

On the variational principle in the unfolded dynamics

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Abstract

The interplay between off-shell and on-shell unfolded systems is analyzed. The formulation of invariant constraints that put an off-shell system on shell is developed by adding new variables and derivation in the target space, that extends the original Q -derivation of the unfolded system to a bicomplex. The analogue of the Euler-Lagrange equations in the unfolded dynamics is suggested. The general class of invariant on-shell equation constraints is defined in cohomological terms. The necessary and sufficient condition for the on-shell equation constraints being Euler-Lagrange for some Lagrangian system is proven. The proposed construction is illustrated by the scalar field example.

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1 Introduction

Unfolded formalism provides a useful tool for the description of physical systems and, in particular, higher-spin gauge theories (see [1] for a review). Unfolded equations are formulated solely in terms of differential forms $W^\Lambda(x)$ in the world-sheet space M , which guarantees their coordinate independence. In application to field theories, unfolded systems involve infinite sets of first-order partial differential equations demanded to obey the compatibility conditions. Furthermore, symmetries of the physical system act geometrically on the differential forms of the unfolded system. The maximally symmetric background geometry is described by the Maurer-Cartan equations representing the simplest unfolded equations [2].

An off-shell unfolded system is a set of conditions expressing higher-rank tensor components in the system via higher-order derivatives of the *ground fields*, that are not expressed via derivatives of the other fields, in a way respecting the compatibility conditions and gauge symmetries at every step. (For recent development of the off-shell higher-spin systems see e.g. [3, 4].) The unfolded off-shell system itself describes the gauge symmetries of the system, but no dynamics, *i.e.*, partial differential equations on the ground fields. Non-trivial dynamics in the on-shell system results from imposing additional constraints on the fields $W^\Lambda(x)$, that effectively restrict some of the derivatives of the ground fields of the off-shell system.

Unfolded equations can be naturally reformulated in terms of a Q -derivation that associates invariant functionals with the Q -cohomology [2]. In the off-shell case, the latter represent various gauge invariant actions.

This approach allows one to reconstruct the already known field-theoretic examples and has been successfully used for the description of the higher-spin gauge theory [5, 6, 7]. However, the algorithm identifying the set of conditions that put the system on-shell was not available directly within the unfolded formalism, except for the case when the action and associated Batalin-Vilkovisky formalism were explicitly known allowing the authors of [9, 10, 11] to reconstruct the associated unfolded systems. The aim of this paper is in a certain sense opposite. We start with an off-shell unfolded system and analyze a possible form of the unfolded Euler-Lagrange equations to establish a relation between the off-shell unfolded systems and the related on-shell systems associated with the action principle. More precisely, our aim is to represent in the unfolded form the constraints themselves that put the system on shell. Let us stress that, starting from a given off-shell unfolded system, we address the question how to introduce constraints on the fields that effectively impose the partial differential equations on the ground fields. The latter may or may not be Euler-Lagrange for some action for the ground fields. This has to be contrasted with the papers [6, 8], where the unfolded system itself is derived from the action principle. It should be emphasized that our approach is applicable to any unfolded systems that may have different form from those of [9, 10, 11], such as those using spinor variables like in [5, 6, 7, 3, 4].

To this end we introduce additional variables δW^Λ , identified with the differentials of the fields W^Λ in the target space \mathcal{M} where W^Λ are local coordinates. Extending the derivation Q that determines the form of the original unfolded equations to \mathcal{Q} , acting on both W and

δW , and introducing the de Rham derivation δ that acts in $\Omega(\mathcal{M})$ we find that they form a bicomplex

$$\mathcal{Q}^2 = 0, \quad \delta^2 = 0, \quad \mathcal{Q}\delta + \delta\mathcal{Q} = 0.$$

In these terms we define the concept of invariant on-shell constraints that put the unfolded system on shell and find the necessary and sufficient conditions at which the resulting system is Lagrangian in terms of the ground fields.

Though the general features of our formalism are somewhat reminiscent of the variational bicomplex described in [12], the details are different and the explicit relation is an interesting question for the future study.

The paper is structured as follows: In Section 2 a brief recollection of the unfolded dynamics is provided with some emphasis on the invariant functionals. In Section 3 variations of fields are introduced and the unfolded system is extended to incorporate them. In Section 4 the variational operator is added, that is shown to form a bicomplex with the extended homological vector field \mathcal{Q} of the original unfolded system. In Section 5 the inverse problem of existence of a Lagrangian for given equations of motion is solved. In Section 6 it is shown that the suggested formalism allows for a natural inclusion of background fields. Section 7 illustrates the proposed construction by a toy example of a scalar field. Section 8 contains brief conclusions.

