SYMPLECTIC ASPECTS OF THE TT*-TODA EQUATIONS

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ABSTRACT. We evaluate explicitly, in terms of the asymptotic data, the ratio of the constant pre-factors in the large and small x asymptotics of the tau functions for global solutions of the tt*-Toda equations. This constant problem for the sinh-Gordon equation, which is the case n=1 of the tt*-Toda equations, was solved by C. A. Tracy [18]. We also introduce natural symplectic structures on the space of asymptotic data and on the space of monodromy data for a wider class of solutions, and show that these symplectic structures are preserved by the Riemann-Hilbert correspondence.

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1. Introduction

Painlevé equations may be formulated as Hamiltonian systems. This has led to an important role in the theory of such equations for concepts from classical mechanics and symplectic geometry, such as canonical coordinates, tau-functions, and moduli spaces of solutions with symplectic structures. The benefit of the symplectic point of view is that it illuminates a path to the study of more general nonlinear differential equations, especially those which are "integrable".

The Painlevé equations themselves are scalar ordinary differential equations of second order, and this facilitates explicit calculations. For systems, or higher-order equations, geometry plays a more essential role.

Indeed, a considerable amount of general theory has been developed, for example by Hitchin [8], Boalch [1], building on earlier work of Schlesinger [17], Jimbo-Miwa-Ueno [12]. On the other hand examples are rather scarce, partly because of the difficulty of carrying out explicit calculations, and partly because of the lack of interesting concrete example for higher rank systems.

The purpose of this article is to explain some symplectic aspects of the tt*-Toda equations, a system of nonlinear ordinary differential equation of "Painlevé type", which is a relatively recent example arising in physics. The tt* equations

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(topological—anti topological fusion equations) arose in the work of Cecotti and Vafa on supersymmetric quantum field theory ([2], [3]), and the tt*-Toda equations are a special case of these equations of "Toda type". The simplest nontrivial case of the tt*-Toda equations is the (radial) sinh-Gordon equation, which is in fact a case of the third Painlevé equation. It was investigated – for similar physical reasons – by McCoy-Tracy-Wu [13], and their work had far-reaching consequences.

More recently, the tt*-Toda equations were investigated in detail by Guest-Its-Lin [6, 7] and by Mochizuki [14, 15], and our motivation was to put some of these results into a symplectic context and investigate them further. We have succeeded to do this only for a certain subset of solutions – an open subset of the moduli space of all solutions – but the results are encouraging, and have already led to a new application, which we shall explain later.

The paper is organized as follows. After a brief review of the tt*-Toda equations in section 2, we give their Hamiltonian formulation in section 3. In section 4 we explain the symplectic structures on the space of solutions that we consider and on a corresponding space of monodromy data. The correspondence between solutions (asymptotic data) and monodromy data (Stokes matrices and connection matrices) is an example of the Riemann-Hilbert correspondence for meromorphic connections with irregular singularities. Our first main result (Theorem 4) is that this correspondence preserves the symplectic structures. This is consistent with the general results of Boalch [1], but we shall go further and give an explicit generating function which relates the corresponding canonical coordinates (Theorem 7).

In section 5 we give an application of these results to the asymptotics of the tau function. For each solution of the tt*-Toda equation there is a corresponding tau function, and it is the properties these tau functions (rather than the solutions themselves) which are important for many applications in physics.

A recent example is the work of Its, Lisovyy, and Tykhyy [10], in which the structure of the tau functions was elucidated using representation theory (conformal blocks), as a consequence of AGT duality in physics. This was used to solve the "constant problem" for Painlevé equations, i.e. the problem of finding the constant which relates the short-distance and long-distance expansions of the tau function. In the case of the (radial) sine-Gordon equation, a rigorous and more direct proof was given by Its and Prokhorov [11]. Our second main result makes use of their method in order to solve this "constant problem" for the tt*-Toda equations. As in [11], we find that the (explicit) generating function plays a crucial role (Theorem 11)

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2. The TT*-Toda equations

Let a positive integer n be fixed. The tt*-Toda equations are

(2.1)
$$2(w_i)_{t\bar{t}} = -e^{2(w_{i+1} - w_i)} + e^{2(w_i - w_{i-1})}, \ w_i : \mathbb{C}^* \to \mathbb{R}, \ i \in \mathbb{Z},$$

where, for all i, $w_i = w_{i+n+1}$, $w_i = w_i(|t|)$ $(t \in \mathbb{C}^*)$, and

$$w_0 + w_n = 0, w_1 + w_{n-1} = 0, \dots$$
 (anti-symmetry condition).

