau,\mathfrak{z}) = \sum_{n=0}^{\infty} \sum_{\ell \in \Lambda} f(n,\ell) q^n \zeta^{\ell}, \quad q = e^{2\pi i \tau}, \ \zeta^{\ell} = e^{2\pi i (\ell,\mathfrak{z})},$$

then it is called a Conway invariant weak Jacobi form of weight k and index t. If φ further satisfies that $f(n,\ell) = 0$ whenever $2nt - (\ell,\ell) < 0$, then it is called a Conway invariant holomorphic Jacobi form. We denote the spaces of Conway invariant weak and holomorphic Jacobi forms of weight kand index t by $J_{k,\Lambda,t}^{\mathrm{w,Co_0}}$ and $J_{k,\Lambda,t}^{\mathrm{Co_0}}$ respectively.

Jacobi forms of index 0 are independent of \mathfrak{z} and degenerate into usual modular forms on $SL_2(\mathbb{Z})$. Since the Leech lattice Λ is unimodular, the theta decomposition (see [17, Corollary 2.6])) yields that every Conway invariant weak Jacobi form of weight k and index 1 is a holomorphic Jacobi form and can be expressed as $g(\tau)\Theta_{\Lambda}(\tau,\mathfrak{z})$, where $g(\tau)$ is a modular form of weight k-12 on $\mathrm{SL}_2(\mathbb{Z})$, and $\Theta_{\Lambda}(\tau,\mathfrak{z})$ is the Jacobi theta function of the Leech lattice defined by

(1.1)
$$\Theta_{\Lambda}(\tau, \mathfrak{z}) = \sum_{\ell \in \Lambda} e^{\pi i(\ell, \ell)\tau + 2\pi i(\ell, \mathfrak{z})}.$$

In this paper we determine the generators of Conway invariant Jacobi forms of indices 2 and 3. Let $J_{*,\Lambda,t}^{\text{w,Co}_0}$ and $J_{*,\Lambda,t}^{\text{Co}_0}$ denote the spaces of Conway invariant weak and holomorphic Jacobi forms of integral weight and given index t respectively. Let $M_*(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{C}[E_4, E_6]$ be the ring of modular forms on $SL_2(\mathbb{Z})$. The following is our main theorem.

Theorem 1.2. As free modules over $M_*(SL_2(\mathbb{Z}))$,

- J_{*,Λ,2}^{w,Co₀} is generated by four forms of weights -4, -2, 0, 0.
 J_{*,Λ,2}^{Co₀} is generated by four forms of weights 12, 12, 14, 16.
 J_{*,Λ,3}^{w,Co₀} is generated by ten forms of weights -14, -12, -12, -12, -10, -8, -6, -4, -2, 0.
 J^{Co₀}_{*,Λ,3} is generated by ten forms of weights 12, 12, 14, 14, 16, 16, 16, 18, 18.

We sketch the main idea of the proof. We first use the differential operators approach in [24] to estimate the minimal weight of weak Jacobi forms of a given index. Then we combine the arguments in [20, 24, 22] to construct generators. We also construct one of the singular-weight generators of $J_{*,\Lambda,t}^{\text{Co}_0}$ as the t-th Fourier–Jacobi coefficient of Borcherds' automorphic form Φ_{12} for the unimodular lattice $II_{26,2}$ (see [4]). The main difficulty of the proof is to calculate the Fourier expansions of generators, because Conway invariant Jacobi forms have unwieldy Fourier expansions in 25 variables. To overcome this difficulty, we write the Fourier expansion of a Jacobi form in terms of $Conway\ orbits$ defined as the Co₀-invariant exponential polynomials

(1.2)
$$\operatorname{orb}(v) = \sum_{\sigma \in \operatorname{Co}_0/(\operatorname{Co}_0)_v} e^{2\pi i (\sigma(v), \mathfrak{z})},$$

where $v \in \Lambda$ and $(Co_0)_v$ is the stabilizer of Co_0 with respect to v. The Conway orbits $\operatorname{orb}(v)$ of type $\frac{1}{2}(v,v) \leq 16$ are available in [12]. In order to calculate the Fourier expansions of products of Jacobi forms, we have to know the decomposition of some products $\operatorname{orb}(v)\operatorname{orb}(u)$ into linear combinations of Conway orbits. We determine such non-trivial decompositions by comparing the Fourier–Jacobi expansion of Φ_{12} and the Borcherds denominator formula for the fake monster Lie algebra (see [2, 4]). Combining these arguments together, we prove the theorem.

This paper is structured as follows. In §2 we collect and prove some basic results on Conway invariant Jacobi forms and orbits of Leech vectors. In §3 we introduce two methods to calculate the Fourier-Jacobi expansion of Borcherds' form Φ_{12} , and determine the product decompositions of $\operatorname{orb}(v_2)\operatorname{orb}(v_2)$ and $\operatorname{orb}(v_2)\operatorname{orb}(v_3)$ by comparing the two methods, where v_2 and v_3 are Leech vectors of types 2 and 3. The main theorem is proved in $\S 4$ for index t=2 and in $\S 5$ for index t=3. In §6 we present many applications of our main results. (i) We find the modular linear differential equations satisfied by some holomorphic Jacobi forms. (ii) We determine more product decompositions of Conway orbits by means of linear relations among Conway invariant holomorphic Jacobi forms of index 3. These results are formulated in Appendix A. (iii) We classify Conway invariant holomorphic Jacobi forms of singular weight 12 and index t < 3 with non-trivial character. (iv) We use the Fourier expansions of our Jacobi forms to determine all conjugate relations among Conway orbits of type $\frac{1}{2}(v,v) \leq 16$ modulo 2Λ and 3Λ . (v) We calculate the pullbacks of Conway invariant Jacobi forms and Conway orbits along Leech vectors of types 2, 3 and 4. The pullbacks of Conway orbits are formulated in Appendix B. In §7 we discuss Conway invariant Jacobi forms of higher index and propose some open questions. We know from Borcherds' thesis [1] that $\Lambda/4\Lambda$ have 31 orbits with respect to Co₀. We give an explicit description of the representative system of minimal length of $\Lambda/4\Lambda$ in Theorem 7.1. Borcherds' result yields that the rank of $J_{*,\Lambda,4}^{\mathrm{w,Co_0}}$ is 31. It seems very difficult to determine and construct the associated 31 generators of index 4.

2. Conway invariant Jacobi forms

In this section we prove some basic properties of Conway invariant Jacobi forms. Most of them are standard in the general theory of Jacobi forms. We also construct some basic Conway invariant holomorphic Jacobi forms.

2.1. **Basic results.** All results in this subsection are known to experts, and their analogues hold if we replace the Leech lattice and the Conway group with any even lattice and its orthogonal group.

Lemma 2.1. There is no nonzero Conway invariant weak Jacobi form of odd weight.

Proof. It follows from
$$-id \in Co_0$$
.

Lemma 2.2. The Fourier expansion of $\varphi \in J_{k,\Lambda,t}^{\mathrm{w,Co_0}}$ with $t \geq 1$ satisfies the following properties.

(1) If
$$2n_1t - (\ell_1, \ell_1) = 2n_2t - (\ell_2, \ell_2)$$
 and $\ell_1 - \ell_2 \in t\Lambda$, then $f(n_1, \ell_1) = f(n_2, \ell_2)$. In addition, $f(n, \ell) = f(n, -\ell)$, for all $n \in \mathbb{N}$ and $\ell \in \Lambda$.

(2) If $f(n, \ell) \neq 0$, then

$$2nt - (\ell, \ell) \ge -\min\{(v, v) : v \in \ell + t\Lambda\}.$$

Proof. It is a particular case of [17, Lemma 2.1]. The proof follows from the quasi-periodicity and the transformation under $-I_2 \in SL_2(\mathbb{Z})$.

The above (2) yields that for any $n \ge 0$ the q^n -term of a Conway invariant weak Jacobi form φ defined by

$$[\varphi]_{q^n} = \sum_{\ell \in \Lambda} f(n,\ell) \zeta^{\ell}, \quad \zeta^{\ell} = e^{2\pi i (\ell,\mathfrak{z})}$$

has only finitely many terms, and is actually an exponential polynomial invariant under Co₀.

Lemma 2.3. Let φ be a non-constant Conway invariant holomorphic Jacobi form of weight k and index $t \geq 1$. Then $k \geq 12$. Moreover, if φ is of weight 12, then its Fourier expansion has the form

$$\varphi(\tau, \mathfrak{z}) = \sum_{n=0}^{\infty} \sum_{\substack{\ell \in \Lambda \\ (\ell, \ell) = 2nt}} f(n, \ell) q^n \zeta^{\ell}.$$

Proof. The claim is proved by the theta decomposition (see e.g. [17, Lemma 2.3, Lemma 2.5]). \Box

The minimal positive weight 12 is called the singular weight of Conway invariant holomorphic Jacobi forms. By the two lemmas above, the Fourier coefficients $f(n,\ell)$ of a Conway invariant holomorphic Jacobi form of singular weight and index t only depend on the Co₀-orbit of the class of ℓ in $\Lambda/t\Lambda$.

We have defined the Conway orbit of any Leech vector in (1.2). Obviously, any q^n -term of a Conway invariant Jacobi form is a \mathbb{C} -linear combination of finitely many Conway orbits. In view of the q^0 -term of a Conway invariant weak Jacobi form of index $t \geq 1$ (see Lemma 2.2), we consider the Leech vectors of minimal norm modulo $t\Lambda$. Let us define

(2.1)
$$S_t = \{ v \in \Lambda : (v, v) \le (u, u), \text{ for all } u \in v + t\Lambda \}.$$

Due to the Conway invariance, we further consider the orbit space S_t/Co_0 . If two orbits $\operatorname{Co}_0 v$ and $\operatorname{Co}_0 u$ in S_t/Co_0 are conjugate modulo $t\Lambda$, namely there exists $\sigma \in \operatorname{Co}_0$ such that $v - \sigma(u) \in t\Lambda$, then (v,v) = (u,u). The conjugate orbits of the same norm in S_t/Co_0 correspond to the same Fourier coefficient in the sense of Lemma 2.2. Thus we define the quotient space $(S_t/\operatorname{Co}_0)/t\Lambda$, and denote the number of its elements by

$$(2.2) s(t) := |(\mathcal{S}_t/\operatorname{Co}_0)/t\Lambda|.$$

In other words, s(t) is the number of Co₀-orbits of the representative system of minimal length of $\Lambda/t\Lambda$. For convenience, we call the Conway orbits

(2.3)
$$\operatorname{orb}(v), \quad v \in \mathcal{S}_t/\operatorname{Co}_0$$

the basic Conway orbits of index t, as they appear in the q^0 -terms of weak Jacobi forms of index t.

Lemma 2.4. The space $J_{*,\Lambda,t}^{\mathrm{w,Co_0}}$ of Conway invariant weak Jacobi forms of integral weight and fixed index $t \geq 1$ is a free module of rank s(t) over $M_*(\mathrm{SL}_2(\mathbb{Z}))$. Moreover, the q^0 -terms of generators are \mathbb{C} -linear combinations of basic Conway orbits of index t, and thus linearly independent over \mathbb{C} . The space $J_{*,\Lambda,t}^{\mathrm{Co_0}}$ is also a free module of the same rank s(t) over $M_*(\mathrm{SL}_2(\mathbb{Z}))$.

Proof. For any $\varphi \in J_{k,\Lambda,t}^{\mathrm{w,Co_0}}$ the product $\Delta^d \varphi$ is a holomorphic Jacobi form for all integers d satisfying

$$d \ge \frac{1}{2t} \cdot \max\{(v, v) : v \in \mathcal{S}_t\}.$$

Thus the weight of Conway invariant weak Jacobi forms is bounded from below. Similar to the proof of [14, Theorem 8.4], we show that $J_{*,\Lambda,t}^{\mathrm{W,Co_0}}$ and $J_{*,\Lambda,t}^{\mathrm{Co_0}}$ are free modules of the same rank over $M_*(\mathrm{SL}_2(\mathbb{Z}))$. Lemma 2.2 yields that the q^0 -term of a Conway invariant weak Jacobi form of index t is a \mathbb{C} -linear combination of the Conway orbits associated with vectors in \mathcal{S}_t . On the other hand, we view Conway invariant weak Jacobi forms of index t as modular forms for the Weil representation

attached to $\Lambda/t\Lambda$ (see [5]). By the obstruction principle in [6], there exist weak Jacobi forms of sufficiently large weight whose q^0 -term is a single basic Conway orbit of index t (up to conjugate modulo $t\Lambda$). It follows that the rank is s(t). The q^0 -terms of generators are linearly independent, otherwise a suitable $\mathbb{C} \cdot E_4^a E_6^b$ -linear combination of generators will give a weak Jacobi form whose q^0 -term is zero, and therefore can be written as the product of Δ and a $\mathbb{C}[E_4, E_6]$ -linear combination of generators. This contradicts the freeness of $J_{*,\Lambda,t}^{\mathrm{w,Co_0}}$ as a $\mathbb{C}[E_4, E_6]$ -module.

Lemma 2.5. For any even integer $k \geq 14$, the following identity holds

(2.4)
$$\dim J_{k,\Lambda,t}^{\text{w,Co}_0} - \dim J_{k,\Lambda,t}^{\text{Co}_0} = \delta_t, \quad t \ge 1,$$

where δ_t is defined as

(2.5)
$$\delta_t = \sum_{v \in (\mathcal{S}_t/\operatorname{Co}_0)/t\Lambda} \epsilon\left(\frac{(v,v)}{2t}\right),$$

here $\epsilon(x) := \min\{n \in \mathbb{Z} : n \ge x\}.$

Proof. The proof is similar to that of [22, Proposition 5.1]. We only mention two essential ingredients of the proof. (a) A Conway invariant weak Jacobi form of index t is a holomorphic Jacobi form if and only if its Fourier expansion does not contain the following terms

$$q^n \operatorname{orb}(v), \quad 0 \le n < \frac{1}{2t}(v, v), \quad v \in (\mathcal{S}_t/\operatorname{Co}_0)/t\Lambda.$$

- (b) When the weight $k \geq 14$, the obstruction principle yields the existence of Conway invariant weak Jacobi forms of index t whose Fourier expansion involves only one of the above terms. \Box
- 2.2. The construction of Jacobi forms. We introduce two standard methods to construct Jacobi forms. The first one is the differential operators which raise the weight of Jacobi forms.

Lemma 2.6. Given a Conway invariant weak Jacobi form of weight k and index $t \geq 1$

$$\varphi(\tau, \mathfrak{z}) = \sum_{n=0}^{\infty} \sum_{r \in \Lambda/\operatorname{Co}_0} f(n, r) q^n \cdot \operatorname{orb}(r).$$

Then $H_k(\varphi)$ is a Conway invariant weak Jacobi form of weight k+2 and index t, where

$$H_k(\varphi)(\tau,\mathfrak{z}) = \mathcal{H}(\varphi)(\tau,\mathfrak{z}) + \frac{12 - k}{12} E_2(\tau) \varphi(\tau,\mathfrak{z}),$$

$$\mathcal{H}(\varphi)(\tau,\mathfrak{z}) = \sum_{n \in \mathbb{N}} \sum_{r \in \Lambda/\operatorname{Co}_0} \left(n - \frac{(r,r)}{2t} \right) f(n,r) q^n \cdot \operatorname{orb}(r),$$

and $E_2(\tau) = 1 - 24 \sum_{n \ge 1} \sigma(n) q^n$ is the Eisenstein series of weight 2 on $SL_2(\mathbb{Z})$.

Proof. The construction of H_k relies heavily on the transformation laws of the heat operator \mathcal{H} with respect to $SL_2(\mathbb{Z})$ which were described in [14, 7]. We refer to [23, Lemma 2.2] for a proof. \square

Subsequently, if there is no confusion, we will write $H\varphi = H_k(\varphi)$, and denote the d-th composition of H by H^d for short.

Lemma 2.7. The q^0 -term of any $\varphi \in J_{0,\Lambda,t}^{\mathrm{w,Co_0}}$ satisfies the following identity

$$\sum_{r \in \Lambda/\operatorname{Co}_0} (2t - (r, r)) f(0, r) |\operatorname{orb}(r)| = 0.$$

Proof. It is a particular case of [16, Proposition 2.6]. It follows from that the constant term of $H_0(\varphi)(\tau,0)$ is zero.

The second method is the Hecke operators which raise the index of Jacobi forms.

