

# The information horizon entropy for quantum dot and a symmetrical bath

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## Abstract

We study the entropy of quantum-dot system in contact with a symmetrical CFT bath living on the boundary of pure AdS3 black hole. The q-dot is localized at the centre of bath system of finite size. We first determine the exact location of the ‘information horizon’ for q-dot and then obtain corresponding generalised entropy of q-dot plus bath system. It is done by finding codim-2 time extremal curve whose end point uniquely determines the information horizon of localised q-dot system. By including the (bulk) entropy contribution of the information horizon the Page curve for the radiation follows. These results can be easily generalized to higher dimensional cases as well.

# 1 Introduction

The holographic principle in string theory [1] has produced easy to understand answers for some of the difficult and intractable questions in strongly coupled quantum field theories. We focus on the phenomenon of information exchange between two identical quantum systems having a common interface. The information sharing is a real time phenomenon as quantum states would always get entangled. In quantum mechanical theories the information contained in a given state cannot be destroyed, cloned or mutated. For example, in the bi-partite systems the information can either be found in one part of the Hilbert space or in its compliment; see [2, 3]. Generally it is believed that the exchange and sharing of quantum information is guided by the unitarity and locality. The time like flows of isolated quantum systems should essentially be Hamiltonian flows. Under similar claims for the black holes the formation and evaporation processes (via Hawking radiation) it is generally expected that the entropy curve for the radiation should bend after half Page-time is crossed [6]. This certainly holds good when a pure state is divided into two smaller systems. But for a mixed state or finite temperature CFT duals to AdS-black holes, it is not straight forward to answer this question. However, in some recent models by coupling holographic CFTs to an external radiation (bath) system, and also by involving nonperturbative techniques such as replica, wormholes and islands [4, 5], some answers to these difficult questions have been attempted.<sup>1</sup>

The recent proposal for generalised entanglement entropy [4] involves an hypothesis of the island ( $I$ ) contribution, including the contribution from the island boundary ( $\partial I$ ), such that the complete quantum entropy of radiation (bath) can be expressed as

$$S_{Rad}^A = \min [ext\{\frac{Area(\partial I)}{4G} + S[A \cup I]\}].$$

It means one needs to pick the lowest contribution out of a set of extremas. This complicated looking formula seemingly reproduces a Page curve for the radiation entropy. However one of the puzzling feature of above proposal entails in the ad hoc appearance of an island in the black hole geometry usually outside of the horizon. The island does not arise by means of a sound dynamical principle. It is merely presumed to be there, perhaps associated with the presence of the bath system. In contrast, in the present work we would like to propose an alternative picture that there will always exist an ‘**information horizon**’ for a quantum-dot living at the interface of a symmetrical CFT (bath) system. The information horizon is always found to be outside of the black hole horizon. Next the information horizon can be dynamically obtained by extremizing a *time*-curve corresponding to the quantum dot situated at the bath interface. Further, the net information processed by the quantum dot in a given time interval remains a well defined

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<sup>1</sup>See a review on information paradox along different paradigms in [7] and for list of major references therein, also in [[8]-[11]].

physical observable. <sup>2</sup> Effectively the net information processed is a measure of unitary operations (optimal count) a quantum computer (like q-dot) would perform in a given time. This can be evaluated holographically by embedding a codimension-2 time curve in asymptotically AdS spacetime, as described in [12]. (The proposal is very much similar to the holographic measure of entanglement entropy [15, 16].) The end point (or cusp-point) of time extremal curve is always unique and it will be identified as the information horizon ( $I_h$ ). Correspondingly we propose that there would be a bulk contribution to the entropy arising out of information horizon of a quantum dot (being in contact with bath). So that total entropy is given by  $S[I_h] + S[A]$ . Therefore for the quantum entropy of radiation the formula may be written as

$$S_{Rad}^A = \min \left\{ \frac{Area[I_h]}{4G}, S[A] \right\}$$

The above expression reproduces the Page curve for the radiation. We indeed find that when a finite temperature bath becomes sufficiently large, it requires q-dot very long time to process entire information and correspondingly the cusp-point tends to merge with the black hole horizon, i.e.  $z_i \rightarrow z_0$  in the large bath limit. So that the entropy of radiation becomes

$$S_{Rad} \rightarrow S_{BH} \tag{1}$$

after a very long time period.