2 Unfolding

The possibility of decreasing the order of a differential equation by introducing new variables is well known. This approach can be generalized to partial differential equations by using the jet formalism [13]. Unfolded formalism is a further extension of this idea appropriate for the gauge theories in the framework of gravity.

A system of unfolded equations [5, 6] (see also [14]) is a generalization of the first-order formulation of a system of ordinary differential equations, achieved by replacing the partial derivative by the de Rham derivative. This implies that the dynamical variables are now space-time differential forms $W^\Omega(x)$, which allows us to rewrite the system of equations in the form

$$dW^\Omega(\xi, x) = G^\Omega(W(\xi, x)), \quad d := \xi^n \frac{\partial}{\partial x^n}, \quad (2.1)$$

where ξ^n is the differential, used as a placeholder for dx^n , and $G^\Omega(W(\xi, x))$ is some function of $W^\Omega(\xi, x)$ containing exterior products of the differential forms $W^\Omega(\xi, x)$ at the same x (no space-time derivatives in $G^\Omega(W)$); wedge product symbols are implicit while the variables ξ^n are anticommuting $\xi^n \xi^m = -\xi^m \xi^n$. The functions $G^\Omega(W(\xi, x))$ cannot be arbitrary, as the de Rham derivation is nilpotent and thus the compatibility condition $dG^\Omega = 0$ must hold. This demands

$$G^\Lambda(W) \frac{\partial G^\Omega(W)}{\partial W^\Lambda} = 0. \quad (2.2)$$

This constraint allows one to show that system (2.1) is invariant under the following gauge transformations

$$\delta_{gauge} W^\Omega(x) = d\epsilon^\Omega(x) + \epsilon^\Lambda(x) \frac{\partial G^\Omega(W(x))}{\partial W^\Lambda(x)}, \quad (2.3)$$

where

$$\deg \epsilon^\Lambda = \deg W^\Lambda - 1$$

with $\deg \omega$ being the degree of the differential form ω .

Generally speaking, gauge invariance only takes place in the so-called universal systems, in which the compatibility conditions hold as a consequence of the system itself, without taking into account the number of space-time dimensions, *i.e.*, not using the fact that $(d+1)$ -forms vanish in d -dimensional space. Indeed, for systems that are not universal, partial derivative $\frac{\partial G^\Omega}{\partial W^\Lambda}$ might not have any sense, leading to a non-zero derivative of a vanishing term, represented by a $(d+1)$ -form. We shall assume that the systems considered in this paper are, indeed, universal as is the case for higher-spin theories.

For universal systems it is possible to introduce a Q -derivation (homological vector field) of the form [2]

$$Q = G^\Omega(W) \frac{\partial}{\partial W^\Omega}, \quad (2.4)$$

which turns out to be nilpotent,

$$Q^2 = 0$$

as a consequence of the compatibility conditions (2.2). In these terms, any universal unfolded system can be rewritten as

$$dF(W(x)) = QF(W(x)), \quad (2.5)$$

where $F(W)$ is an arbitrary function of the differential forms in question.

This form of the unfolded equations implies that they equate the de Rham derivation on the world-sheet space M with coordinates x^a and the derivation Q on the target space \mathcal{M} with local coordinates W^Λ .

Note that the form of the equations (2.1) implies that G^Λ must have degree $p+1$ if W^Λ had degree p . Hence,

$$\deg Q = 1. \quad (2.6)$$

Consider, following [2], a Q -closed Lagrangian function \mathcal{L} ,

$$Q\mathcal{L}(W) = 0 \quad (2.7)$$

along with a $(d-1)$ -form E and a d -form L , such that

$$dE = L - \mathcal{L}(W), \quad (2.8)$$

$$dL = 0. \quad (2.9)$$

The action defined as the integral of L over a d -cycle M^d

$$S = \int_{M^d} L \quad (2.10)$$

is gauge invariant under the gauge transformations associated via (2.3) with the fields E and L with the gauge parameters ϵ_E and ϵ_L , respectively, as well as under the gauge transformation (2.3) with the gauge parameters $\epsilon^\Lambda(x)$ associated with $W^\Lambda(x)$,

$$\delta_{gauge} E = d\epsilon_E + \epsilon_L - \epsilon^\Lambda \frac{\partial \mathcal{L}}{\partial W^\Lambda}, \quad (2.11)$$

$$\delta_{gauge} L = d\epsilon_L. \quad (2.12)$$

The gauge transformation with the parameter $\epsilon_L(x)$ allows us to gauge fix E to zero, giving

$$S = \int_{M^d} \mathcal{L}. \quad (2.13)$$

The conclusion is that invariant functionals for any unfolded system are associated with the Q -cohomology [2]. Indeed, Q -exact Lagrangians describe total derivatives by virtue of (2.5).