Lemma 2.8. Let m be a positive integer and $\varphi \in J_{k,\Lambda,t}^{\mathrm{w},\mathrm{Co}_0}$. Then we have

$$(\varphi|_k T_-(m))(\tau, \mathfrak{z}) = m^{-1} \sum_{\substack{ad = m, a > 0 \\ 0 \le b < d}} a^k \varphi\left(\frac{a\tau + b}{d}, a\mathfrak{z}\right) \in J_{k,\Lambda,mt}^{\text{w,Co}_0},$$

and the Fourier expansion of $\varphi|_kT_-(m)$ is given by

$$(\varphi|_k T_-(m)) (\tau, \mathfrak{z}) = \sum_{\substack{n \in \mathbb{N} \\ \ell \in \Lambda}} \sum_{\substack{d \ge 1 \\ d \mid (n, \ell, m)}} d^{k-1} f\left(\frac{nm}{d^2}, \frac{\ell}{d}\right) q^n \zeta^{\ell},$$

where $f(n,\ell)$ are Fourier coefficients of φ , and $d|(n,\ell,m)$ means that d|n, d|m and $d^{-1}\ell \in \Lambda$. Proof. It is a particular case of [15, Corollary 1 of Proposition 4] or [17, Corollary 2.9].

2.3. Orbits of Leech vectors under the Conway group. Following the ATLAS [12], we define the type of a Leech vector v as half of its norm, i.e. $\frac{1}{2}(v,v)$. One can read the number λ_n of Leech vectors of type n from the theta function of the Leech lattice

(2.6)
$$\theta_{\Lambda}(\tau) = \sum_{\ell \in \Lambda} e^{\pi i(\ell,\ell)\tau} = \sum_{n=0}^{\infty} \lambda_n q^n.$$

We note that $\theta_{\Lambda}(\tau) = \Theta_{\Lambda}(\tau, 0)$ (see (1.1)). It is well-known that θ_{Λ} is a modular form of weight 12 on $SL_2(\mathbb{Z})$, and it can be calculated via (see e.g. [11])

(2.7)
$$\theta_{\Lambda}(\tau) = E_{12}(\tau) - \frac{65520}{691} \Delta(\tau) = 1 + \frac{65520}{691} \sum_{n=1}^{\infty} (\sigma_{11}(n) - \tau(n)) q^{n}.$$

where $E_{12}(\tau) = 1 + O(q)$ is the normalized Eisenstein series of weight 12, and

(2.8)
$$\Delta(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n$$

is the unique normalized cusp form of weight 12 on $SL_2(\mathbb{Z})$.

The Conway group Co₀ acts naturally on Leech vectors. The orbits of Λ/Co_0 were described in ATLAS [12, Page 181] for Leech vectors of type $x \leq 16$. There are in total 44 orbits of type $x \leq 16$ including the orbit of type 0. Among the 44 orbits, the orbits of the same type have distinct numbers of elements, and their numbers of elements are all divisible by p = 65520. Let $O_{x,yp}$ stand for an orbit of type x whose number of elements is yp. If there are multiple orbits of the same type, then we mark them as O_{xa,y_1p} , O_{xb,y_2p} , etc. in the order of the ATLAS. We formulate the 43 nonzero orbits as follows

$O_{2,3p}$	$O_{3,256p}$	$O_{4,6075p}$	$O_{5,70656p}$	$O_{6a,518400p}$
$O_{6b,6900p}$	$O_{7,2861568p}$	$O_{8a,3p}$	$O_{8b,12295800p}$	$O_{8c,141312p}$
$O_{9a,12441600p}$	$O_{9b,32972800p}$	$O_{10a,143078400p}$	$O_{10b,279450p}$	$O_{10c,1430784p}$
$O_{11a,19430400p}$	$O_{11b,393465600p}$	$O_{12a,256p}$	$O_{12b,141312p}$	$O_{12c,12441600p}$
$O_{12d,2049300p}$	$O_{12e,393465600p}$	$O_{12f,667699200p}$	$O_{13a,12441600p}$	$O_{13b,1007271936p}$
$O_{13c,1573862400p}$	$O_{14a,286156800p}$	$O_{14b,5508518400p}$	$O_{14c,13800p}$	$O_{14d,19430400p}$
$O_{14e,49183200p}$	$O_{15a,2861568p}$	$O_{15b,1335398400p}$	$O_{15c,19430400p}$	$O_{15d,8012390400p}$
$O_{15e,3147724800p}$	$O_{16a,6075p}$	$O_{16b,12441600p}$	$O_{16c,286156800p}$	$O_{16d,98366400p}$
$O_{16e,5901984000p}$	$O_{16f,16024780800p}$	$O_{16g,3147724800p}$		

Since the numbers y appearing in the above orbits are very large, for the sake of simplicity, we omit the subscript yp of the orbit. We also use the same notation O_x to denote the corresponding Conway orbit $\sum_{v \in O_x} e^{2\pi i(v,\mathfrak{z})}$. There are 3 non-primitive orbits of type $x \leq 16$, that is, $O_{8a} = \text{orb}(2v_2)$, $O_{12a} = \text{orb}(2v_3)$ and $O_{16a} = \text{orb}(2v_4)$, where v_x is a Leech vector of type x. For future use, we denote by $O_{18a} = \text{orb}(3v_2)$ the non-primitive Conway orbit of type 18.

To describe the representative system of $\Lambda/t\Lambda$, we introduce the following notions.

Definition 2.9. For $v, u \in \Lambda$, we say that v is conjugate to u modulo $t\Lambda$ if $v - u \in t\Lambda$, noted by $v \sim_t u$. The t-weight of a Leech vector v is defined as the number

$$\#\{u \in \Lambda : u \in t\Lambda + v, \quad (u, u) = (v, v)\}.$$

The t-weight of an orbit $Co_0 v$ is defined as the number

$$\#\{u \in \operatorname{Co}_0 v : u \in t\Lambda + v\}.$$

We warn that the t-weight of an orbit may be different from the t-weight of a vector in this orbit (see Theorem 7.1). The following lemma was proved in [9].

Lemma 2.10. Representatives of $\Lambda/2\Lambda$ of minimal norm may be found among vectors of types up to 4, according to the weighted equality (here the 2-weights of orbits and vectors coincide)

$$1 + \frac{|O_2|}{2} + \frac{|O_3|}{2} + \frac{|O_4|}{48} = 2^{24}.$$

The following lemma was proved in [19, Theorem 4.1]. It can also be derived from [1, Figure 2].

Lemma 2.11. Representatives of $\Lambda/3\Lambda$ of minimal norm may be found among vectors of types up to 9, according to the weighted equality (here the 3-weights of orbits and vectors coincide)

$$1 + |O_2| + |O_3| + |O_4| + |O_5| + |O_{6a}| + \frac{|O_{6b}|}{3} + \frac{|O_7|}{2} + \frac{|O_{8b}|}{9} + \frac{|O_{9b}|}{36} = 3^{24}.$$

We know from [19, Lemma 4.3] that O_{8a} , O_{8c} and O_{9a} are respectively conjugate to O_2 , O_5 and O_{6a} modulo 3Λ . It is easy to derive the following result from the lemmas above.

Lemma 2.12.

- (1) The rank of $J_{*,\Lambda,2}^{\mathrm{w,Co_0}}$ over $M_*(\mathrm{SL}_2(\mathbb{Z}))$ is 4. The basic Conway orbits of index 2 are O_0 , O_2 , O_3 , O_4 . The number δ_2 defined in Lemma 2.5 is 5.
- (2) The rank of $J_{*,\Lambda,3}^{\mathrm{w,Co_0}}$ over $M_*(\mathrm{SL}_2(\mathbb{Z}))$ is 10. The basic Conway orbits of index 3 are O_0 , O_2 , O_3 , O_4 , O_5 , O_{6a} , O_{6b} , O_7 , O_{8b} , O_{9b} . The number δ_3 defined in Lemma 2.5 is 19.
- 2.4. Conway invariant holomorphic Jacobi forms. We construct some Conway invariant holomorphic Jacobi forms of low weights. As in [20] we start with the Jacobi theta function of Λ . As we mentioned in the introduction, the Jacobi theta function $\Theta_{\Lambda}(\tau,\mathfrak{z})$ defined in (1.1) is a Conway invariant holomorphic Jacobi form of weight 12 and index 1. By applying the Hecke operators introduced in Lemma 2.8 to $\Theta_{\Lambda}(\tau,\mathfrak{z})$, we can construct Conway invariant holomorphic Jacobi forms of weight 12 and arbitrary index. More precisely, for any $t \geq 2$ we have

(2.9)
$$A_{t}(\tau, \mathfrak{z}) := \frac{1}{\sigma_{11}(t)} (\Theta_{\Lambda} | T_{-}(t))(\tau, \mathfrak{z})$$
$$= 1 + \frac{1}{\sigma_{11}(t)} \sum_{\ell \in \Lambda, (\ell, \ell) = 2t} q \cdot \zeta^{\ell} + O(q^{2}) \in J_{12, \Lambda, t}^{\text{Co}_{0}}.$$

It is easy to see from Lemma 2.8 that every $\sigma_{11}(t)A_t$ has integral Fourier coefficients. For convenience, we will use A_1 instead of Θ_{Λ} from now on. The reduction of A_t is given by

(2.10)
$$A_t(\tau,0) = E_{12}(\tau) - \frac{65520}{691} \frac{\tau(t)}{\sigma_{11}(t)} \Delta(\tau).$$

Using the data of Conway orbits of type $x \leq 16$, we can calculate the q^d -terms of the Fourier expansion of A_t for $d \leq 16/t$. The forms A_2 and A_3 will be used later and we present their Fourier expansions up to q^5 -terms

$$A_{2} = 1 + \frac{1}{2049} \left(O_{2}q + O_{4}q^{2} + (O_{6a} + O_{6b})q^{3} + (2049O_{8a} + O_{8b} + O_{8c})q^{4} + (O_{10a} + O_{10b} + O_{10c})q^{5} \right) + O(q^{6}),$$

$$A_{3} = 1 + \frac{1}{177148} \left(O_{3}q + (O_{6a} + O_{6b})q^{2} + (O_{9a} + O_{9b})q^{3} + (O_{12a} + O_{12b} + O_{12c} + O_{12d} + O_{12e} + O_{12f})q^{4} + (O_{15a} + O_{15b} + O_{15c} + O_{15d} + O_{15e})q^{5} \right) + O(q^{6}).$$

Since A_t has singular weight, the differential operators introduced in Lemma 2.6 acting on them give zero. In order to construct Conway invariant holomorphic Jacobi forms of weight 14 and index $t \geq 2$, we use the trick in [20, 24]. We notice that $A_1(t\tau,t\mathfrak{z})$ is a Conway invariant holomorphic Jacobi form of weight 12 and index t with respect to the congruence subgroup $\Gamma_0(t)$ of $\mathrm{SL}_2(\mathbb{Z})$. Given a modular form $f_k(\tau)$ of weight k on $\Gamma_0(t)$, then $f_k(\tau)A_1(t\tau,t\mathfrak{z})$ defines a Conway invariant holomorphic Jacobi form of weight 12 + k and index t for $\Gamma_0(t)$. The trace operation

(2.12)
$$\operatorname{Tr}_{\operatorname{SL}_2(\mathbb{Z})}(f_k(\tau)A_1(t\tau,t\mathfrak{z})) = \sum_{\gamma \in \Gamma_0(t) \backslash \operatorname{SL}_2(\mathbb{Z})} (f_k(\tau)A_1(t\tau,t\mathfrak{z}))|_{12+k,t}\gamma$$

gives a Conway invariant holomorphic Jacobi form of weight 12 + k and index t, where $|_{k,t}\gamma$ is the slash action of $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ defined via

$$(\phi|_{k,t}\gamma)(\tau,\mathfrak{z}) := (c\tau+d)^{-k} \exp\left(-t\pi i \frac{c(\mathfrak{z},\mathfrak{z})}{c\tau+d}\right) \phi\left(\frac{a\tau+b}{c\tau+d}, \frac{\mathfrak{z}}{c\tau+d}\right).$$

It is well-known that $f_2(\tau) := (tE_2(t\tau) - E_2(\tau))/(t-1) \in M_2(\Gamma_0(t))$. We apply the trace operator to $f_2(\tau)A_1(t\tau,t\mathfrak{z})$ and denote the normalized image by

(2.13)
$$B_t(\tau, \mathfrak{z}) = 1 + O(q) \in J_{14,\Lambda,t}^{\text{Co}_0}, \quad t \ge 2,$$

whose reduction is

(2.14)
$$B_t(\tau,0) = E_{14}(\tau) = E_4(\tau)^2 E_6(\tau).$$

More precisely, for prime t we have

(2.15)
$$B_t(\tau, \mathfrak{z}) = \frac{t^{12}}{t^{12} - 1} \Big(f_2(\tau) A_1(t\tau, t\mathfrak{z}) - \frac{1}{t^{13}} \sum_{i=0}^{t-1} f_2\left(\frac{\tau + j}{t}\right) A_1\left(\frac{\tau + j}{t}, \mathfrak{z}\right) \Big).$$

The Fourier expansions of B_2 and B_3 are as follows

$$B_{2} = 1 + \frac{1}{4095} \Big((98280 - O_{2})q + (98280 - 24O_{2} - 24O_{3} - O_{4})q^{2} + (393120 - 24O_{2} - 96O_{3} - 24O_{4} - 24O_{5} - O_{6a} - O_{6b})q^{3} + (98280 - 96O_{2} - 144O_{3} - 24O_{4} - 96O_{5} - 24O_{6a} - 24O_{6b} - 24O_{7} + 4095O_{8a} - O_{8b} - O_{8c})q^{4} + (589680 - 24O_{2} - 192O_{3} - 96O_{4} - 144O_{5} - 24O_{6a} - 24O_{6b} - 96O_{7} + 98280O_{8a} - 24O_{8b} - 24O_{8c} - 24O_{9a} - 24O_{9b} - O_{10a} - O_{10b} - O_{10c})q^{5} \Big) + O(q^{6}).$$

$$B_{3} = 1 + \frac{1}{531440} \left((6377280 - 12O_{2} - O_{3})q + (19131840 - 84O_{2} - 12O_{3} - 36O_{4} - 12O_{5} - O_{6a} - O_{6b})q^{2} + (6377280 - 96O_{2} - 36O_{3} - 72O_{4} - 84O_{5} - 12O_{6a} - 12O_{6b} - 36O_{7} - 12O_{8a} - 12O_{8b} - 12O_{8c} - O_{9a} - O_{9b})q^{3} + (44640960 - 216O_{2} - 12O_{3} - 180O_{4} - 96O_{5} - 36O_{6a} - 36O_{6b} - 72O_{7} - 84O_{8a} - 84O_{8b} - 84O_{8c} - 12O_{9a} - 12O_{9b} - 36O_{10a} - 36O_{10b} - 36O_{10c} - 12O_{11a} - 12O_{11b} - O_{12a} - O_{12b} - O_{12c} - O_{12d} - O_{12e} - O_{12f})q^{4} + (38263680 - 168O_{2} - 84O_{3} - 144O_{4} - 216O_{5} - 12O_{6a} - 12O_{6b} - 180O_{7} - 96O_{8a} - 96O_{8b} - 96O_{8c} - 36O_{9a} - 36O_{9b} - 72O_{10a} - 72O_{10b} - 72O_{10c} - 84O_{11a} - 84O_{11b} - 12O_{12a} - 12O_{12b} - 12O_{12c} - 12O_{12d} - 12O_{12e} - 12O_{12f} - 36O_{13a} - 36O_{13b} - 36O_{13c} - 12O_{14a} - 12O_{14b} - 12O_{14c} - 12O_{14d} - 12O_{14c} - 12O_{14d} - 12O_{14e} - O_{15a} - O_{15b} - O_{15c} - O_{15d} - O_{15e})q^{5} \right) + O(q^{6}).$$

3. Jacobi forms of singular weight and the fake monster Lie algebra

In 1990 Borcherds [2] constructed a celebrated generalized Kac-Moody algebra whose root lattice is $II_{25,1}$, that is, the unique even unimodular lattice of signature (25,1). This infinite-dimensional Lie algebra describes the physical states of a bosonic string moving on the \mathbb{Z}_2 orbifold of the torus $\mathbb{R}^{25,1}/II_{25,1}$ (see the string theory background in [13]), and is called the *fake monster Lie algebra*. Borcherds proved that the denominator identity of this algebra defines an automorphic form of weight 12 on a symmetric domain of type IV and dimension 26. This form is exceptional, because Scheithauer [21] proved that it is the unique holomorphic Borcherds product of singular weight on unimodular lattices. In this section we construct Conway invariant holomorphic Jacobi forms of singular weight and calculate their Fourier expansions using this remarkable identity.

