Gelfand–Dickey hierarchy, generalized BGW tau-function, and W-constraints

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Abstract

Let $r \geq 2$ be an integer. The generalized BGW tau-function for the Gelfand–Dickey hierarchy of $(r-1)$ dependent variables (aka the r-reduced KP hierarchy) is defined as a particular tau-function that depends on $(r-1)$ constant parameters $d_1, \dots, d_{r-1}$. In this paper we show that this tau-function satisfies a family of linear equations, called the W-constraints of the second kind. The operators giving rise to the linear equations also depend on $(r-1)$ constant parameters. We show that there is a one-to-one correspondence between the two sets of parameters.

1 Introduction

Let $r \geq 2$ be an integer, and let $n = r-1$. The Gelfand–Dickey (GD) hierarchy with n unknown functions is an infinite family of PDEs, defined by

$$\frac{\partial L}{\partial t_i} = \left[\left(L^{i/r} \right)_+, L \right], \quad i \in \mathbb{N} \backslash r\mathbb{N}, \tag{1.1}$$

where

$$L := \partial^r + v_1 \partial^{r-2} + \cdots + v_{r-1} \tag{1.2}$$

is the Lax operator, $L^{i/r}$, $i \in \mathbb{N}\backslash r\mathbb{N}$, denote the fractional powers of L (cf. e.g. [14] for the definition), and ∂ is understood as ∂_{t_1}. This integrable hierarchy can also be viewed as a reduction of the Kadomtsev–Petviashvili (KP) hierarchy (see e.g. [14] or Section 2).

There are many interesting solutions to the GD hierarchy. For example, in the study of Witten's r-spin invariants [23, 26, 41], the so-called *topological solution* [17, 19] to the GD hierarchy plays an important role. For example, for the case $r = 2$, the GD hierarchy is the celebrated Korteweg–de Vries (KdV) hierarchy, and the topological solution is famously known as the Witten–Kontsevich solution (cf. e.g. [18, 19]), governing the integrals of psi-classes over the moduli space of curves [34, 40]. The interest of this paper is on another solution to the GD hierarchy again for any r. Unlike the topological solution, the solution of interest of this paper will depend non-trivially on $r-1$ arbitrary parameters. Again, let us look at the KdV case first (i.e., the case with $r = 2$). For this case, it is known that there exists a solution to the KdV hierarchy, called the generalized BGW solution, depending non-trivially on one arbitrary parameter [3, 18], having bispectral properties [18, 20], and possessing enumerative meanings [33, 38, 42]. This motivates us to generalize the generalized BGW solution to an arbitrary $r \geq 2$. Indeed, let $d_1, \dots, d_n$ be arbitrarily given complex numbers, and define $v_{\mathrm{BGW}}(\mathbf{t})$ as the unique solution in $\mathbb{C}[[\mathbf{t}]]^n$ to the GD hierarchy, satisfying the initial condition

$$v_{\alpha,\mathrm{BGW}}\big(t_1, \mathbf{t}_{\geq 2} = \mathbf{0}\big) = \frac{d_\alpha}{(1-t_1)^{\alpha+1}}, \quad \alpha = 1, \dots, r-1, \tag{1.3}$$

where $\mathbf{t} = (t_i)_{i \in \mathbb{N}\setminus r\mathbb{N}}$. We call $v_{\mathrm{BGW}}(\mathbf{t}; d_1, \dots, d_n)$ the *generalized BGW solution to the GD hierarchy.* The Dubrovin–Zhang type tau-function of this solution (cf. [6, 8, 14, 18, 19]) will be called the *generalized BGW tau-function*, denoted by $\tau_{\mathrm{BGW}} = \tau_{\mathrm{BGW}}(\mathbf{t}; d_1, \dots, d_n)$. We show in Section 3 that the generalized BGW tau-function τ_{BGW} can be chosen such that

$$\sum_{i \in \mathbb{N}\setminus r\mathbb{N}} i\tilde{t}_i \frac{\partial \tau_{\mathrm{BGW}}}{\partial t_i} + \frac{d_1}{r}\tau_{\mathrm{BGW}} = 0, \tag{1.4}$$

where $\tilde{t}_i = t_i - \delta_{i,1}$, moreover, it is unique up to multiplying by a nonzero constant.

The above definition of the generalized BGW tau-function for the GD hierarchy was given in joint work by B. Dubrovin, D. Zagier and the first named author of the present paper in a more general set up, i.e., for the Drinfeld–Sokolov (DS) hierarchy associated to a simple Lie algebra [6], where certain analogues of the triangle numbers on the constants manifold were observed. The GD hierarchy can be considered as the DS hierarchy associated to the A_n type simple Lie algebra under the Wronskian gauge. In [35], certain generalized BGW tau-functions were also given for the DS hierarchy associated to an affine Kac–Moody algebra (the simple Lie algebra case corresponds to the untwisted affine Kac–Moody algebra under a particular choice of vertex in the Dynkin diagram).

For $r = 2$, the generalized BGW tau-function can also be identified with the solution to Virasoro constraints [3, 7, 9]. The goal of this paper is to show that for an arbitrary $r \geq 2$, the generalized BGW tau-function (defined above) satisfies a set of linear constraints, which will be called *W-constraints of the second kind.* To be precise, define a family of operators $W^{\mathrm{red}}_{\alpha,q}$, $\alpha = 1, \dots, n = r-1$, $q \geq 0$, by

$$W^{\mathrm{red}}_{\alpha,q}(\mathbf{t}) := \operatorname{res}_\lambda \lambda^{\alpha+(q-\alpha)r}\Big(\partial_\mu^{\alpha+1}\big(X_{\mathrm{GD}}(\tilde{\mathbf{t}};\lambda,\mu)\big)\Big)\Big|_{\mu=\lambda} d\lambda, \tag{1.5}$$

where $X_{\mathrm{GD}}(\mathbf{t};\lambda,\mu)$ is given by

$$X_{\mathrm{GD}}(\mathbf{t};\lambda,\mu) := e^{\sum_{i\in\mathbb{N}\setminus r\mathbb{N}} t_i(\mu^i-\lambda^i)} \circ e^{\sum_{i\in\mathbb{N}\setminus r\mathbb{N}}\left(\frac{1}{i\lambda^i}-\frac{1}{i\mu^i}\right)\frac{\partial}{\partial t_i}}. \tag{1.6}$$

These operators were given e.g. in [2]; according to [2, 4, 25], they can be expressed by operators coming from the twisted module of the $\mathcal{W}_{A_n}$-algebra [5]. We have the following theorem.

Theorem 1.1 *There exist unique constants $\rho_1, \dots, \rho_{r-1} \in \mathbb{C}$ such that*

$$W^{\mathrm{red}}_{\alpha,q}\big(\tau_{\mathrm{BGW}}\big) = (-1)^\alpha \rho_\alpha \delta_{\alpha,q}\tau_{\mathrm{BGW}}, \qquad \alpha = 1, \dots, r-1,\ q \geq \alpha. \tag{1.7}$$

Moreover, these constants ρ_α are polynomials of $d_1, \dots, d_n$, having the form

$$\rho_\alpha = \frac{d_\alpha}{r} + \omega_\alpha(d_1, \dots, d_{\alpha-1}). \tag{1.8}$$

We refer to (1.7) as *W-constraints of the second kind*. We note that the W-constraints for the topological tau-function [2, 4, 5, 9, 28, 44], being referred to as W-constraints of the first kind, start with $q = 0$ instead of $q = \alpha$ and have the dilaton shift at t_{r-1} instead of at t_1.

The W-constraints of the second kind seem to have common cases with the W-constraints given in [9] in an equivalent way. For certain special common cases, the solutions to the W-constraints of [9] were conjectured by Chidambaram, Garcia–Failde and Giacchetto in a recent letter to the authors of the present paper to be tau-functions for the GD hierarchy. Theorem 1.1 (cf. also Theorem 4.5) should lead to this conjecture; still, it will be interesting to investigate the explicit relationship between $\tau_{\rm BGW}$ and the partition functions defined in [9].

In a subsequent publication, we will consider the analogous open extension of the generalized BGW tau-function for arbitrary $r \geq 2$ (see [43] for the $r = 2$ case; cf. also [8, 11]).

