Derived brackets in bosonic string sigma-model

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Abstract

We study the worldsheet theory of bosonic string from the point of view of the BV formalism. We explicitly describe the derived Poisson structure which arizes when we expand the Master Action near a Lagrangian submanifold. The BV formalism allows us to clarify the mechanism of holomorphic factorization of string amplitudes.

Contents

1 Introduction

In BV formalism we study all gauge fixings simultaneously. The Master Action $S_{\rm BV}$ would probably better be called "Universal Action" because it covers all possible gauge fixings. Gauge fixings correspond to choices of a Lagrangian submanifold L in the BV space. Suppose that we have choosen the Darboux coordinates $\phi^a, \phi^\star_a$ so that L is given by $\phi^\star_a = 0$. In the vicinity of L we can expand $S_{\rm BV}$ as follows:

$$S_{\rm BV} = S_0(\phi) + S_1^a(\phi)\phi^\star_a + S_2^{ab}(\phi)\phi^\star_a\phi^\star_b + \ldots \qquad (1)$$

The coefficients $S_0, S_1, \ldots$ can be equivalently described as ∞-Poisson structures [1], a generalization of Poisson brackets. For functions $F_1(\phi), \ldots F_n(\phi)$ (which depend on ϕ, not $\phi^\star$; they are functions on L) their n-bracket is defined as:

$$\{F_1, \ldots, F_n\} = \{\ldots\{\{S_{\rm BV}, F_1\}_{\rm BV}, F_2\}_{\rm BV}, \ldots F_n\}_{\rm BV}|_{\phi^\star=0} \qquad (2)$$

The derived brackets describe the dependence of $S_{\rm BV}|_L$ on L. Suppose that we deformed L:

$$\phi^\star_a = 0 \stackrel{\rm deform}{\longrightarrow} \phi^\star_a = \frac{\partial\Psi(\phi)}{\partial\phi^a} \qquad (3)$$

Then:

$$S_{\rm BV}|L = S_0 + Q\Psi + \frac{1}{2}\{\Psi, \Psi\} + \frac{1}{6}\{\Psi, \Psi, \Psi\} + \ldots \qquad (4)$$

The Master Equation implies for these brackets some generalized Jacobi identities. It is interesting to observe how they are satisfied in special cases. Especially in the case of bosonic string, where the deformations (3) play crucial role. Indeed, string perturbation theory is defined as integral over families of such deformations [2].

When $S_0 = 0$ the ∞-Poisson structure becomes *strict*. However, it is possible to have a strict Poisson structure with $S_0 \neq 0$. It is sufficient that BV brackets of S_n with S_0 vanish. When a homotopy Poisson structure is not strict, this means that the homotopy Jacobi identities are satisfied only modulo the derivatives of S_0 ("on-shell"). Generally speaking, there is no sense in which the brackets preserve the extremal set of S_0, and therefore no obvious way to turn a non-strict structure into a strict one.

In this paper we will consider the derived Poisson brackets which arize in the worldsheet theory of bosonic string. This theory is of the type "BV-BRST"; the BRST structure corresponds to worldsheet diffeomorphisms, which are gauge symmetries. (In some formulations, there is also Weyl gauge symmetry.) When Lagrangian submanifold is choosen in the usual way, the BRST operator is only nilpotent on-shell, and there is an infinite tower $S_0, S_1^a, S_2^{ab}, \ldots$, generating a homotopy Poisson structure. This "usual" choice of a Lagrangian submanifold is a particular case of the general construction of BRST formalism, where we fix a constraint and choose a Lagrangian submanifold as a conormal bundle of the constraint surface. In this general context, we describe a way to make the homotopy Poisson structure strict, by imposing certain conditions on the Faddeev-Popov ghosts. In the case of bosonic string, it turns out that the differential actually commutes with the derived brackets, and the derived brackets are in some sense constant. We observe that this holds in general, in the BV-BRST formalism, under the condition of vanishing of some cohomological obstacles, see Section 3 and Section 4.

In the case of the bosonic string, the most important example of the deformations of Lagrangian submanifold are those which correspond to variations of the worldsheet complex structure. When the target space is flat, we show that Master Equations imply effective linearization of the deformation of the complex structure, see Section 5. As was explained in [3], this effective linearization plays crucial role in the holomorphic factorization of string amplitudes [4],[5].

2 Brief review of BV-BRST formalism

Bosonic string worldsheet theory belongs to a class of gauge systems which can be called "BV-BRST". It is a particular case of BV formalism, when fields/antifields can be choosen so that the expansion of the Master Action in powers of antifields terminates at the linear term:

$$S_{\rm BV}(\Phi,\Phi^{\star}) = S_{\rm cl}(\Phi) + Q^A(\Phi)\Phi^{\star}_A \tag{5}$$

Moreover, the odd nilpotent vector field Q^A should be of some special form, coming from some gauge symmetry of $S_{\rm cl}(\Phi)$, as we will now review.

2.1 Master Action in BRST case

Let H be the gauge group with gauge Lie algebra $\mathbf{h}$, acting on the space of fields ϕ. We enlarge the space of fields, by adding "ghosts" c parametrizing $\Pi\mathbf{h}$ (the Lie algebra of $\mathbf{h}$ with flipped statistics). The "total" space of fields is now $\Pi\mathbf{h}\times X$, where $\Pi\mathbf{h}$ is parametrized by c and X by ϕ. The Master Action has the form:

$$S_{\rm BV}(\phi,\phi^{\star}) = S_{\rm cl}(\phi) + c^A v^i_A(\phi)\phi^{\star}_i + \frac{1}{2}c^A c^B f^C_{AB} c^{\star}_C \tag{6}$$

In the case of bosonic strings, gauge symmetries are diffeomorphisms, and $\mathbf{h}$ is the algebra of vector fields on the worldsheet.

For quantization we need to restrict $S_{\rm BV}$ to a Lagrangian submanifold, and then take the path integral over the fields which parametrize the Lagrangian submanifold.

2.2 Construction of Lagrangian submanifolds by constraining fields

How do we choose a Lagrangian submanifold? A very naive choice is $\phi^{\star} = 0$, then the restriction would be just $S_{\rm cl}$ — degenerate, can not quantize.

One solution is to use the "conormal bundle" construction, which we will now describe. Let $\mathcal{F}$ denote the space of fields ϕ. Consider some subspace $\mathcal{C}\subset\mathcal{F}$ defined by some equations ("constraints"). We choose:

$$L \;=\; \Pi(T\mathcal{C})^{\perp}\times[c\text{-ghosts}] \;=\; \Pi(T\mathcal{C})^{\perp}\times\Pi\mathbf{h} \tag{7}$$

where $T\mathcal{C}^{\perp}\subset T^*\mathcal{F}$ consists of those linear functionals which vanish on the tangent space to $\mathcal{C}$. This is a Lagrangian submanifold. We want $\mathcal{C}$ to be "sufficiently transverse" to the the gauge orbits, for the resulting action to be sufficiently non-degenerate.

3 Derived Poisson brackets in BRST/BV formalism

Now we will consider the expansion of the Master Action in the vicinity of a Lagrangian submanifold corresponding to a constraint. Such an expansion always defines an ∞-Poisson structure, but in the BRST case we show that it can be made *strict* by imposing certain conditions on ghosts. Moreover, under the condition of vanishing of some cohomological obstacle, the higher brackets can be made essentially constant by a choice of coordinates.