The paper is organized as follows. In section-2 we introduce the information horizon idea and obtain the generalised entropy formulation for pure  $AdS_3$  case. On the boundary we have taken a quantum-dot in contact with finite size symmetrical radiation bath. We then extend our results for the BTZ black holes case in section-3. The last section-4 contains a brief summary of our observations.

## 2 Information processing by a quantum dot

An immediate goal in this section is to know how much information a quantum dot can process in given time interval. We also assume that q-dot is in contact with a symmetrical radiation bath of finite size.

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<sup>2</sup>In a recent work [12] a measure for net exchange of quantum information between two *adjacent* subsystems (having common interface) for the dual field theories of  $AdS_{d+1}$  was discussed. As the quantum systems continuously exchange information a large amount of information will be exchanged over long periods. This leads to overall information growth in time. The information exchange typically grows as  $\propto (\frac{1}{\delta^{d-2}} - \frac{1}{t^{d-2}})$  for extremal  $CFT_d$ . However over finite time interval only optimum information is exchanged. This leads to the formation of ‘information horizon’.

Let us consider pure  $AdS_3$  spacetime geometry

$$ds^2 = \frac{L^2}{z^2}(-dt^2 + dx^2 + dz^2) \quad (2)$$

where  $L$  is the radius of curvature. The coordinate range  $0 \leq z \leq \infty$  represents the full holographic range. The Kaluza-Klein compactification on a circle ( $x \simeq x + 2\pi R$ ) produces a *near*  $AdS_2$  solution, well known as Jackiw-Teitelboim background [13, 14],

$$\begin{aligned} ds_{JT}^2 &= \frac{L^2}{z^2}(-dt^2 + dz^2) \\ e^{-2(\phi-\phi_0)} &= \sqrt{g_{xx}} = \frac{L}{z} \end{aligned} \quad (3)$$

where  $\phi$  is the  $2d$  dilaton field, written in standard convention (effective string coupling vanishes near the boundary). The Newton's constants get related as  $\frac{2\pi R}{G_3} \equiv \frac{1}{G_2}$ , with  $G_2$  being dimensionless. The anti-de Sitter solution without the dilaton remains a topological spacetime with no propagating degrees of freedom.

The  $CFT_2$  lives on entire 2-dimensional noncompact  $(t, x)$  flat boundary of  $AdS_3$  geometry. We consider a CFT subsystem  $[-b, b]$  along spatial  $x$  direction with an interface at the centre  $x = 0$ . At the interface we place a defect system (a quantum dot) that is described by a quantum mechanical theory of its own, see figure (1). To be consistent with the bath  $CFT_2$  the quantum theory is necessarily taken to be "near  $CFT_1$ ", a holographic dual of the JT-gravity as described in (3). Due to this we safely assume that any back reaction of quantum dot on bath CFT is ignorable and vice versa. The entire systems set up is taken in a particular symmetrical way for the convenience. The states of the q-dot and the bath are necessarily entangled.

It is known that the entanglement entropy of extremal  $CFT_2$  of size  $2b$  is ordinarily given by

$$S_{bath} = \frac{L}{2G_3} \ln \frac{2b}{\epsilon} \quad (4)$$

Since there is also a point like quantum system at the centre and its states are entangled with the bath, the information will be processed by the quantum dot based on its independent dynamics. We are assuming a unitary set up here. The net window of time we are interested in is however fixed by the maximum time the radiation takes to travel from one edge of the bath subsystem to the q-dot located at its centre  $x = 0$ . Simply the maximum time window relevant for q-dot (unitary) operations is

$$\Delta t_{max} = b \quad (5)$$

Note the maximum information that can be processed by a quantum dot in given time would always be finite [12]. Holographically, as proposed in [12], the entanglement information processed (through combinations of multiple unitary operations) by any quantum

