

Universal transcendentality limit of BFKL eigenvalue

Mohammad Joubat^(a) and Alex Prygarin^(b)

^(a) Department of Mathematics, Ariel University, Ariel 40700, Israel

^(b) Department of Physics, Ariel University, Ariel 40700, Israel

Abstract

We consider a special limit of the BFKL eigenvalue at $\nu \rightarrow 0$ and odd values of the conformal spin n . We show that in this limit the NLO BFKL eigenvalue can be expressed in terms of a limited set of transcendental constants with rational coefficients. We show that the leading transcendentality term in this limit is universal and does not depend on a specific value of n .

1 Introduction

The Balitsky-Fadin-Kuraev-Lipatov (BFKL) [1–4] equation describes the bound state of two reggeized gluons. The full functional form of the kernel of the BFKL equation in QCD as well as the eigenfunctions and eigenvalues that solve it are currently known at the leading order (LO) and the next-to-leading (NLO) [5] order of the perturbation theory with expansion parameter being the gauge coupling constant enhanced by the corresponding power of the logarithm of the center-of-mass energy. The difference between the leading order and the next-top-leading order is that the latter is suppressed by one power of logarithm of energy. The BFKL equation is formulated for a general color state configuration. The most studied color configurations are the color adjoint and the color singlet states of the two reggeized gluons. The color adjoint state was extensively investigated over past decade in the context of the helicity amplitudes in $N = 4$ super Yang-Mills (SYM) theory [6–22]. The color adjoint BFKL eigenvalue was calculated at higher orders of the perturbation theory in the context of the helicity amplitudes [11, 12, 23, 24]. The situation with the color singlet BFKL eigenvalue is rather different. Its full functional form is known only in the next-to-leading order in both QCD and $N = 4$ SYM, while at higher orders of the perturbation theory only some special limits of it was calculated using Quantum Spectral Curve [25–27] and other techniques [28–34]. The BFKL eigenvalue is a function of two variables, the real continuous ν , related to the anomalous dimension of twist two operators, and the conformal spin n , which takes real integer values.

In this paper we consider the next-to-leading order color singlet BFKL eigenvalue in $N = 4$ SYM in the limit of $\nu \rightarrow 0$ for odd values of n and show that it built of a limited set of transcendental numbers with rational coefficients. We show that the leading transcendentality term is universal in this limit and does not depend on a specific value of n .

2 BFKL eigenvalue

The BFKL eigenvalue theory was first calculated by Fadin and Lipatov using Feynman diagrams in QCD and then was derived in $N = 4$ SYM in accordance with "the maximal transcendentality principle" by Kotikov and Lipatov [35–37]. In $N = 4$ SYM the color singlet BFKL eigenvalue is given by

$$\omega = 4\bar{a} [\chi(n, \gamma) + \bar{a} \delta(n, \gamma)] \quad (1)$$

where $\gamma = \frac{1}{2} - i\nu$ and the leading order (LO) expression is given by

$$\chi(n, \gamma) = 2\psi(1) - \psi(M) - \psi(1 - \bar{M}) \quad (2)$$

and the next-to-leading (NLO) BFKL eigenvalue reads

$$\delta(n, \gamma) = \phi(M) + \phi(1 - \bar{M}) - \frac{\omega_0}{2\bar{a}} (\rho(M) + \rho(1 - \bar{M})). \quad (3)$$

The functional dependence on the twist-2 operators and the conformal spin enters through the complex variables

$$M = \gamma + \frac{|n|}{2}, \quad \bar{M} = \gamma - \frac{|n|}{2}. \quad (4)$$

Here $\psi(z)$ is the Euler ψ -function is given by the logarithmic derivative of the Gamma function $\psi(z) = \frac{d \ln \Gamma(z)}{dz}$, and the coupling $\bar{a} = \frac{g^2 N_c}{16\pi^2}$ is defined in terms of the coupling constant g in the DREG scheme. The functions $\rho(M)$ and $\phi(M)$ are given by

$$\rho(M) = \beta'(M) + \frac{1}{2}\zeta(2) \quad (5)$$

and

$$\phi(M) = 3\zeta(3) + \psi''(M) - 2\Phi_2(M) + 2\beta'(M)(\psi(1) - \psi(M)), \quad (6)$$

where $\beta'(M)$ is the derivative of the Dirichlet beta function

$$\beta'(M) = \frac{1}{4} \left[\psi' \left(\frac{M+1}{2} \right) - \psi' \left(\frac{M}{2} \right) \right] = - \sum_{r=0}^{\infty} \frac{(-1)^r}{(M+r)^2} \quad (7)$$

and $\Phi_2(M)$ reads

$$\Phi_2(M) = \sum_{k=0}^{\infty} \frac{\beta'(k+1)}{k+M} + \sum_{k=0}^{\infty} \frac{(-1)^k \psi'(k+1)}{k+M} - \sum_{k=0}^{\infty} \frac{(-1)^k (\psi(k+1) - \psi(1))}{(k+M)^2}. \quad (8)$$