One distinguishes between off-shell and on-shell unfolded systems. In the on-shell case, the system contains dynamical equations of motion (PDE) on certain ground (primary) fields. An off-shell system just expresses higher-rank tensor components in the system via higher-order derivatives of the ground fields along with the compatibility condition for the latter expressions. It should be stressed that the same universal unfolded equations considered in the world-sheet manifolds M^d of different dimensions may have different interpretations being on-shell in, say, $M^{d'}$ and off-shell in $M^{d''}$ with $d' > d''$ [15].

Unfolded formalism allows for a natural description of background geometry by Maurer–Cartan equations having the form of the simplest unfolded equations. Indeed, let g be a Lie algebra with a basis $\{T_a\}$ and Lie bracket $[\cdot, \cdot]$. Assuming $W = w = w^a T_a$ and $G = -\frac{1}{2}[w, w]$, one can check that equation (2.1) takes the form of the flatness (zero-curvature) condition

$$dw + \frac{1}{2}[w, w] = 0, \quad (2.14)$$

which is just the Maurer–Cartan equation, the solutions to which are known to represent g -invariant backgrounds in the Cartan gravity terms. (For more detail see e.g. [1] and references therein.)

3 Unfolded variations

Consider an off-shell unfolded system. To achieve non-trivial dynamics by restricting some of the lower-order derivatives, some extra conditions have to be imposed on the fields in the system. Let us illustrate how this works in a Lagrangian system. Let the Lagrangian be represented by a form \mathcal{L} obeying the Q -closure condition (2.7). As usual, the dynamics results from extremizing the action functional $S = \int \mathcal{L}$, *i.e.*, imposing the Euler-Lagrange equations

$$\delta S(W) = \int_{M^d} \delta W^\Lambda(x) \frac{\partial \mathcal{L}(W)}{\partial W^\Lambda(x)} = 0, \quad (3.1)$$

where $\delta W^\Lambda(x)$ is the variation of $W^\Lambda(x)$

$$\delta f(W) = \delta W^\Lambda(x) \frac{\partial f(W)}{\partial W^\Lambda(x)}. \quad (3.2)$$

For the future convenience we will assume that δW^Λ has grading shifted by one compared to W^Λ , which will allow us to interpret the variations δW^Λ as one-form differentials in the target space with local coordinates W^Λ . More in detail, in addition to the grading deg identified with the degree p of a world-sheet differential form W^Λ ,

$$deg(W^\Lambda(x)) = p^\Lambda = deg(\delta W^\Lambda(x)), \quad (3.3)$$

we introduce the target space differential form grading

$$Deg W^\Lambda(x) = 0, \quad Deg(\delta W^\Lambda(x)) = 1. \quad (3.4)$$

The commutation properties are determined by the total grading

$$\pi = Deg + deg, \quad fg = (-1)^{\pi(f)\pi(g)} gf. \quad (3.5)$$

Since the fields W^Λ are not independent, being related by the off-shell unfolded equations (2.1), their variations have to obey the conditions resulting from the variation of (2.1)

$$d\delta W^\Lambda(x) = -\delta W^\Omega(x) \frac{\partial G^\Lambda(W)}{\partial W^\Omega(x)}, \quad (3.6)$$

where the extra sign results from the extra grading of δW^Λ compared to W^Λ .

Clearly, since the equation (3.6) is a consequence of the consistent equation (2.1), the system of equations (2.1) and (3.6) is consistent as well. The idea is to unify the fields W^Λ and their variations δW^Ω into an extended set of fields

$$\tilde{W}^A(x) = (W^\Lambda(x), \delta W^\Omega(x)),$$

obeying the system of unfolded equations (2.1), (3.6).