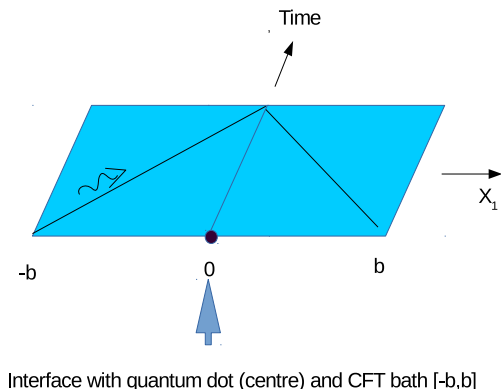


Figure 1: A typical arrangement of quantum dot (at the centre of  $x$ -axis) and 1-dimensional radiation bath  $x = [-b, b]$ . The boundary theory has usual Minkowski space-time. The light signal from the edge of the bath takes time  $b$  to reach at the centre (quantum dot). This is the maximum time window relevant for processing of bath information by a  $q$ -dot.

system may be obtained by extremizing an action functional as

$$I_E = \frac{L}{2G_3} \int_{\epsilon}^{z_i} \frac{dz}{z} \sqrt{1 - (\partial_z t)^2} \quad (6)$$

where  $z = z_i$  is the location of the cusp and  $\epsilon$  is the UV cut-off. The integral expression describes the area of codimension-2 curve, for constant  $x$  (with  $x = 0$ ), in  $AdS_3$  geometry (2). The cusp is an end point of the extremal curve inside the bulk geometry, shown in the figure (2). The equation of an extremal time curve from (6) is given by

$$t' = \frac{z}{z_i \sqrt{1 + \frac{z^2}{z_i^2}}} \quad (7)$$

where b.c.s are:  $t'|_{z=0} = 0$  and at the cusp point  $t'|_{z=z_i} = \frac{1}{\sqrt{2}}$ . By integrating this equation we obtain an exact answer

$$\Delta t = B_0 z_i \quad (8)$$

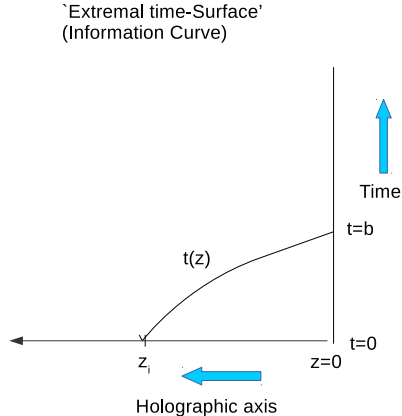


Figure 2: The extremal surface representing the time embedding in the  $AdS_3$  bulk for specific time interval. The cusp point  $z_i$  marks the end point of time extremal curve. It precisely is the information horizon of a quantum dot situated at the bath interface.

where constant ( $B_0 = \sqrt{2} - 1 \simeq .414$ ). Thus the location of the cusp point for the bath parameter  $b$  is precisely

$$z_i = \frac{\Delta t_{max}}{B_0} = \frac{b}{\sqrt{2} - 1} \quad (9)$$

where  $\Delta t_{max}$  is the time required by a signal to reach at the location of the quantum dot at the centre of bath from its boundary. Note that from eq.(9) we get  $z_i > b$  under all situations. The cusp point  $z_i$  may also be taken as an end point of information wedge  $[0, z_i]$  with other end at the AdS boundary. In th next, we now claim that  $z_i$  is the information horizon ( $I_h$ ) of the q-dot system. If the bath size  $2b$  increases correspondingly the location of the information horizon will also change in tandem. The bulk region  $z_0 \geq z > z_i$ , inside information horizon, may then describe an **island** for complementary CFT system ( $[b, \infty]$  plus  $[-b, -\infty]$ ). Although there is practically no need of any physical islands in our bath plus q-dot system set up. *Ultimately the extremality of the information quantity ( $I_E$ ) determines the location of information horizon.* This proposal is distinctly different to the similar set up [4] where maximality of a generalised entropy fixes the location of an

island boundary. Although surprisingly we find that in both the procedures one gets  $z_i > b$ .