The equations (2.1) are equivalent to the flatness of $\nabla := d + \alpha$, i.e. the zero curvature equation $d\alpha + \alpha \wedge \alpha = 0$, where

$$\alpha := \left(w_t + \frac{1}{\lambda}W^T\right)dt + \left(-w_{\bar{t}} + \lambda W\right)d\bar{t},$$

$$w = \begin{pmatrix} w_0 & & & \\ & \ddots & \\ & & w_n \end{pmatrix}, W = \begin{pmatrix} 0 & e^{w_1 - w_0} & & \\ & 0 & \ddots & \\ & & \ddots & e^{w_n - w_{n-1}} \\ e^{w_0 - w_n} & & 0 \end{pmatrix}.$$

The tt*-Toda equations are also equivalent to the isomonodromy condition for the following ordinary differential equation

(2.2)
$$\frac{d\Psi}{d\zeta} = \left(-\frac{1}{\zeta^2}W - \frac{1}{\zeta}xw_x + x^2W^T\right)\Psi,$$

where x := |t|.

Generically, the local solutions near x = 0 of the tt*-Toda equations are parametrized by real numbers γ_i , ρ_i as follows [7, 14, 15]:

(2.3)
$$2w_i(x) = \gamma_i \log x + \rho_i + o(1) \text{ as } x \to 0.$$

We call the parameters γ_i, ρ_i the asymptotic data. "Generically" means $-2 < \gamma_{i+1} - \gamma_i < 2$; the general case has $-2 \le \gamma_{i+1} - \gamma_i \le 2$. We assume the generic condition from now on.

There is another important set of data m_i , $e_i^{\mathbb{R}}$ called the monodromy data. These are eigenvalues of certain matrices M and E, which are related to monodromy data such as Stokes matrices. See [7] or the appendix for details. The proof in [7] is for the case n=3, but exactly the same method provides the results of Theorem 1 and 2 below for general n.

Theorem 1. [7] The monodromy data m_i , $e_i^{\mathbb{R}}$ may be expressed in terms of the asymptotic data as follows:

$$\begin{split} m_i &= -\frac{1}{2} \gamma_i \\ e_i^{\mathbb{R}} &= \begin{cases} e^{\rho_i} 2^{2\gamma_i} \frac{X_{n-i}(\gamma_0, \dots, \gamma_{(n-1)/2}, -\gamma_{(n-1)/2}, \dots, -\gamma_0)}{X_i(\gamma_0, \dots, \gamma_{(n-1)/2}, -\gamma_{(n-1)/2}, \dots, -\gamma_0)} & n: odd \\ e^{\rho_i} 2^{2\gamma_i} \frac{X_{n-i}(\gamma_0, \dots, \gamma_{(n-2)/2}, 0, -\gamma_{(n-2)/2}, \dots, -\gamma_0)}{X_i(\gamma_0, \dots, \gamma_{(n-2)/2}, 0, -\gamma_{(n-2)/2}, \dots, -\gamma_0)} & n: even \end{cases} \end{split}$$

where

$$X_k(\gamma_0, \dots, \gamma_n) := \prod_{j=1}^n \Gamma(\frac{\gamma_k - \gamma_{k+j} + 2j}{2(n+1)}) \ (\gamma_{j+n+1} = \gamma_j).$$

Global solutions can be parametrized only by the γ_i (or only by the m_i), that is, for global solutions the ρ_i are determined by the γ_i :

Theorem 2. [7] For global solutions (i.e. solutions which are smooth for $0 < x < \infty$) we have

$$\rho_i = -(2\log 2)\gamma_i + \log(X_i/X_{n-i}),$$

i.e. $e_i^{\mathbb{R}} = 1$.

3. The Hamiltonian formulation

Next, we introduce a Hamiltonian function and a symplectic form.

Let $\lfloor x \rfloor := \max\{n \in \mathbb{Z} : n \leq x\}$ for $x \in \mathbb{R}$. The tt*-Toda equations can be written as a non-autonomous Hamiltonian system,

(3.1)
$$(w_i)_x = \frac{\partial H}{\partial \tilde{w}_i} = \frac{\tilde{w}_i}{x}$$

$$(3.2) (\tilde{w}_i)_x = -\frac{\partial H}{\partial w_i} = -2x \left(e^{2(w_{i+1} - w_i)} - e^{2(w_i - w_{i-1})} \right),$$

on the phase space $\mathbb{R}^{2\lfloor (n-1)/2\rfloor+2}=\{(w,\tilde{w})\}\ (w=(w_0,\ldots,w_{\lfloor (n-1)/2\rfloor}),\ \tilde{w}=(\tilde{w}_0,\ldots,\tilde{w}_{\lfloor (n-1)/2\rfloor}))$ equipped with the symplectic structure

$$\theta := \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} dw_i \wedge d\tilde{w}_i$$

where the Hamiltonian H is defined by

$$H(w, \tilde{w}; x) := \frac{1}{2x} \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \tilde{w}_i^2 - x \sum_{i=1}^{\lfloor (n-1)/2 \rfloor} e^{2(w_i - w_{i-1})} - \frac{x}{2} \left(e^{-4w_{\lfloor (n-1)/2 \rfloor}} + e^{4w_0} \right).$$