Organization of the paper. In Section 2 we review some basics on KP and GD hierarchies. In Section 3 we give the definition of the generalized BGW tau-function $\tau_{\rm BGW}$ in more details. In Section 4 we prove Theorem 1.1. In Section 5 we present some examples.

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2 Preliminaries

In this section we review tau-functions and wave functions for the KP hierarchy and for the GD hierarchy.

Let $L_{\rm KP}$ denote the pseudo-differential operator

$$L_{\rm KP} := \partial + \sum_{k\geq 1} u_k \partial^{-k}. \tag{2.1}$$

Here $\partial := \partial_x$. Recall that the KP hierarchy [14] is the following commuting system of PDEs for the infinitely many dependent variables $u_1(\mathbf{t}_{\rm KP}), u_2(\mathbf{t}_{\rm KP}), \dots$:

$$\frac{\partial L_{\rm KP}}{\partial t_i} = \Big[\big(L_{\rm KP}^i\big)_+, L_{\rm KP}\Big], \quad i \geq 1. \tag{2.2}$$

Here $\mathbf{t}_{\rm KP} := (t_1, t_2, t_3, \dots)$ denotes the infinite vector of times. The first equation in (2.2) reads

$$\frac{\partial u_k}{\partial t_1} = \frac{\partial u_k}{\partial x}, \quad k \geq 1.$$

Therefore we identify the time t_1 with x. We consider solutions to the KP hierarchy in $\mathbb{C}[[\mathbf{t}_{\rm KP}]]^{\mathbb{N}}$, i.e., $u_k(\mathbf{t}_{\rm KP}) \in \mathbb{C}[[\mathbf{t}_{\rm KP}]]$, $k \geq 1$. Denote for simplicity $u := (u_1, u_2, \cdots)$. It is known (see for example [14]) that for an arbitrary power series solution $u(\mathbf{t}_{\rm KP}) = (u_1(\mathbf{t}_{\rm KP}), u_2(\mathbf{t}_{\rm KP}), \dots)$ to the KP hierarchy, there exists a pseudo-differential operator

$$\Phi(\mathbf{t}_{\rm KP}) = 1 + \sum_{k\geq 1} \phi_k(\mathbf{t}_{\rm KP})\, \partial^{-k}, \quad \phi_k(\mathbf{t}_{\rm KP}) \in \mathbb{C}[[\mathbf{t}_{\rm KP}]], \tag{2.3}$$

called a dressing operator, satisfying

$$L_{\mathrm{KP}} = \Phi \circ \partial \circ \Phi^{-1}, \tag{2.4}$$

$$\frac{\partial \Phi}{\partial t_i} = -\left(L_{\mathrm{KP}}^i\right)_- \circ \Phi, \quad i \geq 1. \tag{2.5}$$

The dressing operator Φ is uniquely determined by the solution u up to the right multiplication by an operator of the form

$$1 + \sum_{k\geq 1} a_k \partial^{-k} \in \mathbb{C}[[\partial^{-1}]],$$

where a_k, $k \geq 1$ are constants. The wave and dual wave functions $\psi(\mathbf{t}_{\mathrm{KP}};\lambda)$, $\psi^*(\mathbf{t}_{\mathrm{KP}};\lambda)$ associated to the solution $u(\mathbf{t}_{\mathrm{KP}})$ are elements in $\mathbb{C}\big(\big(\lambda^{-1}\big)\big)[[\mathbf{t}_{\mathrm{KP}}]]$ defined by

$$\psi(\mathbf{t}_{\mathrm{KP}};\lambda) := \Phi(\mathbf{t}_{\mathrm{KP}};\lambda)\Big(e^{\xi(\mathbf{t}_{\mathrm{KP}};\lambda)}\Big), \quad \psi^*(\mathbf{t}_{\mathrm{KP}};\lambda) = \big(\Phi^*(\mathbf{t}_{\mathrm{KP}};\lambda)\big)^{-1}\Big(e^{-\xi(\mathbf{t}_{\mathrm{KP}};\lambda)}\Big), \tag{2.6}$$

where $\xi(\mathbf{t}_{\mathrm{KP}};\lambda) := \sum_{i\geq 1} t_i \lambda^i$, and Φ^* denotes the formal adjoint operator of Φ, i.e.,

$$\Phi^* := 1 + \sum_{k\geq 1} (-\partial)^{-k} \circ \phi_k. \tag{2.7}$$

They satisfy

$$L_{\mathrm{KP}}(\psi) = \lambda \psi, \quad \frac{\partial \psi}{\partial t_i} = \left(L_{\mathrm{KP}}^i\right)_+(\psi),$$

$$L_{\mathrm{KP}}^*(\psi^*) = \lambda \psi^*, \quad \frac{\partial \psi^*}{\partial t_i} = \left(\left(L_{\mathrm{KP}}^*\right)^i\right)_+(\psi^*)$$

with $L_{\mathrm{KP}}^* := -\partial + \sum_{k\geq 1} (-\partial)^k \circ u_k$. Introduce the following two operators:

$$X(\mathbf{t}_{\mathrm{KP}};\lambda) = e^{\sum_{i\geq 1} t_i \lambda^i} e^{-\sum_{i\geq 1} \frac{1}{i\lambda^i}\partial_i}, \quad X^*(\mathbf{t}_{\mathrm{KP}};\lambda) = e^{-\sum_{i\geq 1} t_i \lambda^i} e^{\sum_{i\geq 1} \frac{1}{i\lambda^i}\partial_i}. \tag{2.8}$$

It was proved in [14] that for an arbitrary solution u in $\mathbb{C}[[\mathbf{t}_{\mathrm{KP}}]]^{\mathbb{N}}$ to the KP hierarchy, there exists a power series $\tau_{\mathrm{KP}}(\mathbf{t}_{\mathrm{KP}}) \in \mathbb{C}[[\mathbf{t}_{\mathrm{KP}}]]$, satisfying

$$\psi(\mathbf{t}_{\mathrm{KP}};\lambda) = \frac{X(\mathbf{t}_{\mathrm{KP}};\lambda)(\tau_{\mathrm{KP}}(\mathbf{t}_{\mathrm{KP}}))}{\tau_{\mathrm{KP}}(\mathbf{t}_{\mathrm{KP}})}, \quad \psi^*(\mathbf{t}_{\mathrm{KP}};\lambda) = \frac{X^*(\mathbf{t}_{\mathrm{KP}};\lambda)(\tau_{\mathrm{KP}}(\mathbf{t}_{\mathrm{KP}}))}{\tau_{\mathrm{KP}}(\mathbf{t}_{\mathrm{KP}})}. \tag{2.9}$$

We call $\tau_{\mathrm{KP}}(\mathbf{t}_{\mathrm{KP}})$ the tau-function of the solution u for the KP hierarchy. We also call $(\Phi, \tau_{\mathrm{KP}})$ a dressing pair associated to u. The dressing pair is uniquely determined by the solution u up to the transformation

$$(\Phi, \tau_{\mathrm{KP}}) \mapsto \left(\Phi \circ e^{-\sum_{i\geq 1} b_i \partial^{-i}}, \quad \tau_{\mathrm{KP}} e^{b_0 + \sum_{i\geq 1} b_i t_i}\right), \quad b_0, b_1, b_2, \dots \in \mathbb{C}.$$

Denote by $\mathcal{A}_u := \mathcal{A}_{u,0}\big[\partial^i(u_k) | i, k \geq 1\big]$ the ring of differential polynomials of u, where $\mathcal{A}_{u,0}$ denotes the ring of smooth functions of u. For a pseudo-differential operator of the form $a = \sum_{i\in\mathbb{Z}} a_i \partial^i$, define $\operatorname{res}_\partial a = a_{-1}$. Define a family of differential polynomials in u by [14]

$$\Omega_{i,j}^{\mathrm{KP}} = \Omega_{i,j}^{\mathrm{KP}}(u, u_x, \dots) := \partial^{-1}\left(\frac{\partial}{\partial t_j} \operatorname{res}_\partial L_{\mathrm{KP}}^i\right) \in \mathcal{A}_u, \quad i, j \geq 1, \tag{2.10}$$

where ∂^{-1} is fixed by the no-integration-constant rule. We call $\Omega_{i,j}$ the two-point correlations functions for the KP hierarchy.