3.1 BV phase space in the vicinity of a constraint surface

Suppose that we can choose coordinates x^m, m^I on $\mathcal{F}$ in the vicinity of $\mathcal{C}$ so that the equation of $\mathcal{C}$ is:

$$m^I = 0 \tag{8}$$

This actually defines a family of surfaces $\mathcal{C}_{(m_0)}$ given by equations:

$$m^I = m_0^I \tag{9}$$

For each $\mathcal{C}_{(m_0)}$, the differentials dm_I form the basis of the fiber of the conormal bundle of $\mathcal{C}_{(m_0)}$. Any element of the fiber can be written as:

$$b_I dm^I \tag{10}$$

The odd symplectic form is:

$$\omega_{\rm BV} \;=\; db_I\, dm^I \;+\; dx^\star_\mu\, dx^\mu \tag{11}$$

The BV action is:

$$\begin{aligned} S_{\rm BV} \; &= S_{\rm cl}(x,m) + c^A v^I_A(x,m) b_I \; + \\ &+ \; c^A v^\mu_A(x,m) x^\star_\mu + \frac{1}{2}[c,c]c^\star \end{aligned} \tag{12}$$

Only the first line contributes to the restriction on $L = \Pi T\mathcal{C} \times \Pi\mathbf{h}$:

$$S_0 = S_{BV}|_L = S_{\rm cl}(x,0) + c^A v^I_A(x,0) b_I \tag{13}$$

3.2 Restriction on ghosts

We allow for the possibility that constraining fields to $\mathcal{C} \subset \mathcal{F}$ does not fix the gauge symmetry completely, but actually fixes it to some subgroup $H_{\mathcal{C}} \subset H$.

For example, in the case of bosonic string, we break diffeomorphisms down to conformal transformations.

We can formally restrict the group of gauge transformations to $H_{\mathcal{C}} \subset H$, in the following way. Consider the subspace $\mathbf{h}_{\mathcal{C}}^\perp \subset \mathbf{h}^*$ consisting of those linear functions which vanish on $\mathbf{h}_{\mathcal{C}}$. Consider the subspace of the space of fields defined by the following constraint:

$$f_A c^A = 0 \quad \forall \quad f \in \mathbf{h}_{\mathcal{C}}^\perp \subset \mathbf{h}^* \tag{14}$$

Notice that Q preserves this constraint, because $\mathbf{h}_{\mathcal{C}}$ is a Lie subalgebra.

At the same time, we consider $c^\star$ mod $\Pi\mathbf{h}_{\mathcal{C}}^\perp$.

This restriction on ghosts might be thought of, in some sense, as a partial on-shell condition (only those EqM which follow from variation over b). We denote the reduced field space $\mathcal{F}_{\rm res}$:

$$\mathcal{F}_{\rm res} \subset \mathcal{F} \tag{15}$$

The subindex "res" stands for "residual", because we are essentially restricting ghosts to the algebra of residual (*i.e.* those remaining after we impose the constraint $\mathcal{C} \subset \mathcal{F}$) gauge transformations.

This restriction on ghosts may be interpreted as a "partial on-shell condition", but only vaguely. Although we have chosen a Lagrangian submanifold L, we are actually working in the BV space (in some vicinity of L). There is no good notion of equations of motion, or "on-shell", in the BV space. What we really use is the fact that ghosts are separate from other fields, as we are in the BRST context. Imposing Eq. (14) is geometrically natural in this context. It is a "BV Hamiltonian reduction" of the BV phase space on the constraint given by Eq. (14), in other words $\Pi T^*\mathcal{F}_{\rm res}$. It happens to coincide with the equations of motion from the variation of b, in restriction to L.

3.3 ∞-Poisson structure

3.3.1 Expansion of $S_{\rm BV}$ in the vicinity of L

We can expand $S_{\rm BV}$ in the vicinity of L:

$$S_{\rm BV} = S_0 + S_1 + S_2 + \ldots \tag{16}$$

where the subindex $0, 1, 2, \ldots$ counts the degree in m (antifield to b) plus the degree in $x^\star$ plus the degree in $c^\star$. Or, once we impose Eq. (14):

$$S_{\rm BV,res} = S_{0,\rm res} + S_{1,\rm res} + S_{2,\rm res} + \ldots \tag{17}$$

Once we impose Eq. (14), the S_0 becomes b-independent (*cp.* Eq. (13)):

$$S_{0,\rm res} = S_{\rm cl}(x, 0) \tag{18}$$

We have, because of the Master Equation:

$$\{S_{0,\rm res}\,,\, S_{1,\rm res}\}_{\rm BV} = 0 \tag{19}$$

Moreover, since $x^\star$ only enters in S_1, and $\frac{\delta}{\delta b} S_{0,\rm res} = 0$, we have:

$$\{S_{0,\rm res}\,,\, S_{n,\rm res}\}_{\rm BV} = 0 \tag{20}$$

Let us denote:

$$Q_{\rm res} = \{S_{1,\rm res}, \_\}_{\rm BV} \tag{21}$$

Then Eq. (20) implies:

$$Q_{\rm res} S_{n,\rm res} + \frac{1}{2}\sum_{k=2}^{n-1} \{S_{k,\rm res}\,,\, S_{n+1-k,\rm res}\}_{\rm BV} = 0 \tag{22}$$

In particular Q_{res} is nilpotent:

$$Q_{\text{res}}^2 = 0 \tag{23}$$

Summary We constructed a tower of brackets π_n on a space parameterized by x, c, b which form an ∞-Poisson structure. The brakets $\pi_{\geq 2}$ only involve derivatives in the b-direction, while $Q = \pi_1$ contains also derivatives w.r.to x and c.

3.4 BRST symmetry

For any Lagrangian submanifold L, the restriction of S_{BV} to L has a fermionic symmetry Q_{BRST} which is defined as follows. Let us introduce Darboux coordinates so that the equation for L is: $\phi_a^\star = 0$. Let us expand S_{BV} in powers of $\phi^\star$:

$$S_{\text{BV}} = S_0(\phi) + Q_{\text{BRST}}^a(\phi)\phi_a^\star + \ldots \tag{24}$$

Master Equation implies that $Q^a(\phi)\frac{\partial}{\partial\phi^a}$ is a symmetry of S_0. It is nilpotent, generally speaking, only on-shell.

3.5 A condition for simplification of ∞-Poisson structure

Consider the Master Action:

$$S_{\text{BV}} \;=\; S_{\text{cl}}(x,m) + c^A v_A^I(x,m) b_I + c^A v_A^\mu(x,m) x_\mu^\star + \frac{1}{2}[c,c]c^\star \tag{25}$$

in the vicinity of the Lagrangian submanifold L given by the equation:

$$m = x^\star = c^\star = 0 \tag{26}$$

We restrict c to a subalgebra of $\mathbf{h}_{\mathcal{C}} \subset \mathbf{h}$ preserving L. Explicitly, the condition on c is:

$$c^A v_A^I(x,0) = 0 \tag{27}$$

Suppose that the following equation is also true, for some x_0:

$$c^A v_A^\mu(x_0,0) = 0 \tag{28}$$

In other words, suppose that $x = x_0$ is a fixed point of $\mathbf{h}_{\mathcal{C}}$. Then we have a *vector field*: $v\langle c\rangle = c^A v_A^I(x,m)\frac{\partial}{\partial m^I} + c^A v_A^\mu(x,m)\frac{\partial}{\partial x^\mu}$ vanishing at $x = m = 0$ and satisfying the Maurer-Cartan equation:

$$[c,c]\frac{\partial}{\partial c}v\langle c\rangle + [v\langle c\rangle, v\langle c\rangle] \tag{29}$$

It **was proven** in [6] that under certain conditions we can adjust the coordinates in such a way that the vector fields v_A become linear in x and m. Potential obstacles are the cohomology groups:

$$H^1\left(\mathbf{h}_{\mathcal{C}}\,,\ \text{Vec}_{\geq 2}(x,m)\right) \tag{30}$$

where $\text{Vec}_{\geq 2}(x,m)$ are polynomial vector fields on (x,m)-space of quadratic and higher order, on which $\mathbf{h}_{\mathcal{C}}$ act by *linear* vector fields v_A. Suppose that the obstacles vanish, and we can indeed make v_A linear in x and m. Then it follows that the b-dependent part of π_n vanishes for $n > 1$. Although higher brackets are nonzero, they are all Q-closed. Moreover, since π_n only contain derivatives in the b direction, they are therefore "essentially constant".

This is what happens in the case of bosonic string, which we will describe in Section 4 — see Eq. (45).