The symplectic form θ is asymptotic to $\sum_{i=0}^{\lfloor (n-1)/2 \rfloor} d(\rho_i/2) \wedge d(\gamma_i/2)$ as $x \to 0$.

Remark 3. The Hamiltonian system may be written in terms of $X := \log x$ as follows:

$$\begin{split} H(w,\tilde{w};X) &:= \frac{1}{2e^X} \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \tilde{w}_i^2 - e^X \sum_{i=1}^{\lfloor (n-1)/2 \rfloor} e^{2(w_i - w_{i-1})} - \frac{e^X}{2} \left(e^{-4w_{\lfloor (n-1)/2 \rfloor}} + e^{4w_0} \right). \\ & (w_i)_X = \frac{\partial e^X H}{\partial \tilde{w}_i} = \tilde{w}_i \\ & (\tilde{w}_i)_X = -\frac{\partial e^X H}{\partial w_i} = -2e^{2X} \left(e^{2(w_{i+1} - w_i)} - e^{2(w_i - w_{i-1})} \right). \end{split}$$

4. Symplectic structures and the Riemann-Hilbert correspondence

Both the asymptotic data γ_i , ρ_i and the monodromy data m_i , $\log e_i^{\mathbb{R}}$ can be considered as defining local charts of the moduli space of solutions. From Theorem 1 we can show that the transformation between two charts via the Riemann-Hilbert correspondence is symplectic with respect to the "obvious" symplectic structure. The symplectic form 2θ we define in section 3 is asymptotic to the left hand side of the equality below as $x \to 0$.

Theorem 4.

$$-\frac{1}{2} \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} d\gamma_i \wedge d\rho_i = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} dm_i \wedge d\log e_i^{\mathbb{R}}.$$

Remark 5. The left hand side is related to the Kirillov-Kostant form on a coadjoint orbit, and the right hand side is related to the Atiyah-Hitchin form on the space of the based rational maps of degree n+1 from $\mathbb{C}P^1$ to itself. Thus, both symplectic forms arise naturally from geometry. We shall present details of these facts elsewhere.

Theorem 4 can be verified by direct calculation, but we prefer to give a proof by showing the existence of a generating function. The generating function will play an important role later.

Definition 6. Let

(4.1)

$$F(\rho_0, \dots, \rho_{\lfloor (n-1)/2 \rfloor}, m_0, \dots, m_{\lfloor (n-1)/2 \rfloor}) := -\sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \rho_i m_i + 2\log 2 \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} m_i^2$$

$$+ \frac{n+1}{2} \sum_{k=0}^{n} \sum_{j=1}^{n} \psi^{(-2)} \left(\frac{m_{k-j} - m_k + j}{n+1} \right)$$

where $m_{j+n+1} = m_j$ and $m_j = -m_{n-j}$. Here $\psi^{(-2)}(z) = \int_0^z \log \Gamma(x) dx = \frac{z(1-z)}{2} + \frac{z}{2} \log 2\pi + z \log \Gamma(z) - \log G(1+z)$, and G is the Barnes G-function.

Theorem 7. The function F is a generating function of the transformation

$$(m_0, \dots, m_{\lfloor (n-1)/2 \rfloor}, \rho_0, \dots, \rho_{\lfloor (n-1)/2 \rfloor}) \mapsto (m_0, \dots, m_{\lfloor (n-1)/2 \rfloor}, \log e_0^{\mathbb{R}}, \dots, \log e_{\lfloor (n-1)/2 \rfloor}^{\mathbb{R}})$$

with respect to the given symplectic forms. More precisely, F satisfies

$$m_i = -\frac{\partial F}{\partial \rho_i}, \ \log e_i^{\mathbb{R}} = -\frac{\partial F}{\partial m_i}.$$