The expressions of $\chi(n, \gamma)$ and $\delta(n, \gamma)$ are real valued functions of real integer n and complex $\gamma = \frac{1}{2} - i\nu$ for real ν .

3 Transcendental constants, transcendental-ity and the loop order

In this section we give a brief review of the transcendental concept we use in our analysis. We start with the definition of the transcendental number and then extend the notion of transcendental to the functions that appear in the BFKL eigenvalue.

The transcendental number is an irrational number that is not a root of any polynomial with rational coefficients. For example, π is transcendental number while $\sqrt{2}$ is irrational, but not transcendental because it is a solution to the polynomial equation $x^2 - 2 = 0$. Another useful number $\ln 2$ is also a transcendental number. In fact, most of numbers are transcendental, but we are interested in those that can appear in the BFKL eigenvalue. Consider the Riemann zeta function $\zeta(n)$, for even values of n it can be written in terms of the powers of π and thus it gives transcendental numbers for even values of n . For odd values of n it gives irrational numbers that are strongly suspected to be transcendental, though a rigorous mathematical proof of this is still to be found. Polylogarithmic functions of half integer argument $\text{Li}_s\left(\frac{1}{2}\right)$ also give irrational numbers that appear in some limits of the BFKL eigenvalue. They are usually treated as transcendental numbers despite the lack of rigorous mathematical proof that they are such. In fact, $\text{Li}_s\left(\frac{1}{2}\right)$ is reducible for $s < 4$ and can be written in terms of transcendental numbers $\ln 2$, π and $\zeta(3)$. For the purpose of the present discussion we treat all of the above-mentioned numbers as transcendental numbers.

Out of infinitely many transcendental numbers there are only two constants π^2 and $\zeta(3)$ that appear in the BFKL eigenvalue together with polygamma functions and their generalizations. The origin of those two constants is better understood if one writes the NLO BFKL eigenvalue in terms of the nested harmonic sums (see Appendix). In the limit of $\nu \rightarrow \infty$ the BFKL eigenvalue must reproduce the expansion of the cusp anomalous dimension, which is expressed in terms of $\zeta(n)$. On the other hand the nested harmonic sums also give the same set of transcendental constants at the infinite value of the argument. This restricts the possible choice of the transcendental numbers that are relevant for the BFKL eigenvalue.

At each loop order of the BFKL eigenvalue the set of possible transcendental numbers is limited by the principle of maximal transcendentality. At the leading order the maximal transcendentality is one, at the next-to-leading or-

der the maximal transcendentality of constants is three, at the next-to-next-to-leading order the maximal transcendentality is five etc. The transcendentality changes by units of two from order to order in the perturbation theory. For example, the set of possible transcendental numbers that can appear at the next-to-next-to-leading BFKL eigenvalue reads

$$\{\pi^2, \zeta(3), \text{Li}_4\left(\frac{1}{2}\right), \zeta(5)\}. \quad (9)$$

All of them appear in the limit of the infinite value of the argument of the nested harmonic sums. The constant $\ln 2$ is also allowed because it appears in the limit of $\lim_{n \rightarrow \infty} S_{-1}(n) = -\ln 2$, but it is absent in the BFKL eigenvalue. This might be related to the fact that in the NLO case as well as in the known NNLO expressions the nested harmonic sums never have index -1 . Most probably, this fact in its turn is related to the cylindrical topology (closed spin chain) of the singlet BFKL equation.

In the next section we consider a new limit of $\nu \rightarrow 0$ for odd values of the conformal spin n in which the NLO BFKL eigenvalue gives a constant that can be written in terms of the transcendental numbers in eq. (9).