3.6 Changes of coordinates

Let us consider the change of coordinates m (which parametrize the choice of constraints):

$$m^I = m^I(\mu) \tag{31}$$

Then:

$$b_I\, dm^I = b_I \frac{\partial m^I}{\partial \mu^a} d\mu^a = \beta_a d\mu^a$$
$$\text{where} \quad \beta_a = b_I \frac{\partial m^I}{\partial \mu^a}$$

This defines the change of Darboux coordinates from b_I, m^I to β_a, μ^a:

$$db_I\, dm^I = d\beta_a\, d\mu^a \tag{32}$$

4 Bosonic string

We will now apply the general BV/BRST formalism to the case of bosonic string worldsheet theory, following [7],[8],[9]. As a constraint, we will choose fixing the complex structure I of the string worldsheet to some particular "c-number" complex structure $I^{(0)}$. This defines some Lagrangian submanifold as described in Section 2.2. Then in Section 4.1.1 we find some BV Darboux coordinates, adjusted to this Lagrangian submanifold, and study the expansion of the Master Action in powers of antifields. Then in Section 4.2 we change the Darboux coordinates so that one of the antifields becomes the Beltrami differential. We find that the derived brackets have a very particular structure: all higher brackets are essentially constant.

4.1 Polyakov gauge in BV formalism

The bosonic string is described by the Polyakov action

$$S_{cl} = \frac{1}{4\pi\alpha'} \int_\Sigma dx \wedge \star dx = \frac{1}{4\pi\alpha'} \int dx \wedge I \cdot dx, \tag{33}$$

where the dynamical fields are D scalar fields $x^0, \ldots, x^{D-1}$ and a complex structure I on the worldsheet. The complex structure is a section of $\mathrm{End}(T\Sigma)$ satisfying $I^2 = -1$ (the integrability condition is automatically satisfied in two dimensions). It defines the Hodge operator $\star$, which takes the 1-form dx to the 1-form $\star dx$.

The action has gauge symmetry under diffeomorphisms. We therefore consider the diffeomorphism ghosts c (corresponding to all smooth vector fields on the worldsheet), the BRST operator Q, as the generating function $\hat{Q}$ of Q on the BV phase space:

$$\hat{Q} = \int ((\mathcal{L}_c x)x^\star + (\mathcal{L}_c I)I^\star + \frac{1}{2}(\mathcal{L}_c c)c^\star), \tag{34}$$

where $x^\star$, $I^\star$ and $c^\star$ are the antifields. A BV action can be defined as

$$S = S_{cl} + \hat{Q}$$

It satisfies the quantum master equation $\{S, S\} = 0$, where the bracket is the BV bracket.

We will choose the Lagrangian submanifold given by $x^\star = 0$, $c^\star = 0$, and $I = I^{(0)}$ for some fixed complex structure $I^{(0)}$. We will construct the BV Darboux coordinates adjusted to this choice of Lagrangian submanifold, in the following way. The "fields" will be x, c, and $I^\star$. The antifields will be $x^\star$, $c^\star$ and something like $I - I^{(0)}$, see Section 4.1.1.

4.1.1 Odd cotangent bundle of the space of complex structures

Consider the space of 2x2 matrices I satisfying $I^2 = -1$. Let $I^\star$ be a 2x2 matrix parametrizing the fiber of the odd cotangent bundle to the space of I's. The odd symplectic form is:

$$\omega = \operatorname{tr}(dI^\star \wedge dI) \tag{35}$$

with the following gauge symmetry of $I^\star$:

$$\delta_\eta I^\star = \eta I + I\eta \tag{36}$$

Let us gauge fix $I^\star$ to:

$$I^\star = \frac{1}{2}\begin{pmatrix} 0 & i\bar{b} \\ -ib & 0 \end{pmatrix} \tag{37}$$

The fields b and $\bar{b}$ are called "b-ghosts".

A generic I satisfying $I^2 = -1$ can be parametrized by $m \in \mathbf{C}$:

$$I = \begin{pmatrix} I^z_z & I^z_{\bar{z}} \\ I^{\bar{z}}_z & I^{\bar{z}}_{\bar{z}} \end{pmatrix} = \begin{pmatrix} i\sqrt{1+m\bar{m}} & im \\ -i\bar{m} & -i\sqrt{1+m\bar{m}} \end{pmatrix}. \tag{38}$$

Notice that in our notations, the reality conditions are:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} I \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \overline{I} \quad , \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} I^\star \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \overline{I^\star} \tag{39}$$

Eq. (35) becomes:

$$\omega = \frac{1}{2} dm\, db + \frac{1}{2} d\bar{m}\, d\bar{b} \tag{40}$$

Therefore, m and b are Darboux coordinates. Remember that I and $I^\star$ are actually fields on the string worldsheet. Including also x, c and their antifields, we get:

$$\omega = \int \left(\delta x \delta x^\star + \delta c \delta c^\star + \delta\bar{c}\delta\bar{c}^\star + \frac{1}{2}\delta m \delta b + \frac{1}{2}\delta\bar{m}\delta\bar{b} \right).$$

The corresponding odd Poisson bracket is:

$$\begin{aligned} &\{b(z), m(z')\} = 2\delta^2(z - z'), \qquad \{b(z), \bar{m}(z')\} = 0, \\ &\{\bar{b}(z), m(z')\} = 0, \qquad \{\bar{b}(z), \bar{m}(z')\} = 2\delta^2(z - z'). \end{aligned} \tag{41}$$

4.1.2 Expansion of the action

Let us consider the expansion of the Master Action:

$$S_{\rm BV} = S_{\rm cl} + \widehat{Q} \tag{42}$$

around the Lagrangian sumbamifold $x^\star = c^\star = m = 0$. The $S_{\rm cl}$ given by Eq. (33)) is written in terms of the $m, \bar{m}$ fields as:

$$S_{cl} = \int d^2z \left(\sqrt{1+m\bar{m}}\partial x \bar{\partial} x + \frac{1}{2} m (\partial x)^2 + \frac{1}{2}\bar{m}(\bar{\partial}x)^2 \right), \tag{43}$$

and the BRST operator can be written in terms of $m, \bar{m}$ and $b, \bar{b}$ as

$$\begin{aligned}\hat{Q} = & \int d^2z \Big(\mathcal{L}_c x x^\star + \frac{1}{2}\mathcal{L}_c c c^\star + \sqrt{1+m\bar{m}}((\bar{\partial}c)b + (\partial\bar{c})\bar{b}) \; + \\ & + \frac{1}{2}\left((c \overset{\leftrightarrow}{\partial} m) b + \bar{\partial}(\bar{c}m) b + (\bar{c} \overset{\leftrightarrow}{\bar{\partial}} \bar{m})\bar{b} + \partial(c\bar{m})\bar{b} \right) \Big)\end{aligned} \tag{44}$$

The derivation of this formula uses the following expressions for the components of the Lie derivative of the complex structure (see Eq. (38)):

$$\begin{aligned}
(\mathcal{L}_c I)^z{}_z &= ic \cdot \partial(\sqrt{1+m\bar{m}}) + i(\bar{\partial}c)\bar{m} + (\partial\bar{c}) i m \\
(\mathcal{L}_c I)^z{}_{\bar{z}} &= ic \cdot \partial(m) + 2i(\bar{\partial}c)\sqrt{1+m\bar{m}} + i(\bar{\partial}\bar{c} - \partial c) m \\
(\mathcal{L}_c I)^{\bar{z}}{}_z &= -ic \cdot \partial(\bar{m}) - 2i(\partial\bar{c})\sqrt{1+m\bar{m}} - i(\partial c - \bar{\partial}\bar{c})\bar{m} \\
(\mathcal{L}_c I)^{\bar{z}}{}_{\bar{z}} &= -ic \cdot \partial(\sqrt{1+m\bar{m}}) - i(\partial\bar{c}) m - i(\bar{\partial}c)\bar{m}
\end{aligned}$$

If we restrict the ghost field to be a conformal Killing vector, *i.e.* $\bar{\partial}c = \partial\bar{c} = 0$, then m transforms as a tensor $m^z_{\bar{z}}$:

$$\{\widehat{Q}, m\}_{\rm BV} = c \overset{\leftrightarrow}{\partial} m + \bar{\partial}(\bar{c}m) \tag{45}$$

— a linear transformation [1]

Now we are ready to play our "field-antifield flip". We call (x, c, b) fields and $(x^\star, c^\star, m)$ antifields. (We call $b, \bar{b}$ the fields of the Polyakov gauge, and let $m, \bar{m}$ be the antifields.) Then we consider the expansion of $S_{\rm cl} + \widehat{Q}$ around the Lagrangian submanifold where $x^\star$, $c^\star$, m are all zero. We have:

$$S = S_0 + S_1 + S_2 + \cdots,$$

where the term S_k is of kth power on $x^\star$, $c^\star$ and $m, \bar{m}$. The antifields $x^\star$ and $c^\star$ only enter linearly, but the dependence on m and $\bar{m}$ is nonlinear.