We first review the modularity of this denominator identity. Let U be an even unimodular lattice of signature (1,1). Then $U \oplus \Lambda$ gives a model of $II_{25,1}$. We further define $II_{26,2} = 2U \oplus \Lambda$. The symmetric domain $\mathcal{D}(II_{26,2})$ of type IV attached to $II_{26,2}$ is one of the two conjugate connected components of the space

$$\{[\mathcal{Z}]\in\mathbb{P}(II_{26,2}\otimes\mathbb{C}):(\mathcal{Z},\mathcal{Z})=0,(\mathcal{Z},\overline{\mathcal{Z}})<0\}.$$

Let $O^+(II_{26,2})$ denote the orthogonal group which preserves $II_{26,2}$ and $\mathcal{D}(II_{26,2})$. We define the affine cone over $\mathcal{D}(II_{26,2})$ as

$$\mathcal{A}(\mathrm{II}_{26,2}) = \{\mathcal{Z} \in \mathrm{II}_{26,2} \otimes \mathbb{C} : [\mathcal{Z}] \in \mathcal{D}(\mathrm{II}_{26,2})\}.$$

In 1995 Borcherds constructed a holomorphic function $\Phi_{12}: \mathcal{A}(\mathrm{II}_{26,2}) \to \mathbb{C}$ which satisfies

$$\Phi_{12}(t\mathcal{Z}) = t^{-12}\Phi_{12}(\mathcal{Z}), \quad \text{for all } t \in \mathbb{C}^{\times},$$

$$\Phi_{12}(g(\mathcal{Z})) = \det(g)\Phi_{12}(\mathcal{Z}), \quad \text{for all } g \in \mathrm{O}^{+}(\mathrm{II}_{26,2}),$$

and vanishes precisely with multiplicity one on hyperplanes orthogonal to roots of $II_{26,2}$. In other words, Φ_{12} is a modular form of minimal weight 12 and determinant character for $O^+(II_{26,2})$. This function has an infinite product expansion. The modular variety $\mathcal{D}(II_{26,2})/O^+(II_{26,2})$ has a unique 0-dimensional cusp and twenty-four 1-dimensional cusps which correspond to the twenty-four Niemeier lattices. We fix a basis $U = \mathbb{Z}e + \mathbb{Z}f$ with $e^2 = f^2 = 0$ and (e, f) = -1. At the 1-dimensional cusp corresponding to the Leech lattice we realize $\mathcal{D}(II_{26,2})$ as the tube domain

$$\mathcal{H}(\Lambda) = \{Z = -\omega e + \mathfrak{z} - \tau f : \tau, \omega \in \mathbb{H}, \ \mathfrak{z} \in \Lambda \otimes \mathbb{C} : (\operatorname{Im}(Z), \operatorname{Im}(Z)) < 0\}.$$

We write $\alpha = ne + \ell + mf \in U \oplus \Lambda$ as (n, ℓ, m) . Note that $\alpha^2 = -2nm + (\ell, \ell)$ and $(\alpha, Z) = n\tau + m\omega + (\ell, \mathfrak{z})$. Borcherds [2, 4] proved that Φ_{12} are represented on $\mathcal{H}(\Lambda)$ by

(3.1)
$$\Phi_{12}(Z) = e^{2\pi i (\rho, Z)} \prod_{\alpha \in \Delta_{+}} \left(1 - e^{2\pi i (\alpha, Z)} \right)^{p_{24}(1 - (\alpha, \alpha)/2)}$$
$$= \sum_{\sigma \in W} \det(\sigma) \sum_{n=1}^{\infty} \tau(n) e^{2\pi i (\sigma(n\rho), Z)},$$

where $\rho = (1,0,0)$ is the Weyl vector, W is the reflection group of $U \oplus \Lambda$, Δ_+ is the set of positive roots of the fake monster Lie algebra which are either positive multiples of ρ or vectors α in $U \oplus \Lambda$ satisfying $(\alpha, \alpha) \leq 2$ and $(\alpha, \rho) < 0$, $\tau(n)$ is the Ramanujan tau function in (2.8), and $p_{24}(n)$ is defined by

$$\Delta(\tau)^{-1} = \sum_{n=0}^{\infty} p_{24}(n)q^{n-1}.$$

This is the so-called Weyl-Kac-Borcherds denominator identity of the fake monster Lie algebra. In terms of Conway invariant Jacobi forms, one can express Φ_{12} as follows (see [4, 16])

(3.2)
$$\Phi_{12}(Z) = \Delta(\tau) \cdot \exp\left(-\sum_{m=1}^{\infty} ((\Delta^{-1}A_1)|T_{-}(m))(\tau, \mathfrak{z})e^{2\pi im\omega}\right)$$
$$= \sum_{m=0}^{\infty} \Phi_{12,m}(\tau, \mathfrak{z})e^{2\pi im\omega},$$

where $T_{-}(m)$ are Hecke operators introduced in Lemma 2.8.

Theorem 3.1. For any $m \in \mathbb{N}$, the function $\Phi_{12,m}$ is a Conway invariant holomorphic Jacobi form of weight 12 and index m. Moreover, its Fourier expansion takes the form

$$\Phi_{12,m}(\tau,\mathfrak{z}) = \sum_{n=0}^{\infty} \sum_{\substack{\ell \in \Lambda \\ (\ell,\ell) = 2nm}} c_m(n,\ell) q^n \zeta^{\ell}$$

and satisfies the following properties.

(1) For any $n, m \in \mathbb{N}$ and $\ell \in \Lambda$ we have

$$c_m(n,\ell) = -c_n(m,\ell).$$

In particular, $c_m(m,\ell) = 0$ for any $m \in \mathbb{N}$ and $\ell \in \Lambda$.

(2) For any $n, m \in \mathbb{N}$ and $\ell \in \Lambda$, we have $c_m(n,\ell) \in \mathbb{Z}$. Let d be the largest positive integer such that $n/d, m/d \in \mathbb{N}$ and $d^{-1}\ell \in \Lambda$. Then we have

$$c_m(n,\ell) \in \{\tau(d), -\tau(d), 0\}.$$

Proof. The function $\Phi_{12,m}$ is in fact the m-th Fourier–Jacobi coefficient of Φ_{12} at the Leech cusp, which follows that it is a Conway invariant holomorphic Jacobi form of weight 12 and index m. Since $\Phi_{12,m}$ has singular weight, Lemma 2.3 yields the claimed Fourier expansion. We notice that Φ_{12} has the determinant character. Thus Φ_{12} is anti-invariant under exchanging τ and ω , which implies the property (1). The property (2) follows from the denominator identity (3.1).

It is clear that $\Phi_{12,0} = \Delta$ and $\Phi_{12,1} = -A_1$. We derive from the above theorem that $c_m(0,0) = -c_0(m,0) = -\tau(m)$ and $c_m(1,\ell) = -c_1(m,\ell) = 1$. Thus when $m \geq 2$ the Fourier expansion of $\Phi_{12,m}$ starts with

(3.3)
$$\Phi_{12,m}(\tau,\mathfrak{z}) = -\tau(m) + \sum_{\ell \in \Lambda, (\ell,\ell) = 2m} q \cdot \zeta^{\ell} + O(q^2),$$

and the reduction of $\Phi_{12,m}$ is given by

(3.4)
$$\Phi_{12,m}(\tau,0) = -\tau(m)E_{12}(\tau) + \frac{65520}{691}\sigma_{11}(m)\Delta(\tau).$$

By comparing reductions of A_t and $\Phi_{12,t}$, we prove the following lemma.

Lemma 3.2. When $t \geq 2$, the forms A_t and $\Phi_{12,t}$ are linearly independent. In particular, we have

$$\dim J_{12,\Lambda,t}^{\operatorname{Co}_0} \geq 2$$
, for $t \geq 2$.

By Theorem 3.1 and the reduction to modular forms of weight 12 on $SL_2(\mathbb{Z})$, we can unambiguously determine the Fourier expansion of $\Phi_{12,t}$ to very high q orders. For example, we find

$$\Phi_{12,2} = 24 + O_{2}q + O_{6b}q^{3} + (24O_{8a} + O_{8c})q^{4} + (O_{10b} + O_{10c})q^{5} + (24O_{12a} + O_{12b} + O_{12c})q^{6} + (O_{14c} + O_{14d} + O_{14e})q^{7} + (24O_{16a} + O_{16b} + O_{16c})q^{8} + O(q^{9}),$$

$$\Phi_{12,3} = -252 + O_{3}q - O_{6b}q^{2} + (O_{12a} + O_{12b} - O_{12d})q^{4} + (O_{15a} - O_{15c})q^{5} + O(q^{6}),$$

$$\Phi_{12,4} = 1472 + O_{4}q - q^{2}(24O_{8a} + O_{8c}) - q^{3}(O_{12a} + O_{12b} - O_{12d}) + O(q^{5}).$$

Let us briefly show how to find the above $\Phi_{12,2}$. We first determine its reduction by (3.4) as

(3.6)
$$\Phi_{12,2}(\tau,0) = 24 + 196560q + 452088000q^3 + 9263479680q^4 + 112054531680q^5 + 824834949120q^6 + 4496467248000q^7 + 19573719984000q^8 + O(q^9).$$

Since $\Phi_{12,2}$ is a holomorphic Jacobi form of index 2 and singular weight, its q^n -term is a linear combination of Conway orbits of type 2n. Clearly, the q^1 -term has to be O_2 . To determine the q^3 -term, we write $[\Phi_{12,2}]_{q^3} = xO_{6a} + yO_{6b}$. Then

$$[\Phi_{12,2}]_{q^3}(\tau,0) = x|O_{6a}| + y|O_{6b}|,$$
 i.e. $33965568000x + 452088000y = 452088000.$

By Theorem 3.1, x, y can only be ± 1 or 0. Obviously, the only solution is x = 0 and y = 1. Following this procedure, we uniquely determine the Fourier expansion of $\Phi_{12,2}$ up to q^8 -term. For $[\Phi_{12,2}]_{q^8}$, the similar linear equation has two solutions: the one in (3.5) and $24(O_{16a} + O_{16b})$, but the second possibility is ruled out knowing only orbits of type $\operatorname{orb}(2v)$ have coefficient 24 which can only be O_{16a} . We also use the same trick to find the Fourier expansions of $\Phi_{12,3}$ up to q^5 -term and $\Phi_{12,4}$ up to q^4 -term.

It turns out that these $\Phi_{12,t}$ are very useful to determine the product decomposition of Conway orbits, which would be very difficult to compute in a brutal way due to the huge size of the orbits. For example, from (3.2) it is straightforward to compute

$$\Phi_{12,2} = \frac{1}{2} \frac{A_1^2}{\Delta} - \Delta \left[\left(\frac{A_1}{\Delta} \right) \middle| T_{-}(2) \right]$$

$$= 24 + O_2 q + \left(\frac{1}{2} O_2 \otimes O_2 - 98280 - 2300 O_2 - 276 O_3 - 23 O_4 - O_5 - \frac{1}{2} O_{8a} \right) q^3 + (12 O_2 \otimes O_2 + O_2 \otimes O_3 - 2358720 - 102304 O_2 - 17802 O_3 - 2600 O_4 - 299 O_5 - 24 O_{6a} - 24 O_{6b} - O_7 + 12 O_{8a}) q^4 + O(q^5).$$

By comparing the q^3 -, q^4 -terms of (3.5) and (3.7), we can neatly determine the decompositions of orbit products $O_2 \otimes O_2$ and $O_2 \otimes O_3$. We then achieve the following result.

Lemma 3.3. The decompositions of $O_2 \otimes O_2$ and $O_2 \otimes O_3$ in Appendix A hold.

By comparing the q^5 -term of $\Phi_{12,2}$ in (3.5) and (3.7), we further determine the decomposition of $O_3 \otimes O_3 + 2O_2 \otimes O_4$, but cannot separate this combination. Using the q^6 -term of $\Phi_{12,2}$, we

determine $O_3 \otimes O_4 + O_2 \otimes O_5$, but cannot separate them either. Utilizing these constraints, we can compute from (3.2) that

$$\Phi_{12,3} = -\frac{1}{6} \frac{A_1^3}{\Delta^2} + A_1 \left[\left(\frac{A_1}{\Delta} \right) \middle| T_-(2) \right] - \Delta \left[\left(\frac{A_1}{\Delta} \right) \middle| T_-(3) \right]$$

$$= -252 + O_3 q - O_{6b} q^2 + \frac{1}{3} \left(2O_2 \otimes O_5 - O_2 \otimes O_{6b} + O_2 \otimes O_{8a} - 89609O_2 - 151800O_3 - 93196O_4 - 38350O_5 - 11868O_{6a} - 9484O_{6b} - 2804O_7 + 4600O_{8a} - 484O_{8b} - 276O_{8c} - 23O_{9a} - 66O_{9b} - 3O_{10a} + 43O_{10b} + 3O_{12a} - O_{18a} \right) q^4 + O(q^5).$$

By comparing the q^4 -term of $\Phi_{12,3}$ in (3.5) and (3.8), we determine the decomposition of the combination $O_2 \otimes (2O_5 - O_{6b} + O_{8a})$. These partially determined decompositions of orbit products can be remedied by the technique of Conway invariant Jacobi forms of index 3, which will be discussed later. The new technique even allows us to determine more product decompositions of orbits, which does not appear in the combinations here, such as $O_2 \otimes O_{6a}$.

4. Conway invariant Jacobi forms of index 2

In this section we prove parts (1) and (2) of Theorem 1.2. We first determine the space of Conway invariant holomorphic Jacobi forms of singular weight and index 2.

Lemma 4.1.

$$\dim J_{12,\Lambda,2}^{\text{Co}_0} = 2.$$

Proof. Suppose dim $J_{12,\Lambda,2}^{\text{Co}_0} > 2$. We notice that there is only one Conway orbit of type 2. By virtue of A_2 and $\Phi_{12,2}$, we can cancel the q^0 - and q^1 -terms of the third holomorphic Jacobi form of weight 12 and index 2. Then Lemma 2.3 yields that there exists a Jacobi form

$$\phi(\tau,\mathfrak{z}) = q^n \sum_{\ell \in \Lambda, (\ell,\ell) = 4n} f(n,\ell) \zeta^{\ell} + O(q^{n+1})$$

for some positive integer $n \geq 2$. Since ϕ/Δ^2 defines a weak Jacobi form of weight -12 and index 2, we have $\phi(\tau,0)=0$. By Lemma 2.2, the q^n -term of ϕ is a linear combination of basic Conway orbits of type 2n and index 2. Therefore, Lemma 2.12 implies that n=2 and up to nonzero scalar $\phi(\tau,\mathfrak{z})=q^2O_4+O(q^3)$, which contradicts with $\phi(\tau,0)=0$.

Theorem 4.2. The free $\mathbb{C}[E_4, E_6]$ -module $J_{*, \Lambda, 2}^{\text{Co}_0}$ is generated by A_2 , $\Phi_{12,2}$, B_2 and HB_2 .

Proof. We know that the minimal weight of non-constant Jacobi forms in $J_{*,\Lambda,2}^{\text{Co}_0}$ is 12. According to Lemma 2.12, $J_{*,\Lambda,2}^{\text{Co}_0}$ has four generators. By the above lemma, there are exactly two generators of weight 12. Clearly, B_2 has to be a generator of weight 14. The image of B_2 under the differential operator introduced in Lemma 2.6 gives a Conway invariant holomorphic Jacobi form of weight 16 and index 2. We verify that HB_2 is independent of E_4A_2 and $E_4\Phi_{12,2}$. Therefore, HB_2 is a generator of weight 16. We have found all four generators, and thus proved the theorem.

We remark that A_1^2 can be expressed by $A_2, \Phi_{12,2}, B_2, HB_2$ as

(4.1)
$$A_1^2 = 2\Delta\Phi_{12,2} + \frac{455}{1024}E_4(E_6B_2 - 3E_4HB_2) + \frac{683}{18432}(5E_4^3 + 4E_6^2)A_2.$$

To determine the free module of weak Jacobi forms, we first estimate the minimal weight of weak Jacobi forms using the differential operators approach established in [24].