Let us introduce the operator Q' for the right hand side of the equations (3.6) (from here onwards we shall be omitting the x dependencies if it does not create ambiguity)

$$Q' := -\delta W^\Omega \frac{\partial G^\Lambda(W)}{\partial W^\Omega} \frac{\partial}{\partial \delta W^\Lambda}. \quad (3.7)$$

Since Q and Q' act only on their respective variables, they can be combined to rewrite equations in a uniform way

$$d\tilde{W}^A(x) = \mathcal{Q}\tilde{W}^A(x), \quad \mathcal{Q} := Q + Q'. \quad (3.8)$$

It is not difficult to see that \mathcal{Q} is also nilpotent as a result of the compatibility conditions for the equations (2.1),

$$\mathcal{Q}^2 = 0. \quad (3.9)$$

This simple fact plays the key role in our construction.

Note that, since Q' acts trivially on the δW^Λ -independent functions, cohomologies of Q are also cohomologies of \mathcal{Q}

$$H(Q) \subset H(\mathcal{Q}). \quad (3.10)$$

Clearly,

$$\deg \mathcal{Q} = 1, \quad \text{Deg } \mathcal{Q} = 0. \quad (3.11)$$

Since δW^Λ are treated as new fields in the unfolded system they have their own gauge parameters $\delta\epsilon$ and a gauge transformation law

$$\delta_{\text{gauge}} \delta W^\Omega = d(\delta\epsilon^\Omega) - (\delta\epsilon^\Lambda) \frac{\partial G^\Omega(W)}{\partial W^\Lambda} - (-1)^{p^\Lambda} \epsilon^\Lambda \delta W^\Phi \frac{\partial^2 G^\Omega(W)}{\partial W^\Phi \partial W^\Lambda}. \quad (3.12)$$

The gauge transformations (2.3) for the fields W^Ω remain unchanged.

4 Target space bicomplex

Let us now introduce another nilpotent operator

$$\delta := \delta W^\Lambda \frac{\partial}{\partial W^\Lambda}, \quad \delta^2 = 0, \quad (4.1)$$

which is the de Rham derivation in the target space \mathcal{M} of W^Λ , that acts on $\Omega(\mathcal{M})$ and has gradings

$$\deg \delta = 0, \quad \text{Deg } \delta = 1. \quad (4.2)$$

A remarkable fact is that \mathcal{Q} and δ anticommute:

$$\{\mathcal{Q}, \delta\} = 0, \quad (4.3)$$

as one can see by an elementary straightforward computation

$$\begin{aligned} \left\{ G^\Lambda(W) \frac{\partial}{\partial W^\Lambda} - \delta W^\Omega \frac{\partial G^\Lambda(W)}{\partial W^\Omega} \frac{\partial}{\partial \delta W^\Lambda}, \delta W^\Phi \frac{\partial}{\partial W^\Phi} \right\} &= -\delta W^\Omega \frac{\partial G^\Lambda(W)}{\partial W^\Omega} \frac{\partial}{\partial W^\Lambda} + \\ &+ \delta W^\Phi \frac{\partial G^\Lambda(W)}{\partial W^\Phi} \frac{\partial}{\partial W^\Lambda} - \left(\delta W^\Phi \frac{\partial}{\partial W^\Phi} \delta W^\Omega \frac{\partial G^\Lambda(W)}{\partial W^\Omega} \right) \frac{\partial}{\partial \delta W^\Lambda} = 0, \end{aligned} \quad (4.4)$$

where all terms with two differentiations on the *r.h.s.* are dropped since they vanish in the anticommutator of any odd vector fields.

This result is not too surprising however, as \mathcal{Q} can be seen as the Lie derivation along a vector field G^Λ via the Cartan formula:

$$\mathcal{Q} := [i_G, \delta] = \left[G^\Omega \frac{\partial}{\partial \delta W^\Omega}, \delta W^\Lambda \frac{\partial}{\partial W^\Lambda} \right], \quad (4.5)$$

which makes (4.3) obvious by virtue of the second relation in (4.1).

Equation (4.3) implies that the total derivation

$$Q_{tot} := \mathcal{Q} + \delta \tag{4.6}$$

is also nilpotent

$$Q_{tot}^2 = 0. \tag{4.7}$$

The derivations \mathcal{Q} and δ form a bicomplex.

According to (3.2) δ is nothing else but the variation operator. As a consequence of (4.3) for every \mathcal{Q} -closed \mathcal{L} , $\delta\mathcal{L}$ is \mathcal{Q} -closed. Hence for every invariant functional of the original unfolded system, its variation is \mathcal{Q} -closed. In these terms the Euler-Lagrange equations demand \mathcal{L} to be δ -closed, up to a total derivative in the world-sheet space M^d . Since, by virtue of the unfolded equations (3.8), space-time derivative equates to the action of \mathcal{Q} , it implies that

$$\delta\mathcal{L} + \mathcal{Q}\Phi = 0, \tag{4.8}$$

where Φ is adjusted in such a way that the *l.h.s.* of (4.8) only contains the variation of the ground fields among W^Λ . Recall that, similarly to the fact that only a few ground fields among W^Λ can be treated as independent while all others are expressed via their space-time derivatives by virtue of (2.1), only the variations of the ground fields are independent, while all others are their descendants to be eliminated by an appropriate $\mathcal{Q}\Phi$. This mechanism is illustrated by the scalar field example in Section 7.