Proof. The first identity is obvious. We show the second identity. Let

$$\tilde{K}(m_0,\ldots,m_n) := (n+1) \sum_{i=0}^n \sum_{j=1}^n \psi^{(-2)}(\frac{m_i - m_{i+j} + j}{n+1}),$$

where $m_{j+n+1} = m_j$. Let

$$K(m_0,\ldots,m_{\lfloor (n-1)/2\rfloor}):=\frac{1}{2}\tilde{K}(m_0,\ldots,m_{\lfloor (n-1)/2\rfloor},-m_{\lfloor (n-1)/2\rfloor},\ldots,-m_0).$$

This K is the last term of F in (4.1). From the definition of $\log e_i^{\mathbb{R}}$ and F, it suffices to show that

$$\frac{\partial K}{\partial m_k}(m_0, \dots, m_{\lfloor (n-1)/2 \rfloor})$$

$$= \log \left(\frac{X_k(m_0, \dots m_{\lfloor (n-1)/2 \rfloor}, -m_{\lfloor (n-1)/2 \rfloor}, \dots, -m_0)}{X_{n-k}(m_0, \dots m_{\lfloor (n-1)/2 \rfloor}, -m_{\lfloor (n-1)/2 \rfloor}, \dots, -m_0)} \right).$$

We can easily obtain that $X_{n-k}(-m_n, \ldots, -m_0) = \prod_{j=1}^n \Gamma(\frac{-m_k + m_{k-j} + j}{n+1})$ and that $\tilde{K}(m_0, \ldots, m_n) = (n+1) \sum_{k=0}^n \sum_{j=1}^n \psi^{(-2)}(\frac{m_{k-j} - m_k + j}{n+1})$. Then we obtain

$$\frac{\partial \tilde{K}}{\partial m_k} = \log(X_k(m_0, \dots, m_n) / X_{n-k}(-m_n, \dots, -m_0)).$$

Hence we have

$$\frac{\partial K}{\partial m_k}(m_0, \dots, m_{\lfloor (n-1)/2 \rfloor})$$

$$= \frac{1}{2} \left(\frac{\partial \tilde{K}}{\partial m_k}(m_0, \dots, m_{\lfloor (n-1)/2 \rfloor}, -m_{\lfloor (n-1)/2 \rfloor}, \dots, -m_0) \right)$$

$$- \frac{\partial \tilde{K}}{\partial m_{n-k}}(m_0, \dots, m_{\lfloor (n-1)/2 \rfloor}, -m_{\lfloor (n-1)/2 \rfloor}, \dots, -m_0) \right)$$

$$= \frac{1}{2} (\log X_k(m_0, \dots, m_{\lfloor (n-1)/2 \rfloor}, -m_{\lfloor (n-1)/2 \rfloor}, \dots, -m_0)$$

$$- \log X_{n-k}(m_0, \dots, m_{\lfloor (n-1)/2 \rfloor}, -m_{\lfloor (n-1)/2 \rfloor}, \dots, -m_0)$$

$$- \log X_n(m_0, \dots, m_{\lfloor (n-1)/2 \rfloor}, -m_{\lfloor (n-1)/2 \rfloor}, \dots, -m_0)$$

$$+ \log X_k(m_0, \dots, m_{\lfloor (n-1)/2 \rfloor}, -m_{\lfloor (n-1)/2 \rfloor}, \dots, -m_0)$$

$$- \log X_{n-k}(m_0, \dots, m_{\lfloor (n-1)/2 \rfloor}, -m_{\lfloor (n-1)/2 \rfloor}, \dots, -m_0)$$

$$- \log X_{n-k}(m_0, \dots, m_{\lfloor (n-1)/2 \rfloor}, -m_{\lfloor (n-1)/2 \rfloor}, \dots, -m_0)$$

This completes the proof.

5. Tau functions and the constant problem

In this section, we assume for simplicity that n=3, so $w=(w_0,w_1,w_2,w_3)$ with $w_2=-w_1$, $w_3=-w_0$. For general n the same method applies. We consider only the global solutions, i.e., we assume $\log e_i^{\mathbb{R}}=0$, which means that the ρ_i 's are determined by the γ_j 's, as in Theorem 2. The following calculation is motivated by Theorem 1 in [11]. See also [16] for further details.

Definition 8. Let us define the tau function of a global solution w by

$$\log \tau^{w}(x_{1}, x_{2}) = \int_{x_{1}}^{x_{2}} H(w_{i}(x), \tilde{w}_{i}(x), x) dx$$

where H is the Hamiltonian function.

Remark 9. Usually the tau function is defined (up to a multiplicative constant) by $\log \tau^w(x) = \int^x H(w_i(x), \tilde{w}_i(x), x) dx$. In that notation we have $\tau^w(x_1, x_2) = \tau^w(x_2)/\tau^w(x_1)$.