4 The uniform transcendentality limit

We consider the limit of the NLO eigenvalue $\delta(n, \gamma)$ in which $\nu \rightarrow 0$ and the conformal spin n takes special odd values given by $n = 4k + 1$ ($k = 0, 1, 2, 3, \dots$). In this limit the variable $\gamma = \frac{1}{2} - i\nu$ equals one half and the

first fifteen terms of $\delta(n, \gamma)$ read

$$\begin{aligned}
\delta\left(1, \frac{1}{2}\right) &= 0 \\
\delta\left(5, \frac{1}{2}\right) &= 6 + \pi^2 \\
\delta\left(9, \frac{1}{2}\right) &= \frac{725}{108} + \frac{25\pi^2}{18} \\
\delta\left(13, \frac{1}{2}\right) &= \frac{31213}{4500} + \frac{49\pi^2}{30} \\
\delta\left(17, \frac{1}{2}\right) &= \frac{86873597}{12348000} + \frac{761\pi^2}{420} \\
\delta\left(21, \frac{1}{2}\right) &= \frac{7089858149}{1000188000} + \frac{7381\pi^2}{3780} \\
\delta\left(25, \frac{1}{2}\right) &= \frac{9479055229969}{1331250228000} + \frac{86021\pi^2}{41580} \\
\delta\left(29, \frac{1}{2}\right) &= \frac{20886136478164093}{2924756750916000} + \frac{1171733\pi^2}{540540} \\
\delta\left(33, \frac{1}{2}\right) &= \frac{167423375594334947}{23398054007328000} + \frac{2436559\pi^2}{1081080} \\
\delta\left(37, \frac{1}{2}\right) &= \frac{274577795894479279537}{38318213112667488000} + \frac{14274301\pi^2}{6126120} \\
\delta\left(41, \frac{1}{2}\right) &= \frac{75413751118241146234843}{10512984949591452007680} + \frac{55835135\pi^2}{23279256} \\
\delta\left(45, \frac{1}{2}\right) &= \frac{75475515884701985008123}{10512984949591452007680} + \frac{6364399\pi^2}{2586584} \\
\delta\left(49, \frac{1}{2}\right) &= \frac{918900025283615884381544531}{127911487881679196577442560} + \frac{1347822955\pi^2}{535422888} \\
\delta\left(53, \frac{1}{2}\right) &= \frac{2873035558259746167005637883471}{399723399630247489304508000000} + \frac{34395742267\pi^2}{13385572200} \\
\delta\left(57, \frac{1}{2}\right) &= \frac{232812878912645811878247805986151}{32377595370050046633665148000000} + \frac{315404588903\pi^2}{120470149800}.
\end{aligned} \tag{10}$$

A similar structure persists for higher values of the conformal spin n we have considered. One can notice that the most transcendental terms of ei-

ther $\zeta(3)$ or $\zeta(2)\ln 2$ allowed at this perturbative order are absent. The π^2 terms are related to $\zeta(2) = \frac{\pi^2}{6}$ and $\zeta(-2) = -\frac{\pi^2}{12}$, which are of the lower transcendentality. We have checked other values of the conformal spin in this limit and found that the most transcendental terms of the color singlet BFKL eigenvalue in either QCD or $N = 4$ SYM are absent for $\nu \rightarrow 0$ and $n = 4k + 1$, ($k = 0, 1, 2, 3, \dots$). The list of the expression for $\delta(n, \gamma)$ in this limit for the first thousand values of $n = 4k + 1$ ($k = 0, 1, 2, 3, \dots$) is given in the attached file.

Next we consider a similar limit of $\nu \rightarrow 0$ for $n = 4k + 3$, ($k = 0, 1, 2, 3, \dots$). In this case the most transcendental term is non-zero and has a universal coefficient for all terms we have calculated. The first fifteen terms of the color singlet NLO BFKL eigenvalue in the limit of $\nu \rightarrow 0$ for $n = 4k + 3$, ($k = 0, 1, 2, 3, \dots$) are given by

$$\begin{aligned}
\delta\left(3, \frac{1}{2}\right) &= 4\zeta(3) + \frac{2\pi^2}{3} & (11) \\
\delta\left(7, \frac{1}{2}\right) &= 4\zeta(3) + \frac{5}{3} + \frac{11\pi^2}{9} \\
\delta\left(11, \frac{1}{2}\right) &= 4\zeta(3) + \frac{49}{24} + \frac{137\pi^2}{90} \\
\delta\left(15, \frac{1}{2}\right) &= 4\zeta(3) + \frac{787}{360} + \frac{121\pi^2}{70} \\
\delta\left(19, \frac{1}{2}\right) &= 4\zeta(3) + \frac{81917}{36288} + \frac{7129\pi^2}{3780} \\
\delta\left(23, \frac{1}{2}\right) &= 4\zeta(3) + \frac{10424089}{4536000} + \frac{83711\pi^2}{41580} \\
\delta\left(27, \frac{1}{2}\right) &= 4\zeta(3) + \frac{16562497}{7128000} + \frac{1145993\pi^2}{540540}
\end{aligned}$$