5 Inverse problem

Let $\mathcal{E}(\delta W, W)$ be a one-form in the target space \mathcal{M} and d -form in M^d

$$Deg \mathcal{E} = 1, \quad deg \mathcal{E} = d. \tag{5.1}$$

Let us address the question whether the condition

$$\int_{M^d} \mathcal{E}(\delta W(x), W(x)) = 0 \tag{5.2}$$

with arbitrary $\delta W(x)$ and $W(x)$, subject to their unfolded equations (3.6) and (2.1), respectively, can describe some gauge invariant equations. Analogously to the case of actions considered in [2], the integral (5.2) is gauge invariant iff \mathcal{E} is \mathcal{Q} -closed

$$\mathcal{Q}\mathcal{E}(\delta W, W) = 0. \tag{5.3}$$

Note that Euler-Lagrange equations (4.8) fulfill this condition. \mathcal{Q} -exact $\mathcal{E}(\delta W, W)$ trivializes upon integration in (5.2) due to (3.8). Therefore, nontrivial gauge invariant equations are represented by $Deg = 1$ $H(\mathcal{Q})$.

By (4.8) the necessary condition for \mathcal{E} to represent Euler-Lagrange equations therefore is

$$\delta\mathcal{E} = \mathcal{Q}\delta\Phi \tag{5.4}$$

with some Φ . If $\delta\mathcal{E} \neq \mathcal{Q}\delta\Phi$, the equations (5.2) associated with \mathcal{E} are not Lagrangian.

If equation (5.4) holds true, \mathcal{E} can be shown to represent Lagrangian equations. Indeed, using the Poincaré lemma for the target space de Rham derivation δ , (5.4) can be rewritten in the form

$$\mathcal{E} + \mathcal{Q}\Phi = \delta\chi. \quad (5.5)$$

The fact that \mathcal{E} is \mathcal{Q} -closed yields

$$\delta\mathcal{Q}\chi = 0. \quad (5.6)$$

Since χ and hence $\mathcal{Q}\chi$ are target-space zero-forms, it follows from (5.6) that $\mathcal{Q}\chi$ is W independent. However, for a non-zero $\mathcal{Q}\chi$, $\deg(\mathcal{Q}\chi) > 0$ which is not possible for $\mathcal{Q}\chi$ being a constant. From this it follows that $\mathcal{Q}\chi$ is indeed zero, *i.e.*,

$$\chi = \mathcal{L} + \mathcal{Q}\phi \quad (5.7)$$

with some \mathcal{L} representing $H(\mathcal{Q})$. This gives

$$\mathcal{E} = \delta\mathcal{L} + \mathcal{Q}(\Phi + \delta\phi), \quad (5.8)$$

which is just the variation (4.8) with the appropriately redefined Φ . Hence we arrive at the following

Theorem:

The equations (5.2) associated with \mathcal{E} are Euler-Lagrange iff the condition (5.4) holds true. This is one of the central results of the paper.

After being reduced to the form

$$\int_{M^d} \delta W_{gr}^i(x) e_i(W) = 0, \quad (5.9)$$

with the help of (3.6) and addition of total derivatives in the form of \mathcal{Q} -exact terms (*i.e.*, expressing all δW^Λ via derivatives of δW_{gr}^i by virtue of (3.6)), equation (5.2) yields constraints on the fields W^Λ and hence $W_{gr}^i(x)$,

$$e_i(W) = 0, \quad (5.10)$$

with some $e_i(W)$. These are usual Euler-Lagrange equations.

It should be stressed that the resulting equations (5.10) are by construction invariant under the gauge transformations of the original off-shell system which leave invariant the on-shell submanifold (5.10) of \mathcal{M} . Among these gauge transformations some may themselves be proportional to $e_i(W)$. These act trivially on the fields on-shell analogously to on-shell-trivial gauge transformations that appear in the so-called open algebras.