Lemma 4.3.

$$J_{k,\Lambda,2}^{\text{w,Co_0}} = \{0\} \text{ if } k < -4.$$

Proof. Suppose that there exists a nonzero $\phi_k \in J_{k,\Lambda,2}^{\mathrm{w,Co_0}}$ for some k < -4 whose q^0 -term is nonzero. We can assume that k = -8 or -6, otherwise we multiply ϕ_k with a modular form in $\mathbb{C}[E_4, E_6]$. We write the q^0 -term of ϕ_k as

$$[\phi_k]_{q^0} = c_0 O_0 + c_2 O_2 + c_3 O_3 + c_4 O_4, \quad c_0, c_2, c_3, c_4 \in \mathbb{C}.$$

Applying the differential operators to ϕ_k we construct weak Jacobi forms $H\phi_k$, $H^2\phi_k$, ... of weights k+2, k+4, For each weak Jacobi form of negative weight including ϕ_k , its reduction to $\mathfrak{z}=0$ is identically zero, and in particular its q^0 -term reduces to zero by taking $\mathfrak{z}=0$, which yields a linear equation with four unknowns c_i . For example, for ϕ_k we have $c_0+c_2|O_2|+c_3|O_3|+c_4|O_4|=0$. For the weak Jacobi form of weight 0, there is a similar linear equation obtained from Lemma 2.7. In this way, we build a system of |k|/2+1 linear equations with 4 unknowns c_i . A direct calculation shows that this linear system has only trivial solution for k=-6 and -8. This contradicts our assumption that the q^0 -term of ϕ_k is nonzero.

In the following we construct four weak Jacobi forms $\varphi_{k,2}$ of weight k and index 2 in terms of holomorphic generators and present their q^0 - and q^1 -terms.

$$\varphi_{-4,2} = \frac{1}{32\Delta^2} \Big(E_4^2 (2813277A_2 - 2048\Phi_{12,2}) + 1842750(E_6B_2 + 15E_4HB_2) \Big)$$

$$= (8491392000 - 2025O_3 + 64O_4) + (543449088000 - 2073600O_2$$

$$- 380700O_3 + 33792O_4 - 2025O_5 + 64O_{6a})q + O(q^2).$$

$$\varphi_{-2,2} = 6H\varphi_{-4,2}$$

$$= \frac{1}{8\Delta^2} \left(E_4(921375E_4B_2 - 2813277E_6A_2 + 2048E_6\Phi_{12,2}) - 11056500E_6HB_2 \right)$$

$$= (67931136000 + 2025O_3 - 256O_4) + (5977939968000 - 16588800O_2$$

$$- 1514700O_3 + 55296O_4 + 2025O_5 - 256O_{6a})q + O(q^2).$$

$$\varphi_{0,2} = 3H\varphi_{-2,2}$$

$$= \frac{1}{16\Delta^2} \Big((3E_4^3 + 2E_6^2)(2813277A_2 - 2048\Phi_{12,2}) + 1842750E_4(21E_4HB_2 - 4E_6B_2) \Big)$$

$$= (237758976000 - 2025O_3 + 640O_4) + (33150394368000 - 58060800O_2$$

$$- 3199500O_3 + 49152O_4 - 2025O_5 + 640O_{6a})q + O(q^2).$$

(4.5)
$$\psi_{0,2} = \frac{683}{\Delta} (\Phi_{12,2} - 24A_2) = 675O_2 + 8(2025O_2 - O_4)q + O(q^2).$$

Theorem 4.4. The free $\mathbb{C}[E_4, E_6]$ -module $J_{*,\Lambda,2}^{\mathrm{w,Co_0}}$ is generated by $\varphi_{-4,2}$, $\varphi_{-2,2}$, $\varphi_{0,2}$ and $\psi_{0,2}$.

Proof. As we mentioned in the proof of Lemma 2.4, for any $\phi_k \in J_{k,\Lambda,2}^{\mathrm{w,Co_0}}$, the product $\Delta^2 \phi_k$ is a holomorphic Jacobi form of weight k+24, and therefore lies in the $\mathbb{C}[E_4, E_6]$ -module generated by A_2 , $\Phi_{12,2}$, B_2 and HB_2 . In this way, we find that the vector space $J_{k,\Lambda,2}^{\mathrm{w,Co_0}}$ for k=-4, -2 and 0 has dimension 1, 1 and 3 respectively. Thus there are one generator of weight -4, one generator of weight -2 and two generators of weight 0. We then finish the proof.

We remark that the argument in the proof of Lemma 4.3 also yields the estimations

$$\dim J_{-4,\Lambda,2}^{\text{w,Co}_0} \le 1$$
, $\dim J_{-2,\Lambda,2}^{\text{w,Co}_0} \le 2$, $\dim J_{0,\Lambda,2}^{\text{w,Co}_0} \le 3$.

This gives another proof of the above theorem. As far as the authors know, $J_{*,\Lambda,2}^{\text{w,Co}_0}$ is the first free module of weak Jacobi forms on an irreducible lattice which has multiple generators of weight 0.

In summary, the generating series of Conway invariant weak Jacobi forms of index 2 is given by

$$\sum_{k \in \mathbb{Z}} \dim J_{k,\Lambda,2}^{\mathrm{w,Co_0}} x^k = \frac{x^{-4} + x^{-2} + 2}{(1 - x^4)(1 - x^6)}$$

$$= x^{-4} + x^{-2} + 3 + 2x^2 + 4x^4 + 4x^6 + 5x^8 + 5x^{10} + 7x^{12} + 6x^{14} + 8x^{16} + 8x^{18} + 9x^{20} + 9x^{22} + 11x^{24} + 10x^{26} + 12x^{28} + 12x^{30} + O(x^{32}).$$

5. Conway invariant Jacobi forms of index 3

In this section we prove parts (3) and (4) of Theorem 1.2.

5.1. Conway invariant holomorphic Jacobi forms of index 3. We first estimate the minimal weight of Conway invariant weak Jacobi forms of index 3 in a similar way to the case of index 2.

Lemma 5.1.

$$J_{k,\Lambda,3}^{\text{w,Co_0}} = \{0\} \quad \text{if } k < -14.$$

Proof. Suppose that there exists a nonzero $\phi_k \in J_{k,\Lambda,3}^{\text{w,Co_0}}$ for some k < -14 whose q^0 -term is nonzero. As explained in the proof of Lemma 4.3, we only need to consider the cases of k = -18 and -16. We represent the q^0 -term of ϕ_k as

$$[\phi_k]_{q^0} = c_0 O_0 + c_2 O_2 + c_3 O_3 + c_4 O_4 + c_5 O_5 + c_{6a} O_{6a} + c_{6b} O_{6b} + c_7 O_7 + c_8 O_{8b} + c_9 O_{9b}, \quad c_i \in \mathbb{C}.$$

Again, by applying the differential operators to ϕ_k we construct Jacobi forms $H\phi_k$, $H^2\phi_k$, ... of weights k+2, k+4, By taking $\mathfrak{z}=0$ and considering the reductions of weak Jacobi forms of non-positive weights, we construct a system of |k|/2+1 linear equations with 10 unknowns c_i . By direct calculation, we find that the only non-trivial solution of this linear system is

$$c_0, c_2, c_3, c_4, c_5, c_7, c_8, c_9 = 0, \quad c_{6a}|O_{6a}| + c_{6b}|O_{6b}| = 0.$$

Thus the q^0 -term of ϕ_k has the form

$$\phi_k = |O_{6b}|O_{6a} - |O_{6a}|O_{6b} + O(q).$$

We notice that $\Delta^2 \phi_k$ defines a Conway invariant holomorphic Jacobi form of weight k + 24 and index 3. It follows that $k + 24 \ge 12$, i.e. $k \ge -12$. This contradicts our assumption on the weight k. We have thus proved the lemma.

For calculation purposes, we need to determine the decompositions of $O_2 \otimes O_4$ and $O_3 \otimes O_3$. The following identity is useful.

Lemma 5.2. The following identity holds.

$$H(A_1B_2) = A_1HB_2 - \frac{E_4}{1326780} \left(44287(7E_4^3 + 9E_6^2)A_3 + 797160E_4E_6B_3 - 1296A_1\Phi_{12,2} - 3888\Delta\Phi_{12,3} \right) + \frac{292}{243} \left(2E_4^3 + E_6^2 \right) HB_3 - \frac{584}{81} E_4 E_6 H^2 B_3 + \frac{2}{4095} \left(2049H^2(A_1A_2) - 4H^2(A_1\Phi_{12,2}) \right).$$

Proof. By means of the decomposition of $O_2 \otimes O_2$ which was determined in Lemma 3.3, we are able to calculate the Fourier expansions of the basic forms A_1A_2 , $A_1\Phi_{12,2}$, A_1B_2 and A_1HB_2 up to (and including) q^3 -terms. These data are enough to verify that both sides of (5.1) have the same Fourier coefficients up to q^3 -terms. Therefore, the quotient of their difference by Δ^4 gives a Conway invariant weak Jacobi form of weight -20 and index 3, which has to be zero by Lemma 5.1. We then prove the desired identity.

Lemma 5.3. The decompositions of $O_2 \otimes O_4$ and $O_3 \otimes O_3$ in Appendix A hold.

Proof. As mentioned in §3, we can determine the product decomposition

$$O_{3} \otimes O_{3} + 2O_{2} \otimes O_{4} = 16773120 + 1140156O_{2} + 354800O_{3} + 97152O_{4} + 22356O_{5}$$

$$+ 4094O_{6a} + 4598O_{6b} + 552O_{7} + 48O_{8b} + 2O_{9a} + 2O_{9b}$$

$$+ 2O_{10b} + 2O_{10c} + O_{12a}.$$

The q^4 -terms of A_1A_2 , A_1B_2 and A_1HB_2 contain $O_2 \otimes O_4$, but not $O_3 \otimes O_3$. To determine the decomposition of $O_2 \otimes O_4$, one first needs to make an ansatz. From the triangle inequality, we see

(5.3)
$$O_p \otimes O_q = \sum c_{n*} O_{n*}, \qquad \frac{1}{2} (\sqrt{2p} - \sqrt{2q})^2 \le n \le \frac{1}{2} (\sqrt{2p} + \sqrt{2q})^2.$$

Then $O_2 \otimes O_4$ at most contains orbits of type from 2 to 11, while $O_3 \otimes O_3$ at most contains orbits of type from 0 to 12. In fact, we can do better knowing only the orbits in $O_3 \otimes O_3 + 2O_2 \otimes O_4$ can appear in the ansatz. In summary, we have the following ansatz:

(5.4)
$$O_2 \otimes O_4 = \sum x_* O_*, \qquad O_3 \otimes O_3 = 16773120 + O_{12a} + \sum y_* O_*.$$

The above sums take over orbits $O_2, O_3, O_4, O_5, O_{6a}, O_{6b}, O_7, O_{8b}, O_{9a}, O_{9b}, O_{10b}, O_{10c}$. Substitute the above ansatz for $O_2 \otimes O_4$ into A_1A_2, A_1B_2, A_1HB_2 , then the q^4 -term of (5.1) solves

$$(5.5) x_2 = 93150, x_3 = 48600, x_4 = 16192, x_5 = 4050, x_{6a} = 759, x_{6b} = 891, x_7 = 100, x_{8b} = 8.$$

It remains to determine $x_{9a}, x_{9b}, x_{10b}, x_{10c}$. Using $|O_2 \otimes O_4| = |O_2| \times |O_4|$, we can solve

(5.6)
$$x_{10c} = \frac{26175}{2944} - \frac{200}{23}x_{9a} - \frac{5600}{243}x_{9b} - \frac{25}{128}x_{10b}.$$

As all x_* and y_* need to be non-negative integers, (5.2) and (5.4) require that $x_{9a}, x_{9b}, x_{10b}, x_{10c}$ can only be 0 or 1. It is then easy to find that the only solution is

$$(5.7) x_{9a} = 1, x_{9b} = 0, x_{10b} = 1, x_{10c} = 0.$$

This solves the decomposition of $O_2 \otimes O_4$, and by extension $O_3 \otimes O_3$.

Thanks to the decompositions of $O_2 \otimes O_2$, $O_2 \otimes O_3$ and $O_2 \otimes O_4$, we are able to calculate the Fourier expansions of A_1A_2 , $A_1\Phi_{12,2}$, A_1B_2 and A_1HB_2 up to (and including) q^4 -terms. This allows us to construct a new Conway invariant holomorphic Jacobi form of weight 12 and index 3, and prove that this new form generates $J_{12,\Lambda,3}^{\text{Co}_0}$ together with A_3 and $\Phi_{12,3}$.

Lemma 5.4. The following function lies in the space $J_{12.\Lambda.3}^{\text{Co}_0}$.

$$\Psi_{12,3} = 84909\Phi_{12,3} + \frac{1}{3888\Delta} \left(42576(486\psi_{24,3} + 33215(E_4E_6B_3 - 6E_4^2HB_3 + 12E_6H^2B_3)) + 10345968A_1(17\Phi_{12,2} - 2049A_2) + 44287A_3(18153E_4^3 + 10231E_6^2) \right)$$

$$= (48384O_{6b} - 644O_{6a})q^2 + (-644O_{9a} + 243O_{9b})q^3 + q^4(-644O_{12c} + 48384O_{12d} - 644O_{12e} + 243O_{12f}) + q^5(-644O_{15b} + 48384O_{15c} + 243O_{15d} - 644O_{15e}) + O(q^6),$$

where

(5.9)
$$\psi_{24,3} = \frac{1}{E_4} \Big(8190 A_1 H B_2 + 2049 H^2 (A_1 A_2) - 16 H^2 (A_1 \Phi_{12,2}) \Big).$$

Proof. We first calculate the Fourier coefficients of $\Psi_{12,3}$ up to q^3 -terms by definition. If $\Psi_{12,3}$ has no poles, then we conclude from its Fourier expansion that $\Psi_{12,3}$ is a holomorphic Jacobi form.

We now prove that $\Psi_{12,3}$ has no poles. This is equivalent to show that $\psi_{24,3}$ is holomorphic, which follows from the following identity

$$E_6\psi_{24,3} = -\frac{A_1}{24}(59421E_6A_2 + 151515E_4B_2 + 424E_6\Phi_{12,2}) - \frac{44287}{314928}A_3(85041E_4^3E_6 + 19583E_6^3)$$

$$-\frac{33215}{13122}B_3\left(2199E_4^4 + 4340E_4E_6^2\right) - \frac{66430}{2187}H^2B_3\left(3249E_4^3 + 3290E_6^2\right) - 53\Delta E_6\Phi_{12,3}$$

$$-55323H^3(A_1A_2) + \frac{217192885}{2187}E_4^2E_6HB_3 + \frac{34833}{2}E_4H(A_1A_2).$$

We explain why the above identity holds. Let $\psi_{30,3}$ denote the right hand side of this identity. We check that $E_6E_4\psi_{24,3} - E_4\psi_{30,3} = O(q^5)$. Thus the quotient by Δ^5 defines a Conway invariant weak Jacobi form of weight -26 and index 3, which has to be zero by Lemma 5.1. This proves our claim, and thus proves the lemma. To calculate the q^4 - and q^5 -terms of $\Psi_{12,3}$, we solve the linear equations (defined by the reductions of q^4 - and q^5 -terms to $\mathfrak{z}=0$)

$$\sum c_* |O_{12*}| = 0$$
 and $\sum c_* |O_{15*}| = 0$.

These coefficients have to be 0, -644, 48384 or 243, as the orbits of types 12 and 15 are conjugate to O_3 , O_{6a} , O_{6b} or O_{9b} modulo 3Λ .

Lemma 5.5. The space $J_{12,\Lambda,3}^{\text{Co}_0}$ is generated by A_3 , $\Phi_{12,3}$ and $\Psi_{12,3}$.

Proof. Suppose dim $J_{12,\Lambda,3}^{\text{Co}_0} > 3$. Similar to Lemma 4.1, by means of A_3 , $\Phi_{12,3}$ and $\Psi_{12,3}$, we can cancel the q^0 -, q^1 - and q^2 -terms of the fourth Jacobi form of singular weight and index 3 denoted ϕ . Then its Fourier expansion takes the form $q^3O_{9b} + O(q^4)$, which contradicts $\phi(\tau,0) = 0$.