The term S_0 is the string worldsheet action in the Polyakov gauge:

$$S_0 = \int d^2z \left(\partial x \bar{\partial} x - b\bar{\partial}c - \bar{b}\partial\bar{c} \right) \tag{46}$$

[1] This is because we do not impose any equations on m; it can be any function of $z, \bar{z}$. Had we imposed, say, some wave equations, obstacles to linearization could have appeared as in [6].

The first order term S_1 is:

$$S_1 = \int d^2z \Big(\mathcal{L}_c x x^\star + \frac{1}{2}\mathcal{L}_c c c^\star + \frac{1}{2} m (\partial x)^2 + \frac{1}{2}\bar{m}(\bar\partial x)^2 - \frac{1}{2} b(c\partial + \bar{c}\bar\partial - (\partial c) + (\bar\partial \bar{c}))m - \frac{1}{2}\bar{b}(c\partial + \bar{c}\bar\partial - (\bar\partial\bar{c}) + (\partial c))\bar{m}\Big). \tag{47}$$

It defines the BRST operator in Polyakov gauge, which generates the BRST symmetry of S_0:

$$\begin{aligned} Q_{\rm BRST} x &= \mathcal{L}_c x \\ Q_{\rm BRST} c &= \frac{1}{2}\mathcal{L}_c c \\ Q_{\rm BRST} b &= -(\partial x)^2 + 2(\partial c) b + (c\partial + \bar{c}\bar\partial) b = -(\partial x)^2 + \mathcal{L}_c b & (48) \\ Q_{\rm BRST} \bar{b} &= -(\bar\partial x)^2 + 2(\bar\partial \bar{c}) \bar{b} + (c\partial + \bar{c}\bar\partial) \bar{b} = -(\bar\partial x)^2 + \overline{\mathcal{L}_c b} & (49) \end{aligned}$$

Notice that $Q_{\rm BRST} b$ contains the term $\bar{c}\bar\partial b$, and $Q_{\rm BRST}\bar{b}$ contains $c\partial\bar{b}$, which both are zero on-shell. But, if we omitted them, $Q_{\rm BRST}$ would not be a symmetry of the action. In this sense, it is not completely accurate to say that BRST variation of b is the energy-momentum tensor.

4.1.3 On-shell restriction of ghosts

The only equations of motion required for the nilpotence of this $Q_{\rm BRST}$ are $\partial\bar{c} = 0$ and $\bar\partial c = 0$. (The derivation of the nilpotence uses $\mathcal{L}_c \partial x = \partial \mathcal{L}_c x$, which is only true when c in a conformal Killing vector field.) Furthermore, if we restrict the ghosts to satisfy $\partial\bar{c} = 0$ and $\bar\partial c = 0$, then m and $\bar{m}$ transform as sections of $T^{1,0} \otimes \Omega^{0,1}$ and $T^{0,1}\otimes\Omega^{1,0}$, respectively, in the following sense:

$$\{S_{\rm BV}, m\} = c \overleftrightarrow{\partial} m + \bar\partial(\bar{c} m) \tag{50}$$

$$\{S_{\rm BV}, \bar{m}\} = \bar{c} \overleftrightarrow{\bar\partial} \bar{m} + \partial(c\bar{m}) \tag{51}$$

In particular $\bar{m}m$ transforms as a scalar:

$$\{S_{\rm BV}, |m|^2\} = (c\partial + \bar{c}\bar\partial)|m|^2 \tag{52}$$

(This is true only when c and $\bar{c}$ are restricted to $\partial\bar{c} = 0$ and $\bar\partial c = 0$.)

4.1.4 Expansion in powers of m

The quadratic terms are:

$$S_2 = \int d^2z \left(\partial x \bar\partial x - b\bar\partial c - \bar{b}\partial\bar{c}\right) \frac{m\bar{m}}{2} \tag{53}$$

The term of the order $2n$ is, when $n \geq 2$:

$$S_{2n} = \int d^2z \left(\partial x\bar\partial x - b\bar\partial c - \bar{b}\partial\bar{c}\right)(-1)^{n+1}\frac{(2n-3)!}{2^{2n-2} n!(n-2)!}(m\bar{m})^n \tag{54}$$

4.2 Beltrami differential

In order to find the Darboux coordinates adjusted to our choice of Lagrangian submanifold, it was useful to parametrize the complex structures by $m, \bar{m}$, see Eq. (38). But it is more useful to work, instead of m, with Beltrami differentials μ, which are holomorphic coordinates on the space of complex structures. We will now change the Darboux coordinates from $m, b, \bar{m}, \bar{b}$ to $\mu, \beta, \bar{\mu}, \bar{\beta}$, and describe the expansion of the Master Action in powers of $\mu, \bar{\mu}$.

4.2.1 Deformation of Dolbeault operators

In $z\bar{z}$ coordinates we can write the complex structure locally as

$$I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

so it has eigenvalues $\pm i$.

We define the Dolbeault operators as

$$\partial := dz\frac{\partial}{\partial z}, \qquad \bar{\partial} := \ d\bar{z}\frac{\partial}{\partial \bar{z}},$$

which satisfy $d = \partial + \bar{\partial}$, where d is the de Rham operator. Their action on the X fields are eigenfunctions of the complex structures, as

$$I\partial X = i\partial X, \qquad I\bar{\partial} X = -i\bar{\partial} X.$$

Then we can write Polyakov action in terms of ∂ and $\bar{\partial}$, which gives

$$S = \frac{-i}{2\pi\alpha'}\int \partial X \wedge \bar{\partial} X,$$

which has the form of the Polyakov action with a flat metric.

To parameterize locally the possible gauge fixing choices we can fix the complex structures to different values. It should be equivalent to fixing the complex structure to be flat in different coordinates.

Let's parameterize the family of complex structures by functions μ and $\bar{\mu}$, and call the complex structure $I^{[\mu]}$. For $\mu = \bar{\mu} = 0$ we get the flat one, $I^{[0]}$. The Dolbeault operators ∂ and $\bar{\partial}$ are eigenvectors of $I^{[0]}$. We can then define deformed Dolbeault operators by

$$\begin{aligned} I^{[\mu]}\partial^{[\mu]} =& i\partial^{[\mu]}, \\ I^{[\mu]}\bar{\partial}^{[\mu]} =& -i\bar{\partial}^{[\mu]}. \end{aligned}$$

We fix their normalization by asking that

$$d = \partial^{[\mu]} + \bar{\partial}^{[\mu]} \tag{55}$$

continues to hold.