The Hamiltonian function is

$$H(x, w_0, w_1, \tilde{w}_0, \tilde{w}_1) = \frac{1}{2x} (\tilde{w}_0^2 + \tilde{w}_1^2) - xe^{2(w_1 - w_0)} - \frac{x}{2} \left(e^{-4w_1} + e^{4w_0} \right)$$

and is quasihomogeneous, that is,

$$H(w, \lambda \tilde{w}; \lambda x) = \lambda H(w, \tilde{w}; x)$$
 for any $\lambda > 0$.

It follows that

$$\sum_{i=0}^{1} \tilde{w}_i \frac{\partial H}{\partial \tilde{w}_i} + x \frac{\partial H}{\partial x} = H.$$

For the solution $(w_0(x), w_1(x), \tilde{w}_0(x), \tilde{w}_1(x))$ of (3.1) and (3.2), we have

$$\sum_{i=0}^{1} \tilde{w}_{i}(x) \frac{\partial H}{\partial \tilde{w}_{i}}(x, w_{0}(x), w_{1}(x), \tilde{w}_{0}(x), \tilde{w}_{1}(x)) = \sum_{i=0}^{1} \tilde{w}_{i}(x) (w_{i})_{x} (x)$$

$$x \frac{\partial H}{\partial x}(x, w_{0}(x), w_{1}(x), \tilde{w}_{0}(x), \tilde{w}_{1}(x)) = -H(w_{0}(x), w_{1}(x), \tilde{w}_{0}(x), \tilde{w}_{1}(x)) + \frac{dx H(x, w_{0}(x), w_{1}(x), \tilde{w}_{0}(x), \tilde{w}_{1}(x))}{dx}.$$

The first equality is obvious. The second equality follows from $\frac{dxH}{dx} = x\frac{dH}{dx} + H$ and

$$\begin{split} & \frac{dH(x,w_{0}(x),w_{1}(x),\tilde{w}_{0}(x),\tilde{w}_{1}(x))}{dx} \\ & = \frac{\partial H}{\partial x}(x,w_{0}(x),w_{1}(x),\tilde{w}_{0}(x),\tilde{w}_{1}(x)) + (w_{i})_{x}(x)\frac{\partial H}{\partial w_{i}}(x,w_{0}(x),w_{1}(x),\tilde{w}_{0}(x),\tilde{w}_{1}(x)) \\ & + (\tilde{w}_{i})_{x}(x)\frac{\partial H}{\partial \tilde{w}_{i}}(x,w_{0}(x),w_{1}(x),\tilde{w}_{0}(x),\tilde{w}_{1}(x)) \\ & = \frac{\partial H}{\partial x}(x,w_{0}(x),w_{1}(x),\tilde{w}_{0}(x),\tilde{w}_{1}(x)) \\ & - (w_{i})_{x}(x)(\tilde{w}_{i})_{x}(x) + (\tilde{w}_{i})_{x}(x)(w_{i})_{x}(x) \\ & = \frac{\partial H}{\partial x}(x,w_{0}(x),w_{1}(x),\tilde{w}_{0}(x),\tilde{w}_{1}(x)). \end{split}$$

Then it follows that

Proposition 10.
$$H = \tilde{w}_0(w_0)_x + \tilde{w}_1(w_1)_x - H + \frac{d}{dx}(xH)$$
.

Let $S(x_1, x_2) := \int_{x_1}^{x_2} \left(\sum_{i=0}^{1} \tilde{w}_i \left(w_i \right)_x - H \right) dx$, which is called the classical action, the functional from which we can derive the Euler-Lagrange equation using the fundamental lemma of calculus of variations. We obtain

$$\frac{\partial S(x_1, x_2)}{\partial \gamma_j} = \int_{x_1}^{x_2} \left(\sum_{i=0}^1 \left((\tilde{w}_i)_{\gamma_j} (w_i)_x + \tilde{w}_i ((w_i)_x)_{\gamma_j} \right) - (H)_{\gamma_j} \right) dx$$

$$= \int_{x_1}^{x_2} \left(\sum_{i=0}^1 \left((\tilde{w}_i)_{\gamma_j} \frac{\partial H}{\partial \tilde{w}_i} + \tilde{w}_i ((w_i)_x)_{\gamma_j} \right) - \sum_{i=0}^1 \left(\frac{\partial H}{\partial w_i} (w_i)_{\gamma_j} + \frac{\partial H}{\partial \tilde{w}_i} (\tilde{w}_i)_{\gamma_j} \right) \right) dx$$