Lemma 2.1 *Let u be an arbitrary solution in $\mathbb{C}[[\mathbf{t}_{\rm KP}]]^{\mathbb{N}}$ to the KP hierarchy, and $\tau_{\rm KP}\in\mathbb{C}[[\mathbf{t}_{\rm KP}]]$ the tau-function of u. Then the following formulae hold true:*

$$\frac{\partial^2\log\tau_{\rm KP}}{\partial t_i\partial t_j}=\Omega^{\rm KP}_{i,j},\quad\forall\, i,j\geq 1.\tag{2.11}$$

Proof Let $(\Phi,\tau_{\rm KP})$ be the dressing pair associated to u, and ψ the corresponding wave function. It was shown in [14] that for given $i,j\geq 1$,

$$\frac{\partial^2\log\tau_{\rm KP}}{\partial t_i\partial t_j}=\operatorname{res}_z z^i\left(-\sum_{\ell\geq 1}z^{-\ell-1}\partial_{t_\ell}+\frac{\partial}{\partial z}\right)\left(-\frac{\left(L^j_{\rm KP}\right)_-(\psi)}{\psi}\right)\tag{2.12}$$

Note that $\left(L^j_{\rm KP}\right)_-$ can be rewritten into the form

$$\left(L^j_{\rm KP}\right)_-=\sum_{k\geq 1}a_{j,k}L_{\rm KP}^{-k},\tag{2.13}$$

where $a_{j,k}\in\mathcal{A}_u$ satisfying $a_{j,k}|_{u=u_x=u_{xx}=\cdots=0}=0$. Combining (2.12) with (2.13) we find that

$$\frac{\partial^2\log\tau_{\rm KP}}{\partial t_i\partial t_j}=ia_{j,i}+\sum_{k=1}^{i-1}\frac{\partial a_{j,k}}{\partial t_{i-k}}.$$

In particular, observing that $a_{j,1}=\operatorname{res}_\partial L^j_{\rm KP}$, we have

$$\frac{\partial^2\log\tau}{\partial t_1\partial t_j}=\operatorname{res}_\partial L^j_{\rm KP}.$$

Taking the derivative with respect to t_i on the both sides of the above identity, and then by using the definition (2.10), the lemma is proved. □

Introduce the following operator:

$$X\left(\mathbf{t}_{\rm KP};\lambda,\mu\right):=e^{\sum_{i\geq 1}t_i\left(\mu^i-\lambda^i\right)}\circ e^{\sum_{i\geq 1}\left(\frac{1}{i\lambda^i}-\frac{1}{i\mu^i}\right)\frac{\partial}{\partial t_i}}.\tag{2.14}$$

It is shown in [12] that

$$\begin{aligned}&\operatorname{res}_\nu\frac{X(\mathbf{t}_{\rm KP};\nu)\circ X(\mathbf{t}_{\rm KP};\lambda,\mu)\left(\tau_{\rm KP}(\mathbf{t}_{\rm KP})\right)}{\tau_{\rm KP}(\mathbf{t}_{\rm KP})}\frac{X^*(\mathbf{t}'_{\rm KP};\nu)\left(\tau_{\rm KP}(\mathbf{t}'_{\rm KP})\right)}{\tau_{\rm KP}(\mathbf{t}'_{\rm KP})}d\nu\\=&(\lambda-\mu)\psi(\mathbf{t}_{\rm KP};\mu)\psi^*(\mathbf{t}'_{\rm KP};\lambda),\end{aligned}\tag{2.15}$$

where $\mathbf{t}_{\rm KP}=(t_1,t_2,\dots)$ and $\mathbf{t}'_{\rm KP}=(t'_1,t'_2,\dots)$.

Following [39] and [1], introduce the following operator:

$$M:=\Phi\circ\left(\sum_{i\geq 1}it_i\partial^{i-1}\right)\circ\Phi^{-1}\in\mathbb{C}[[\mathbf{t}_{\rm KP}]]\otimes_{\mathbb{C}}\mathbb{C}((\partial^{-1})).\tag{2.16}$$

By using Lemma 3.2 of [1] we have

$$\operatorname{res}_\partial M^i\circ L^k_{\rm KP}=\operatorname{res}_\lambda\lambda^k\psi^*(\mathbf{t}_{\rm KP};\lambda)\partial^i_\lambda(\psi(\mathbf{t}_{\rm KP};\lambda)),\quad\forall\, i,k\geq 0.\tag{2.17}$$

Then it follows from (2.15) that

$$\operatorname{res}_\partial M^i \circ L_{\rm KP}^k = \frac{1}{i+1}\partial\left(\frac{\operatorname{res}_\lambda \lambda^k \partial_\mu^{i+1}\circ X(\mathbf{t}_{\rm KP};\lambda,\mu)\left(\tau_{\rm KP}(\mathbf{t}_{\rm KP})\right)|_{\mu=\lambda}}{\tau_{\rm KP}(\mathbf{t}_{\rm KP})}\right). \tag{2.18}$$

Denote by $\mathbf{t} = (t_i)_{i\in\mathbb{N}\backslash r\mathbb{N}}$ the infinite time vector for the GD hierarchy.

For an arbitrary solution $v(\mathbf{t}) = \big(v_1(\mathbf{t}),\dots,v_{r-1}(\mathbf{t})\big)$ in $\mathbb{C}[[\mathbf{t}]]^{r-1}$ to the GD hierarchy, we associate to it an infinite sequence of power series in $\mathbb{C}[[\mathbf{t}]]$ defined by

$$u_k = u_k(\mathbf{t}) := \operatorname{res}_\partial\Big(L^{1/r}\circ\partial^{k-1}\Big),\quad k\geq 1.$$

In other words,

$$\partial + \sum_{k\geq 1} u_k\partial^{-k} = L^{1/r}. \tag{2.19}$$

Obviously, $u = (u_1, u_2, \dots)$ satisfies the KP hierarchy, namely, for all $i\in\mathbb{N}$,

$$\frac{\partial L^{1/r}}{\partial t_i} = \Big[\Big(L^{i/r}\Big)_+,\, L^{1/r}\Big]. \tag{2.20}$$

Let $\tau_{\rm KP}$ be the tau-function of the solution u to the *KP hierarchy*. By the definition (2.10), we know $\Omega^{\rm KP}_{ir,jr} = 0$ for $i,j\geq 1$ when $L_{\rm KP} = L^{1/r}$. It then follows from Lemma 2.1 that $\tau_{\rm KP}$ satisfies

$$\frac{\partial^2 \log\tau_{\rm KP}}{\partial t_{ir}\partial t_{jr}} = 0,\quad i,j\geq 1.$$

This means that, there exist constants $a_1, a_2, \dots,$ such that

$$\frac{\partial}{\partial t_{ir}}\left(\sum_{j\geq 1} a_j t_{jr} + \log\tau_{\rm KP}\right) = 0,\quad i\geq 1.$$

Let $\tau = \tau(\mathbf{t}) := \tau_{\rm KP}(\mathbf{t}_{\rm KP})\exp\Big(\sum_{j\geq 1} a_j t_{jr}\Big)$. Then τ is still a KP tau function. We call this particular chosen τ *the tau-function of the solution v for the GD hierarchy reduced from the KP hierarchy.*

Let us proceed to give a second definition of tau-function for the GD hierarchy.

Lemma 2.2 *For an arbitrary solution $v = v(\mathbf{t})$ in $\mathbb{C}[[\mathbf{t}]]^{r-1}$ to the GD hierarchy* (1.1)*, there exists a power series $\tau_{\rm DZ} = \tau_{\rm DZ}(\mathbf{t})\in\mathbb{C}[[\mathbf{t}]]$ satisfying*

$$\frac{\partial^2\log\tau_{\rm DZ}}{\partial t_i\partial t_j} = \Omega^{\rm GD}_{i,j},\qquad \forall\, i,j\in\mathbb{N}\backslash r\mathbb{N}, \tag{2.21}$$

where $\Omega^{\rm GD}_{i,j}$ are differential polynomials in v defined as

$$\Omega^{\rm GD}_{i,j} = \Omega^{\rm GD}_{i,j}(v, v_x, \dots) := \partial^{-1}\left(\frac{\partial \operatorname{res}_\partial L^{i/r}}{\partial t_j}\right). \tag{2.22}$$

Here ∂^{-1} is again fixed by the no-integration-constant rule. We call $\tau_{\rm DZ}$ the Dubrovin–Zhang type tau-function of the solution v for the GD hierarchy.