The gravitational entropy corresponding to information horizon located at  $z = z_i$  can be written as

$$S_{bulk}[I_h] = \frac{l_x L}{2G_3 z_i} + constant = \frac{l_x L B_0}{2G_3 b} + constant \quad (10)$$

Here we take  $l_x \gg b$  as it describes the IR scale of  $CFT_2$ . The  $[-l_x, l_x]$  covers the entire range of CFT  $x$  coordinate. (If  $x \sim x + 2\pi R$  is compactified then we must take  $l_x = \pi R$ .) Thus the entropy contribution due to quantum dot corresponding to its information horizon is given by

$$S_{dot} \equiv S[I_h]. \quad (11)$$

A generalised entropy of the bath and quantum dot together may be written as

$$\begin{aligned} S_{gen} &= S_{dot} + S_{bath} \\ &= \frac{L}{2G_3} \left( \frac{l_x B_0}{b} + \ln \frac{2b}{\epsilon} \right) + constt. \end{aligned} \quad (12)$$

where  $\epsilon$  is UV cut-off term for both q-dot system and the bath. It is rather useful to define a dimensionless variable  $\tilde{b} = b/l_x$ , keeping in mind the hierarchy of scales  $l_x \gg b > \epsilon$ , we reexpress

$$S_{gen} = \frac{L}{2G_3} \left( \frac{B_0}{\tilde{b}} + \ln 2\tilde{b} \right) + S_0. \quad (13)$$

The overall constant  $S_0 \approx O(\ln \frac{l_x}{\epsilon})$  and contains other parameters. Obviously the generalised entropy varies with bath parameter  $b$  and there is a unique minimum for the total entropy at  $\tilde{b} = B_0$ . However, our claim is that the quantum entropy of the radiation would be only given by (for any  $b$ ) by the following selection rule

$$S_{Rad} =_{min} \{S_{I_h}, S_{bath}\} \quad (14)$$

This expression makes the quantum radiation entropy proposal complete and no further extremization is necessary. Interestingly, the gravitational entropy of information horizon (relevant for quantum-dot) always dominates for small bath size, whereas the bath term dominates when  $b$  becomes sufficiently large. For larger bath sizes the information horizon ( $I_h$ ) contribution becomes subleading. A plot has been drawn for various entropy components in figure (3) There is a crossover point where two contributions become exactly equal. Hence there is a Page curve phenomenon here for  $S_{Rad}$  provided we keep the contribution only of the smaller component in  $\{S_{dot}, S_{bath}\}$ . The Page curve [6] for

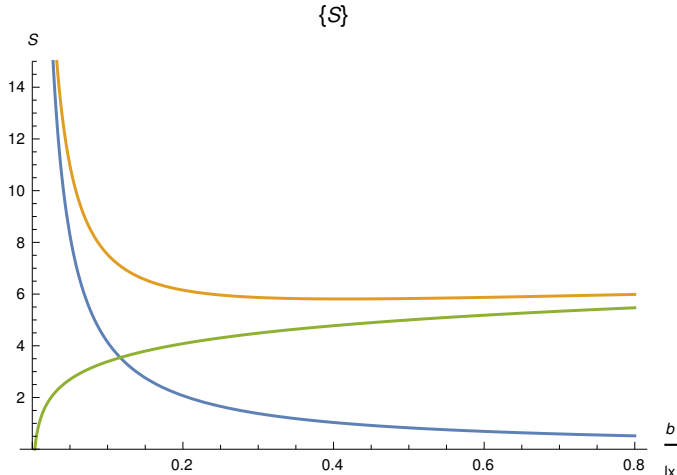


Figure 3: The blue curve is for information horizon entropy for quantum dot. The rising graph (green) is for the radiation entropy. The topmost graph represent total of two entropies. We set  $l_x = 1$ ,  $\frac{L}{2G_3} = 1$ . There is a minimum at  $\tilde{b} = B_0 \sim .414$  for the total entropy.

radiation follows from the standard principle that the minimum entropy is to be favoured. That is so far for extremal AdS case of zero temperature CFT with a quantum dot in a symmetrical bath set up.