The way we want to parameterize $\partial^{[\mu]}$ and $\bar{\partial}^{[\mu]}$ is the following. Define the vector fields

$$
\begin{aligned}
v^{[\mu]} &= \frac{\partial}{\partial z} + \bar{\mu}\frac{\partial}{\partial \bar{z}}, \\
\bar{v}^{[\mu]} &= \frac{\partial}{\partial \bar{z}} + \mu\frac{\partial}{\partial z}.
\end{aligned} \tag{56}
$$

The Dolbeault operators are mixed 2-tensors, so we define them as

$$
\partial^{[\mu]} = u^{[\mu]}v^{[\mu]}, \qquad \bar{\partial}^{[\mu]} = \bar{u}^{[\mu]}\bar{v}^{[\mu]},
$$

for a family of forms $u^{[\mu]}$ and $\bar{u}^{[\mu]}$. Upon the definition of eq. (56), the forms are fixed by Eq. (55). They are given by

$$
u^{[\mu]} = \frac{dz - \mu d\bar{z}}{1 - \mu\bar{\mu}}, \qquad \bar{u}^{[\mu]} = \frac{d\bar{z} - \bar{\mu}dz}{1 - \mu\bar{\mu}}.
$$

The action of the Dolbeault operators on functions is given by:

$$
\begin{aligned}
\partial^{[\mu]} f &= \frac{1}{1 - \mu\bar{\mu}}(dz - \mu d\bar{z})\left(\frac{\partial}{\partial z} + \bar{\mu}\frac{\partial}{\partial \bar{z}}\right) f, \\
\bar{\partial}^{[\mu]} f &= \frac{1}{1 - \mu\bar{\mu}}(d\bar{z} - \bar{\mu}dz)\left(\frac{\partial}{\partial \bar{z}} + \mu\frac{\partial}{\partial z}\right) f.
\end{aligned} \tag{57}
$$

Relation between μ and m The Polyakov action is defined in terms of the complex structure $I^{[\mu]}$ as

$$
S_{cl} = \frac{1}{4\pi\alpha'}\int dx \wedge I^{[\mu]}dx,
$$

where we make the dependence on μ and $\bar{\mu}$ explicit. Then it is written in terms of the Dolbeault operators as

$$
\begin{aligned}
S_{cl} &= \frac{1}{4\pi\alpha'}\int (\partial^{[\mu]}x + \bar{\partial}^{[\mu]}x) \wedge I^{[\mu]}(\partial^{[\mu]}x + \bar{\partial}^{[\mu]}x) \\
&= \frac{-i}{2\pi\alpha'}\int \partial^{[\mu]}x \wedge \bar{\partial}^{[\mu]}x.
\end{aligned}
$$

Using eq. (57) to write the dependence on $\mu, \bar{\mu}$ explicitly, we get

$$
S_{cl} = \frac{-i}{2\pi\alpha'}\int\left(\frac{1 + \mu\bar{\mu}}{1 - \mu\bar{\mu}}\partial x\bar{\partial}x + \frac{dzd\bar{z}}{1 - \mu\bar{\mu}}\left(m\left(\frac{\partial x}{\partial z}\right)^2 + \bar{m}\left(\frac{\partial x}{\partial \bar{z}}\right)^2\right)\right).
$$

Then we compare it with the Polyakov action in terms of the $m, \bar{m}$ coordinates (eq. (43)), which can be written in terms of the Dolbeault operators as

$$
S_{cl} = \frac{-i}{2\pi\alpha'}\left(\sqrt{1 + m\bar{m}}\partial x\bar{\partial}x + \frac{1}{2}dzd\bar{z}\left(m\left(\frac{\partial x}{\partial z}\right)^2 + \bar{m}\left(\frac{\partial x}{\partial \bar{z}}\right)^2\right)\right).
$$

We can then relate the coordinates $m, \bar{m}$ with the coordinates $\mu, \bar{\mu}$. The relation is

$$m = \frac{2\mu}{1-\mu\bar{\mu}} \tag{58}$$

$$\mu = \frac{m}{1+\sqrt{1+|m|^2}}$$

$$I = \frac{i}{1-|\mu|^2}\begin{pmatrix} 1+|\mu|^2 & 2\mu \\ -2\bar{\mu} & -1-|\mu|^2 \end{pmatrix} \tag{59}$$

4.2.2 Beltrami differentials μ are holomorphic coordinates on the space of complex structures

To describe a complex structure on Σ is equivalent to saying which functions are holomorphic. From this point of view, the definition of Beltrami differential μ is very natural. Namely, we say that the function f is holomorphic, if:

$$\left(\frac{\partial}{\partial\bar{z}} + \mu(z,\bar{z})\frac{\partial}{\partial z}\right) f(z,\bar{z}) = 0 \tag{60}$$

Therefore, the definition of holomorphicity depends as a parameter on a complex-valued $\mu(z,\bar{z})$, therefore $\mu(z,\bar{z})$ defines a holomorphic coordinate on the infinite-dimensional space of complex structures.

Notice that m is *not* a holomorphic coordinate. We started with m because it allowed a straightforward construction of Darboux coordinates. But it is better to use μ, because it is a holomorphic coordinate.

In the rest of this Section we will explain how to replace $m, b, \bar{m}, \bar{b}$ with new Darboux coordinates $\mu, \beta, \bar{\mu}, \bar{\beta}$ which agree with the complex structure. It turns out that those simplifications which we observed in coordinates $m, b, \bar{m}, \bar{b}$ persist in $\mu, \beta, \bar{\mu}, \bar{\beta}$.

4.2.3 Expansion in μ

Expansion of $S_{\rm cl}$

$$\begin{aligned} S_{\rm cl} &= \int d^2z \left(\frac{1+|\mu|^2}{1-|\mu|^2}\partial x\bar{\partial}x + \frac{\mu}{1-|\mu|^2}(\partial x)^2 + \frac{\bar{\mu}}{1-|\mu|^2}(\bar{\partial}x)^2\right) = \\ &= \int d^2z \left(\partial x\bar{\partial}x + \sum_{k>0} 2|\mu|^{2k}\partial x\bar{\partial}x + \sum_{k\geq 0}\mu^{k+1}\bar{\mu}^k(\partial x)^2 + \sum_{k\geq 0}\mu^k\bar{\mu}^{k+1}(\bar{\partial}x)^2\right) \end{aligned}$$

Expansion of $\hat{Q}$ It is useful to rewrite Eq. (44) as follows:

$$\begin{aligned} \hat{Q} = \int d^2z \Big(& \mathcal{L}_c x x^\star + \frac{1}{2}\mathcal{L}_c c c^\star + \sqrt{1+m\bar{m}}((\bar{\partial}c)b + (\partial\bar{c})\bar{b}) \; + \\ & + \frac{1}{2}(-\partial c + \bar{\partial}\bar{c})(mb - \bar{m}\bar{b}) \\ & + \frac{1}{2}\left(c(b\partial m + \bar{b}\partial\bar{m}) + \bar{c}(b\bar{\partial}m + \bar{b}\bar{\partial}\bar{m})\right)\Big) \end{aligned} \tag{61}$$

We have:

$$
\begin{aligned}
b \ &= \beta\frac{\partial\mu}{\partial m} + \bar\beta\frac{\partial\bar\mu}{\partial m} = \frac{1}{2}\frac{1-|\mu|^2}{1+|\mu|^2}(\beta - \bar\mu^2\bar\beta) \\
\bar b \ &= \bar\beta\frac{\partial\bar\mu}{\partial\bar m} + \beta\frac{\partial\mu}{\partial\bar m} = \frac{1}{2}\frac{1-|\mu|^2}{1+|\mu|^2}(\bar\beta - \mu^2\beta)
\end{aligned}
$$

In the first line of Eq. (61) we have:

$$
\sqrt{1+|m|^2}((\bar\partial c)b + (\partial\bar c)\bar b) = \frac{1}{2}\left((\bar\partial c)(\beta - \bar\mu^2\bar\beta) + (\partial\bar c)(\bar\beta - \mu^2\beta)\right) \tag{62}
$$

In the middle line:

$$
\frac{1}{2}(-\partial c + \bar\partial\bar c)\left(\beta\left(m\frac{\partial}{\partial m} - \bar m\frac{\partial}{\partial\bar m}\right)\mu + \bar\beta\left(m\frac{\partial}{\partial m} - \bar m\frac{\partial}{\partial\bar m}\right)\bar\mu\right) = \frac{1}{2}(-\partial c + \bar\partial\bar c)(\beta\mu - \bar\beta\bar\mu) \tag{63}
$$

In the last line, we observe:

$$
c(b\partial m + \bar b\partial\bar m) + \bar c(b\bar\partial m + \bar b\bar\partial\bar m) \ = \ c(\beta\partial\mu + \bar\beta\partial\bar\mu) + \bar c(\beta\bar\partial\mu + \bar\beta\bar\partial\bar\mu) \tag{64}
$$