$$\begin{aligned}
\delta\left(31, \frac{1}{2}\right) &= 4\zeta(3) + \frac{14879171647}{6356750400} + \frac{1195757\pi^2}{540540} \\
\delta\left(35, \frac{1}{2}\right) &= 4\zeta(3) + \frac{119646854891}{50854003200} + \frac{42142223\pi^2}{18378360} \\
\delta\left(39, \frac{1}{2}\right) &= 4\zeta(3) + \frac{6124929706001}{2593554163200} + \frac{275295799\pi^2}{116396280} \\
\delta\left(43, \frac{1}{2}\right) &= 4\zeta(3) + \frac{10609392277369}{4479775372800} + \frac{18858053\pi^2}{7759752} \\
\delta\left(47, \frac{1}{2}\right) &= 4\zeta(3) + \frac{14152043794450139}{5962581021196800} + \frac{444316699\pi^2}{178474296} \\
\delta\left(51, \frac{1}{2}\right) &= 4\zeta(3) + \frac{125408026665021137}{52745909033664000} + \frac{34052522467\pi^2}{13385572200} \\
\delta\left(55, \frac{1}{2}\right) &= 4\zeta(3) + \frac{107963113217545418747}{45345428666201664000} + \frac{312536252003\pi^2}{120470149800} \\
\delta\left(59, \frac{1}{2}\right) &= 4\zeta(3) + \frac{2485984260732243834301}{1042944859322638272000} + \frac{9227046511387\pi^2}{3493634344200}
\end{aligned}$$

One can note the most transcendental term $\zeta(3)$ has a universal coefficient of 4 that does not depend on a specific value of n . This pattern persists for all other values of $n = 4k + 3$, ($k = 0, 1, 2, 3, \dots$) we have considered. Despite the lack of a general proof of the universality of the leading transcendental term, the direct calculation shows that this coefficient is the same for $\nu \rightarrow 0$ and $n = 4k + 3$, ($k = 0, 1, 2, 3, \dots$).

In our calculations we used the nested harmonic sums analytically continued from the even positive integer values of the argument to the complex plane. The details of the calculations are given in the Appendix.

Two other limits of $a) \nu \rightarrow 0$ for $n = 1$ in which the BFKL eigenvalue vanishes, and $b) \nu \rightarrow \infty$ that recovers the coefficients of the cusp anomalous dimension are known for many years and used for either defining constraints or cross checking the results in calculations using Quantum Spectral Curve [25–27] and other novel techniques [28–34]. The new universal limit of $\nu \rightarrow 0$ for odd values of n supplements existing constraints in fixing free coefficient of general ansatz for higher order BFKL eigenvalues. At this point we are unable to comment on recent results for the BFKL eigenvalue derived using the Quantum Spectral Curve and other techniques

due to the complexity of the underlying expressions. We believe that a similar universality of the most transcendental terms in the limit $\nu \rightarrow 0$ for odd values of the conformal spin n also holds beyond NLO order of the BFKL eigenvalue in QCD and $N = 4$ SYM.

A few words to be said about a similar limit of $\nu \rightarrow 0$ for even values of the conformal spin n . In this limit the NLO BFKL eigenvalue will give a new set of transcendental constants (Catalan number C_n , polylogs of complex argument $\text{Li}_s\left(\frac{1}{2} + \frac{i}{2}\right)$ etc.). Those constants do not appear in the functional basis of nested harmonic sums and their relations should be investigated further in more details. In particular, it is not clear what would be the irreducible and linear independent ¹ set of those constants for any given transcendentality.