Finally, let us note that instead of imposing the on-shell conditions (5.2) one can quotient them away dropping all terms proportional to such expressions. Though, naively, the two schemes are equivalent, in the infinite-dimensional case they may be different. For instance, in the higher-spin theory of [16] (see also [1]) the quotient approach was applied for a certain ideal in the HS algebra, while the constraint approach was considered in [17].

6 Background fields

So far it was assumed that all fields in the system are dynamical. This is the case in theories involving gravity but not true in presence of background fields that describe background space. For instance, to describe Minkowski space in a coordinate independent way the background geometry is described in the unfolded Maurer-Cartan form for the Poincaré algebra:

$$de^n - \omega^{nm}e_m = 0, \quad d\omega^{nm} - \omega^{nk}\omega_k{}^m = 0, \quad (6.1)$$

where e^n and ω^{mn} are the vierbein and Lorentz connection one-forms, respectively. In Cartesian coordinate system $e^m = dx^m$ and $\omega^{mn} = 0$.

Generally, we distinguish between the background fields $w^\alpha(x)$ and the dynamical fields $W^\Omega(x)$, assuming that unfolded equations have the form

$$dw^\alpha(x) = g^\alpha(w(x)), \quad (6.2)$$

$$dW^\Omega(x) = G^\Omega(w(x), W(x)). \quad (6.3)$$

The system (6.2) is autonomous,

$$\frac{\partial g^\alpha}{\partial W^\Omega} = 0 \quad (6.4)$$

with the background fields not varied

$$\delta w^\alpha = 0. \quad (6.5)$$

The system for dynamical fields (6.3) depends on both background fields and dynamical fields. Different choices of the background fields are usually equivalent. For instance, different solutions to equations (6.1) with non-degenerate frame e^n describe the same Minkowski space-time in different coordinate systems.

The variation problem in presence of the background fields is set with respect to the dynamical fields with the variations δW^Ω and no variations for w^α . Other details are not affected by the presence of the background fields due to the triangular form of the equations (6.2), (6.3). Namely, using the compatibility conditions

$$g^\alpha(w) \frac{\partial g^\beta(w)}{\partial w^\alpha} = 0, \quad (6.6)$$

$$g^\alpha(w) \frac{\partial G^\Omega(w, W)}{\partial w^\alpha} + G^\Lambda(w, W) \frac{\partial G^\Omega(w, W)}{\partial W^\Lambda} = 0 \quad (6.7)$$

along with (6.5) one can easily check that

$$Q = g^\alpha(w) \frac{\partial}{\partial w^\alpha} + G^\Omega(w, W) \frac{\partial}{\partial W^\Omega}, \quad Q^2 = 0, \quad (6.8)$$

$$\mathcal{Q} = g^\alpha(w) \frac{\partial}{\partial w^\alpha} + G^\Omega(w, W) \frac{\partial}{\partial W^\Omega} - \delta W^\Omega \frac{\partial G^\Lambda(w, W)}{\partial W^\Omega} \frac{\partial}{\partial \delta W^\Lambda}, \quad \mathcal{Q}^2 = 0. \quad (6.9)$$

Since background fields are not varied, it is also easy to check that δ introduced in the standard way

$$\delta = \delta W^\Omega \frac{\partial}{\partial W^\Omega}, \quad \delta^2 = 0 \quad (6.10)$$

respects the bicomplex condition (4.3).

7 Scalar field example

Let us now illustrate the developed technique by a specific example of a scalar field, for which the unfolded formalism is well known [18, 2], leading to equations

$$dC(y, x) = dx^m \frac{\partial}{\partial y^m} C(y, x) \quad (7.1)$$

with the generating function $C(y, x)$

$$C(y, x) = \sum_{n=0}^{\infty} \frac{1}{n!} C_{m_1 \dots m_n}(x) y^{m_1} \dots y^{m_n}. \quad (7.2)$$

It is not hard to see that this system is off-shell [2]. Here the genuine scalar field $C(x) := C(0, x)$ is the ground field while all other descendent components in the generating function (7.2) are expressed via derivatives of $C(x)$ by virtue of (7.1).

This system can be put on-shell by demanding

$$C_m{}^m = 0, \quad (7.3)$$

which imposes the Klein-Gordon equation $\square C(x) = 0$ on the ground field.