This results in the following expression for $\widehat{Q}$, which is perhaps simpler than expected:

$$
\begin{aligned}
\widehat{Q} \ = \ &\int d^2z\left(\mathcal{L}_c x x^\star + \frac{1}{2}\mathcal{L}_c c c^\star + \frac{1}{2}\left((c \overleftrightarrow{\partial} \mu)\beta + \bar\partial(\bar c\mu)\beta + (\bar c \overleftrightarrow{\bar\partial} \bar\mu)\bar\beta + \partial(c\bar\mu)\bar\beta\right)\right) \ + \\
&+ \int d^2z \ \left(\frac{1}{2}(\bar\partial c)(\beta - \bar\mu^2\bar\beta) + \frac{1}{2}(\partial\bar c)(\bar\beta - \mu^2\beta)\right)
\end{aligned}
$$

After imposing the condition $\bar\partial c = \partial\bar c = 0$, the second line drops out. Therefore, the BRST operator is linear in *both* b, m and β, μ coordinates. [2]

5 Derived brackets and holomorphic factorization

Correlation functions tend to factorize as a product of holomorphic and antiholomorphic function of the moduli. In the language of Bertrami differentials this was explained in [3]. Here we will give a BV explanation of this phenomenon.

5.1 Relation between contact terms and derived bracket

Let Ψ_1 and Ψ_2 be two "gauge fermions", *i.e.* local operators of x, c, b. The corresponding infinitesimal deformations of the action are:

$$
S_0 \longrightarrow S_0 + \epsilon_1\int_\Sigma Q_{\mathrm{BRST}}\Psi_1 + \epsilon_2\int_\Sigma Q_{\mathrm{BRST}}\Psi_2
$$

[2] This actually follows from Eq. (52). Remember that the action of $\{S_{\mathrm{BV}}, \_\}$ is essentially an infinitesimal conformal transformations (with the parameter c). The difference between $m, \bar m$ and $\mu, \bar\mu$ is in a factor, which is a function of $|m|^2$. Eq. (52) says that $|m|^2$ transforms under conformal transformations as a function. Therefore both m and μ transform as sections of $T^{1,0} \otimes \Omega^{0,1}\ \Sigma$.

5.1.1 A relation between quadratic order and OPE of first order deformations

Contact terms are delta-function terms in the OPE $\mathcal{O}_1(z,\bar z)\mathcal{O}_2(0,0)$:

$$\mathcal{O}_1(z,\bar z)\mathcal{O}_2(0,0) = \delta(z,\bar z)\Phi(0,0) + \dots \tag{65}$$

Consider the special case when both operators are BRST-exact: $\mathcal{O}_j = Q_{\rm BRST}\Psi_j$. In this case, the contact term is related to the derived bracket:

$$\Phi \simeq \{\Psi_1,\Psi_2\} + Q_{\rm BRST}(\dots) \tag{66}$$

Indeed:

$$\begin{aligned}(Q_{\rm BRST}\Psi_1(z,\bar z))\;(Q_{\rm BRST}\Psi_2(0,0)) \;&= Q_{\rm BRST}\left(Q_{\rm BRST}\Psi_1(z,\bar z)\;\Psi_2(0,0)\right)\\ &-\;\left(Q_{\rm BRST}^2\Psi_1(z,\bar z)\right)\Psi_2(0,0)\end{aligned}$$

The first term is BRST exact. The second term is proportional to Equations of Motion:

$$Q^2_{\rm BRST}\Psi_1 = \{S_0,\Psi_1\} \tag{67}$$

where $\{\_,\_\}$ is the derived bracket. Under the path integral, this results in the contact term $\{\Psi_1,\Psi_2\}$:

$$\int [d\phi]e^{S_0}\{S_0,\Psi_1\}\Psi_2\dots = \int [d\phi]\left\{e^{S_0},\Psi_1\right\}\Psi_2\dots = \int [d\phi]e^{S_0}\{\Psi_1,\Psi_2\} \tag{68}$$

5.1.2 Contact terms and higher orders

Let us deform the Lagrangian submanifold by letting it flow along $e^{t\{\Psi,\_\}_{\rm BV}}$ where Ψ is some gauge fermion which is a functional of ϕ (and does not contain $\phi^\star$). Then:

$$S_0 \mapsto S_0 + tQ_{\rm BRST}\Psi + \frac{t^2}{2}\{\Psi,\Psi\} + \frac{t^3}{6}\{\Psi,\Psi,\Psi\} + \dots \tag{69}$$

Eq. (66) implies that $t^2\{\Psi,\Psi\}$ cancels against the contact terms in the OPE $tQ_{\rm BRST}\Psi$. Moreover, we will now argue that the total effect of all the contact terms is to cancel $t^2\{\Psi,\Psi\}$, $t^3\{\Psi,\Psi,\Psi\}$, … and replace $t\Psi$ with some $\widetilde{\Psi}(t)$ in the BRST-exact term $Q_{\rm BRST}t\Psi$. Roughly speaking:

$$\exp\left(tQ_{\rm BRST}\Psi + \frac{t^2}{2}\{\Psi,\Psi\} + \frac{t^3}{6}\{\Psi,\Psi,\Psi\} + \dots\right) = {}^{\tt x}_{\tt x}\exp\left(Q_{\rm BRST}\widetilde{\Psi}(t)\right){}^{\tt x}_{\tt x} \tag{70}$$

Here ${}^{\tt x}_{\tt x}\dots{}^{\tt x}_{\tt x}$ means dropping contact terms, as we now explain.

5.1.3 The case of bosonic string

Let us consider a particular case when Ψ is the b-ghost:

$$\Psi = \int mb + \int \bar m\bar b \tag{71}$$

In this case:

$$S_0 \mapsto S_0 + (Q_{\mathrm{BRST}}\Psi)_{(2,0)+(0,2)} + \frac{1}{2}\{\Psi,\Psi\}_{(1,1)} + \frac{1}{6}\{\Psi,\Psi,\Psi,\Psi\}_{(1,1)} + \dots \tag{72}$$

where the lower index stands for the conformal dimension. All higher order terms in the deformation have conformal dimension $(1,1)$, and only the linear term has dimensions $(2,0)$ and $(0,2)$. (As we explained in Section 3.5, this actually happens in general BRST formalism under some cohomological condition.)

5.1.4 CFT considerations

In this case, there is an additional CFT-based argument (inspired by [10]), showing that the second order term $m\bar{m}\partial X\bar{\partial}X$ must indeed be the contact term between $m(\partial X)^2$ and $\bar{m}(\bar{\partial}X)^2$. The argument goes as follows. We know that the deformed theory is conformally inviariant (althought the action of conformal transformations changes, as we change the worldsheet complex structure). Consider the exponential vertex operator:

$$V_k(z,\bar{z}) = \lim_{\epsilon\to 0} \epsilon^{k^2} \exp\left(\int_{D_\epsilon} d^2z\ (k\cdot X(z,\bar{z}))\right) \tag{73}$$

where D_ϵ is the disk $|z|<\epsilon$.

This operator remains finite when we deform the complex structure.

Let us study the effects of turning on μ on this operator. Actually, for the conservation of momentum, we have to insert other vertex operators $V_{p_1},\dots,V_{p_N}$ and consider the coefficient of $\delta(k+p_1+\dots p_N)$. We can consider μ and $\bar{\mu}$ with finite support, localized around the insertion of V_k, so that $\mu=\bar{\mu}=0$ at the points of insertion of $V_{p_1},\dots,V_{p_N}$. When we insert $\int\mu(\partial X)^2\int\bar{\mu}(\bar{\partial}X)^2$, there is logarithmic divergence due to the contact term. It should cancel with the logarithmic divergence from the insertion of $\int\mu\bar{\mu}(\partial X\bar{\partial}X)$.