5 Conclusions

We discussed a new limit of the BFKL eigenvalue at the leading order (LO) and the next-to-leading (NLO) perturbative order. We show that in the limit of $\nu \rightarrow 0$ for odd values of the conformal spin n the NLO BFKL eigenvalue is a non-zero constant that can be expressed in terms of transcendental and rational numbers. We show that for special odd values of the conformal spin given by $n = 4k + 3$, ($k = 0, 1, 2, 3, \dots$) the NLO BFKL eigenvalue gives the maximally transcendental term that is equal to $4\zeta(3)$ and does not depend on a specific value of n . For another values of odd conformal spin $n = 4k + 1$, ($k = 0, 1, 2, 3, \dots$) in the limit of $\nu \rightarrow 0$ the maximally transcendental term is absent in the NLO BFKL eigenvalue, i.e. the coefficient of either $\zeta(3)$ or $\pi^2 \ln 2$ is zero and also does not depend on specific value of the conformal spin. This universality in coefficients of the most transcendental terms for odd values of the conformal spin suggests that there might be some other non-trivial limits of the BFKL eigenvalue at higher orders of the perturbation theory.

¹By linear independence of two transcendental numbers we mean that they cannot be related to each other using only rational coefficients.

6 Acknowledgments

We are indebted to Sergey Bondarenko and Ian Balitsky for enlightening discussions and valuable comments.

Appendix

The next-to-leading order BFKL eigenvalue includes some complicated functions that build $\phi(M)$ in eq. (6). Those functions are functions of one complex variable and we found it useful to express $\phi(M)$ in terms of the nested harmonic sums analytically continued from even integer values of the argument to the complex plane.

The nested harmonic sums are defined [38–41] in terms of nested summation for $n \in \mathbb{N}$

$$S_{a_1, a_2, \dots, a_k}(n) = \sum_{n \geq i_1 \geq i_2 \geq \dots \geq i_k \geq 1} \frac{\text{sign}(a_1)^{i_1}}{i_1^{|a_1|}} \dots \frac{\text{sign}(a_k)^{i_k}}{i_k^{|a_k|}}. \quad (12)$$

The harmonic sums are defined for non-zero real integer values of a_i , that build the alphabet of the possible negative and positive indices uniquely labeling $S_{a_1, a_2, \dots, a_k}(n)$. It is useful to define two important characteristics for the harmonic sums. Namely, the *depth* k that denotes the number of the nested summations and the *weight* $w = \sum_{i=1}^k |a_i|$ that is related to the transcendentality of a given sum. The nested harmonic sums are defined for positive integer values of the argument and require analytic continuation to the complex plane done using their integral representation. There are two different analytic continuations *a)* from even integer values of the argument and *b)* from the odd integer values of the argument to the complex plane. The choice of the analytic continuation is a matter of convenience and we choose the continuation from the even integers denoted by \tilde{S}^+ consistent with recent works on the BFKL eigenvalue. The details of two analytic continuations can be found in the work of Kotikov and Velizhanin [42]. The analytic properties of the nested harmonic sums in the context of the BFKL eigenvalues were recently studied in a series of publications [43–47].

The most complicated part of the NLO BFKL eigenvalue is given by the function $\phi(M)$ in eq. (6)

$$\phi(M) = 3\zeta(3) + \psi''(M) - 2\Phi_2(M) + 2\beta'(M) (\psi(1) - \psi(M)), \quad (13)$$

where $\Phi_2(M)$ reads

$$\begin{aligned} \Phi_2(M) = & \sum_{k=0}^{\infty} \frac{\beta'(k+1)}{k+M} + \sum_{k=0}^{\infty} \frac{(-1)^k \psi'(k+1)}{k+M} \\ & - \sum_{k=0}^{\infty} \frac{(-1)^k (\psi(k+1) - \psi(1))}{(k+M)^2}. \end{aligned} \quad (14)$$

The complex variable $M = \gamma + \frac{|n|}{2}$ is defined in terms $\gamma = \frac{1}{2} - i\nu$ for real continuous ν and real integer n .

The main objective of the present study is to consider the NLO BFKL eigenvalue in the limit of $\nu \rightarrow 0$ and odd values of the conformal spin n . In this limit the variable $M = \frac{1}{2} + \frac{|n|}{2}$ is a positive integer and its convenient to express the function $\phi(M)$ in terms of the nested harmonic sums that are easy to compute.

The expression of $\phi(M)$ written in terms of the nested harmonic sums is given by

$$\phi(M) = -4S_{-2,1} + 2S_{-3} + \frac{\pi^2}{3}S_1 + 4S_{-2}S_1 + 2S_3, \quad (15)$$

where for simplicity each sums S_a denotes the analytically continued nested harmonic sum from the even integer values of the argument to the complex plane, namely $\tilde{S}_{-2,1}^+(M-1)$. The shift of the argument by unity is due to the respective definitions of the harmonic sums and the functions of $\phi(M)$. The rest of terms in the NLO eigenvalue in $\delta(n, \gamma)$ are easily calculable by Mathematica using built-in libraries of special functions.