Note that τ_{DZ} is uniquely determined by v up to multiplying by a factor the form

$$\exp\left(b_0 + \sum_{i\in\mathbb{N}\backslash r\mathbb{N}} b_i t_i\right), \tag{2.23}$$

where b's are arbitrary constants.

Theorem 2.3 *Let v be an arbitrary solution in $\mathbb{C}[[\mathbf{t}]]^{r-1}$ to the GD hierarchy, and τ_{DZ} and τ be the Dubrovin–Zhang type tau-function and the tau-function reduced from the KP hierarchy of v, respectively. Then there exist constants $b_0, b_1, b_2, \ldots$ such that*

$$\tau = \tau_{\mathrm{DZ}} \exp\left(b_0 + \sum_{i\in\mathbb{N}\backslash r\mathbb{N}} b_i t_i\right). \tag{2.24}$$

Proof It suffices to show τ satisfies (2.21). Let u be the solution to the KP hierarchy determined by v via (2.19). By the definition of $\Omega^{\mathrm{KP}}_{i,j}$ and $\Omega^{\mathrm{GD}}_{i,j}$, we have

$$\Omega^{\mathrm{KP}}_{i,j}(u, u_x, \ldots) = \Omega^{\mathrm{GD}}_{i,j}(v, v_x, \ldots), \quad \forall\, i, j \in \mathbb{N}\backslash r\mathbb{N}.$$

On the other hand, by using Lemma 2.1 and using the definition of τ, we know τ satisfies

$$\frac{\partial^2 \log \tau}{\partial t_i \partial t_j} = \Omega^{\mathrm{KP}}_{i,j}(u, u_x, \ldots), \quad \forall\, i, j \in \mathbb{N}\backslash r\mathbb{N},$$

where $\Omega^{\mathrm{KP}}_{i,j}$ is given by (2.10). □

3 The generalized BGW tau-function for the GD hierarchy

In this section we give more details about the definition of the generalized BGW tau-function.

Introduce a gradation on $\mathcal{A}_v$ by assigning the degree:

$$\deg\, \partial^k(v_\alpha) = \alpha + 1 + k, \qquad \alpha = 1, \ldots, r-1, \quad k \geq 0.$$

It is easy to verify that

$$\deg\, \mathrm{res}_\partial \left(L^{i/r}\partial^{-k}\right) = i - k, \qquad i \geq 1, \quad k \leq i-2. \tag{3.1}$$

This implies the GD hierarchy (1.1) has the form

$$\frac{\partial v_\alpha}{\partial t_i} = X^i_\alpha(v, v_x, \ldots), \qquad \alpha = 1, \ldots, r-1, \quad i \in \mathbb{N}\backslash r\mathbb{N},$$

with $X^i_\alpha = X^i_\alpha(v, v_x, \ldots) \in \mathcal{A}_v$ having the degree

$$\deg X^i_\alpha = \alpha + i + 1. \tag{3.2}$$

Proposition 3.1 *The generalized BGW solution v_{BGW} satisfies the following linear equations:*

$$\sum_{i\in\mathbb{N}\backslash r\mathbb{N}} i\tilde{t}_i \frac{\partial v_{\alpha,\mathrm{BGW}}}{\partial t_i} + (\alpha+1) v_{\alpha,\mathrm{BGW}} = 0. \tag{3.3}$$

Proof For simplicity, we denote $v_\alpha = v_{\alpha,\mathrm{BGW}}$, and denote

$$f_\alpha(\mathbf{t}) := \sum_{i\in\mathbb{N}\setminus r\mathbb{N}} i\tilde{t}_i \frac{\partial v_\alpha(\mathbf{t})}{\partial t_i} + (\alpha+1)v_\alpha(\mathbf{t}). \tag{3.4}$$

We are to show $f_\alpha = 0$. Firstly, from the initial condition (1.3), it is easy to see that

$$f_\alpha(t_1 = x, t_2 = 0, \dots) = (x-1)\partial_x(v_\alpha(x,\mathbf{0})) + (\alpha+1)v_\alpha(x,\mathbf{0}) = 0. \tag{3.5}$$

Taking the derivative of (3.4) with respect to t_j, we have

$$\begin{aligned}
\frac{\partial f_\alpha}{\partial t_j} &= \sum_{i\in\mathbb{N}\setminus r\mathbb{N}} i\tilde{t}_i \frac{\partial^2 v_\alpha}{\partial t_i \partial t_j} + (\alpha+j+1)\frac{\partial v_\alpha}{\partial t_j} \\
&= \sum_{i\in\mathbb{N}\setminus r\mathbb{N}} i\tilde{t}_i \sum_{\beta=1}^{r-1}\sum_{k\geq 0} \partial^k\big(X^i_\beta\big)\frac{\partial X^j_\alpha}{\partial v^{(k)}_\beta} + (\alpha+j+1)\frac{\partial v_\alpha}{\partial t_j} \\
&= \sum_{\beta=1}^{r-1}\sum_{k\geq 0}\Big(\partial_x^k(f_\beta) - (\beta+k+1)v^{(k)}_\beta\Big)\frac{\partial X^j_\alpha}{\partial v^{(k)}_\beta} + (\alpha+1+j)X^j_\alpha \\
&= \sum_{\beta=1}^{r-1}\sum_{k\geq 0}\partial_x^k(f_\beta)\frac{\partial X^j_\alpha}{\partial v^{(k)}_\beta}.
\end{aligned}$$

Here, $v^{(k)}_\beta = \partial^k(v_\beta)$, $k\geq 0$, the second equality is due to

$$\frac{\partial}{\partial t_i} = \sum_{\beta=1}^{r-1}\sum_{k\geq 0}\partial^k\big(X^i_\beta\big)\frac{\partial}{\partial v^{(k)}_\beta},$$

the third equality can be obtained by applying ∂^k to (3.4), and the last equality is due to (3.2). Hence the identity (3.5) implies

$$\frac{\partial f_\alpha}{\partial t_j}(x,\mathbf{0}) = 0.$$

By induction on m, we have for arbitrary $j_1,\dots,j_m \in \mathbb{N}\setminus r\mathbb{N}$,

$$\frac{\partial^m f_\alpha}{\partial t_{j_1}\cdots\partial t_{j_m}}(x,\mathbf{0}) = 0.$$

The proposition is proved. □

Theorem 3.2 *The generalized BGW tau-function τ_{BGW} can be chosen to satisfy* (1.4)*.*

Proof By using (2.22) and (3.1), one can verify that

$$\deg \Omega^{\mathrm{GD}}_{j_1,j_2} = j_1 + j_2, \quad j_1, j_2 \in \mathbb{N}\setminus r\mathbb{N}.$$

Then it follows from Proposition 3.1 that

$$\sum_{i\in\mathbb{N}\backslash r\mathbb{N}} i\tilde{t}_i \frac{\partial\Omega^{\rm GD}_{j_1,j_2}}{\partial t_i} = \sum_{i\in\mathbb{N}\backslash r\mathbb{N}} i\tilde{t}_i \sum_{\alpha=1}^{r-1}\sum_{k\geq 1} \partial_{t_i}\big(v_\alpha^{(k)}\big) \frac{\partial\Omega^{\rm GD}_{j_1,j_2}}{\partial v_\alpha^{(k)}}$$
$$= -\sum_{\alpha=1}^{r-1}\sum_{k\geq 1}(\alpha+k+1) v_\alpha^{(k)} \frac{\partial\Omega^{\rm GD}_{j_1,j_2}}{\partial v_\alpha^{(k)}} = -(j_1+j_2)\Omega^{\rm GD}_{j_1,j_2}.$$

Therefore, by using (2.21), we have

$$\frac{\partial^2}{\partial t_{j_1}\partial t_{j_2}}\left(\sum_{i\in\mathbb{N}\backslash r\mathbb{N}} i\tilde{t}_i \frac{\partial \log\tau_{\rm BGW}}{\partial t_i}\right) = 0, \quad j_1, j_2 \in \mathbb{N}\backslash r\mathbb{N}.$$

