

ABOUT THE COLEMAN INSTANTONS IN D DIMENSIONS

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Abstract

We identify an infinite class of unbounded potentials for which the Coleman instantons do not exist in D -dimensional spacetime. For such potentials, the decay of a false vacuum is described by the new instantons introduced in [5, 6, 4]. For each spacetime dimension D , we also construct the two different classes of potentials for which the instanton equations are exactly solvable.

1 Obstacles to the existence of the Coleman instantons

In 1978, S. Coleman, V. Glaser and A. Martin [1] found a large class of scalar field potentials $V(\varphi)$ in $D > 2$ dimensions for which the Coleman instanton describing the false vacuum decay exists and has a minimal action in the class of all solutions with maximal $O(D)$ symmetry. Namely, they proved that if for the continuous differentiable potential with a local minimum at $\varphi = 0$ there exist positive numbers a, b, α and β , such that

$$\beta < \alpha < 2D/(D - 2), \quad (1)$$

and

$$V(\varphi) \geq a|\varphi|^\beta - b|\varphi|^\alpha \quad (2)$$

for all φ , then the Coleman instanton exists¹.

In the present work, we somewhat compliment the result of [1] and consider the problem of false vacuum decay in D -dimensional scalar field theory under the assumption that the potential $V(\varphi)$ has a maximum at $\varphi = 0$ and has no any local minima at positive φ , is unbounded from below, and satisfies the inequality opposite to (2), i.e.

$$V(\varphi) < a|\varphi|^\beta - b|\varphi|^\alpha. \quad (3)$$

We will prove that in this case the Coleman instanton, which is supposed to describe the decay of the false vacuum at the absolute local minimum at $\varphi_f < 0$, does not exist regardless of the form of the potential at negative φ . Moreover, we will identify a broader class of unbounded potentials for which the instantons with the Coleman boundary conditions do not exist.

For this purpose, we will consider the $O(D)$ Euclidean solutions of the equation for the scalar field in D -dimensional spacetime

$$\frac{\partial^2 \varphi(t, \mathbf{x})}{\partial t^2} - \Delta \varphi(t, \mathbf{x}) + \frac{dV(\varphi)}{d\varphi} = 0. \quad (4)$$

Having performed a Wick rotation, i.e. using the Euclidian time $\tau = it$, and considering $O(D)$ -invariant solutions, for which $\varphi(\tau, \mathbf{x}) = \varphi(\varrho)$, $\varrho \equiv \sqrt{\tau^2 + \mathbf{x}^2}$, the equation (4) reduces to the ordinary differential equation [2, 1]

$$\ddot{\varphi} + \frac{D-1}{\varrho} \dot{\varphi} - V' = 0, \quad (5)$$

where $V' \equiv \frac{dV(\varphi)}{d\varphi}$ and $\dot{\varphi} \equiv \frac{d\varphi}{d\varrho}$. The Coleman instanton satisfies the two boundary conditions, namely,

$$\varphi(\varrho \rightarrow \infty) = \varphi_f, \quad \dot{\varphi}(\varrho = 0) = 0. \quad (6)$$

While the first condition must obviously be fulfilled, the second condition is imposed to avoid a singularity at the ‘‘center of the bubble’’, which would lead to a divergent infinite action. To prove the absence of instantons with these boundary conditions for the unbounded potentials satisfying (3), we construct the following class of nonlocal integrals for the equation (5):

$$E(\alpha) = \varrho^{\frac{2(\alpha(D-2)-2)}{\alpha+2}} \left(\varrho^2 \left(\frac{1}{2} \dot{\varphi}^2 - V \right) + \frac{2(D-1)}{\alpha+2} \varrho \dot{\varphi} \dot{\varphi} - \frac{(D-1)(\alpha(D-2)-2D)}{(\alpha+2)^2} \varphi^2 \right) \quad (7)$$

$$+ \frac{2(D-1)}{\alpha+2} \int_0^\varrho d\bar{\varrho} \bar{\varrho}^{\frac{\alpha(2D-5)-6}{\alpha+2}} \left[\frac{(\alpha(D-2)-2D)(\alpha(D-2)-2)}{(\alpha+2)^2} \varphi^2 + \bar{\varrho}^2 (\alpha V - \varphi V') \right],$$