*Some Comments:* Our result seems to provide an alternative way to understand radiation bath models developed recently, see ref.[4], where the presence of a bulk island ( $I$ ) was expected but the location of island boundary ( $\partial I$ ) was largely kept arbitrary. Although there appears to be a lack of clear holographic understanding in determining the bulk island. As an alternative we have obtained the information horizon ( $I_h$ ) holographically by extremizing codim-2 time-curve for the q-dot information processing. The island models rely primarily on the assumption that island will be there, but anywhere in front of the horizon (even if sometime behind the horizon). Furthermore this requires to extremize a generalised entropy (bulk entropy together with q-dot plus bath entropy) all by hand (not following holographic or dynamical principle). In contrast we have proposed here that the ‘information horizon’ holographically appears when we extremize the information processed by the quantum dot. The information time-curve entirely encodes the information processing capacity (through unitary operations) of a quantum mechanical device (described by dual JT gravity) and in contact with finite size bath.



## 2.1 An equilibrium in time?

Since the expression (14) is true for any size ( $b$ ) of the bath CFT, and that  $\Delta t_{max} = b$ , we may convert above result (10) into a time dependent expression, namely

$$S_{gen} = \frac{l_x L B_0}{2G_3 \Delta t_{max}} + \frac{L}{2G_3} \ln\left(\frac{2 \Delta t_{max}}{\epsilon}\right) \quad (15)$$

Using this a Page time graph can be deduced ( $l_x > \Delta t > \epsilon$ ). For small time interval the gravitational contribution (q-dot) leads whereas the entropy of the radiation (bath) remains smaller, but the radiation contribution starts rising for large time gap and keeps on growing until  $T_{Page}$ . After the Page time radiation entropy crosses the geometric contribution of the information horizon. The crossover point may not generally be the point where the total entropy will be the lowest. The crossover may actually be described as the point of equilibrium where information processing by the dot system is in equilibrium with the bath system.

## 3 Radiation entropy at finite temperature

For finite temperature entropy the spacetime AdS geometry will be taken with a horizon

$$ds^2 = \frac{L^2}{z^2}(-f dt^2 + \frac{dz^2}{f} + dx^2) \quad (16)$$

The function  $f(z) = (1 - \frac{z^2}{z_0^2})$  and  $z = z_0$  is location of black hole horizon. There is finite temperature in the field theory at boundary.<sup>3</sup> Now the quantum dot is taken in thermal equilibrium with symmetrical bath and is located at the centre  $x = 0$  of CFT subsystem of size  $[-b, b]$ .

We embed the time coordinate of the quantum dot system inside the bulk geometry (16). Correspondingly the information action for q-dot located at  $x = 0$  is given by

$$I_E = \frac{L}{4G_3} \int_{\epsilon}^{z_i} \frac{dz}{z} \sqrt{\frac{1}{f} - f(\partial_z t)^2} \quad (17)$$

From this we get the following equation describing an extremal time curve

$$t' = \frac{z/z_i}{f \sqrt{f + (z/z_i)^2}} \quad (18)$$

where  $z = z_i$  is the cusp point and it corresponds to value  $t(z_i) = 0$  at the boundary. Especially the slope of extremal curve at the cusp point is

$$t'|_{z=z_i} = \frac{1}{(1 - \frac{z_i^2}{z_0^2}) \sqrt{2 - \frac{z_i^2}{z_0^2}}} \geq \frac{1}{\sqrt{2}} \quad (19)$$

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<sup>3</sup>One may set coordinate  $x = L\phi$ , with range  $0 \leq \phi \leq 2\pi$  for BTZ black hole.

Note that this slope in the black hole case is always larger than  $\frac{1}{\sqrt{2}}$ , a value we have found for extremal  $AdS_3$  background in the previous section. The slope vanishes near the boundary  $z = 0$ . From here we get an exact expression for  $z_i$  in terms of interval  $\Delta t$ . By integrating (18), it is given by

$$\tanh\left(\frac{\Delta t}{z_0}\right) = \frac{\frac{z_i}{z_0}(\sqrt{2 - \frac{z_i^2}{z_0^2}} - 1)}{1 - \frac{z_i^2}{z_0^2}\sqrt{2 - \frac{z_i^2}{z_0^2}}} \quad (20)$$

and equally well, by knowing  $\Delta t_{max} = b$ , it can be written as

$$\tanh\left(\frac{b}{z_0}\right) = \frac{\frac{z_i}{z_0}(\sqrt{2 - \frac{z_i^2}{z_0^2}} - 1)}{1 - \frac{z_i^2}{z_0^2}\sqrt{2 - \frac{z_i^2}{z_0^2}}} \quad (21)$$

A plot for  $\Delta t$  Vs  $z_i$  has been provided in the figure (4) for the convenience.