References

- [1] L. N. Lipatov, “Reggeization of the Vector Meson and the Vacuum Singularity in Nonabelian Gauge Theories,” *Sov. J. Nucl. Phys.* **23**, 338-345 (1976)
- [2] V. S. Fadin, E. A. Kuraev and L. N. Lipatov, “On the Pomeranchuk Singularity in Asymptotically Free Theories,” *Phys. Lett. B* **60**, 50-52 (1975) doi:10.1016/0370-2693(75)90524-9
- [3] E. A. Kuraev, L. N. Lipatov and V. S. Fadin, “Multi - Reggeon Processes in the Yang-Mills Theory,” *Sov. Phys. JETP* **44**, 443-450 (1976)

- [4] I. I. Balitsky and L. N. Lipatov, “The Pomeron singularity in Quantum Chromodynamics,” *Sov. J. Nucl. Phys.* **28**, 822-829 (1978)
- [5] V. S. Fadin and L. N. Lipatov, “BFKL pomeron in the next-to-leading approximation,” *Phys. Lett. B* **429**, 127 (1998) doi:10.1016/S0370-2693(98)00473-0 [hep-ph/9802290].
- [6] A. Prygarin, M. Spradlin, C. Vergu and A. Volovich, “All Two-Loop MHV Amplitudes in Multi-Regge Kinematics From Applied Symbology,” *Phys. Rev. D* **85**, 085019 (2012) doi:10.1103/PhysRevD.85.085019 [arXiv:1112.6365 [hep-th]].
- [7] J. Bartels, A. Kormilitzin, L. N. Lipatov and A. Prygarin, “BFKL approach and $2 \rightarrow 5$ maximally helicity violating amplitude in $\mathcal{N} = 4$ super-Yang-Mills theory,” *Phys. Rev. D* **86**, 065026 (2012) doi:10.1103/PhysRevD.86.065026 [arXiv:1112.6366 [hep-th]].
- [8] J. Bartels, L. N. Lipatov and A. Prygarin, “Collinear and Regge behavior of $2 \rightarrow 4$ MHV amplitude in $N = 4$ super Yang-Mills theory,” [arXiv:1104.4709 [hep-th]].
- [9] J. Bartels, L. N. Lipatov and A. Prygarin, “Integrable spin chains and scattering amplitudes,” *J. Phys. A* **44**, 454013 (2011) doi:10.1088/1751-8113/44/45/454013 [arXiv:1104.0816 [hep-th]].
- [10] J. Bartels, L. N. Lipatov and A. Prygarin, “MHV amplitude for $3 \rightarrow 3$ gluon scattering in Regge limit,” *Phys. Lett. B* **705**, 507-512 (2011) doi:10.1016/j.physletb.2011.09.061 [arXiv:1012.3178 [hep-th]].
- [11] L. N. Lipatov and A. Prygarin, “BFKL approach and six-particle MHV amplitude in $N=4$ super Yang-Mills,” *Phys. Rev. D* **83**, 125001 (2011) doi:10.1103/PhysRevD.83.125001 [arXiv:1011.2673 [hep-th]].
- [12] L. N. Lipatov and A. Prygarin, “Mandelstam cuts and light-like Wilson loops in $N=4$ SUSY,” *Phys. Rev. D* **83**, 045020 (2011) doi:10.1103/PhysRevD.83.045020 [arXiv:1008.1016 [hep-th]].
- [13] V. S. Fadin and L. N. Lipatov, “BFKL equation for the adjoint representation of the gauge group in the next-to-leading approximation at $N=4$ SUSY,” *Phys. Lett. B* **706**, 470-476 (2012) doi:10.1016/j.physletb.2011.11.048 [arXiv:1111.0782 [hep-th]].