¹Please note that in order to simplify the comparison of the results with results of our previous work [3], we have changed some notations compared to [1]. In particular, we use the standard notation for the scalar field potential $V(\varphi)$ instead of $U(\Phi)$, have swapped α and β compared to [1], and denote the number of dimensions by D instead of N .

which is parametrized by α . To verify that it is indeed the integral of motion for any value of α , one can take its derivative with respect to ϱ :

$$\frac{dE(\alpha)}{d\varrho} = \varrho^{\frac{\alpha(2D-3)-2}{\alpha+2}} \left(\ddot{\varphi}(\varrho) + \frac{D-1}{\varrho} \dot{\varphi}(\varrho) - V' \right) \left(\varrho \dot{\varphi} + \frac{2(D-1)}{\alpha+2} \varphi \right), \quad (8)$$

which vanishes when equation (5) is satisfied.

We note that integral (7) is invariant under the transformation: $V \Rightarrow V + \text{constant}$. Therefore, without loss of generality we can normalize the potential at the maximum such that $V(\varphi = 0) = 0$, and assume that it has the deepest local minimum (false vacuum) at $\varphi = \varphi_f < 0$, has no any minima for $\varphi > 0$ and is unbounded from below, otherwise V can be completely arbitrary.

Let us assume that the ‘‘initial condition’’ $\varphi(\varrho \rightarrow \infty) = \varphi_f$ is satisfied, and that there is the corresponding Coleman instanton that allows the tunnelling from the false vacuum at $\varphi_f < 0$ to $\varphi > 0$. For this solution, the field $\varphi(\varrho)$ must vanish at a *finite* value of $\varrho = \varrho_0$, i.e. $\varphi(\varrho_0) = 0$. Now, for $\alpha \geq \frac{2D}{D-2}$ we calculate the values of the integral $E(\alpha)$ (7) at $\varrho = 0$

$$E(\alpha) = 0 \quad (9)$$

and at $\varrho = \varrho_0$

$$E(\alpha) = \frac{1}{2} \varrho_0^{\frac{2\alpha(D-1)}{\alpha+2}} \dot{\varphi}(\varrho_0)^2 + \frac{2(D-1)}{\alpha+2} \int_0^{\varrho_0} d\bar{\varrho} \bar{\varrho}^{\frac{\alpha(2D-5)-6}{\alpha+2}} \left[\frac{(\alpha(D-2) - 2D)(\alpha(D-2) - 2)}{(\alpha+2)^2} \varphi^2 + \bar{\varrho}^2 \varphi^{\alpha+1} v'_\alpha \right], \quad (10)$$

then compare the corresponding results, where the rescaled potential $v_\alpha(\varphi)$ is defined as:

$$v_\alpha(\varphi) \equiv -\varphi^{-\alpha} V(\varphi). \quad (11)$$

If the Coleman instanton exists, the results of these two different ways of calculating $E(\alpha)$ must be consistent, but they certainly are not if

$$v'_\alpha \equiv \frac{dv_\alpha(\varphi)}{d\varphi} > 0. \quad (12)$$

In fact, in this case the right-side of (10) is obviously positive, $E(\alpha) > 0$, which contradicts to (9) imposed by the Coleman boundary condition $\dot{\varphi}(\varrho = 0) = 0$. This implies that the field $\varphi(\varrho)$ cannot cross $\varphi = 0$ to propagate towards φ_f , so it remains positive for all finite values of ϱ , and hence the Coleman instanton does not exist.