We further construct a form $\Psi_{14,3} \in J^{\text{Co}_0}_{14,\Lambda,3}$ which is linearly independent of B_3 , and a form $\Psi_{16,3} \in J^{\text{Co}_0}_{16,\Lambda,3}$ which is linearly independent of $E_4A_3, E_4\Phi_{12,3}, E_4\Psi_{12,3}, HB_3$ and $H\Psi_{14,3}$.

$$\begin{split} \Psi_{14,3} &= \frac{1}{1109777760\Delta} \Big(14E_4(E_4(99645B_3E_4 + 88574A_3E_6) + 597870(2E_4H^2B_3 - E_6HB_3)) \\ &- 1990170A_1B_2 + 729(4098H(A_1A_2) - H(A_1\Phi_{12,2})) \Big) \\ &= 1 + \frac{1}{6850480} \Big(2(41102880 - 3102O_2 + 29O_3)q + (246617280 - 43428O_2 \\ &+ 696O_3 - 948O_4 - 129O_5 - 5O_{6b} - 11O_{6a})q^2 + (82205760 - 49632O_2 \\ &+ 2088O_3 - 1896O_4 - 993O_5 - 60O_{6b} - 132O_{6a} - 462O_7 - 6204O_{8a} \\ &- 129O_{8c} - 156O_{8b} - 11O_{9a} - 14O_{9b})q^3 \Big) + O(q^4). \end{split}$$

$$\Psi_{16,3} = 2660529E_4\Phi_{12,3} + \frac{1}{3888\Delta} \Big(-69984 \Big(1825659H^2(A_1A_2) + 123364H^2(A_1\Phi_{12,2}) \Big) \\ &+ 1594320 \Big(278111E_4^2E_6B_3 - 6(174161E_4^3 + 103950E_6^2)HB_3 + 3337332E_4E_6H^2B_3 \Big) \\ &- 3888A_1 \Big(107980251E_4A_2 - 1239877\Phi_{12,2}E_4 - 1360490040HB_2 \Big) \\ &+ 44287E_4A_3 \Big(3734541E_4^3 + 5165011E_6^2 \Big) \Big) \\ = 616 \Big(3974400O_2 - 186300O_3 + 11776O_4 - 675O_5 + 1728O_{6b} \Big)q^2 \\ &+ (9792921600O_2 - 1032847200O_3 + 101556224O_4 - 7068600O_5 - 3060288O_{6a} \\ &+ 242694144O_{6b} - 30800O_7 + 21504O_{8b} - 415800O_{8c} - 2673O_{9b} \Big)q^3 + O(q^4). \end{split}$$

Lemma 5.6. The space $J_{14,\Lambda,3}^{\text{Co}_0}$ is generated by B_3 and $\Psi_{14,3}$.

Proof. We know that $E_4A_3, E_4\Phi_{12,3}, E_4\Psi_{12,3}, \Psi_{16,3}, HB_3$ and $H\Psi_{14,3}$ are linearly independent. Suppose dim $J_{14,\Lambda,3}^{\text{Co}_0} \geq 3$. Then dim $J_{20,\Lambda,3}^{\text{Co}_0} \geq 9$. Therefore, there exists a nonzero $\phi \in J_{20,\Lambda,3}^{\text{Co}_0}$ whose Fourier expansion takes the form

$$\phi = q^2(c_0 + c_2O_2 + c_3O_3) + O(q^3).$$

This gives a weak Jacobi form

$$\psi := \Delta^{-2} \phi = c_0 + c_2 O_2 + c_3 O_3 + O(q) \in J_{-4 \text{ A.3}}^{\text{w,Co}_0}$$

We apply the differential operators to construct $H\psi$ and $H^2\psi$. The reductions to $\mathfrak{z}=0$ of these two Jacobi forms together with ψ yields a system of 3 linear equations with 3 unknowns c_0 , c_2 and c_3 . We find that this system has only trivial solution. Thus ψ/Δ gives a nonzero Conway invariant weak Jacobi form of weight -16 and index 3, which contradicts Lemma 5.1.

Theorem 5.7. The free $\mathbb{C}[E_4, E_6]$ -module $J^{\text{Co}_0}_{*,\Lambda,3}$ is generated by ten forms

$$A_3, \Phi_{12,3}, \Psi_{12,3}, B_3, \Psi_{14,3}, HB_3, H\Psi_{14,3}, \Psi_{16,3}, H^2B_3, H^2\Psi_{14,3}$$

which have weights 12, 12, 12, 14, 14, 16, 16, 16, 18, 18 respectively.

Proof. We know from Lemma 2.12 that this free module has ten generators. By Lemma 5.5 and Lemma 5.6, there are exactly three generators of weight 12 and two generators of weight 14. Since $\Psi_{16,3}$, HB_3 and $H\Psi_{14,3}$ are linearly independent with E_4A_3 , $E_4\Phi_{12,3}$ and $E_4\Psi_{12,3}$, they have to be generators of weight 16. We also verify that H^2B_3 and $H^2\Psi_{14,3}$ are linearly independent of E_6A_3 , $E_6\Phi_{12,3}$, $E_6\Psi_{12,3}$, E_4B_3 and $E_4\Psi_{14,3}$, so they are generators of weight 18. We have found all ten generators. This completes the proof of the theorem.

5.2. Conway invariant weak Jacobi forms of index 3. We now construct Conway invariant weak Jacobi forms of index 3 in terms of the ten holomorphic generators. For the lowest weight weak Jacobi form $\varphi_{-14,3}$, we normalize the constant term in its q^0 -term to be 1.

$$\varphi_{-14,3} = \frac{1}{8692012800\Delta^{3}} \left(-\frac{E_{4}E_{6}}{4881764160} (6737646323408A_{3} + 333264204\Phi_{12,3} + 62607\Psi_{12,3}) \right.$$

$$\left. -\frac{E_{4}^{2}}{36} (324193B_{3} + 25407\Psi_{14,3}) + \frac{E_{6}}{5765760} (3\Psi_{16,3} - 26961014080HB_{3} + 32854902080H\Psi_{14,3}) + E_{4}(35843H^{2}B_{3} + 166557H^{2}\Psi_{14,3}) \right),$$

$$\left. = \left(1 - \frac{O_{2}}{8190} + \frac{O_{3}}{199680} - \frac{O_{4}}{3159000} + \frac{O_{5}}{55111680} - \frac{O_{7}}{5208053760} + \frac{O_{8b}}{38362896000} - \frac{O_{9b}}{540094464000} \right) + O(q).$$

We remark that this q^0 -term is consistent with the unique solution of the linear system built in a similar way to Lemma 5.1 for k = -14. For weight -12, two obvious weak Jacobi forms are

$$(5.12) H\varphi_{-14,3} = \left(\frac{13}{6} - \frac{O_2}{5460} + \frac{7O_3}{1198080} - \frac{O_4}{3790800} + \frac{O_5}{110223360} + \frac{O_7}{31248322560} - \frac{O_{8b}}{76725792000} + \frac{O_{9b}}{648113356800}\right) + O(q),$$

and

(5.13)
$$\psi_{-12,3} = \frac{\Psi_{12,3}}{\Delta^2} = (48384O_{6b} - 644O_{6a}) + O(q).$$

We construct the third independent weak Jacobi form of weight -12 such that its q^0 -term does not contain O_0 and O_{6a} :

$$\begin{split} \phi_{-12,3} &= \frac{1}{579467520\Delta^3} \Big(\frac{E_4^3}{3973760} \big(16703964459728A_3 + 277004364\Phi_{12,3} + 178351\Psi_{12,3} \big) \\ &- \frac{E_6^2}{35763840} \big(202943145966736A_3 + 25403868\Phi_{12,3} + 2307555\Psi_{12,3} \big) \\ &- 91E_4E_6 \big(248711B_3 + 27289\Psi_{14,3} \big) + \frac{E_4^2}{3080} \big(79754795120HB_3 \\ &+ 54331156880H\Psi_{14,3} + 3\Psi_{16,3} \big) + 364E_6 \big(1682869H^2B_3 - 8469H^2\Psi_{14,3} \big) \Big) \\ &= \Big(O_2 - \frac{19O_3}{256} + \frac{17O_4}{2025} - \frac{25O_5}{23552} + \frac{13O_{6b}}{1380} - \frac{11O_7}{953856} + \frac{O_{8b}}{1366200} - \frac{O_{9b}}{32972800} \Big) + O(q). \end{split}$$

Theorem 5.8. The free $\mathbb{C}[E_4, E_6]$ -module $J_{*,\Lambda,3}^{\mathrm{w,Co_0}}$ is generated by ten forms

$$\varphi_{-14,3}, \quad \phi_{-12,3}, \quad \psi_{-12,3}, \quad H^d \varphi_{-14,3}, \quad 1 \le d \le 7$$

whose weights are respectively -14, -12, -12, -14+2d, for $1 \le d \le 7$.

Proof. For any $\phi_k \in J_{k,\Lambda,3}^{\mathrm{w},\mathrm{Co_0}}$ we have $\Delta^3 \phi_k \in J_{36+k,\Lambda,3}^{\mathrm{Co_0}}$. We note that the Fourier expansions of ten holomorphic generators can be calculated up to (and including) q^3 -terms. These data are sufficient to determine the subspace

$$\mathcal{J}_k := \{ f \in J_{36+k,\Lambda,3}^{\text{Co}_0} : f = O(q^3) \}$$

which is isomorphic to $J_{k,\Lambda,3}^{\mathrm{w},\mathrm{Co_0}}$ via the map $f\mapsto f/\Delta^3$. Therefore, for any even $-14\leq k\leq 0$ we can find a basis of $J_{k,\Lambda,3}^{\mathrm{w},\mathrm{Co_0}}$ with known q^0 -terms. We then pick up the ten weak generators from these bases and determine their weights. Notice the q^0 -term of $\varphi_{-14,3}$ does not contain orbits of type 6, then the q^0 -terms of all $H^d\varphi_{-14,3}$ do not contain orbits of type 6. On the other hand, the q^0 -terms of both $\varphi_{-12,3}$ and $\psi_{-12,3}$ do contain orbits of type 6 and are linearly independent. Then it is easy to verify that the q^0 -terms of the ten forms are linearly independent over $\mathbb C$. Therefore, the ten forms are indeed generators. This concludes the proof.

In summary, the generating series of Conway invariant weak Jacobi forms of index 3 is given by

$$\sum_{k \in \mathbb{Z}} \dim J_{k,\Lambda,3}^{\mathrm{w},\mathrm{Co}_0} x^k = \frac{x^{-14} + 3x^{-12} + x^{-10} + x^{-8} + x^{-6} + x^{-4} + x^{-2} + 1}{(1 - x^4)(1 - x^6)}$$

$$= x^{-14} + 3x^{-12} + 2x^{-10} + 5x^{-8} + 6x^{-6} + 7x^{-4} + 9x^{-2} + 12 + 11x^2$$

$$+ 15x^4 + 16x^6 + 17x^8 + 19x^{10} + 22x^{12} + 21x^{14} + 25x^{16} + 26x^{18}$$

$$+ 27x^{20} + 29x^{22} + 32x^{24} + 31x^{26} + 35x^{28} + 36x^{30} + O(x^{32}).$$

By Theorem 5.8, the two-dimensional vector space of Conway invariant weak Jacobi forms of weight -10 and index 3 are spanned by $E_4\varphi_{-14,3}$ and $H^2\varphi_{-14,3}$. Thus $H\phi_{-12,3}$ should be linearly dependent of them. Indeed, we find

(5.16)
$$H\phi_{-12,3} = 4680H^2\varphi_{-14,3} - 20280E_4\varphi_{-14,3}.$$

6. Applications

6.1. Modular linear differential equations. Modular forms on $SL_2(\mathbb{Z})$ satisfy some modular linear differential equations (MLDEs) composed of Serre derivatives. As a generalization, Jacobi forms can satisfy MLDEs composed of differential operators introduced in Lemma 2.6. The MLDEs

for holomorphic Jacobi forms are particularly interesting. For example, all holomorphic Jacobi forms ϕ of singular weight (including $A_t, \Phi_{12,t}$ and $\Psi_{12,3}$) satisfy the first-order MLDE

$$(6.1) H\phi = 0.$$

We further study MLDEs for holomorphic Jacobi forms of non-singular weight. We first conclude that $B_2 \in J_{14,\Lambda,2}^{\text{Co}_0}$ satisfies the following second-order MLDE

(6.2)
$$\left[H^2 - \frac{E_4}{18} \right] B_2 = 0.$$

We also find that the two Conway invariant holomorphic Jacobi forms of weight 14 and index 3, that is, B_3 and $\Psi_{14,3}$ satisfy the same third-order MLDE

(6.3)
$$\left[H^3 - \frac{E_4}{12} H + \frac{E_6}{72} \right] \phi = 0.$$

For any even positive-definite unimodular lattice L and any fixed index $t \geq 2$, we can define the similar holomorphic Jacobi form B_t of weight $\frac{1}{2} \operatorname{rank}(L) + 2$ and index t for L which is invariant under the orthogonal group of L. We remark that the MLDE of B_t does not depend on the lattice L. In fact, the MLDE of B_t is completely determined by the MLDE of the modular form of weight 2 involved in the definition of B_t . Indeed, Sakai's Jacobi forms B_2 and B_3 for root lattice E_8 (see [20]) satisfy the same MLDEs as (6.2) and (6.3) respectively.

The Conway invariant holomorphic Jacobi form $\Psi_{16,3}$ of weight 16 and index 3 satisfies the following fifth-order MLDE

$$\left[H^5 - \frac{5E_4}{12}H^3 + \frac{5E_6}{24}H^2 - \frac{5E_4^2}{144}H\right]\Psi_{16,3} = 0.$$

This MLDE is integrable. In fact, we find

(6.5)
$$\left[H^4 - \frac{5E_4}{12}H^2 + \frac{5E_6}{72}H \right] \Psi_{16,3} = \frac{88\Delta}{887} \left(85562484(252A_3 + \Phi_{12,3}) - 3359\Psi_{12,3} \right).$$

Due to the singular weight property and the equality $H(\Delta \phi) = \Delta H(\phi)$, the *H*-image of the above right hand side is indeed zero.

The MLDEs for the products of two Jacobi forms usually have high orders. For example, we find that $A_1\Phi_{12,2}$ satisfies the following MLDE of order ten

(6.6)
$$\left[H^{10} + \mu_1 E_4 H^8 + \mu_2 E_6 H^7 + \mu_3 E_4^2 H^6 + \mu_4 E_4 E_6 H^5 + (\mu_5 E_6^2 + \mu_6 E_4^3) H^4 + \mu_7 E_4^2 E_6 H^3 + (\mu_8 E_4 E_6^2 + \mu_9 E_4^4) H^2 + (\mu_{10} E_6^3 + \mu_{11} E_4^3 E_6) H \right] (A_1 \Phi_{12,2}) = 0$$

where

(6.7)
$$\mu_{1} = -\frac{25}{6}, \quad \mu_{2} = \frac{235}{36}, \quad \mu_{3} = -\frac{35}{16}, \quad \mu_{4} = -\frac{805}{144}, \quad \mu_{5} = \frac{7325}{1728}, \quad \mu_{6} = \frac{1075}{216}, \\ \mu_{7} = -\frac{23275}{3456}, \quad \mu_{8} = \frac{2525}{1728}, \quad \mu_{9} = \frac{1475}{1296}, \quad \mu_{10} = -\frac{25}{432}, \quad \mu_{11} = -\frac{11675}{31104}.$$

To prove this MLDE, it is sufficient to verify the Fourier expansion up to q^4 -term. We remark that the MLDEs for A_1A_2 , A_1B_2 and A_1HB_2 have orders 13, 14 and 15. In the next subsection, we will also encounter systems of MLDEs, which we also call modular linear relations.