$$= \int_{x_1}^{x_2} \left(\sum_{i=0}^1 \left((\tilde{w}_i)_{\gamma_j} \frac{\partial H}{\partial \tilde{w}_i} - (\tilde{w}_i)_x (w_i)_{\gamma_j} \right) - \sum_{i=0}^1 \left(\frac{\partial H}{\partial w_i} (w_i)_{\gamma_j} + \frac{\partial H}{\partial \tilde{w}_i} (\tilde{w}_i)_{\gamma_j} \right) \right) dx$$

$$+ \left(\sum_{i=0}^1 \tilde{w}_i (w_i)_{\gamma_j} \right) \Big|_{x_1}^{x_2}$$

$$= \left(\tilde{w}_0 (w_0)_{\gamma_j} + \tilde{w}_1 (w_1)_{\gamma_j} \right) \Big|_{x_1}^{x_2}.$$

The second equality follows from (3.1) and the chain rule, the third from integration by parts, and the fourth from (3.2).

From the proposition and the definition of the τ function, we obtain

$$\frac{\partial}{\partial \gamma_{j}} \log \tau^{w}(x_{1}, x_{2}) = \frac{\partial}{\partial \gamma_{j}} \int_{x_{1}}^{x_{2}} \left(\sum_{i=0}^{1} \tilde{w}_{i} (w_{i})_{x} - H + \frac{d}{dx} (xH) \right) dx$$

$$= \frac{\partial S(x_{1}, x_{2})}{\partial \gamma_{j}} + (x_{2}H(x_{2}) - x_{1}H(x_{1}))_{\gamma_{j}}$$

$$= \left(\sum_{i=0}^{1} \tilde{w}_{i} (w_{i})_{\gamma_{j}} \right) \Big|_{x_{1}}^{x_{2}} + (x_{2}H(x_{2}) - x_{1}H(x_{1}))_{\gamma_{j}}.$$
(5.1)

At x = 0 the form of (2.3) is

$$w_i(x) = \frac{\gamma_i}{2} \log x + \frac{\rho_i}{2} + O(x^{\epsilon_i}), \quad x \to 0$$

for some $\varepsilon_i > 0$ (which depends on γ_0 and γ_1); this can be shown as in Theorem 14.1 of [4] for the case n = 1. This formula is differentiable in x and the γ_i 's. Therefore

$$\tilde{w}_i = \frac{\gamma_i}{2} + O(x^{\varepsilon_i}),$$

$$(w_i)_{\gamma_j} = \frac{\delta_{i,j}}{2} \log x + \frac{1}{2} (\rho_i)_{\gamma_j} + O(x^{\varepsilon_i} \log x),$$

$$(\tilde{w}_i)_{\gamma_j} = \frac{\delta_{i,j}}{2} + O(x^{\varepsilon_i} \log x)$$

as $x \to 0$.

At $x = \infty$, from [6], if $s_1^{\mathbb{R}} \neq 0$,

(5.2)
$$w_i(x) = -s_1^{\mathbb{R}} 2^{-\frac{7}{4}} (\pi x)^{-\frac{1}{2}} e^{-2\sqrt{2}x} + O(x^{-1} e^{-2\sqrt{2}x}) \quad \text{as } x \to \infty,$$
where $s_1^{\mathbb{R}} = -2\cos\frac{\pi}{4}(\gamma_0 + 1) - 2\cos\frac{\pi}{4}(\gamma_1 + 3)$. If $s_1^{\mathbb{R}} = 0$, we have
$$w_0(x) = s_2^{\mathbb{R}} 2^{-\frac{5}{2}} (\pi x)^{-\frac{1}{2}} e^{-4x} + O(x^{-1} e^{-4x}) \sim O(x^{-1} e^{-2\sqrt{2}x})$$

$$w_1(x) = -s_2^{\mathbb{R}} 2^{-\frac{5}{2}} (\pi x)^{-\frac{1}{2}} e^{-4x} + O(x^{-1} e^{-4x}) \sim O(x^{-1} e^{-2\sqrt{2}x}),$$

so the equation (5.2) holds for any generic (γ_0, γ_1) . The equation (5.2) is also differentiable in x and the γ_i 's, so

$$\tilde{w}_{i}(x) = s_{1}^{\mathbb{R}} 2^{-\frac{1}{4}} \sqrt{\pi} x^{\frac{1}{2}} e^{-2\sqrt{2}x} + O(e^{-2\sqrt{2}x}),$$

$$(w_{i})_{\gamma_{j}} = -\left(s_{1}^{\mathbb{R}}\right)_{\gamma_{j}} 2^{-\frac{7}{4}} (\pi x)^{-\frac{1}{2}} e^{-2\sqrt{2}x} + O(x^{-1}e^{-2\sqrt{2}x}),$$

$$(\tilde{w}_{i})_{\gamma_{j}} = \left(s_{1}^{\mathbb{R}}\right)_{\gamma_{i}} 2^{-\frac{1}{4}} \sqrt{\pi} x^{\frac{1}{2}} e^{-2\sqrt{2}x} + O(e^{-2\sqrt{2}x})$$

as $x \to \infty$.