The net information processed by a quantum dot in given time interval  $\Delta t_{max} = b$  remains always finite [12]. Consequently this leads to the existence of an information horizon at  $z = z_i$ . The gravitational entropy corresponding to the information horizon is

$$S[I_h] = \frac{L}{2G_3} \frac{l_x}{z_i} + const. \quad (22)$$

and this is the entropy of the quantum dot

$$S_{dot} = S[I_h] , \quad (23)$$

where  $z_i$  is to be determined from eq.(21) for given  $b$ . The range  $[-l_x, l_x]$  determines the box size of  $x$  coordinate. (But  $l_x = \pi L$  for using it for BTZ case.) The entropy of the finite temperature radiation bath of size  $2b$  is given by

$$S_{bath} = \frac{L}{4G_3} \ln \sinh^2\left(\frac{2b}{z_0}\right) \quad (24)$$

Hence the generalised entropy at finite temperature for the combined system will be

$$S_{gen} = \frac{L}{2G_3} \left( \frac{l_x}{z_i} + \ln \sinh \frac{2b}{z_0} \right). \quad (25)$$

While the quantum entropy of the radiation would be the minimum of the two values

$$S_{Rad} = \min \{S_{dot}, S_{bath}\} \quad (26)$$

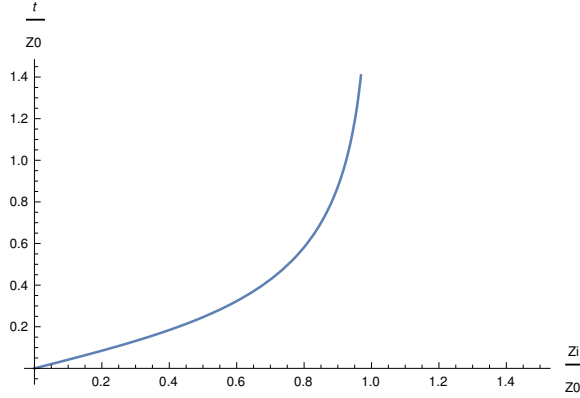


Figure 4: The  $\Delta t$  Vs  $z_i$  plot for BTZ black hole. When  $z_i \rightarrow z_0$  the time required for information processing by quantum dot indeed becomes very large. We have taken  $z_0 = 1$  here.

For small bath cases, i.e.  $b \ll z_0$ , from eq.(21) one obtains  $b \simeq z_i(\sqrt{2}-1) + O(\frac{z_i^3}{z_0^3})$ . This gives zero temperature result once higher orders are neglected and it is expected. In other words the black hole horizon does not play significant role in the entanglement for very small bath subsystems. The quantum entropy of the radiation can still be approximated as  $\sim \frac{L}{2G_3} \ln \frac{2b}{z_0}$ . However under large bath limit  $b \rightarrow \infty$ , equally for large time intervals (as  $\Delta t_{max} \rightarrow \infty$ ), we indeed find from (21) that it leads to  $z_i \rightarrow z_0$ , the information horizon tends to merge with the BH horizon. Thus the geometric entropy contribution of information horizon becomes (for  $b \rightarrow \infty$ )

$$S_{dot} \rightarrow \frac{\pi L^2}{2G_3 z_0} \equiv S_{BH} \quad (27)$$

which is the lowest value of the entanglement entropy at finite temperature. Thus from (26) for large bath case the quantum entropy of radiation would become

$$Lim_{b \rightarrow \infty} S_{Rad} \simeq S_{BH}. \quad (28)$$

## 4 Summary

We have explored the entanglement dynamics of a quantum mechanical system in contact with 1-dimensional radiation bath. The bath has an interface at  $x = 0$  where the q-dot is located. The strongly coupled quantum system is living on the boundary of  $AdS_3$  spacetime, including the black holes. There is an entanglement and information exchange (sharing) between two subsystems at the interface. We first calculated the precise location of an information horizon inside the bulk geometry. We do it by extremizing codim-2 time-curve whose end point in the bulk uniquely determines the location of the information