6.2. The product decomposition of Conway orbits. In previous sections, we have determined the decompositions of $O_2 \otimes O_2$, $O_2 \otimes O_3$, $O_2 \otimes O_4$ and $O_3 \otimes O_3$. We now use the technique of Conway invariant Jacobi forms of index 3 to determine more product decompositions of Conway orbits. As we mentioned in §3, the q^6 -term of $\Phi_{12,2}$ enables us to determine

$$O_{3} \otimes O_{4} + O_{2} \otimes O_{5} = 4194304O_{2} + 1612875O_{3} + 565248O_{4} + 177400O_{5} + 48576O_{6a}$$

$$+ 47104O_{6b} + 11177O_{7} + 2048O_{8b} + 2300O_{8c} + 276O_{9a}$$

$$+ 276O_{9b} + 24O_{10a} + O_{11a} + O_{11b} + O_{12b} + O_{12c}.$$

This allows us to make ansatz

(6.9)
$$O_2 \otimes O_5 = \sum_{O \in S_1} x_* O_*, \qquad O_3 \otimes O_4 = \sum_{O \in S_1} y_* O_*,$$

where S_1 is the set of 16 orbits

$$O_2, O_3, O_4, O_5, O_{6a}, O_{6b}, O_7, O_{8b}, O_{8c}, O_{9a}, O_{9b}, O_{10a}, O_{11a}, O_{11b}, O_{12b}, O_{12c}$$

On the other hand, the q^5 -terms of A_1A_2 , A_1B_2 , A_1HB_2 and $A_1\Phi_{12,2}$ contain orbit products $O_2\otimes O_5$, $O_2\otimes O_{6a}$ and $O_2\otimes O_{6b}$. We also need to make ansatz for the last two of them. From bound (5.3), we know $O_2\otimes O_6$ at most contains orbits of type from 2 to 14. Furthermore, the q^7 -term of $\Phi_{12,2}$ enables us to determine the decomposition of $O_4\otimes O_4+2O_3\otimes O_5+2O_2\otimes (O_{6a}+O_{6b})$. From these two constraints we obtain the following most general ansatz

(6.10)
$$O_2 \otimes O_{6a} = \sum_{O \in S_2} z_*^a O_*, \qquad O_2 \otimes O_{6b} = \sum_{O \in S_2} z_*^b O_*,$$

where S_2 is the set of 26 orbits $O_2, O_3, O_4, O_5, O_{6a}, O_{6b}, O_7, O_{8a}, O_{8b}, O_{8c}, O_{9a}, O_{9b}, O_{10a}, O_{10b}, O_{10c}, O_{11a}, O_{11b}, O_{12d}, O_{12e}, O_{12f}, O_{13a}, O_{13b}, O_{13c}, O_{14c}, O_{14d}, O_{14e}$. Now we can compute A_1A_2 , A_1B_2 , A_1B_2 and $A_1\Phi_{12,2}$ up to (and including) q^5 -terms with 68 parameters x_*, z_*^a, z_*^b involved. We consider the modular linear relations among the following holomorphic Jacobi forms

$$(6.11) A_3, \Phi_{12,3}, \Psi_{12,3}, H^i B_3, H^i (A_1 A_2), H^i (A_1 B_2), H^i (A_1 H B_2), H^i (A_1 \Phi_{12,2}).$$

Due to MLDE (6.3), for H^iB_3 we only need to consider i = 0, 1, 2. Two modular linear relations at weight 28 have been given in (5.1) and (5.8). We find five more modular linear relations at weight 30, among which we display two examples here

(6.12)
$$0 = 4098E_6A_1A_2 + E_4(5915A_1B_2 - 3415H(A_1A_2) + 52H(A_1\Phi_{12,2})) + 8196H^3(A_1A_2) - 64H^3(A_1\Phi_{12,2}) - 8190H^2(A_1B_2) + 49140H(A_1HB_2),$$

$$0 = 44287E_6A_3(16695E_4^3 + 11689E_6^2) - 574776A_1(61425E_4B_2 + 2E_6(14343A_2 + 65\Phi_{12,2}))$$

$$(6.13) \qquad -1296\Delta E_6(223767\Phi_{12,3} + \Psi_{12,3}) + 21288(66430E_4E_6^2B_3 - 81E_4(10245H(A_1A_2) + 64H(A_1\Phi_{12,2})) - 398580E_4^2E_6HB_3 + 2730(2187H(A_1HB_2) + 292E_6^2H^2B_3)).$$

These identities are established by checking their Fourier expansions up to q^3 -terms. Then the q^5 -terms of all these seven modular linear relations give many constraints and together determine 60 of the 68 parameters, but left with 8 ones of the large orbits $x_{11a}, x_{11b}, z_{13a}^a, z_{13b}^a, z_{13c}^a, z_{14c}^b, z_{14d}^b, z_{14e}^b$. It turns out that all these 8 left parameters can be uniquely fixed by the condition $|O \otimes O| = |O| \times |O|$ and the requirement that all x_*, z_*^a, z_*^b are non-negative integers. For example, $|O_2 \otimes O_5| = |O_2| \times |O_5|$ implies

Clearly, the only non-negative integer solution is $x_{11a} = 1, x_{11b} = 0$. By this process, we completely determine the product decompositions of $O_2 \otimes O_5$, $O_2 \otimes O_{6a}$, $O_2 \otimes O_{6b}$ and by extension $O_3 \otimes O_4$. The results are collected in Appendix A.

Furthermore, as we mentioned in §3, the q^4 -term of $\Phi_{12,3}$ enables us to find

$$O_{2} \otimes (2O_{5} - O_{6b} + O_{8a}) = 89609O_{2} + 151800O_{3} + 93196O_{4} + 38350O_{5} + 11868O_{6a}$$

$$+ 9484O_{6b} + 2804O_{7} - 4600O_{8a} + 484O_{8b} + 276O_{8c} + 23O_{9a}$$

$$+ 66O_{9b} + 3O_{10a} - 43O_{10b} + 3O_{12b} - 3O_{12d} + O_{18a}.$$

This allows us to also fix the decomposition of $O_2 \otimes O_{8a}$, which is presented in Appendix A.

In conclusion, we have proved the decompositions for nine products of Conway orbits which are formulated in Appendix A.

To determine the q^5 -terms of holormorphic generators $\Psi_{14,3}$ and $\Psi_{16,3}$ and more orbit products decomposition, we need to compute $A_3, B_3, A_1A_2, A_1\Phi_{12,2}, A_1B_2, A_1HB_2$ to q^6 -terms. This of course cannot be achieved completely due to the lack of classification on Conway orbits of types 17 and 18. However, we can introduce two pseudo-orbits $O_{17',49597544448p}, O_{18b',93053764668p}$ besides O_{18a} , in the sense that we do not distinguish the orbits of type 17 and the orbits of type 18 except O_{18a} . This is legitimate for our current purpose, since the orbits of the same type are uniformly transformed under the differential operators H. Therefore, our following computations and results are rigorous for all orbits of type smaller than 17 and also O_{18a} . Note that the two pseudo-orbits appearing in the orbit product decomposition do not necessarily have integer coefficients.

With the two pseudo-orbits, we can compute A_3 and B_3 up to their q^6 -terms. The q^6 -terms of A_1A_2 , $A_1\Phi_{12,2}$, A_1B_2 , A_1HB_2 involve the orbit products $O_2 \otimes O_7$, $O_2 \otimes O_{8b}$, $O_2 \otimes O_{8c}$, $O_3 \otimes O_5$, $O_3 \otimes O_{6a}$, $O_3 \otimes O_{6b}$ and $O_4 \otimes O_4$. To write ansatz for the decompositions of these orbit products, we first make use of the Fourier expansion of $\Phi_{12,2}$. The q^7 -term of $\Phi_{12,2}$ gives us

$$O_{4} \otimes O_{4} + 2O_{3} \otimes O_{5}$$

$$= 398034000 + 45742400O_{2} + 20165112O_{3} + 8269144O_{4} + 3135550O_{5} + 1087440O_{6a}$$

$$+ 1095120O_{6b} + 339000O_{7} + 187358O_{8a} + 92614O_{8b} + 89606O_{8c} + 21298O_{9a}$$

$$+ 21384O_{9b} + 3940O_{10a} + 4512O_{10b} + 4400O_{10c} + 548O_{11a} + 536O_{11b} + 42O_{12d}$$

$$+ 46O_{12e} + 48O_{12f} + 2O_{13b} + 2O_{13c} + 2O_{14d} + 2O_{14e} + O_{16a}.$$

This enables us to make ansatz for $O_4 \otimes O_4$ and $O_3 \otimes O_5$ using the orbits appearing in the right hand side of the above decomposition. The q^8 -term of $\Phi_{12,2}$ gives us

$$O_{4} \otimes O_{5} + O_{2} \otimes O_{7} + O_{3} \otimes O_{6a} + O_{3} \otimes O_{6b}$$

$$= 99532800O_{2} + 48897678O_{3} + 22892544O_{4} + 10131156O_{5} + 4200352O_{6a} + 4194304O_{6b}$$

$$+ 1612875O_{7} + 565248O_{8b} + 570078O_{8c} + 177399O_{9a} + 177399O_{9b} + 48576O_{10a}$$

$$+ 47104O_{10b} + 47104O_{10c} + 11178O_{11a} + 11178O_{11b} + 2300O_{12b} + 2300O_{12c} + 2048O_{12d}$$

$$+ 2048O_{12e} + 2048O_{12f} + 276O_{13a} + 276O_{13b} + 276O_{13c} + 24O_{14a} + 24O_{14b} + O_{15a}$$

$$+ O_{15b} + O_{15c} + O_{15d} + O_{15e} + O_{16b} + O_{16c}.$$

This enables us to make ansatz for $O_2 \otimes O_7$, $O_3 \otimes O_{6a}$ and $O_3 \otimes O_{6b}$. Note that the q^5 -term of $\Phi_{12,3}$ relates $O_2 \otimes O_7$ and $O_3 \otimes O_{6b}$ by

(6.17)
$$O_{2} \otimes O_{7} - O_{3} \otimes O_{6b}$$

$$= -64722O_{3} + 32481O_{5} + 30912O_{6a} + 35840O_{6b} + 17875O_{7} + 7680O_{8b} + 7128O_{8c}$$

$$+ 2300O_{9a} + 2664O_{9b} + 660O_{10a} + 704O_{10c} + 130O_{11b} - 275O_{12b} + 23O_{12c} + 24O_{12f}$$

$$- 23O_{13a} + O_{13b} + O_{13c} + O_{15a} - O_{15c}.$$

For $O_2 \otimes O_{8b}$ and $O_2 \otimes O_{8c}$, by bound (5.3) they at most contain orbits of type from 2 to 18. With the ansatz for all these seven orbit products, we can finally compute the q^6 -terms of A_1A_2 , $A_1\Phi_{12,2}$, A_1B_2 and A_1HB_2 with parameters.

Consider the q^6 -terms of the two weight 28 modular linear relations (5.1) and (5.8) and the five weight 30 modular linear relations including (6.12) and (6.13). In these relations, when $\Phi_{12,3}$ and $\Psi_{12,3}$ appear, they are always multiplied by Δ (e.g. see (6.13)). Therefore, we can compute the Fourier expansion of each monomial in these relations up to $O(q^7)$. Then the q^6 -terms of all the modular linear relations give a large number of constraints on the ansatz. Together with the requirement on the non-negative integrity of the decomposition coefficients, we are able to determine the following combinations of orbit products:

$$\begin{aligned} O_{4} \otimes O_{4} + O_{3} \otimes O_{6a} + O_{2} \otimes O_{8b} \\ &= 398034000 + 36936000O_{2} + 17512200O_{3} + 8238232O_{4} + 3653100O_{5} + 1515010O_{6a} \\ &+ 1518264O_{6b} + 581900O_{7} + 93150O_{8a} + 204030O_{8b} + 202500O_{8c} + 63756O_{9a} \\ &+ 64152O_{9b} + 17525O_{10a} + 16192O_{10b} + 17525O_{10c} + 4050O_{11a} + 4032O_{11b} + 759O_{12c} \\ &+ 890O_{12d} + 758O_{12e} + 728O_{12f} + 100O_{13b} + 100O_{13c} + 9O_{14b} + 9O_{14e} + O_{15b} + O_{15e} \\ &+ O_{16a} + O_{16b} + O_{16d}. \end{aligned}$$

$$O_{3} \otimes O_{5} + O_{2} \otimes O_{7} \\ &= 6476800O_{2} + 3945834O_{3} + 1884160O_{4} + 802406O_{5} + 310224O_{6a} + 311040O_{6b} \\ &+ 107625O_{7} + 47104O_{8a} + 32768O_{8b} + 31681O_{8c} + 8395O_{9a} + 8505O_{9b} + 1782O_{10a} \\ &+ 2048O_{10b} + 1804O_{10c} + 274O_{11a} + 286O_{11b} + 23O_{12c} + 24O_{12e} + 36O_{12f} + 2O_{13b} \\ &+ 2O_{13c} + O_{14a} + O_{14d} + O_{15a}. \end{aligned}$$

$$O_{3} \otimes O_{6b} + O_{2} \otimes O_{8c} \\ &= 76452O_{3} + 47104O_{4} + 26875O_{5} + 12696O_{6a} + 11264O_{6b} + 5402O_{7} + 47104O_{8a} \\ &+ 2048O_{8b} + 4325O_{8c} + 1012O_{9a} + 561O_{9b} + 198O_{10a} + 2048O_{10b} + 200O_{10c} \\ &+ 274O_{11a} + 36O_{11b} + 552O_{12a} + 550O_{12b} + 23O_{12c} + 24O_{12e} + 46O_{13a} + O_{13b} \\ &+ O_{13c} + 2O_{14a} + 2O_{14d} + 3O_{15c} + c_{17}O_{17}, \end{aligned}$$

where $c_{17'} = 1/350979$. This $c_{17'}$ suggests that at type 17 there probably exists an orbit $O_{17,141312p}$ such that its coefficient in the above decomposition is 1.

Unfortunately, we cannot separate the above decomposition of orbit products. There may be some opportunity to use Conway invariant Jacobi forms of index 4 to separate these combinations. The above (6.18), (6.19) and (6.20) appear in pairs in the Fourier expansion of any product of A_1 and a Conway invariant Jacobi form of index 2, since O_4 is conjugate to O_{6a} and O_{8b} , and O_5 is conjugate to O_7 , and O_{6b} is conjugate to O_{8c} , in the sense of modulo 2Λ (see §6.4). Therefore, the above combinations are sufficient for us to fix $\Psi_{14,3}$ and $\Psi_{16,3}$ up to $O(q^6)$, and $A_1A_2, A_1\Phi_{12,2}, A_1B_2, A_1HB_2$ up to $O(q^7)$. The Fourier expansions of these holomorphic Jacobi forms are very useful in determining the conjugate relations among Conway orbits modulo 2Λ and 3Λ , which will be discussed in §6.4.

6.3. Conway invariant Jacobi forms of singular weight and non-trivial character. Let v_{η} be the multiplier system of the Dedekind eta function

$$v_{\eta}(A) = \frac{\eta(A \cdot \tau)}{\eta(\tau)\sqrt{c\tau + d}}, \quad \eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n), \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

We classify singular-weight Conway invariant holomorphic Jacobi forms of index $t \leq 3$ and character v_{η}^{d} , where d is an even integer which only depends on the conjugate class modulo 24. Through the theta decomposition [17], such forms are one-to-one corresponding to invariants of the Weil representation of $SL_2(\mathbb{Z})$ attached to $\Lambda/t\Lambda$ which are invariant under Co₀. Similar to the case of trivial character, such a form takes the Fourier expansion of the form

(6.21)
$$\Phi_{12,t;a}(\tau,\mathfrak{z}) = \sum_{n \in \frac{d}{24} + \mathbb{Z}} \sum_{\substack{\ell \in \Lambda \\ (\ell,\ell) = 2nt}} f(n,\ell) q^n \zeta^{\ell},$$

where a = 24/(24, d) indicates the order of the character v_n^d .

Clearly, the singular-weight Conway invariant holomorphic Jacobi forms of index 1 and nontrivial character do not exist. When index t=2, this type of form does not exist either. Otherwise, if such a form exists, then d can only be 12, and the Fourier expansion has to be

$$\Phi_{12,2;2}(\tau,\mathfrak{z}) = q^{\frac{3}{2}}O_3 + O(q^{\frac{5}{2}}),$$

which contradicts the reduction $\Phi_{12,2;2}(\tau,0)=0$ because $\Phi_{12,2;2}/\eta^{36}\in J_{-6,\Lambda,2}^{\mathrm{w,Co_0}}$

When t = 3, this type of Jacobi form does exist. We construct a Conway invariant holomorphic Jacobi form of weight 16 and index 3

$$F_{16,3} = \frac{E_4(3359\Psi_{12,3} - 85562484(252A_3 + \Phi_{12,3}))}{7311718400} + \frac{6251063}{920}(HB_3 - H\Psi_{14,3}) - \frac{3\Psi_{16,3}}{90675200},$$

and find that

$$\Phi_{12,3;3} := \eta^{-8} F_{16,3}$$

$$= \frac{q^{\frac{2}{3}}}{8096} \Big(8096O_2 + 77O_5q + (8096O_{8a} - 4O_{8b} + 77O_{8c})q^2 + (77O_{11a} - 4O_{11b})q^3 + (77O_{14a} - 4O_{14b} + 8096O_{14c} + 77O_{14d} - 4O_{14e})q^4 + O(q^5) \Big).$$

By the product decomposition in Appendix A, we are able to calculate the q^n -terms in the above bracket for $n \leq 3$. This implies that $\Phi_{12,3;3}$ is a singular-weight Conway invariant holomorphic Jacobi form of index 3 and character v_{η}^{16} . We then determine the q^4 -term by solving the linear equation

$$x_a|O_{14a}| + x_b|O_{14b}| + x_c|O_{14c}| + x_d|O_{14d}| + x_e|O_{14e}| = 11420136000,$$

where these coefficients x_* can only be 8096, 77 or -4, and the constant on the right hand side is determined by the $q^{14/3}$ -term of the reduction $\Phi_{12,3;3}(\tau,0)$. Besides, the q^4 -term can also be computed by using the q^5 -terms of holomorphic generators which are determined at the end of the previous subsection. We have checked that the two types of calculation are consistent. We then arrive at the following classification.