With background fields obeying the Maurer-Cartan equations for the Poincaré algebra (6.1) equation (7.1) can be rewritten in the following coordinate-independent form:

$$dC(y, x) = (\omega_{nm} y^n \frac{\partial}{\partial y_m} + e^m \frac{\partial}{\partial y^m}) C(y, x). \quad (7.4)$$

The extension to the variations yields

$$d\delta C(y, x) = -\delta C_m{}^m(y, x) (\omega_n{}^m y^n + e^m), \quad (7.5)$$

where

$$\delta C_m{}^m(y, x) = \delta \frac{\partial}{\partial y^m} C(y, x). \quad (7.6)$$

The form of the invariant action for the original off-shell system is also known [2]

$$\mathcal{L} = e^{n_1} e^{n_2} \dots e^{n_d} \epsilon_{n_1 \dots n_d} l(C, C_n, C_{nm}, \dots), \quad (7.7)$$

where l is an arbitrary Lorentz invariant function, that is demanded by the condition that all ω^{nm} -dependent terms have to cancel upon the action of Q , which by (6.1),(7.4) has the form

$$QC(y, x) = (\omega_{nm}y^n \frac{\partial}{\partial y_m} + e^m \frac{\partial}{\partial y^m})C(y, x), \quad (7.8)$$

$$Qe^n = \omega^{nm}e_m, \quad (7.9)$$

$$Q\omega^{nm} = \omega^{nk}\omega_k{}^m. \quad (7.10)$$

The terms with the frame field vanish upon the action of Q since \mathcal{L} already contains d frame-fields. In usual terms, l is an arbitrary Lorentz invariant function built from the derivatives of the field $C(x)$.

Taking into account variations $\delta C(y, x)$, which have similar equations, we can also write down the \mathcal{Q} , which acts as Q on C , e^n , ω^{nm} and

$$\mathcal{Q}\delta C(y, x) = -\delta C_m(y, x)(\omega_n{}^m y^n + e^m). \quad (7.11)$$

Now let us consider a familiar Lorentz invariant action for the scalar field:

$$l = C_m C^m.$$

As is well known its variation yields equations of motion upon integrating out a total derivative. The latter is equivalent to the addition of an appropriate \mathcal{Q} -exact term.

More in detail, after taking the variation (working for simplicity in Cartesian coordinates)

$$\begin{aligned} (\delta C_m)e^{n_1}e^{n_2}\dots e^{n_d}\epsilon_{n_1\dots n_d}C^m &= d(\delta C_m)e^m e^{n_2}\dots e^{n_d}\epsilon_{n_1\dots n_d}C^{n_1} = \\ &- \mathcal{Q}(d(\delta C)e^{n_2}\dots e^{n_d}\epsilon_{n_1\dots n_d}C^{n_1}) + d(\delta C)e^m e^{n_2}\dots e^{n_d}\epsilon_{n_1\dots n_d}C_m{}^{n_1}, \end{aligned} \quad (7.12)$$

which is clearly \mathcal{Q} -closed (d is space-time dimension). Also, it is by construction δ -exact by virtue of anticommutativity of both e^i and δC .

Discarding the \mathcal{Q} -exact term, the variation takes the form

$$\mathcal{E} = (\delta C)e^{n_1}e^{n_2}\dots e^{n_d}\epsilon_{n_1\dots n_d}C_m^m, \quad (7.13)$$

imposing the Klein-Gordon equation in the form (7.3).

It should be noted, however, that, as discussed in [2], generally, the choice of the Lagrangian l in (7.7) was arbitrary. Lagrangians from the same \mathcal{Q} -cohomology class give rise to equivalent equations of motion, while those from different classes result in different equations of motion, featuring, for instance, mass terms, non-trivial potential, higher-derivative and/or interaction terms. The next simplest example, $l = C_k C^k + m^2 C^2$, can be easily checked to reproduce the equations of motion for the massive scalar, while $l = C_{kl}C^{kl}$ leads to the higher-derivative equation $\square^2 C(x) = 0$.

To illustrate general results of Section 5 consider the following equations of motion for two scalar fields

$$\mathcal{E} = e^{n_1}e^{n_2}\dots e^{n_d}\epsilon_{n_1\dots n_d}((\delta C_1)C_{1m}^m + (\delta C_2)C_{2m}^m + \delta C_2 C_1). \quad (7.14)$$

So defined \mathcal{E} is Poincaré invariant and, hence, \mathcal{Q} -closed. Its variation is

$$\delta\mathcal{E} = e^{n_1} e^{n_2} \dots e^{n_d} \epsilon_{n_1, \dots, n_d} ((\delta C_{1m}^m)(\delta C_1) + (\delta C_{2m}^m)(\delta C_2) + \delta C_1 \delta C_2). \quad (7.15)$$

Since both δC_1 and δC_2 are ground fields, no Φ satisfying (5.4) exists, which means in accordance with Section 5 that the equations are non-Lagrangian, which is indeed true. Note that the replacement of $\delta C_2 C_1$ by $\delta C_2 C_1 + \delta C_1 C_2$ makes the equation (7.14) Lagrangian.