- [14] V. S. Fadin, “NLO BFKL kernels for the adjoint representations of the gauge group,” AIP Conf. Proc. **1523**, no.1, 239-242 (2013) doi:10.1063/1.4802158
- [15] V. S. Fadin, R. Fiore, L. N. Lipatov and A. Papa, “Moebius invariant BFKL equation for the adjoint representation in N=4 SUSY,” Nucl. Phys. B **874**, 230-242 (2013) doi:10.1016/j.nuclphysb.2013.05.024 [arXiv:1305.3395 [hep-th]].
- [16] V. S. Fadin and R. Fiore, “On Hermitian separability of the next-to-leading order BFKL kernel for the adjoint representation of the gauge group in the planar $N = 4$ SYM,” Eur. Phys. J. C **76**, no.5, 230 (2016) doi:10.1140/epjc/s10052-016-4046-4 [arXiv:1512.08471 [hep-th]].
- [17] L. J. Dixon, J. M. Drummond, C. Duhr and J. Pennington, “The four-loop remainder function and multi-Regge behavior at NNLLA in planar $N = 4$ super-Yang-Mills theory,” JHEP **06**, 116 (2014) doi:10.1007/JHEP06(2014)116 [arXiv:1402.3300 [hep-th]].
- [18] V. Del Duca, L. J. Dixon, C. Duhr and J. Pennington, “The BFKL equation, Mueller-Navelet jets and single-valued harmonic polylogarithms,” JHEP **02**, 086 (2014) doi:10.1007/JHEP02(2014)086 [arXiv:1309.6647 [hep-ph]].
- [19] L. J. Dixon, J. M. Drummond and J. M. Henn, “Bootstrapping the three-loop hexagon,” JHEP **11**, 023 (2011) doi:10.1007/JHEP11(2011)023 [arXiv:1108.4461 [hep-th]].
- [20] S. Bondarenko and A. Prygarin, “Hermitian separability and transition from singlet to adjoint BFKL equations in $\mathcal{N} = 4$ super Yang-Mills Theory,” [arXiv:1510.00589 [hep-th]].
- [21] S. Bondarenko and A. Prygarin, “On a residual freedom of the next-to-leading BFKL eigenvalue in color adjoint representation in planar $\mathcal{N} = 4$ SYM,” JHEP **07**, 081 (2016) doi:10.1007/JHEP07(2016)081 [arXiv:1603.01093 [hep-th]].
- [22] A. Prygarin, “Reflection identities of harmonic sums up to weight three,” [arXiv:1808.09307 [hep-th]].

- [23] L. J. Dixon, M. von Hippel and A. J. McLeod, “The four-loop six-gluon NMHV ratio function,” *JHEP* **01**, 053 (2016) doi:10.1007/JHEP01(2016)053 [arXiv:1509.08127 [hep-th]].
- [24] S. Caron-Huot, L. J. Dixon, F. Dulat, M. von Hippel, A. J. McLeod and G. Papathanasiou, “Six-Gluon amplitudes in planar $\mathcal{N} = 4$ super-Yang-Mills theory at six and seven loops,” *JHEP* **08**, 016 (2019) doi:10.1007/JHEP08(2019)016 [arXiv:1903.10890 [hep-th]].
- [25] N. Gromov, F. Levkovich-Maslyuk and G. Sizov, “Pomeron Eigenvalue at Three Loops in $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory,” *Phys. Rev. Lett.* **115**, no. 25, 251601 (2015) doi:10.1103/PhysRevLett.115.251601 [arXiv:1507.04010 [hep-th]].
- [26] M. Alfimov, N. Gromov and G. Sizov, “BFKL spectrum of $\mathcal{N} = 4$: non-zero conformal spin,” *JHEP* **07**, 181 (2018) doi:10.1007/JHEP07(2018)181 [arXiv:1802.06908 [hep-th]].
- [27] M. Alfimov, N. Gromov and V. Kazakov, “ $N = 4$ SYM Quantum Spectral Curve in BFKL regime,” [arXiv:2003.03536 [hep-th]].
- [28] V. N. Velizhanin, “BFKL pomeron in the next-to-next-to-leading approximation in the planar $N=4$ SYM theory,” [arXiv:1508.02857 [hep-th]].
- [29] S. Caron-Huot and M. Herranen, “High-energy evolution to three loops,” *JHEP* **1802**, 058 (2018) doi:10.1007/JHEP02(2018)058 [arXiv:1604.07417 [hep-ph]].
- [30] S. Caron-Huot, “Resummation of non-global logarithms and the BFKL equation,” *JHEP* **03**, 036 (2018) doi:10.1007/JHEP03(2018)036 [arXiv:1501.03754 [hep-ph]].
- [31] S. Caron-Huot, E. Gardi, J. Reichel and L. Vernazza, “Two-parton scattering amplitudes in the Regge limit to high loop orders,” *JHEP* **08**, 116 (2020) doi:10.1007/JHEP08(2020)116 [arXiv:2006.01267 [hep-ph]].
- [32] E. Gardi, S. Caron-Huot, J. Reichel and L. Vernazza, “The High-Energy Limit of 2-to-2 Partonic Scattering Amplitudes,” *PoS RAD-COR2019*, 050 (2019) doi:10.22323/1.375.0050 [arXiv:1912.10883 [hep-ph]].