Hence there exist constants a_0 and a_i, $i\in\mathbb{N}\backslash r\mathbb{N}$, such that

$$\sum_{i\in\mathbb{N}\backslash r\mathbb{N}} i\tilde{t}_i \frac{\partial \log\tau_{\rm BGW}}{\partial t_i} = \sum_{i\in\mathbb{N}\backslash r\mathbb{N}} a_i\tilde{t}_i + a_0.$$

Let us modify $\tau_{\rm BGW}$ as $\tau_{\rm BGW}\exp\big(-\sum_{i\in\mathbb{N}\backslash r\mathbb{N}} a_i t_i\big)$. Then $\tau_{\rm BGW}$ is still a Dubrovin–Zhang type tau-function, and satisfies

$$\sum_{i\in\mathbb{N}\backslash r\mathbb{N}} i\tilde{t}_i \frac{\partial \log\tau_{\rm BGW}}{\partial t_i} - a_0 = 0. \tag{3.6}$$

It remains to show $a_0 = -d_1/r$. The above (3.6) implies

$$\left.\frac{\partial^2 \log\tau_{\rm BGW}}{\partial t_1^2}\right|_{\mathbf{t}=\mathbf{0}} = \left.\frac{\partial \log\tau_{\rm BGW}}{\partial t_1}\right|_{\mathbf{t}=\mathbf{0}} = -a_0.$$

Hence by using (2.21), (2.22) and the initial condition (1.3), we have

$$\frac{\partial^2 \log\tau_{\rm BGW}}{\partial t_1^2}(\mathbf{0}) = \Omega^{\rm GD}_{1,1}\,|_{\mathbf{t}=\mathbf{0}} = \frac{v_1(\mathbf{0})}{r} = \frac{d_1}{r}.$$

The theorem is proved. □

4 W-constraints of the second kind

In this section, we show that the generalized BGW tau-function $\tau_{\rm BGW}$ for the GD hierarchy satisfies the W-constraints of the second kind. Let $v_{\rm BGW}$ be the generalized BGW solution to the GD hierarchy, and $u_{\rm BGW}$ the corresponding solution to the KP hierarchy (cf. (2.19)). By Theorem 2.3 we know that $\tau_{\rm BGW}$ can be regarded as a tau-function of $u_{\rm BGW}$ to the KP hierarchy.

Let

$$L_{\rm KP} = \partial + u_{1,{\rm BGW}}\partial^{-1} + u_{2,{\rm BGW}}\partial^{-2} + \cdots$$

be the Lax operator for $u_{\rm BGW}$, and let Φ be the dressing operator for $u_{\rm BGW}$ such that Φ and $\tau_{\rm BGW}$ form a dressing pair. The identities in the following lemma are analogues of (2.18).

Lemma 4.1 *The following identities hold true:* $\forall\, i,k\geq 0$,

$$\operatorname{res}_\partial (M-1)^i\circ L^{\frac{k}{r}} = \frac{1}{i+1}\partial\left(\frac{\operatorname{res}_\lambda \lambda^k\partial_\mu^{i+1}\circ X(\tilde{\mathbf{t}}_{\rm KP};\lambda,\mu)\,(\tau_{\rm BGW})\,\big|_{\mu=\lambda}}{\tau_{\rm BGW}}\right), \tag{4.1}$$

where the operator M *is given by* (2.16), *and* $\tilde{\mathbf{t}}_{\rm KP}=(t_1-1,t_2,t_3,\dots)$.

Proof Recalling the definition (2.14), we have

$$\partial_\mu^{i+1}\circ X\big(\tilde{\mathbf{t}}_{\rm KP};\lambda,\mu\big) = \sum_{j=0}^{i+1}\binom{i+1}{j}(-1)^{i+1-j}\partial_\mu^j\circ X(\mathbf{t}_{\rm KP};\lambda,\mu). \tag{4.2}$$

By using (2.18), one can then write the right-hand side of (4.1) into

$$\frac{1}{i+1}\sum_{j=1}^{i+1}\binom{i+1}{j}(-1)^{i+1-j}jM^{j-1}\circ L_{\rm KP}^k = \operatorname{res}_\partial (M-1)^i\circ L_{\rm KP}^k. \tag{4.3}$$

By noticing that $L_{\rm KP}=L^{\frac{1}{r}}$, the lemma is proved. □

Lemma 4.2 *The following identities hold true:*

$$\left((M-1)^i\circ L^{k+\frac{i}{r}}\right)_-=0,\quad \forall\, i,k\geq 0. \tag{4.4}$$

Proof The case $i=0$ is obvious. For $i\geq 1$, let us first show $\big((M-1)L^{1/r}\big)_-=0$. Observe that

$$\sum_{i\geq 1} it_i\frac{\partial}{\partial t_i}-\frac{\partial}{\partial t_1}+\frac{d_1}{r} = \operatorname{res}_\lambda\left(\frac{\lambda}{2}\partial_\mu^2 X(\mathbf{t}_{\rm KP};\lambda,\mu)-\lambda\partial_\mu X(\mathbf{t}_{\rm KP};\lambda,\mu)+\frac{d_1}{\lambda r}\right)\bigg|_{\mu=\lambda} d\lambda =: G(\mathbf{t}_{\rm KP}). \tag{4.5}$$

By identity (1.4), we have $G(\mathbf{t}_{\rm KP})(\tau_{\rm BGW})=0$. Therefore,

$$\begin{aligned} 0 &= \operatorname{res}_\nu \frac{X(\mathbf{t}_{\rm KP};\nu)\circ G(\mathbf{t}_{\rm KP})(\tau_{\rm BGW}(\mathbf{t}_{\rm KP}))}{\tau_{\rm BGW}(\mathbf{t}_{\rm KP})}\frac{X^*(\mathbf{t}'_{\rm KP};\nu)\,(\tau_{\rm BGW}(\mathbf{t}'_{\rm KP}))}{\tau_{\rm BGW}(\mathbf{t}'_{\rm KP})}d\nu \\ &= -\operatorname{res}_\lambda\big(\lambda\partial_\lambda(\psi(\mathbf{t}_{\rm KP};\lambda))\psi^*(\mathbf{t}'_{\rm KP};\lambda)-\lambda\psi(\mathbf{t}_{\rm KP};\lambda)\psi^*(\mathbf{t}'_{\rm KP};\lambda)\big)\,d\lambda. \end{aligned} \tag{4.6}$$

Here the second equality used the identity (2.15). By using the facts

$$M(\psi)=\partial_\lambda(\psi),\quad L_{\rm KP}(\psi)=\lambda\psi,$$

as well as the following identity [12]: for arbitrary x,x' and arbitrary $t_2=t_2',t_3=t_3',\dots$,

$$\begin{aligned} &\big(U(\mathbf{t}_{\rm KP})V(\mathbf{t}'_{\rm KP})\big)_-\,\big(\delta(x-x')\big) \\ &\quad = -\operatorname{res}_z U(\mathbf{t}_{\rm KP})\left(e^{xz+\sum_{i\geq 2}t_iz^i}\right)V^*(\mathbf{t}'_{\rm KP})\left(e^{-x'z-\sum_{i\geq 2}t_iz^i}\right)dz\,H(x-x') \end{aligned} \tag{4.7}$$

with $U(\mathbf{t}_{\mathrm{KP}}), V(\mathbf{t}'_{\mathrm{KP}})$ being arbitrary pseudo-differential operators whose coefficients are power series of their arguments, we have

$$
\begin{aligned}
0 &= -\operatorname{res}_\lambda \big(\lambda\partial_\lambda(\psi(\mathbf{t}_{\mathrm{KP}};\lambda))\psi^*(\mathbf{t}'_{\mathrm{KP}};\lambda) - \lambda\psi(\mathbf{t}_{\mathrm{KP}};\lambda)\psi^*(\mathbf{t}'_{\mathrm{KP}};\lambda)\big)\, d\lambda\, H(x-x') \\
&= (M\circ L_{\mathrm{KP}} - L_{\mathrm{KP}})_- \big(\delta(x-x')\big) \\
&= \sum_{i\geq 1} \mathrm{Coef}_{\partial^{-i}}\Big(M\circ L_{\mathrm{KP}} - L_{\mathrm{KP}}\Big)\frac{(x-x')^{i-1}}{(i-1)!} H(x-x'). \qquad (4.8)
\end{aligned}
$$

Here $\delta(x)$ is the Dirac delta function, and $H(x)$ is the Heaviside unit step function. Therefore, $\big((M-1)\circ L_{\mathrm{KP}}\big)_- = 0$.