By substituting the above asymptotic expansions into (5.1) we obtain

$$\frac{\partial}{\partial \gamma_i} \log \tau^w(x_1, x_2) = -\frac{\gamma_i}{4} \log x_1 - \sum_{k=0}^{1} \frac{\gamma_k}{4} (\rho_k)_{\gamma_i} - \frac{\gamma_i}{4} + O(x_1^{\varepsilon_i} \log x_1) + O(x_2^{\frac{3}{2}} e^{-2\sqrt{2}x_2})$$

as $x_1 \to 0$, $x_2 \to \infty$.

In our situation we have:

$$\tau^{w}(1,x) = C_0 x^{\frac{1}{8}(\gamma_0^2 + \gamma_1^2)} (1 + O(x^{\varepsilon})), \quad x \to 0,$$

$$\tau^{w}(1,x) = C_{\infty} e^{-x^2} (1 + O(x^{1/2} e^{-2\sqrt{2}x})), \quad x \to \infty.$$

Then

$$\log \tau^w(x_1, x_2) = \log \frac{C_\infty}{C_0} - x_2^2 - \frac{1}{8} (\gamma_0^2 + \gamma_1^2) \log x_1 + O(x_1^{\varepsilon}) + O(x_2^{1/2} e^{-2\sqrt{2}x_2}).$$

Let

$$C := \log \frac{C_{\infty}}{C_0} = \lim_{\substack{x_1 \to 0 \\ x_2 \to \infty}} \left(\log \tau^w(x_1, x_2) + x_2^2 + \frac{\gamma_0^2 + \gamma_1^2}{8} \log x_1 \right).$$

Then we obtain

$$\frac{\partial C}{\partial \gamma_i} = \lim_{\substack{x_1 \to 0 \\ x_2 \to \infty}} \left(\frac{\partial}{\partial \gamma_i} \left(\log \tau^w(x_1, x_2) + x_2^2 + \frac{\gamma_0^2 + \gamma_1^2}{8} \log x_1 \right) \right) = -\frac{\gamma_i}{4} - \sum_{k=0}^{1} \frac{\gamma_k}{4} \left(\rho_k \right)_{\gamma_i}$$
that is,

(5.3)
$$C = -\sum_{i=0}^{1} \frac{\gamma_i^2}{8} - \frac{1}{4} \sum_{k=0}^{1} \gamma_k \rho_k + \frac{1}{4} \int \sum_{k=0}^{1} \rho_k d\gamma_k.$$

Note that $\frac{\partial K}{\partial m_i} = -2\frac{\partial K}{\partial \gamma_i}$, where K is the function defined in the proof of theorem 7, and

$$\int \sum_{k=0}^{1} \rho_k d\gamma_k = -(\log 2) \sum_{k=0}^{1} \gamma_k^2 - 2K + \text{const.}$$

The constant above is independent of the γ_i 's. By substituting $\gamma_0 = \gamma_1 = 0$, which corresponds to the trivial solution $w_0 \equiv w_1 \equiv 0$, into (5.3), we obtain $C = -4 \left(\psi^{(-2)}(1/4) + \psi^{(-2)}(2/4) + \psi^{(-2)}(3/4)\right) + \text{const.}$ On the other hand, the tau function $\tau^w(x_1, x_2)$ corresponding to the trivial solution is $\exp(x_1^2 - x_2^2)$, so C = 0 in this case. In conclusion we have the following result:

Theorem 11.

$$C = -\frac{1}{8} \left(\gamma_0^2 + \gamma_1^2 \right) - \frac{1}{2} \left(\gamma_0 \rho_0 + \gamma_1 \rho_1 \right) - \frac{1}{2} F + 4 \left(\psi^{(-2)}(1/4) + \psi^{(-2)}(2/4) + \psi^{(-2)}(3/4) \right).$$

The function F in the theorem, which is the generating function, is given in Definition 6.