- [33] L. Vernazza, S. Caron-Huot, E. Gardi and J. Reichel, “The Regge Limit and infrared singularities of QCD scattering amplitudes to all orders,” PoS **LL2018**, 038 (2018) doi:10.22323/1.303.0038
- [34] S. Caron-Huot, E. Gardi and L. Vernazza, “Two-parton scattering in the high-energy limit,” JHEP **06**, 016 (2017) doi:10.1007/JHEP06(2017)016 [arXiv:1701.05241 [hep-ph]].
- [35] A. V. Kotikov and L. N. Lipatov, “NLO corrections to the BFKL equation in QCD and in supersymmetric gauge theories,” Nucl. Phys. B **582**, 19-43 (2000) doi:10.1016/S0550-3213(00)00329-1 [arXiv:hep-ph/0004008 [hep-ph]].
- [36] A. V. Kotikov and L. N. Lipatov, “DGLAP and BFKL evolution equations in the $N=4$ supersymmetric gauge theory,” [arXiv:hep-ph/0112346 [hep-ph]].
- [37] A. V. Kotikov and L. N. Lipatov, “DGLAP and BFKL equations in the $N = 4$ supersymmetric gauge theory,” Nucl. Phys. B **661**, 19-61 (2003) [erratum: Nucl. Phys. B **685**, 405-407 (2004)] doi:10.1016/S0550-3213(03)00264-5 [arXiv:hep-ph/0208220 [hep-ph]].
- [38] A. Gonzalez-Arroyo, C. Lopez, and F.J. Yndurain, Nucl. Phys. **B153** (1979) 161;
A. Gonzalez-Arroyo and C. Lopez, Nucl. Phys. **B166** (1980) 429.
- [39] J. A. M. Vermaseren, “Harmonic sums, Mellin transforms and integrals,” Int. J. Mod. Phys. A **14**, 2037 (1999) doi:10.1142/S0217751X99001032 [hep-ph/9806280].
- [40] J. Blumlein and S. Kurth, “Harmonic sums and Mellin transforms up to two loop order,” Phys. Rev. D **60**, 014018 (1999) doi:10.1103/PhysRevD.60.014018 [hep-ph/9810241].
- [41] E. Remiddi and J. A. M. Vermaseren, Int. J. Mod. Phys. A **15**, 725 (2000) doi:10.1142/S0217751X00000367 [hep-ph/9905237].
- [42] A. V. Kotikov and V. N. Velizhanin, “Analytic continuation of the Mellin moments of deep inelastic structure functions,” hep-ph/0501274.

- [43] A. Prygarin, “Reflection identities of harmonic sums of weight four,” *Universe* **5**, 77 (2019) doi:10.3390/universe5030077 [arXiv:1809.06696 [math.NT]].
- [44] A. Prygarin, “BFKL Eigenvalue and Maximal Alternation of Harmonic Sums,” *Nucl. Phys. B* **947**, 114734 (2019) doi:10.1016/j.nuclphysb.2019.114734 [arXiv:1901.05248 [hep-th]].
- [45] M. Joubat and A. Prygarin, “The analytic structure of the BFKL equation and reflection identities of harmonic sums at weight five,” *Int. J. Mod. Phys. A* **34**, no.11, 1950064 (2019) doi:10.1142/S0217751X19500647 [arXiv:1903.06773 [hep-th]].
- [46] M. Joubat and A. Prygarin, “Hermitian separability of BFKL eigenvalue in Bethe–Salpeter approach,” *Eur. Phys. J. C* **80**, no.12, 1183 (2020) doi:10.1140/epjc/s10052-020-08727-9 [arXiv:2007.15388 [hep-th]].
- [47] M. Joubat and A. Prygarin, “Reflection Identities of Harmonic Sums and pole decomposition of BFKL eigenvalue,” *Int. J. Mod. Phys. A* **36**, no.04, 2150025 (2021) doi:10.1142/S0217751X21500251 [arXiv:2011.08095 [hep-ph]].