5.2 Holomorphic factorization

Consider the correlation function:

$$\left\langle {}^{\times}_{\times} e^{p_1 x(z_1,\bar{z}_1)}\cdots e^{p_m x(z_1,\bar{z}_m)} \exp\left(\int d^2z Q_{\mathrm{BRST}}(\mu b+\bar{\mu}\bar{b})\right) {}^{\times}_{\times} \right\rangle \tag{74}$$

It is proportional to $\delta(p_1+\dots+p_m)$. Therefore we can assume that $p_1+\dots+p_m=0$. Then:

$$e^{p_1 x(z_1,\bar{z}_1)}\cdots e^{p_m x(z_1,\bar{z}_m)} = \exp\left(\sum_k \left(p_k \int_{(0,0)}^{z_k,\bar{z}_k} dz\partial x\right)\right) \exp\left(\sum_k \left(p_k \int_{(0,0)}^{z_k,\bar{z}_k} d\bar{z}\bar{\partial} x\right)\right) \tag{75}$$

Under ${}^{\times}_{\times}\dots{}^{\times}_{\times}$, ∂x only talks to T, and $\bar{\partial}x$ only to $\bar{T}$. Therefore we have a product of an expression depending only on μ and an expression depending only on $\bar{\mu}$.

5.3 The procedure of dropping contact terms

We will now explain what we mean by "dropping contact terms".

Suppose that we deform the action: $S_0 \mapsto S_0 + \int d^2z\ \rho(z,\bar{z})\ \mathcal{O}(z,\bar{z})$ where ρ is some function and $\mathcal{O}$ some operators. This may be thought of as introducing $z, \bar{z}$-dependent coupling constants. Consider the correlation function of some operators in the deformed theory:

$$\langle V_1(z_1,\bar{z}_1)\cdots V_m(z_m,\bar{z}_m)\rangle_\rho := \left\langle V_1(z_1,\bar{z}_1)\cdots V_m(z_m,\bar{z}_m)\ \exp\left(\int d^2z\ \rho(z,\bar{z})\ \mathcal{O}(z,\bar{z})\right)\right\rangle_0 \tag{76}$$

where $\langle\ldots\rangle_0$ is the correlation function in the undeformed theory. Let us expand it in powers of ρ. Suppose that the N-th power of the expansion can be written as a sum of multiple integrals:

$$\int d^2z_1\cdots\int d^2z_N\ F_N(z_1,\ldots,z_N,\bar{z}_1,\ldots,\bar{z}_N)\ \rho(z_1,\bar{z}_1)\cdots\rho(z_N,\bar{z}_N)\ + \tag{77}$$

$$\int d^2z_1\cdots\int d^2z_{N-1}\ G_N(z_1,\ldots,z_{N-1},\bar{z}_1,\ldots,\bar{z}_{N-1})\ (\rho(z_1,\bar{z}_1))^2\cdots\rho(z_{N-1},\bar{z}_{N-1})\ + \tag{78}$$

$$\int d^2z_1\cdots\int d^2z_{N-1}\ \tilde{G}_N(z_1,\ldots,z_{N-1},\bar{z}_1,\ldots,\bar{z}_{N-1})\ (\rho(z_1,\bar{z}_1)\partial\rho(z_1,\bar{z}_1))\cdots\rho(z_{N-1},\bar{z}_{N-1})\ + \tag{79}$$

$$\int d^2z_1\cdots\int d^2z_{N-2}\ H_N(z_1,\ldots,z_{N-2},\bar{z}_1,\ldots,\bar{z}_{N-2})\ (\rho(z_1,\bar{z}_1))^3\cdots\rho(z_{N-2},\bar{z}_{N-2})\ + \tag{80}$$

$$\ldots$$

Then we define

$$\left\langle V_1(z_1,\bar{z}_1)\cdots V_m(z_m,\bar{z}_m)\exp\left(\int d^2z\ \rho(z,\bar{z})\ \mathcal{O}(z,\bar{z})\right)\right\rangle^{\text{top}} := \\ = \Sigma_N \int d^2z_1\cdots\int d^2z_N\ F_N(z_1,\ldots,z_N,\bar{z}_1,\ldots,\bar{z}_N)\ \rho(z_1,\bar{z}_1)\cdots\rho(z_N,\bar{z}_N)$$

In other words, we pick only the terms with the maximal number of integrations over the positions of ρ. Equivalently, we expand the exponential in powers of $\rho(z,\bar{z})$ and drop the delta-functions in the OPEs $\mathcal{O}\mathcal{O}$ and $V\mathcal{O}$:

$$\left\langle V_1(z_1,\bar{z}_1)\cdots V_m(z_m,\bar{z}_m)\exp\left(\int d^2z\ \rho(z,\bar{z})\ \mathcal{O}(z,\bar{z})\right)\right\rangle^{\text{top}} := \\ = \left\langle {}^{\times}_{\times} V_1(z_1,\bar{z}_1)\cdots V_m(z_m,\bar{z}_m)\exp\left(\int d^2z\ \rho(z,\bar{z})\ \mathcal{O}(z,\bar{z})\right) {}^{\times}_{\times}\right\rangle$$

5.4 BRST invariance

Since Q_{BRST} is a local symmetry of S_0, this prescription is BRST-invariant:

$$\left\langle (Q_{\text{BRST}}V_1)(z_1,\bar{z}_1)\cdots V_m(z_m,\bar{z}_m)\exp\left(\int d^2z\;\rho(z,\bar{z})\;\mathcal{O}(z,\bar{z})\right)\right\rangle^{\text{top}} +$$
$$\ldots +$$
$$\left\langle V_1(z_1,\bar{z}_1)\cdots (Q_{\text{BRST}}V_m)(z_m,\bar{z}_m)\exp\left(\int d^2z\;\rho(z,\bar{z})\;\mathcal{O}(z,\bar{z})\right)\right\rangle^{\text{top}} +$$
$$\left\langle V_1(z_1,\bar{z}_1)\cdots V_m(z_m,\bar{z}_m)\int d^2z\;\rho(z,\bar{z})\;Q_{\text{BRST}}\mathcal{O}(z,\bar{z})\;\exp\left(\int d^2z\;\rho(z,\bar{z})\;\mathcal{O}(z,\bar{z})\right)\right\rangle^{\text{top}} =$$
$$= 0$$

Moreover, when $\mathcal{O} = Q_{\text{BRST}}\Psi$, the deformed correlation function

$$\langle V_1\cdots V_m\rangle_{\text{deformed}} \;:=\; \left\langle V_1\cdots V_m\exp\left(\int d^2z\;\rho(z,\bar{z})\;\mathcal{O}(z,\bar{z})\right)\right\rangle^{\text{top}} \tag{81}$$

is BRST-invariant:

$$\langle Q_{\text{BRST}}V_1\cdots V_m\rangle_{\text{deformed}} + \ldots + \langle V_1\cdots Q_{\text{BRST}}V_m\rangle_{\text{deformed}} \;=\; 0 \tag{82}$$

This follows from the BRST invariance of $\langle\ldots\rangle^{\text{top}}$ and the fact that Q_{BRST} is nilpotent on-shell. They key point is that the terms proportional to the equations of motion make multiple integrals collapse to the lower multiple integrals, and therefore do not contribute to the $\langle\cdots\rangle^{\text{top}}$.

5.5 Relation between prescriptions

Exists operator $\mathcal{O}_\rho$:

$$\mathcal{O}_\rho(z,\bar{z}) = \rho(z,\bar{z})\mathcal{O}(z,\bar{z}) + O(\rho^2) \tag{83}$$

and a map $\mathcal{W}_\rho$ on the space of operators:

$$V \mapsto \mathcal{W}_\rho[V]$$
$$\mathcal{W}_\rho[V] = V + O(\rho)$$

such that:

$$\left\langle V_1\cdots V_m\exp\left(\int d^2z\;\rho(z,\bar{z})\;\mathcal{O}(z,\bar{z})\right)\right\rangle^{\text{top}} =$$
$$= \left\langle \mathcal{W}_\rho[V_1](z_1,\bar{z}_1)\cdots\mathcal{W}_\rho[V_m](z_m,\bar{z}_m)\exp\left(\int d^2z\;\mathcal{O}_\rho(z,\bar{z})\right)\right\rangle \tag{84}$$

Both $\mathcal{O}_\rho(z,\bar{z})$ and $\mathcal{W}_\rho[V](z,\bar{z})$ depend on $\rho(z,\bar{z})$ and its derivatives with respect to z, $\bar{z}$. The BRST operator induces a map Q_ρ on the space of operators such that:

$$Q_\rho\mathcal{W}_\rho[V] = \mathcal{W}_\rho[Q_{\text{BRST}}V]$$
$$Q_\rho\mathcal{W}_\rho[V] = Q_{\text{BRST}}V + O(\rho)$$

If $\mathcal{O} = Q_{\rm BRST}\Psi$, then by construction Q_ρ is a symmetry of the correlation functions, and therefore it corresponds to a conserved charge. The Q_ρ is the BRST transformation (defined in Section 3.4) of the theory deformed by the insertion of $\exp\left(\int d^2z\, \mathcal{O}_\rho(z,\bar{z})\right)$ — the second line of Eq. (84).