APPENDIX A. MONODROMY DATA

At $\zeta = 0$ we have a formal solution $\Psi_f^{(0)} = e^{-w}\Omega(I + \sum_{i \geq 1} \Psi_i^{(0)} \zeta^i)e^{\frac{1}{\zeta}d_{n+1}}$ of (2.2), where

$$d_{n+1} = \begin{pmatrix} 1 & & & \\ & \omega & & \\ & & \ddots & \\ & & & \omega^n \end{pmatrix}, \ \Omega = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1\\ 1 & \omega & \omega^2 & \cdots & \omega^n\\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2n}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & \omega^n & \omega^{2n} & \cdots & \omega^{n^2} \end{pmatrix}.$$

We define the sector

$$\Omega_1^{(0)} := \begin{cases} \left(-\left(\frac{1}{n+1} + \frac{1}{2}\right)\pi, \frac{\pi}{2}\right) & (n+1 \in 2\mathbb{Z}) \\ \left(-\left(\frac{1}{2(n+1)} + \frac{1}{2}\right)\pi, \left(\frac{1}{2(n+1)} + \frac{1}{2}\right)\pi\right) & (n+1 \in 2\mathbb{Z} + 1) \end{cases},$$

where we use the notation $(a,b) := \{ \zeta \in \mathbb{C}^* | a < \arg \zeta < b \}.$

We let $\Omega_{k+\frac{1}{n+1}}^{(0)}=e^{-\frac{\pi}{n+1}\sqrt{-1}}\Omega_k^{(0)}$ $(k\in\frac{1}{4}\mathbb{Z})$ in the universal covering $\tilde{\mathbb{C}}^*$.

Let $\Psi_k^{(0)}$ be the fundamental solution such that $\Psi_k^{(0)} \sim \Psi_f^{(0)}$ on $\Omega_k^{(0)}$. Similarly, at $\zeta = \infty$, we have the formal solution $\Psi_f^{(\infty)} = e^w \Omega^{-1} (I + \sum_{i \geq 1} \Psi_i^{(\infty)} \zeta^{-i}) e^{x^2 \zeta d_{n+1}}$. and the sectors

$$\Omega_1^{(\infty)} := \begin{cases}
(-\frac{\pi}{2}, (\frac{1}{n+1} + \frac{1}{2})\pi) & (n+1 \in 2\mathbb{Z}) \\
(-(\frac{1}{2(n+1)} + \frac{1}{2})\pi, (\frac{1}{2(n+1)} + \frac{1}{2})\pi) & (n+1 \in 2\mathbb{Z} + 1)
\end{cases}$$

$$\Omega_{k+\frac{1}{n+1}}^{(\infty)} := e^{\frac{\pi}{n+1}\sqrt{-1}}\Omega_k^{(\infty)}.$$

Let $\Psi_k^{(\infty)}$ be the fundamental solution such that $\Psi_k^{(\infty)} \sim \Psi_f^{(\infty)}$ on $\Omega_k^{(\infty)}$. We define the Stokes matrices $S_k^{(0)}$, $S_k^{(\infty)}$ by $\Psi_{k+1}^{(0)} = \Psi_k^{(0)} S_k^{(0)}$, $\Psi_{k+1}^{(\infty)} = \Psi_k^{(\infty)} S_k^{(\infty)}$. We define the Stokes factors $Q_k^{(0)}$, $Q_k^{(\infty)}$ by $\Psi_{k+\frac{1}{n+1}}^{(0)} = \Psi_k^{(0)} Q_k^{(0)}$, $\Psi_{k+\frac{1}{n+1}}^{(\infty)} = \Psi_k^{(\infty)} Q_k^{(\infty)}$

 $\begin{array}{c} \Psi_k^{(\infty)}Q_k^{(\infty)}.\\ \text{Let } M:=Q_1^{(0)}Q_{1+\frac{1}{n+1}}^{(0)}\Pi \text{ where} \end{array}$

$$\Pi = \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{pmatrix}.$$

Let m_i be the eigenvalues of M. It is proved in [6, 7] that the m_i determine all

We define the connection matrices E_k by $\Psi_k^{(\infty)} = \Psi_k^{(0)} E_k$. Let E_1^{global} be $\frac{1}{n+1} A Q_{\frac{n}{n+1}}^{(\infty)}$ where

$$A = \begin{pmatrix} 1 & & & \\ & & & 1 \\ & & \ddots & \\ & 1 & & \end{pmatrix}.$$

It is known that $E_1=E_1^{\mathrm{global}}$ for global solutions w. Let $e_i^{\mathbb{R}}$ be the eigenvalues of $E:=E_1\left(E_1^{\mathrm{global}}\right)^{-1}$. We have $e_i^{\mathbb{R}}e_{n-i}^{\mathbb{R}}=1$. It is proved in [7] that the $e_i^{\mathbb{R}}$ determine

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