horizon. The information horizon is found to be located always outside the BH horizon. We have calculated the generalized entropy of q-dot and the bath. It is proposed that quantum radiation entropy follows the principle that

$$S_{Rad} =_{min} \{S_{dot}, S_{bath}\}$$

We also find that under large bath limit the quantum entropy of radiation tends to become

$$S_{Rad} \simeq S_{BH}$$

and thus realising the Page curve for the entropy of thermal radiation. These results can be generalized for higher dimensional CFT cases also.

## References

- [1] J. Maldacena, Adv. Theor. Math. Phys. 2, 231 (1998) [arXiv:9711200\[hep-th\]](#); S. Gubser, I. Klebanov and A.M. Polyakov, Phys. Lett. B 428, 105 (1998) [arXiv:9802109\[hep-th\]](#); E. Witten, Adv. Theor. Math. Phys. 2, 253 (1998) [arXiv:9802150\[hep-th\]](#).
- [2] A.K. Pati and S.L. Braunstein, NATURE **404**, 164 (2000); S.L. Braunstein and A.K. Pati, Phys. Rev. Lett. 98, 080502 (2007).
- [3] J. R. Samal, A. K. Pati, and Anil Kumar, Phys. Rev. Lett. 106, 080401 (2011).
- [4] A. Almheiri, R. Mahajan and J. Maldacena, “Islands outside the horizon”, [arXiv:1910.11077\[hep-th\]](#).
- [5] G. Penington, S. H. Shenker, D. Stanford and Z. Yang, “Replica wormholes and the black hole interior”, [arXiv:1911.11977\[hep-th\]](#); A. Almheiri, T. Hartman, J. Maldacena, E. Shaghoulian and A. Tajdini, “Replica Wormholes and the Entropy of Hawking Radiation”, JHEP05 (2020) 013, [arXiv:1911.12333](#).
- [6] D. N. Page, “Average entropy of a subsystem”, Phys. Rev. Lett. 71 (1993) 1291–1294, [\[gr-qc/9305007\]](#).
- [7] S. Raju, “Lessons from the Information Paradox”, [arXiv:2012.05770](#).
- [8] L. Susskind, [arXiv:1810.11563\[hep-th\]](#); Fortschr. Phys. 64 (2016) 24; D. Stanford and L. Susskind, Phys. Rev. D90 (2014) 126007.

- [9] A.R. Brown , D.A. Robert, L. Susskind, B. Swingle, and Y. Zhao, Phys. Rev. Lett 116 (2016) 191301; A.R. Brown , D.A. Robert, L. Susskind, B. Swingle, and Y. Zhao, Phys. Rev. Lett 93 (2016) 086006;
- [10] A. Bernamonti et.al, Phys. Rev. Lett. 123 (2019) 081601.
- [11] K. Hashimoto, N. Iizuka, and S. Sugishita, Phys. Rev D96 (2017) 126001; K. Hashimoto, N. Iizuka, and S. Sugishita, [arXiv:1805.04226 \[hep-th\]](#)
- [12] H. Singh, "Holography and quantum information exchange between systems", to appear in IJMPA, [arXiv:2101.02513 \[hep-th\]](#).
- [13] R. Jackiw, "Lower Dimensional Gravity", Nucl. Phys. B252, 343 (1985).
- [14] C. Teitelboim, " Gravitational and Hamiltonian Structure in Two Space-Time Dimensions", Phys. Lett. 126B, 41 (1983).
- [15] S. Ryu and T. Takayanagi, Phys. Rev. Lett. **86** (2006) 181602, [arXiv:0603001 \[hep-th\]](#); S. Ryu and T. Takayanagi, "Aspects of Holographic Entanglement Entropy", JHEP 0608 (2006) 045, [arXiv:0605073 \[hep-th\]](#).
- [16] V. E. Hubeny, M. Rangamani and T. Takayanagi, JHEP 0707 (2007) 062, [arXiv:0705.0016 \[hep-th\]](#).