Proposition 6.1. The form $\Phi_{12,3;3}$ is the unique (up to scalar) singular-weight Conway invariant holomorphic Jacobi form of index $t \leq 3$ and non-trivial character.

Proof. It suffices to prove that there is no other singular Jacobi form of index 3 and character v_{η}^{16} which is linearly independent of $\Phi_{12,3,3}$, and there is no singular Jacobi form of index 3 and character v_n^8 . Suppose that the second singular-weight Jacobi form of index 3 and character v_n^{16} exists. Then its Fourier expansion will have the form

$$\Psi_{12,3:3} = q^{\frac{5}{3}}O_5 + O(q^{\frac{8}{3}})$$
 or $q^{\frac{8}{3}}O_{8b} + O(q^{\frac{11}{3}})$,

which contradicts the reduction $\Psi_{12,3;3}(\tau,0)=0$. Suppose that the singular-weight Jacobi form of index 3 and character v_n^8 exists. Then its Fourier expansion takes the form

$$\Psi_{12,3;3} = q^{\frac{4}{3}}O_4 + O(q^{\frac{7}{3}}) \quad \text{or} \quad q^{\frac{7}{3}}O_7 + O(q^{\frac{10}{3}}),$$

which contradicts the reduction $\Psi_{12,3;3}(\tau,0)=0$ again. We then complete the proof.

6.4. Conjugate classes of orbits modulo $t\Lambda$. According to Lemma 2.2, in the Fourier expansion of a Conway invariant weak Jacobi form of index t, the coefficients $f(n_1, \ell_1)$ and $f(n_2, \ell_2)$ are equal if $2n_1t - (\ell_1, \ell_1) = 2n_2t - (\ell_2, \ell_2)$ and $\ell_1 - \ell_2 \in t\Lambda$. From this basic fact, we can determine whether or not two orbits of $\Lambda/\operatorname{Co}_0$ are conjugate modulo $t\Lambda$ by observing the Fourier expansions of some nice Jacobi forms. We first consider the case of t = 2. The orbits O_0 , O_2 , O_3 and O_4 form a minimal length representative system of $(\Lambda/\operatorname{Co}_0)/2\Lambda$. Clearly, every orbit of odd type is conjugate to O_3 modulo 2Λ . The following result describes the conjugate classes of orbits of even type $x \leq 16$, which can be found in [12, Page 181]. We give it a simple proof based on Jacobi forms.

Proposition 6.2. For orbits of even type $x \leq 16$, the modulo 2Λ conjugate classes are as follows

```
O_0: O_{8a}, O_{12a}, O_{16a};
```

 $O_2: O_{6b}, O_{8c}, O_{10b}, O_{10c}, O_{12b}, O_{12c}, O_{14c}, O_{14d}, O_{14e}, O_{16b}, O_{16c};$

 $O_4: O_{6a}, O_{8b}, O_{10a}, O_{12d}, O_{12e}, O_{12f}, O_{14a}, O_{14b}, O_{16d}, O_{16e}, O_{16f}, O_{16q}.$

Proof. The proof follows from the Fourier expansion (3.5) of $\Phi_{12,2}$ (does not rely on the conjugate relations modulo 2Λ), where the terms q^0O_0 , qO_2 and q^2O_4 have distinct coefficients 24, 1, 0.

We now consider the case of t = 3. We prove the following theorem.

Proposition 6.3. For orbits of type $x \leq 16$, the modulo 3Λ conjugate classes are as follows

 $O_0: \emptyset;$

 $O_2: O_{8a}, O_{14c};$

 $O_3: O_{12a}, O_{12b}, O_{15a};$

 $O_4: O_{10b}, O_{13a}, O_{16a}, O_{16b}, O_{16d};$

 $O_5: O_{8c}, O_{11a}, O_{14a}, O_{14d};$

 $O_{6a}: O_{9a}, O_{12c}, O_{12e}, O_{15b}, O_{15e};$

 $O_{6b}: O_{12d}, O_{15c};$

 $O_7: O_{10a}, O_{10c}, O_{13b}, O_{13c}, O_{16c}, O_{16e}, O_{16f}, O_{16g};$

 $O_{8b}: O_{11b}, O_{14b}, O_{14e};$

 $O_{9b}: O_{12f}, O_{15d}.$

Proof. We first determine the conjugate classes of orbits of type 3x. We derive the conjugate classes of O_0 , O_3 and O_{6b} from the Fourier expansion of $\Phi_{12,3}$ given in (3.5), because the terms q^0O_0 , qO_3 and q^2O_{6b} have distinct coefficients -252, 1 and -1. We further determine the conjugate classes of O_{6a} and O_{9b} from the Fourier expansion of $\Psi_{12,3}$ given in (5.8), because the terms q^2O_{6a} and q^3O_{9b} have distinct coefficients -644 and 243.

We then consider the conjugate classes of orbits of type 3x + 2. We read the information from the Fourier expansion of $\Phi_{12,3;3}$ given in (6.22), where the terms $q^{2/3}O_2$, $q^{5/3}O_5$ and $q^{8/3}O_{8b}$ have distinct coefficients 1, 77/8096 and -4/8096.

We observe the conjugate classes of orbits of type 3x + 1 from the Fourier expansion of A_1A_2 , where the terms q^2O_4 and q^3O_7 have distinct coefficients 1/2049 and 0.

6.5. Pullbacks of Conway invariant Jacobi forms to Jacobi forms of Eichler–Zaiger. We study the pullback of Conway invariant Jacobi forms to classical Jacobi forms of Eichler–Zaiger [14] along a Leech vector. The Fourier coefficients of these pullback forms are very useful to characterize the intersection of Leech vectors, which are not easy to obtain by brutal computations.

Let v_a be a Leech vector of type a and $z \in \mathbb{C}$. For any $f_{k,t} \in J_{k,\Lambda,t}^{\mathrm{w,Co_0}}$, the pullback $f_{k,t}(\tau,zv_a)$ along v_a is a weak Jacobi form of weight k and index at in the sense of Eichler–Zagier [14]. The Conway invariance yields that $f_{k,t}(\tau,zv_a)$ depends only on the Co₀-orbit of v_a . We express $f_{k,t}(\tau,zv_a)$ in terms of the standard generators

(6.23)
$$\phi_0(\tau, z) = u^{\pm 1} + 10 + O(q), \quad \phi_{-2}(\tau, z) = u^{\pm 1} - 2 + O(q)$$

of the polynomial algebra of classical weak Jacobi forms of even weight and integral index over $\mathbb{C}[E_4, E_6]$, where

$$u^{\pm n} := e^{2\pi i n z} + e^{-2\pi i n z}, \text{ for } n \in \mathbb{N}.$$

For two Leech vectors x and y, we define their intersection number as the inner product (x, y). By means of the expression of $f_{k,t}(\tau, zv_a)$, we calculate the pullbacks of Conway orbits

(6.24)
$$\operatorname{orb}(r)(zv_a) = \sum_{v \in \operatorname{Co}_0 \cdot r} e^{2\pi i(v,v_a)z},$$

which describe the intersection numbers of Leech vectors. We next consider some examples.

We first determine the pullback of Conway invariant Jacobi form A_1 along v_2 . From the obvious fact $A_1(\tau, zv_2) = 1 + O(q^2)$, we derive

$$A_{1}(\tau, zv_{2}) = \frac{1}{1728} \left((7E_{4}^{3} + 5E_{6}^{2})\phi_{0}^{2} - 24E_{4}^{2}E_{6}\phi_{0}\phi_{-2} + 3E_{4}(E_{4}^{3} + 3E_{6}^{2})\phi_{-2}^{2} \right)$$

$$= 1 + q^{2} \left(u^{\pm 4} + 4600u^{\pm 2} + 47104u^{\pm 1} + 93150 \right)$$

$$+ q^{3} \left(47104u^{\pm 3} + 953856u^{\pm 2} + 4147200u^{\pm 1} + 6476800 \right) + O(q^{4}).$$

The above q^2 -term characterizes the intersection between O_2 and v_2 . More precisely, among all 196560 Leech vectors of type 2 there are exactly 1, 4600, 47104 vectors whose intersection numbers with v_2 are respectively 4, 2, 1, and there are exactly 93150 vectors orthogonal to v_2 . Similarly, we read the intersections between O_3 , O_4 , O_5 , O_7 and v_2 from the q^3 -, q^4 -, q^5 -, q^7 -terms of the above pullback respectively. However, we cannot determine from $A_1(\tau, zv_2)$ itself the intersection between other Conway orbits and v_2 , because there are multiple orbits of the same type.

We further consider the pullback of $\Phi_{12,2}$ along v_2 . We find that its q^0 - and q^2 -terms are sufficient to determine

$$\begin{split} \Phi_{12,2}(\tau,zv_2) &= \frac{1}{82944} \Big(\phi_0^4 \left(511 E_4^3 - 415 E_6^2 \right) - 384 E_4^2 E_6 \phi_0^3 \phi_{-2} + 2 E_4 \phi_0^2 \phi_{-2}^2 \left(779 E_6^2 - 491 E_4^3 \right) \\ &\quad + 48 E_6 \phi_0 \phi_{-2}^3 \left(7 E_4^3 - 15 E_6^2 \right) + 3 E_4^2 \phi_{-2}^4 \left(61 E_4^3 - 29 E_6^2 \right) \Big) \\ &= 24 + q \left(u^{\pm 4} + 4600 u^{\pm 2} + 47104 u^{\pm 1} + 93150 \right) + q^3 \left(4600 u^{\pm 6} + 2049300 u^{\pm 4} \right) \\ &\quad + 12953600 u^{\pm 3} + 51791400 u^{\pm 2} + 95385600 u^{\pm 1} + 127719000 \right) + O(q^4). \end{split}$$

The intersection between O_{6b} and v_2 corresponds to the above q^3 -term. We note that the intersection between O_{8a} and v_2 follows from the intersection between the primitive orbit O_2 and v_2 . Combining the two pullbacks together, we further determine the intersections between O_{6a} , O_{8a} , O_{8b} , O_{8c} , O_{10a} and v_2 . The intersections between O_{9a} , O_{9b} and v_2 can be determined by further computing the pullback of $\Psi_{14,3}$, of which we omit the expression here. We also calculate the pullbacks of A_1 and $\Phi_{12,2}$ along v_3 and v_4 , and use these Fourier expansions to determine the intersections of orbits of type less than 9 with v_3 and v_4 . The exponential polynomials to express these intersections are formulated in Appendix B.

Certainly, by calculating the pullbacks of more Jacobi forms along v_a , we can determine more intersections between Conway orbits and v_a . For any $v, u \in \Lambda / \operatorname{Co}_0$, we have the following relation

between pullbacks of Conway orbits defined by (6.24)

$$|\operatorname{orb}(u)| \cdot \operatorname{orb}(v)(zu) = |\operatorname{orb}(v)| \cdot \operatorname{orb}(u)(zv).$$

Thus the known intersections are useful to calculate the pullbacks of Conway invariant Jacobi forms of large index along Leech vectors of large type.

7. Conway invariant Jacobi forms of higher index

In this section we discuss Conway invariant Jacobi forms of index larger than 3 and propose some open questions. We first determine the minimal-norm representatives of $\Lambda/4\Lambda$. The following theorem is derived from Borcherds' thesis [1].

Theorem 7.1. Representatives of $\Lambda/4\Lambda$ of minimal norm may be found among vectors of types up to 16, according to the weighted equality

(7.1)
$$4^{24} = 1 + |O_{2}| + |O_{3}| + |O_{4}| + |O_{5}| + |O_{6a}| + |O_{6b}| + |O_{7}| + \frac{1}{2}|O_{8a}| + |O_{8b}| + |O_{8c}| + |O_{9a}| + |O_{9b}| + |O_{10a}| + \frac{1}{2}|O_{10b}| + |O_{10c}| + \frac{1}{2}|O_{11a}| + |O_{11b}| + \frac{1}{2}|O_{12a}| + |O_{12c}| + \frac{1}{4}|O_{12d}| + \frac{1}{2}|O_{12e}| + |O_{12f}| + \frac{1}{2}|O_{13b}| + \frac{1}{3}|O_{13c}| + \frac{1}{6}|O_{14b}| + \frac{1}{4}|O_{14d}| + \frac{1}{8}|O_{14e}| + \frac{1}{14}(|O_{15b}| + |O_{15d}|) + \frac{1}{48}|O_{16a}| + \frac{1}{32}|O_{16e}|.$$

In the above equality, the weights are the 4-weights of vectors in the orbits (see Definition 2.9). Moreover, we have the following facts.

- (i) The orbits O_{15b} and O_{15d} are conjugate modulo 4Λ , but they are not conjugate to other orbits of type 15.
- (ii) Every orbit O_x in (7.1) not of type 15 is not conjugate to any orbit O_y of type $y \le x$ modulo 4Λ . Thus the 4-weight of O_x coincides with the 4-weight of any vector in O_x .
- (iii) The 4-weights of O_{15b} and O_{15d} are respectively 2 and 12, which are different from the 4-weight of any vector in O_{15b} or O_{15d} , i.e. 14.

Proof. We know from Borcherds' thesis that the Co₀-orbits of $\Lambda/4\Lambda$ correspond one-to-one to the non-empty Dynkin diagrams related to orbits of vectors in $II_{25,1}$ of height 4, and the size of the Dynkin diagram describes the 4-weight of any vector in the orbit, that is the number of Leech vectors of type x which are conjugate (modulo 4Λ) to a given vector of type x (see Definition 2.9 for notions). We refer to [1, Figure 2, Page 91] for details. More precisely, the minimal-norm representative system of $\Lambda/4\Lambda$ consists of the following orbits.

- (1) All orbits of type $x \le 11$. For these orbits, their 4-weights coincide with the 4-weights of vectors. There are unique orbits of types 8, 10 and 11 which have 4-weight 2. The other orbits have 4-weight 1.
- (2) Five orbits of type 12, whose vectors have 4-weights 1, 1, 2, 2 and 4.
- (3) Two orbits of type 13, whose vectors have 4-weights 2 and 3.
- (4) Three orbits of type 14, whose vectors have 4-weights 4, 6 and 8.
- (5) One orbit of type 15, whose any vector has 4-weight 14.
- (6) Two orbits of type 16, whose vectors have 4-weights 32 and 48.

Firstly, it is easy to determine the orbits of type $x \le 11$ and weight 2 by observing their shapes in ATLAS [12]. For example, O_{11a} has shape $u_3 + 2v_2$, and then $u_3 + 2v_2 \sim_4 u_3 - 2v_2$, which implies that O_{11a} has 4-weight 2 and thus O_{11b} has 4-weight 1.

For orbits of type 12, we see from their shapes that $O_{12b} \sim_4 O_{8c}$. Thus any two of the remaining 5 orbits of type 12 are not conjugate modulo 4Λ , and they are the orbits of minimal length in (2).

Clearly, the 4-weight of the non-primitive orbit O_{12a} equals the 2-weight of O_3 which is 2. Then the other 4 orbits must have 4-weights 1, 1, 2 and 4. We need to consider 12 possibilities in (2).

Again, we see from their shapes that $O_{13a} \sim_4 O_{9a}$. We cannot determine the weights of the other two orbits of type 13. Thus we need to consider 2 possibilities in (3).

There are 5 orbits of type 14. By the shapes in the ATLAS, $O_{14a} \sim_4 O_{10a}$, $O_{14c} \sim_4 O_{6b}$. Thus the other 3 orbits are the orbits in (4) with 4-weights 4, 6 and 8. There are 6 possibilities in (4).

We have $O_{15a} \sim_4 O_7$ by their shapes. One or multiple of O_{15b} , O_{15c} , O_{15d} and O_{15e} will be the orbits of minimal length modulo 4Λ whose vectors have 4-weight 14. Thus we need to consider all $2^4 = 16$ possibilities in (5).

We first conclude from the shapes that $O_{16b} \sim_4 O_{12c}$. Since O_4 has minimal norm modulo 2Λ , the non-primitive orbit $O_{16a} = \text{orb}(2v_4)$ has minimal norm modulo 4Λ . Moreover, the 4-weight of O_{16a} equals the 2-weight of O_4 which is 48. Clearly, if $v \sim_4 u$ then $v \sim_2 u$. By Proposition 6.2, $O_{16c} \not\sim_4 O_{16a}$, O_{16d} , O_{16e} , O_{16f} , O_{16g} . We cannot distinguish the orbits of type 16 and 4-weight 32, which may be either O_{16c} , or one or multiple of O_{16d} , O_{16e} , O_{16f} and O_{16g} . Thus we need to consider $1 + 2^4 = 17$ possibilities in (6).