5.6 Explicit computation using Wick theorem

We will now do an explicit computation for the matter part of the theory.

5.6.1 Action and propagator

The undeformed path integral is:

$$\int [dx] \exp\left(-\frac{1}{\pi\alpha'}\int d^2z\, (\bar{\partial}x\partial x)\right) = \int [dx] \exp\left(-\frac{1}{2}\frac{1}{\pi\alpha'}\int d^2z\, (\partial x, \bar{\partial}x)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \partial x \\ \bar{\partial}x \end{pmatrix}\right) \tag{85}$$

With this action:

$$\left\langle \bar{\partial}x(z,\bar{z})\partial x(0,0)\right\rangle = \frac{\pi\alpha'}{2}\delta^2(z,\bar{z}) \tag{86}$$

with notations: $z = x + iy$, $\delta^2(z,\bar{z}) = \delta(x)\delta(y)$.

$$\left\langle \begin{pmatrix} \partial x(z,\bar{z}) \\ \bar{\partial}x(z,\bar{z}) \end{pmatrix} \big(\partial x(0,0)\,,\, \bar{\partial}x(0,0)\big)\right\rangle_{\texttt{contact terms}} = \frac{\pi\alpha'}{2}\begin{pmatrix} 0 & \delta^2(z,\bar{z}) \\ \delta^2(z,\bar{z}) & 0 \end{pmatrix} \tag{87}$$

Complex structure deformation corresponds to the insertion of:

$$\exp\left(-\frac{1}{2}\frac{1}{\pi\alpha'}\int d^2z\, (\partial x, \bar{\partial}x)\begin{pmatrix} m & -1+\sqrt{1+|m|^2} \\ -1+\sqrt{1+|m|^2} & \bar{m} \end{pmatrix}\begin{pmatrix} \partial x \\ \bar{\partial}x \end{pmatrix}\right) \tag{88}$$

5.6.2 Normal ordering of exponentials of quadratic expressions

Consider a linear space with coordinates $x^1, \ldots, x^N$ and a symmetric matrices A and G. Notice that:

$$\exp\left(\frac{1}{2}G^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}\right)\exp\left(-\frac{1}{2}A_{ij}x^ix^j\right) = \exp\left(-\frac{1}{2}[A(\mathbf{1}+GA)^{-1}]_{ij}x^ix^j\right) \tag{89}$$

$$\begin{aligned} A &= \frac{1}{\pi\alpha'}\begin{pmatrix} m & -1+\sqrt{1+|m|^2} \\ -1+\sqrt{1+|m|^2} & \bar{m} \end{pmatrix} \\ G &= \frac{\pi\alpha'}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

$$\left(1 + \frac{1}{2}\begin{pmatrix} -1+\sqrt{1+|m|^2} & \bar{m} \\ m & -1+\sqrt{1+|m|^2} \end{pmatrix}\right)^{-1} =$$

$$= \begin{pmatrix} 2 & -\frac{2\bar{m}}{1+\sqrt{1+|m|^2}} \\ -\frac{2m}{1+\sqrt{1+|m|^2}} & 2 \end{pmatrix}$$

$$
\begin{aligned}
& A\left(1+\frac{1}{2}\begin{pmatrix} -1+\sqrt{1+|m|^2} & \bar{m} \\ m & -1+\sqrt{1+|m|^2} \end{pmatrix}\right)^{-1} = \\
&= \frac{2}{\pi\alpha'}\begin{pmatrix} \frac{m}{1+\sqrt{1+|m|^2}} & 0 \\ 0 & \frac{\bar{m}}{1+\sqrt{1+|m|^2}} \end{pmatrix} \\
&= \frac{2}{\pi\alpha'}\begin{pmatrix} \frac{-1+\sqrt{1+|m|^2}}{\bar{m}} & 0 \\ 0 & \frac{-1+\sqrt{1+|m|^2}}{m} \end{pmatrix}
\end{aligned}
\tag{90}
$$

Therefore:

$$
\exp\left[-\frac{1}{\pi\alpha'}\int d^2z\left((-1+\sqrt{1+|m|^2})(\bar{\partial}x\partial x)+\frac{1}{2}m(\partial x)^2+\frac{1}{2}\bar{m}(\bar{\partial}x)^2\right)\right] =
$$

$$
= {}^{\times}_{\times}\exp\left[-\frac{1}{\pi\alpha'}\int d^2z\left(\frac{m}{1+\sqrt{1+|m|^2}}(\partial x)^2+\frac{\bar{m}}{1+\sqrt{1+|m|^2}}(\bar{\partial}x)^2\right)\right]{}^{\times}_{\times} \tag{91}
$$

$$
= {}^{\times}_{\times}\exp\left[-\frac{1}{\pi\alpha'}\int d^2z\left(\mu(\partial x)^2+\bar{\mu}(\bar{\partial}x)^2\right)\right]{}^{\times}_{\times} \tag{92}
$$

where ${}^{\times}_{\times}\ldots{}^{\times}_{\times}$ means dropping contact terms, and μ is defined in Eq. (56). The appearance of μ in Eq. (92) can be understood in the following way. In the deformed theory, we know that there should be no contact terms in the OPE of $\partial x+\bar{\mu}\bar{\partial}x$ with $\partial x+\bar{\mu}\bar{\partial}x$. (This is because $\partial^{[\mu]}x = f(z,\bar{z})(\partial+\bar{\mu}\bar{\partial})x$.) The contractions between $\bar{\mu}\bar{\partial}x$ and ∂x cancel against the contraction with $\frac{1}{2}\bar{\mu}(\bar{\partial}x)^2$ brought down from the action. [3]

Eq (91) shows that not only at the second order in m, but to all orders, the term $(\bar{\partial}x\partial x)$ can be interpreted as the effect of contact terms. Notice that Eq. (89) can be "inverted":

$$
\exp\left(-\frac{1}{2}A_{ij}x^ix^j\right) = \exp\left(-\frac{1}{2}G^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}\right)\exp\left(-\frac{1}{2}[A(\mathbf{1}+GA)^{-1}]_{ij}x^ix^j\right) \tag{95}
$$

5.7 Summary

The worldsheet metric depends on the complex structure, which can be parametrized by the Beltrami differentials μ and $\bar{\mu}$. Here we have shown that the dependence of correlation functions on μ and $\bar{\mu}$ can be effectively computed by inserting the exponential of a linear function of μ and $\bar{\mu}$:

$$
\exp(\int \mu T + \bar{\mu}\overline{T}) \tag{96}
$$

[3] Essentially, we used:

$$
\det\begin{pmatrix} \sqrt{1+|m|^2} & \bar{m} \\ m & \sqrt{1+|m|^2} \end{pmatrix} = 1 \tag{93}
$$

For any 2×2 matrix U such that $\det U = 1$:

$$
\mathrm{tr}\frac{\mathbf{1}-U}{\mathbf{1}+U} = 0 \tag{94}
$$

implying the vanishing of the off-diagonal terms in Eq. (90). Notice that the transformation $U\mapsto\frac{\mathbf{1}-U}{\mathbf{1}+U}$ squares to identity.

and dropping some contact terms. We explained this simplification from the point of view of BV/BRST formalism.

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