By using the fact that

$$
(M-1)^{i+1}\circ L^{k+\frac{i+1}{r}} = M^i\circ L^{k+\frac{i}{r}}\circ(M-1)\circ L - (kr+i)\,(M-1)^i\circ L^{k+\frac{i}{r}}, \quad \forall\, i,k\geq 0,
$$

we can prove by induction that identities (4.4) are true. □

From Lemmas 4.1 and 4.2, it follows that, for $\alpha = 1, 2, \ldots, r-1$ and $q\geq\alpha$,

$$
\partial_x\left(\frac{\operatorname{res}_\lambda \lambda^{\alpha+(q-\alpha)r}\Big(\partial_\mu^{\alpha+1}\circ X\big(\tilde{\mathbf{t}}_{\mathrm{KP}};\lambda,\mu\big)\Big)\,(\tau_{\mathrm{BGW}}(\mathbf{t}))\Big|_{\mu=\lambda}}{\tau_{\mathrm{BGW}}(\mathbf{t})}\right) = 0. \qquad (4.9)
$$

Denote

$$
X_{\mathrm{GD}}(\mathbf{t};\lambda,\mu) := e^{\sum_{i\in\mathbb{N}\setminus r\mathbb{N}} t_i(\mu^i-\lambda^i)} e^{\sum_{i\in\mathbb{N}\setminus r\mathbb{N}}\left(\frac{1}{i\lambda^i}-\frac{1}{i\mu^i}\right)\frac{\partial}{\partial t_i}}. \qquad (4.10)
$$

We have that

$$
\begin{aligned}
&\partial_\mu^{\alpha+1}\circ X_{\mathrm{GD}}(\mathbf{t};\lambda,\mu)\circ e^{\sum_{j\geq 1}\frac{1}{jr}\left(\frac{1}{\lambda^{jr}}-\frac{1}{\mu^{jr}}\right)\frac{\partial}{\partial t_{jr}}} \\
=&\partial_\mu^{\alpha+1}\circ e^{-\sum_{i\geq 1} t_{ir}(\mu^{ir}-\lambda^{ir})}\circ X(\mathbf{t}_{\mathrm{KP}};\lambda,\mu) \\
=&\sum_{k_1=0}^{\alpha+1}\binom{\alpha+1}{k_1}\partial_\mu^{k_1}\left(e^{-\sum_{j\geq 1} t_{jr}(\mu^{jr}-\lambda^{jr})}\right)\partial_\mu^{\alpha+1-k_1}\circ X(\mathbf{t}_{\mathrm{KP}};\lambda,\mu) \\
=&\sum_{k_1=0}^{\alpha+1}\sum_{p\geq 1} f_{k_1,p}\mu^{pr-k_1}\partial_\mu^{\alpha+1-k_1}\circ X(\mathbf{t}_{\mathrm{KP}};\lambda,\mu),
\end{aligned}
$$

where $f_{k_1,p} = f_{k_1,p}(t_r, t_{2r},\ldots)\in\mathbb{C}[[t_r,t_{2r},\ldots]]$. By using this identity, one can obtain that

$$
\begin{aligned}
&\frac{1}{\tau_{\mathrm{BGW}}(\mathbf{t})}\operatorname{res}_{\lambda=\infty}\lambda^{\alpha+(q-\alpha)r}\Big(\partial_\mu^{\alpha+1}\circ X_{\mathrm{GD}}\big(\tilde{\mathbf{t}};\lambda,\mu\big)\big(\tau_{\mathrm{BGW}}(\mathbf{t})\big)\Big)\Big|_{\mu=\lambda} d\lambda \\
=&\frac{1}{\tau_{\mathrm{BGW}}(\mathbf{t})}\sum_{j=0}^{\alpha+1}\sum_{p\geq 1} f_{j,p}\operatorname{res}_{\lambda=\infty}\lambda^{\alpha-j+(q-\alpha+p)r}\Big(\partial_\mu^{\alpha-j+1}\circ X\big(\tilde{\mathbf{t}}_{\mathrm{KP}};\lambda,\mu\big)\big(\tau_{\mathrm{BGW}}(\mathbf{t})\big)\Big)\Big|_{\mu=\lambda} d\lambda.
\end{aligned}
$$

Together with (4.9), it follows that

$$
\partial_x\left(\frac{W^{\mathrm{red}}_{\alpha,q}(\mathbf{t})\,(\tau_{\mathrm{BGW}}(\mathbf{t}))}{\tau_{\mathrm{BGW}}(\mathbf{t})}\right) = 0, \qquad \alpha = 1,\ldots,r-1, \quad q\geq\alpha, \qquad (4.11)
$$

where the operators $W^{\rm red}_{\alpha,q}(\mathbf{t})$ are defined in (1.5).

We are ready to prove Theorem 1.1.

Proof of Theorem 1.1 From formula (4.11), we know that there exist power series $c_{\alpha,q}(\mathbf{t})$, independent of $x = t_1$, such that

$$W^{\rm red}_{\alpha,q}(\mathbf{t})\left(\tau_{\rm BGW}(\mathbf{t})\right) = c_{\alpha,q}(\mathbf{t})\tau_{\rm BGW}(\mathbf{t}). \tag{4.12}$$

By using Theorem 3.2 and the fact that $W^{\rm red}_{1,1}(\mathbf{t}) = \sum_{i\in\mathbb{N}\backslash r\mathbb{N}} i\tilde{t}_i\frac{\partial}{\partial t_i}$, we have $\rho_1 = d_1/r$. By the definition (1.5), one can verify that

$$\left[W^{\rm red}_{1,1}(\mathbf{t}), W^{\rm red}_{\alpha,q}(\mathbf{t})\right] = -(q-\alpha)rW^{\rm red}_{\alpha,q}(\mathbf{t}).$$

Applying the both sides of this identity onto $\tau_{\rm BGW}(\mathbf{t})$, we obtain that

$$W^{\rm red}_{1,1}(\mathbf{t})\left(c_{\alpha,q}(\mathbf{t})\right) = -(q-\alpha)rc_{\alpha,q}(\mathbf{t}).$$

This implies that $c_{\alpha,q}(\mathbf{t})$ are power series with non-positive degrees if we let the degree of t_i be assigned with i, $i \in \mathbb{Z}_{\geq 2}\backslash r\mathbb{N}$. So $c_{\alpha,q}(\mathbf{t})$ must be constants, moreover, these constants vanish if $q > \alpha$.

Let us proceed to prove the property (1.8). To this end, we first prove the following lemma.

Lemma 4.3 *Denote*

$$S_{\alpha,q} := \frac{1}{\alpha+1}\,{\rm res}_\lambda\,\lambda^{\alpha+(q-\alpha)r} : \left(\sum_{j\in\mathbb{N}\backslash r\mathbb{N}} j\lambda^{j-1}\tilde{t}_j + \sum_{j\in\mathbb{N}\backslash r\mathbb{N}} \lambda^{-j-1}\frac{\partial}{\partial t_j}\right)^{\alpha+1} :, \tag{4.13}$$

where $\alpha = 1, \ldots, n$, $q \geq 0$, and ": :" denotes the normal ordering (defined by putting the operators $\frac{\partial}{\partial t_j}$ on the right of operators $\tilde{t}_i$). The constraints (1.7) *can be equivalently written as*

$$S_{\alpha,q}\big(\tau_{\rm BGW}\big) = (-1)^\alpha\sigma_\alpha\delta_{\alpha,q}\tau_{\rm BGW}, \qquad \alpha = 1, \ldots, r-1,\ q \geq \alpha, \tag{4.14}$$

where $\sigma_1, \ldots, \sigma_{r-1}$ are certain polynomials of $\rho_1, \ldots, \rho_{r-1}$.