In total, we need to check $12 \times 2 \times 6 \times 16 \times 17 = 39168$ possibilities. By scanning the weighted equality of form (7.1), we find that there is a unique solution which is exactly (7.1). The properties (1)–(4) follow from this equality and the above discussion.

The above theorem yields that O_{16c} is conjugate to some orbit of smaller type modulo 4Λ . By Proposition 6.2, O_{16c} is conjugate to O_{8c} or O_{12c} modulo 4Λ . We see from the Fourier expansion of $\Phi_{12,4}$ given in (3.5) that $O_{16c} \nsim_4 O_{8c}$. Thus $O_{16c} \sim_4 O_{12c} \sim_4 O_{16b}$.

The above theorem yields the following facts on Conway invariant Jacobi forms of index 4.

Proposition 7.2. The rank of $J_{*,\Lambda,4}^{w,Co_0}$ is 31. The basic Conway orbits of index 4 are the 32 orbits in (7.1). The number δ_4 defined in Lemma 2.5 is 85.

Besides, we conclude from $\Delta^4 J^{\mathrm{w,Co_0}}_{*,\Lambda,4} \subset J^{\mathrm{Co_0}}_{*,\Lambda,4}$ that $J^{\mathrm{w,Co_0}}_{k,\Lambda,4} = \{0\}$ for k < -36. We check that A_4 , $\Phi_{12,4}, \ A_1(\tau,2\mathfrak{z}), \ \Phi_{12,2}|T_-(2)$ are linearly independent. Thus $\dim J^{\mathrm{Co_0}}_{12,\Lambda,4} \geq 4$. A similar argument to Lemma 5.5 gives upper bound $\dim J^{\mathrm{Co_0}}_{12,\Lambda,4} \leq 9$. We do not know the exact dimension. We note that $\dim J^{\mathrm{Co_0}}_{12,\Lambda,6} \geq 5$, because the following five Jacobi forms are linearly independent.

$$A_6$$
, $\Phi_{12.6}$, $\Phi_{12.2}|T_-(3)$, $\Phi_{12.3}|T_-(2)$, $\Psi_{12.3}|T_-(2)$.

By Theorem 1.2, there exist Conway invariant weak Jacobi forms of weight 4 and index 2 whose q^0 -terms are a single orbit O_0 , O_2 , O_3 or O_4 . Similarly, there exist Conway invariant weak Jacobi forms of weight 4 and index 3 whose q^0 -terms are a single orbit O_0 , O_2 , O_3 , O_4 , O_5 , O_{6a} , O_{6b} , O_7 , O_{8b} or O_{9b} . Therefore, each of the following orbit products

$$\{O_0, O_2, O_3, O_4\} \otimes \{O_0, O_2, O_3, O_4, O_5, O_{6a}, O_{6b}, O_7, O_{8b}, O_{9b}\}$$

can be regarded as the q^0 -term of a Conway invariant weak Jacobi form of weight 8 and index 5. Thus every orbit appearing in the decompositions of these products has minimal norm modulo 5Λ , and if two orbits of the same type have distinct coefficients in a given product decomposition, then they are not conjugate modulo 5Λ . In this way, from the decompositions of orbit products determined in this paper we derive that all orbits of type $x \leq 15$, O_{16a} , O_{16b} , O_{16c} , O_{16d} and O_{16e} have minimal norm modulo 5Λ , and any two of them are not conjugate modulo 5Λ . To prove this claim, we also use the simple fact that if O_x has minimal norm modulo $t\Lambda$ then it has minimal norm modulo $s\Lambda$ for all $s \geq t$. We do not know if O_{16f} and O_{16g} have minimal norm modulo 5Λ .

In fact, we can also determine some orbits of minimal norm modulo 4Λ from the decomposition of orbit products $O_2 \otimes O_2$, $O_2 \otimes O_3$, $O_2 \otimes O_4$, $O_3 \otimes O_3$ and $O_3 \otimes O_4$. These minimal-length orbits are consistent with Theorem 7.1.

Question 7.3. We formulate some questions related to Conway invariant Jacobi forms.

- (1) What is the minimal weight of Conway invariant weak Jacobi forms of given index? We guess that $J_{k,\Lambda,t}^{\text{w,Co}_0} = \{0\}$ if k < -6t.
- (2) To find a set of 24 algebraically independent Conway orbits.
- (3) To give a formula to describe the rank of $J_{*,\Lambda,t}^{\mathrm{w,Co_0}}$, i.e. the number of Co₀-orbits of $\Lambda/t\Lambda$.
- (4) To give a dimensional formula for the space of Conway invariant holomorphic Jacobi forms of singular weight and given index.
- (5) Is the ring $J_{*,\Lambda,*}^{\text{w,Co}_0}$ finitely generated? If so, what is the maximal index of the generators?

Acknowledgements The authors thank Daniel Allcock and Kimyeong Lee for useful discussions, and thank Brandon Williams for pointing out [19] and for fruitful discussions on Borcherds' thesis. The authors also thank Richard Borcherds for valuable comments. KS is supported by KIAS Grant (QP081001). HW thanks Max Planck Institute for Mathematics in Bonn for its hospitality where this work was started. HW is supported by the Institute for Basic Science (IBS-R003-D1). The authors also thank the referee for many valuable comments that make this paper better.

APPENDIX A. PRODUCT DECOMPOSITIONS OF CONWAY ORBITS

We determine the following orbit product decompositions:

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\begin{array}{c} O_2 \otimes O_2 = 196560O_0 \oplus 4600O_2 \oplus 552O_3 \oplus 46O_4 \oplus 2O_5 \oplus 2O_{6b} \oplus O_{8a} \\ O_2 \otimes O_3 = 47104O_2 \oplus 11178O_3 \oplus 2048O_4 \oplus 275O_5 \oplus 24O_{6a} \oplus O_7 \oplus O_{8c} \\ O_2 \otimes O_4 = 93150O_2 \oplus 48600O_3 \oplus 16192O_4 \oplus 4050O_5 \oplus 759O_{6a} \oplus 891O_{6b} \oplus 100O_7 \\ & \oplus 8O_{8b} \oplus O_{9a} \oplus O_{10b} \\ O_3 \otimes O_3 = 16773120O_0 \oplus 953856O_2 \oplus 257600O_3 \oplus 64768O_4 \oplus 14256O_5 \oplus 2576O_{6a} \\ & \oplus 2816O_{6b} \oplus 352O_7 \oplus 32O_{8b} \oplus 2O_{9b} \oplus 2O_{10c} \oplus O_{12a} \\ O_2 \otimes O_5 = 47104O_2 \oplus 75900O_3 \oplus 47104O_4 \oplus 19450O_5 \oplus 6072O_{6a} \oplus 5632O_{6b} \oplus 1452O_7 \\ & \oplus 256O_{8b} \oplus 275O_{8c} \oplus 23O_{9a} \oplus 33O_{9b} \oplus 2O_{10a} \oplus O_{11a} \oplus O_{12b} \\ O_3 \otimes O_4 = 4147200O_2 \oplus 1536975O_3 \oplus 518144O_4 \oplus 157950O_5 \oplus 42504O_{6a} \oplus 41472O_{6b} \\ & \oplus 9725O_7 \oplus 1792O_{8b} \oplus 2025O_{8c} \oplus 253O_{9a} \oplus 243O_{9b} \oplus 22O_{10a} \oplus O_{11b} \oplus O_{12c} \\ O_2 \otimes O_{6a} = 48600O_3 \oplus 64768O_4 \oplus 44550O_5 \oplus 21252O_{6a} \oplus 20736O_{6b} \oplus 7800O_7 \oplus 2240O_{8b} \\ & \oplus 2025O_{8c} \oplus 506O_{9a} \oplus 486O_{9b} \oplus 77O_{10a} \oplus 100O_{10c} \oplus 8O_{11b} \oplus O_{12e} \oplus O_{13a} \\ O_2 \otimes O_{6b} = 4600O_2 \oplus 1012O_4 \oplus 550O_5 \oplus 276O_{6a} \oplus 1782O_{6b} \oplus 100O_7 \oplus 4600O_{8a} \oplus 28O_{8b} \\ & \oplus 275O_{8c} \oplus 23O_{9a} \oplus O_{10a} \oplus 44O_{10b} \oplus 2O_{11a} \oplus 3O_{12d} \oplus O_{14c} \\ O_2 \otimes O_{8a} = O_2 \oplus 2O_{6b} \oplus O_{10b} \oplus O_{12b} \oplus O_{8c} \oplus O_{14c} \oplus O_{18a} \\ \end{array}
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APPENDIX B. INTERSECTION BETWEEN CONWAY ORBITS AND LEECH VECTORS For the intersection between O_* and any Leech vector of type 2, we have

```
O_2: u^{\pm 4} + 4600u^{\pm 2} + 47104u^{\pm 1} + 93150
O_3: 512(92u^{\pm 3} + 1863u^{\pm 2} + 8100u^{\pm 1} + 12650)
O_4: 4050(23u^{\pm 4} + 1024u^{\pm 3} + 8096u^{\pm 2} + 23552u^{\pm 1} + 32890)
O_5: 23552(2u^{\pm 5} + 275u^{\pm 4} + 4050u^{\pm 3} + 19450u^{\pm 2} + 45100u^{\pm 1} + 58806)
O_{6a}: 518400(8u^{\pm 5} + 253u^{\pm 4} + 2024u^{\pm 3} + 7176u^{\pm 2} + 14352u^{\pm 1} + 17894)
O_{6b}: 2300(2u^{\pm 6} + 891u^{\pm 4} + 5632u^{\pm 3} + 22518u^{\pm 2} + 41472u^{\pm 1} + 55530)
O_7: 953856(u^{\pm 6} + 100u^{\pm 5} + 1452u^{\pm 4} + 7900u^{\pm 3} + 22825u^{\pm 2} + 41152u^{\pm 1} + 49700)
O_{8a}: u^{\pm 8} + 4600u^{\pm 4} + 47104u^{\pm 2} + 93150
O_{8b}: 16394400(2u^{\pm 6} + 64u^{\pm 5} + 567u^{\pm 4} + 2368u^{\pm 3} + 5902u^{\pm 2} + 9856u^{\pm 1} + 11622)
O_{8c}: 47104(u^{\pm 7} + 275u^{\pm 5} + 2300u^{\pm 4} + 9153u^{\pm 3} + 24576u^{\pm 2} + 37675u^{\pm 1} + 48600)
O_{9a}: 4147200(u^{\pm 7} + 23u^{\pm 6} + 529u^{\pm 5} + 3059u^{\pm 4} + 10879u^{\pm 3} + 24035u^{\pm 2} + 37743u^{\pm 1} + 44022)
O_{9b}: 32972800(11u^{\pm 6} + 162u^{\pm 5} + 1053u^{\pm 4} + 3586u^{\pm 3} + 8019u^{\pm 2} + 12636u^{\pm 1} + 14586)
O_{10a}: 47692800(2u^{\pm 7} + 78u^{\pm 6} + 814u^{\pm 5} + 3993u^{\pm 4} + 11882u^{\pm 3} + 24266u^{\pm 2} + 36454u^{\pm 1} + 41582)
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For the intersection between O_* and any Leech vector of type 3, we have

$$\begin{aligned} O_2 &: 6(92u^{\pm 3} + 1863u^{\pm 2} + 8100u^{\pm 1} + 12650) \\ O_3 &: u^{\pm 6} + 11178u^{\pm 4} + 257600u^{\pm 3} + 1536975u^{\pm 2} + 3934656u^{\pm 1} + 5292300 \\ O_4 &: 6075(8u^{\pm 5} + 253u^{\pm 4} + 2024u^{\pm 3} + 7176u^{\pm 2} + 14352u^{\pm 1} + 17894) \\ O_5 &: 276(275u^{\pm 6} + 14256u^{\pm 5} + 157950u^{\pm 4} + 743600u^{\pm 3} + 1986525u^{\pm 2} + 3434400u^{\pm 1} + 4099108) \\ O_{6a} &: 16200(3u^{\pm 7} + 322u^{\pm 6} + 5313u^{\pm 5} + 33396u^{\pm 4} + 115115u^{\pm 3} + 256542u^{\pm 2} + 403857u^{\pm 1} \\ &\quad + 467544) \\ O_{6b} &: 6900(11u^{\pm 6} + 162u^{\pm 5} + 1053u^{\pm 4} + 3586u^{\pm 3} + 8019u^{\pm 2} + 12636u^{\pm 1} + 14586) \\ O_7 &: 11178(u^{\pm 8} + 352u^{\pm 7} + 9725u^{\pm 6} + 84800u^{\pm 5} + 376750u^{\pm 4} + 1053504u^{\pm 3} + 2075603u^{\pm 2} \\ &\quad + 3053600u^{\pm 1} + 3464450) \\ O_{8a} &: 6(92u^{\pm 6} + 1863u^{\pm 4} + 8100u^{\pm 1} + 12650) \\ O_{8b} &: 1536975(u^{\pm 8} + 56u^{\pm 7} + 728u^{\pm 6} + 4264u^{\pm 5} + 14924u^{\pm 4} + 36024u^{\pm 3} + 64744u^{\pm 2} + 90728u^{\pm 1} \\ &\quad + 101222) \\ O_{8c} &: 552(u^{\pm 9} + 2025u^{\pm 7} + 22528u^{\pm 6} + 137700u^{\pm 5} + 476928u^{\pm 4} + 1151700u^{\pm 3} + 2073600u^{\pm 2} \\ &\quad + 2902878u^{\pm 1} + 3238400) \end{aligned}$$

For the intersection between O_* and any Leech vector of type 4, we have

$$O_2: 2(23u^{\pm 4} + 1024u^{\pm 3} + 8096u^{\pm 2} + 23552u^{\pm 1} + 32890)$$

$$O_3: 256(8u^{\pm 5} + 253u^{\pm 4} + 2024u^{\pm 3} + 7176u^{\pm 2} + 14352u^{\pm 1} + 17894)$$

$$O_4: u^{\pm 8} + 16192u^{\pm 6} + 518144u^{\pm 5} + 4595032u^{\pm 4} + 19171328u^{\pm 3} + 47829696u^{\pm 2} + 79794176u^{\pm 1} + 94184862$$

$$O_5: 23552(2u^{\pm 7} + 78u^{\pm 6} + 814u^{\pm 5} + 3993u^{\pm 4} + 11882u^{\pm 3} + 24266u^{\pm 2} + 36454u^{\pm 1} + 41582)$$

$$O_{6a}: 256(253u^{\pm 8} + 14168u^{\pm 7} + 184368u^{\pm 6} + 1078792u^{\pm 5} + 3779498u^{\pm 4} + 9114072u^{\pm 3} + 16396432u^{\pm 2} + 22954184u^{\pm 1} + 25634466)$$

$$O_{6b}: 92(11u^{\pm 8} + 512u^{\pm 7} + 6864u^{\pm 6} + 39936u^{\pm 5} + 139854u^{\pm 4} + 337920u^{\pm 3} + 606832u^{\pm 2} + 850432u^{\pm 1} + 949278)$$

$$O_7: 11776(4u^{\pm 9} + 389u^{\pm 8} + 6776u^{\pm 7} + 48532u^{\pm 6} + 200772u^{\pm 5} + 564135u^{\pm 4} + 1181756u^{\pm 3} + 1943044u^{\pm 2} + 2592004u^{\pm 1} + 2846536)$$

$$O_{8a}: 2(23u^{\pm 8} + 1024u^{\pm 6} + 8096u^{\pm 4} + 23552u^{\pm 2} + 32890)$$

$$O_{8b}: 4048(4u^{\pm 10} + 896u^{\pm 9} + 23011u^{\pm 8} + 209664u^{\pm 7} + 1038804u^{\pm 6} + 3398784u^{\pm 5} + 8194512u^{\pm 4} + 15480192u^{\pm 3} + 23860008u^{\pm 2} + 30652288u^{\pm 1} + 33300674)$$

$$O_{8c}: 47104(u^{\pm 9} + 22u^{\pm 8} + 209u^{\pm 7} + 1024u^{\pm 6} + 3356u^{\pm 5} + 8096u^{\pm 4} + 15292u^{\pm 3} + 23552u^{\pm 2} + 30294u^{\pm 1} + 32868)$$

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