8 Conclusion

In this paper, an extended unfolded system is suggested by introducing extra variables corresponding to variations δW^Ω of the original fields W^Ω and interpreted as differentials in the target space where W^Ω serve as local coordinates. This allows us to define a new \mathcal{Q} -derivation in the extended system, giving rise to an action and the equations of motion resulting from the action variation. Remarkably, \mathcal{Q} and the de Rham derivation δ in the target space form a bicomplex in terms of which the structure of invariant field equations can be analyzed. In particular, we found a necessary and sufficient condition for the equation constraints on the ground (dynamical) fields to be Lagrangian, *i.e.*, resulting from the action principle.

It should be stressed that our approach is entirely formulated within the general unfolded dynamics formalism with no reference to its specific realization and/or external ingredients like for instance those used in [9, 10, 11]. This allows us to address all kind of dynamical questions within unfolded dynamics. On the other hand, the variation fields δW^Λ introduced in this paper are somewhat analogous to the antifields in the Batalin-Vilkovisky formalism [19]. It would be interesting to make the relation between the two approaches more explicit. The same concerns the variational bicomplex of [12].

There are many questions for the future study. In particular, it would be nice to find a way for reconstruction of the form of Lagrangian from the Lagrangian field constraint rather than just proving the existence theorem. Also the interpretation of the higher cohomologies of the operator \mathcal{Q} and the (\mathcal{Q}, δ) bicomplex need further analysis. These questions are particularly interesting in the context of higher-spin gauge theory.

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References

- [1] X. Bekaert, S. Cnockaert, C. Iazeolla and M. A. Vasiliev, [arXiv:hep-th/0503128 [hep-th]].
- [2] M. A. Vasiliev, *Int. J. Geom. Meth. Mod. Phys.* **3** (2006), 37-80 [arXiv:hep-th/0504090 [hep-th]].
- [3] N. Misuna, *Phys. Lett. B* **798** (2019), 134956 [arXiv:1905.06925 [hep-th]].
- [4] N. G. Misuna, [arXiv:2012.06570 [hep-th]].
- [5] M. A. Vasiliev, *Phys. Lett. B* **209** (1988) 491.
- [6] M. A. Vasiliev, *Annals Phys.* **190** (1989) 59.
- [7] M. A. Vasiliev, *Phys. Lett. B* **285** (1992), 225-234
- [8] N. Boulanger and P. Sundell, *J. Phys. A* **44** (2011), 495402 [arXiv:1102.2219 [hep-th]].
- [9] G. Barnich, M. Grigoriev, A. Semikhatov and I. Tipunin, *Commun. Math. Phys.* **260** (2005), 147-181 [arXiv:hep-th/0406192 [hep-th]].
- [10] G. Barnich and M. Grigoriev, *JHEP* **01** (2011), 122 [arXiv:1009.0190 [hep-th]].
- [11] M. Grigoriev, *JHEP* **12** (2012), 048 [arXiv:1204.1793 [hep-th]].
- [12] I.M. Anderson (1992) Introduction to the Variational Bicomplex. *Contemporary Mathematics*, 132, 51-73.
- [13] Vinogradov, A.M. Geometry of nonlinear differential equations. *J Math Sci* 17, 1624–1649 (1981). <https://doi.org/10.1007/BF01084594>
- [14] N. Boulanger, C. Iazeolla and P. Sundell, *JHEP* **07** (2009), 013 [arXiv:0812.3615 [hep-th]].
- [15] M. A. Vasiliev, *J. Phys. A* **46** (2013), 214013 [arXiv:1203.5554 [hep-th]].
- [16] M. A. Vasiliev, *Phys. Lett. B* **567** (2003), 139-151 [arXiv:hep-th/0304049 [hep-th]].
- [17] A. Sagnotti, E. Sezgin and P. Sundell, [arXiv:hep-th/0501156 [hep-th]].
- [18] O. V. Shaynkman and M. A. Vasiliev, *Theor. Math. Phys.* **123** (2000), 683-700 [arXiv:hep-th/0003123 [hep-th]].
- [19] I. A. Batalin and G. A. Vilkovisky, *Phys. Lett. B* **69** (1977), 309-312