Finally, using the equations (11) and (12), one can conclude, that for any unbounded potential, which for positive φ can be represented as

$$V(\varphi) = -\varphi^\alpha \int^\varphi d\bar{\varphi} v'_\alpha(\bar{\varphi}), \quad (13)$$

where

$$\alpha \geq \frac{2D}{D-2}, \quad (14)$$

and

$$v'_\alpha \geq 0, \quad (15)$$

the Coleman instanton does not describe the decay of the deepest false vacuum at $\varphi_f < 0$ regardless of the form of the potential for negative values of φ . In particular, the potentials that satisfy the condition (3) and have no minima at positive φ belong to the class of the potentials (13), so that the Coleman instanton does not exist in this case. Other examples of such potentials are presented in [3] and we refer the reader to that work. Therefore, the new regularized instantons [4] must necessarily be used for such potentials for which the false vacuum is obviously unstable.

2 Integrable potentials in D dimensions

Using the non-local integrals (7), we can determine two classes of integrable potentials in D - dimensions. For this purpose, we first note that the first term under the integral in (7) vanishes if either $\alpha = 2D/(D - 2)$ or $\alpha = 2/(D - 2)$. In these cases, the expression under the integral in (7) for the potential $V(\varphi) = \lambda \varphi^\alpha$ vanishes and the integral becomes local. Therefore, one can expect such potential to be integrable. Let us start with the potential

$$V(\varphi) = \lambda (\varphi - \varphi_0)^{\frac{2D}{D-2}} + V_0, \quad (16)$$

for which equation (5) becomes

$$\ddot{\varphi} + \frac{D-1}{\varrho} \dot{\varphi} - \lambda \frac{2D}{D-2} (\varphi - \varphi_0)^{\frac{D+2}{D-2}} = 0. \quad (17)$$

If instead of ϱ we introduce the new variable $\eta = \ln \varrho$, for the rescaled field $\phi = e^{\frac{D-2}{2}\eta} (\varphi - \varphi_0)$, we get the following equation

$$\frac{d^2}{d\eta^2} \phi - \frac{(D-2)^2}{4} \phi - \lambda \frac{2D}{D-2} \phi^{\frac{D+2}{D-2}} = 0, \quad (18)$$

which has an obvious first integral.

For the potential

$$V(\varphi) = \lambda (\varphi - \varphi_0)^{\frac{2}{D-2}} + V_0, \quad (19)$$

the equation

$$\ddot{\varphi} + \frac{D-1}{\varrho} \dot{\varphi} - \lambda \frac{2}{D-2} (\varphi - \varphi_0)^{\frac{4-D}{D-2}} = 0, \quad (20)$$

reduces to an autonomous integrable equation, if one replaces ϱ by $\eta = \varrho^{D-2}$ and then for the rescaled field $\phi = \eta (\varphi - \varphi_0)$ the above equation becomes

$$\frac{d^2}{d\eta^2} \phi - \lambda \frac{2}{(D-2)^3} \phi^{\frac{4-D}{D-2}} = 0. \quad (21)$$

The parameters λ and V_0 in the potentials (16) and (19) are arbitrary and can be negative or positive. One could use these potentials to form piecewise exactly solvable bounded and unbounded potentials with false and true vacua for arbitrary dimensions $D > 2$. In particular, the case of $D = 4$, where the exactly solvable potentials are φ^4 and φ , was considered in detail in [5, 6]. For $D = 3$ the exactly solvable potentials are φ^6 and φ^2 .

3 Conclusions

We have considered the instantons for any $D > 2$ and proved that for the unbounded potentials which have no any local minima after a maximum and do not satisfy the condition (2) derived in [1], the instantons satisfying the Coleman boundary conditions (6) do not exist. In these cases, the instability of the false vacuum is described by the new instantons [5, 6, 3]. Moreover, using the class of non-local integrals (7), we have identified two classes of exactly solvable potentials in D dimensions. These potentials can be used to build piecewise exactly solvable potentials with the required properties in any number of dimensions $D > 2$.

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