Proof We denote

$$a(\mathbf{t};\lambda) := \sum_{i\in\mathbb{N}\backslash r\mathbb{N}} \lambda^i t_i, \quad b(\mathbf{t};\lambda) := -\sum_{i\in\mathbb{N}\backslash r\mathbb{N}} \frac{1}{i\lambda^i}\frac{\partial}{\partial t_i},$$

and denote

$$P_i(\mathbf{t};\lambda) := \partial_\mu^{i+1}\left(e^{a(\mathbf{t};\mu)-a(\mathbf{t};\lambda)}e^{b(\mathbf{t};\mu)-b(\mathbf{t};\lambda)}\right)\Big|_{\mu=\lambda}, \quad i \geq 0. \tag{4.15}$$

It is easy to see that $P_i(\mathbf{t};\lambda)$ satisfy following the recursion relations:

$$P_i(\mathbf{t};\lambda) = \frac{\partial a(\mathbf{t};\lambda)}{\partial\lambda}\circ P_{i-1}(\mathbf{t};\lambda) + P_{i-1}(\mathbf{t};\lambda)\circ\frac{\partial b(\mathbf{t};\lambda)}{\partial\lambda} + \frac{\partial P_{i-1}(\mathbf{t};\lambda)}{\partial\lambda}. \tag{4.16}$$

By using the above (4.16), one can prove the following identity by induction:

$$P_i(\mathbf{t};\lambda) = \sum_{j=0}^{i-1}\frac{\partial^j}{\partial\lambda^j}\left(:\big(\partial_\lambda(a(\mathbf{t};\lambda)) + \partial_\lambda(b(\mathbf{t};\lambda))\big)^{i+1-j}:\right), \quad i \geq 1. \tag{4.17}$$

Then we have that, for $\alpha = 1, \dots, r-1$ and $q \geq \alpha$,

$$
\begin{aligned}
&\operatorname{res}_\lambda \lambda^{\alpha+(q-\alpha)r} : (\partial_\lambda a(\mathbf{t};\lambda) + \partial_\lambda b(\mathbf{t};\lambda))^{i+1} : d\lambda \\
=& \operatorname{res}_\lambda \lambda^{\alpha+(q-\alpha)r} P_\alpha(\mathbf{t};\lambda) - \sum_{j=1}^{\alpha-1} \operatorname{res}_\lambda \lambda^{\alpha+(q-\alpha)r} : \partial_\lambda^j (\partial_\lambda a(\mathbf{t};\lambda) + \partial_\lambda b(\mathbf{t};\lambda))^{\alpha+1-j} : d\lambda \\
=& \operatorname{res}_\lambda \lambda^{\alpha+(q-\alpha)r} P_\alpha(\mathbf{t};\lambda) d\lambda \\
&- \sum_{j=1}^{\alpha-1} \frac{(-1)^j (\alpha+(q-\alpha)r)!}{(\alpha+(q-\alpha)r-j)!} \operatorname{res}_\lambda \lambda^{\alpha-j+(q-\alpha)r} : (\partial_\lambda a(\mathbf{t};\lambda) + \partial_\lambda b(\mathbf{t};\lambda))^{\alpha+1-j} : d\lambda.
\end{aligned}
$$

By using the definition (4.13) and by noticing that $W^{\mathrm{red}}_{\alpha,q}(\mathbf{t})$ can be rewritten as

$$
W^{\mathrm{red}}_{\alpha,q}(\mathbf{t}) = \frac{1}{\alpha+1} \operatorname{res}_\lambda \lambda^{\alpha+(q-\alpha)r} P_\alpha(\mathbf{t};\lambda), \tag{4.18}
$$

we have

$$
S_{\alpha,q} = W^{\mathrm{red}}_{\alpha,q} - \sum_{j=1}^{\alpha-1} \frac{(-1)^j (\alpha-j+1)(\alpha+(q-\alpha)r)!}{(\alpha+(k-\alpha)r-j)!} S_{\alpha-j,q-j}. \tag{4.19}
$$

Therefore, by using Theorem 1.1 we obtain (1.7), where the constants $\sigma_1, \dots, \sigma_{r-1}$ can be uniquely determined by

$$
\sigma_\alpha = \rho_\alpha + \frac{\alpha!}{\alpha+1} \sum_{j=1}^{\alpha-1} \frac{(-1)^j}{j!} \sigma_j, \quad \alpha = 1, \dots, r-1. \tag{4.20}
$$

The lemma is proved. □

The following lemma will also be needed, and will also have other important applications. For simplicity, we denote

$$
\langle \tau_{i_1} \cdots \tau_{i_k} \rangle^\bullet := \left. \frac{\partial^k \tau_{\mathrm{BGW}}}{\partial t_{i_1} \dots \partial t_{i_k}} \right|_{\mathbf{t}=\mathbf{0}}, \quad \langle \tau_{i_1} \cdots \tau_{i_k} \rangle := \left. \frac{\partial^k \log \tau_{\mathrm{BGW}}}{\partial t_{i_1} \dots \partial t_{i_k}} \right|_{\mathbf{t}=\mathbf{0}}.
$$

Lemma 4.4 *The system* (1.7) *(or equivalently* (4.14)*) has a unique solution in* $\mathbb{C}[[\mathbf{t}]]$ *with initial value* 1.

Proof The existence of the solution is already proved. To show the uniqueness, we use the argument similar to that in [2, 9, 36]. By (4.13), we know that

$$
\begin{aligned}
S_{\alpha,q} =& \sum_{j=0}^{\alpha} \frac{\alpha!}{j!(\alpha+1-j)!} \sum_{\substack{k_{j+1}+\cdots+k_{\alpha+1} \\ -k_1-\cdots-k_j=(q-\alpha)r}} k_1 \cdots k_j \tilde{t}_{k_1} \cdots \tilde{t}_{k_j} \frac{\partial^{\alpha-j}}{\partial t_{k_{j+1}} \cdots \partial t_{k_{\alpha+1}}} \\
=& \sum_{j=0}^{\alpha} (-1)^j \binom{\alpha}{j} \sum_{p=0}^{\alpha-j} \sum_{\substack{k_{p+1}+\cdots+k_{\alpha-j+1} \\ -k_1-\cdots-k_p=(q-\alpha)r+j}} k_1 \cdots k_p t_{k_1} \cdots t_{k_p} \frac{\partial^{\alpha-j-p}}{\partial t_{k_{p+1}} \cdots \partial t_{k_{\alpha-j+1}}},
\end{aligned}
$$

where $\alpha = 1, \ldots, r-1$ and $q \geq \alpha$. Hence equations (4.14) can be recast to

$$\frac{\partial \tau_{\rm BGW}}{\partial t_{\alpha+(q-\alpha)r}} = \sigma_\alpha \delta_{q,\alpha} \tau_{\rm BGW} + \sum_{j=0}^{\alpha} (-1)^{\alpha-j} \binom{\alpha}{j} \sum_{p=0}^{\alpha-j} \sum_{\substack{k_{p+1}+\cdots+k_{\alpha-j+1} \\ -k_1-\cdots-k_p=(q-\alpha)r+j}} \frac{\partial^{\alpha-j-p} \tau_{\rm BGW}}{\partial t_{k_{p+1}} \cdots \partial t_{k_{\alpha-j+1}}} \prod_{a=1}^{p} k_a t_{k_a}. \tag{4.21}$$

In terms of $\langle \tau_{i_1} \cdots \tau_{i_k} \rangle^\bullet$, we have the recursion relations:

$$\begin{aligned} &\langle \tau_A \tau_{\alpha+(m-\alpha)r} \rangle^\bullet \\ =& \langle \tau_A \rangle^\bullet + \sum_{j=0}^{\alpha} (-1)^{\alpha-j} \binom{\alpha}{j} \sum_{p=0}^{\alpha-j} \sum_{\substack{k_{p+1}+\cdots+k_{(\alpha-j)+1} \\ -k_1-\cdots-k_p=(m-\alpha)r+j}} p! k_1 \cdots k_p \langle \tau_{A \setminus \{k_1,\ldots,k_p\} \cup \{k_{p+1},\ldots,k_{\alpha-j+1}\}} \rangle^\bullet. \end{aligned} \tag{4.22}$$

Here $\tau_A := \tau_{a_1} \cdots \tau_{a_N}$ for $A = \{a_1, \ldots, a_N\}$. In this way it is clear that all the coefficients of $\langle \tau_{i_1} \cdots \tau_{i_k} \rangle^\bullet$ can be uniquely determined. The lemma is proved. □

(The uniqueness statement in the above Lemma 4.4 can also be proved directly from (1.7).)

By using Lemma 4.3 and formula (4.22), we obtain that

$$\langle \tau_1 \tau_\alpha \rangle = \langle \tau_\alpha \rangle = c_\alpha(\sigma_1, \ldots, \sigma_\alpha) = \sigma_\alpha + \gamma_\alpha(\sigma_1, \ldots, \sigma_{\alpha-1}), \tag{4.23}$$

where c_α are certain polynomials of $\sigma_1, \ldots, \sigma_\alpha$. The property (1.8) then follows from (4.20) and the relation

$$\frac{\partial \log \tau_{\rm BGW}}{\partial t_1 \partial t_\alpha} = \Omega_{1,\alpha}(v_{\rm BGW}, v_{x,{\rm BGW}}, \ldots) = \operatorname{res}_\partial L^{\frac{\alpha}{r}}. \tag{4.24}$$

The theorem is proved. □

We note that, the constants c_α in the above proof are initial values of the *normal coordinates* $\Omega_{1,\alpha} = \operatorname{res}_\partial L^{\alpha/r}$ for the GD hierarchy, i.e., of the corresponding Dubrovin–Zhang hierarchy [19] (cf. also [8]).

By using Theorem 1.1, Lemma 4.3 and 4.4, we arrive at the following theorem.

Theorem 4.5 *A power series $\tau \in \mathbb{C}[[\mathbf{t}]]$ satisfies* (1.7) *if and only if τ is the tau-function for the Gelfand–Dickey hierarchy satisfying* (1.4).

5 Examples

In this section, we use Theorem 1.1 (in particular Lemma 4.3) to compute $\tau_{\rm BGW}$ and $\log \tau_{\rm BGW}$.

Example 5.1 *For the case with $r = 2$, the constraints* (4.14) *give the following relations:*

$$\begin{aligned} \langle \prod_{i=1}^{N} \tau_{2a_i+1} \tau_{2m+1} \rangle^\bullet =& \sum_{i=1}^{N} (2a_i+1) \langle \tau_{2a_i+2m+1} \prod_{j \neq i} \tau_{2a_j+1} \rangle^\bullet + \frac{1}{2} \sum_{k_1+k_2=m-1} \langle \tau_{2k_1+1} \tau_{2k_2+1} \prod_{i=1}^{N} \tau_{2a_i+1} \rangle^\bullet \\ &+ \delta_{m,0} c_1 \langle \prod_{i=1}^{N} \tau_{2a_i+1} \rangle^\bullet, \end{aligned}$$

where $N, a_1, \ldots, a_N, m \geq 0$. The constants $c_1, d_1, \sigma_1, \rho_1$ are related by

$$\rho_1 = \sigma_1 = c_1, \quad d_1 = 2c_1. \tag{5.1}$$

We have

$$\langle \tau_1 \rangle^\bullet = c_1, \quad \langle \tau_1^2 \rangle^\bullet = c_1(c_1+1), \quad \langle \tau_3 \rangle^\bullet = \frac{1}{2}c_1(c_1+1), \quad \langle \tau_1^3 \rangle^\bullet = c_1(c_1+1)(c_1+2),$$
$$\langle \tau_1 \rangle = c_1, \quad \langle \tau_1^2 \rangle = c_1, \quad \langle \tau_3 \rangle = \frac{1}{2}c_1(c_1+1), \quad \langle \tau_1^3 \rangle = 2c_1, \quad \langle \tau_3\tau_1 \rangle = \frac{3}{2}c_1(c_1+1).$$

Example 5.2 *For the case with $r = 3$, the constants $c_\alpha, d_\alpha, \sigma_\alpha, \rho_\alpha$ are related by*

$$\begin{aligned}
&\rho_1 = \sigma_1 = c_1, \quad d_1 = 3c_1, \\
&\rho_2 = c_2 + \frac{2}{3}c_1, \quad \sigma_2 = c_2, \quad d_2 = \frac{3}{2}c_2 + 3c_1.
\end{aligned}$$

We have

$$\begin{aligned}
&\langle \tau_1 \rangle^\bullet = c_1, \quad \langle \tau_2 \rangle^\bullet = c_2, \quad \langle \tau_1^2 \rangle^\bullet = c_1(c_1+1), \\
&\langle \tau_2\tau_1 \rangle^\bullet = (c_1+2)\frac{c_2}{2}, \quad \langle \tau_1^3 \rangle^\bullet = c_1(c_1+1)(c_1+2), \\
&\langle \tau_4 \rangle^\bullet = c_2(c_1+2), \quad \langle \tau_2^2 \rangle^\bullet = c_2^2 - 2c_1(c_1+1), \\
&\langle \tau_2\tau_1^2 \rangle^\bullet = c_2(c_1+2)(c_1+3), \quad \langle \tau_1^4 \rangle^\bullet = c_1(c_1+1)(c_1+2)(c_1+3), \\
&\langle \tau_1 \rangle = c_1, \quad \langle \tau_2 \rangle = c_2, \quad \langle \tau_1^2 \rangle = c_1, \quad \langle \tau_2\tau_1 \rangle = 2c_2, \quad \langle \tau_1^3 \rangle = 2c_1, \\
&\langle \tau_4 \rangle = c_2(c_1+2), \quad \langle \tau_2^2 \rangle = -2c_1(c_1+1), \quad \langle \tau_2\tau_1^2 \rangle = 6c_2, \quad \langle \tau_1^4 \rangle = 6c_1.
\end{aligned}$$

Example 5.3 *Similarly, for the case with $r = 4$, we have*

$$\begin{aligned}
&\rho_1 = \sigma_1 = c_1, \quad d_1 = 4c_1, \\
&\rho_2 = c_2 + \frac{2}{3}c_1, \quad \sigma_2 = c_2 \quad d_2 = 4c_2 + 8c_1, \\
&\rho_3 = c_3 - \frac{3}{4}c_2 - \frac{3}{2}c_1^2, \quad \sigma_3 = c_3 - \frac{3}{2}c_1^2 - \frac{3}{2}c_1 \quad d_3 = \frac{4}{3}c_3 + 3c_2 + 2c_1^2 + 10c_1,
\end{aligned}$$

and

$$\begin{aligned}
&\langle \tau_1 \rangle^\bullet = c_1, \quad \langle \tau_2 \rangle^\bullet = c_2, \quad \langle \tau_1^2 \rangle^\bullet = c_1(c_1+1), \quad \langle \tau_3 \rangle^\bullet = c_3, \\
&\langle \tau_2\tau_1 \rangle^\bullet = c_2(c_1+2), \quad \langle \tau_1^3 \rangle^\bullet = c_1(c_1+1)(c_1+2), \\
&\langle \tau_3\tau_1 \rangle^\bullet = c_3(c_1+3), \quad \langle \tau_2^2 \rangle^\bullet = 4c_3 + c_2^2 - 2c_1(c_1+1), \\
&\langle \tau_2\tau_1^2 \rangle^\bullet = c_2(c_1+2)(c_1+3), \quad \langle \tau_1^4 \rangle^\bullet = c_1(c_1+1)(c_1+2)(c_1+3), \\
&\langle \tau_1 \rangle = c_1, \quad \langle \tau_2 \rangle = c_2, \quad \langle \tau_1^2 \rangle = c_1, \quad \langle \tau_3 \rangle = c_3, \quad \langle \tau_2\tau_1 \rangle = 2c_2, \quad \langle \tau_1^3 \rangle = 2c_1, \\
&\langle \tau_3\tau_1 \rangle = 3c_3, \quad \langle \tau_2^2 \rangle = 4c_3 - 2c_1(c_1+1), \quad \langle \tau_2\tau_1^2 \rangle = 6c_2, \quad \langle \tau_1^4 \rangle = 6c_1.
\end{aligned}$$

Remark 5.4 *Define*

$$\langle \tau_{i_1} \cdots \tau_{i_k} \rangle_\infty = \lim_{r\to\infty} \langle \tau_{i_1} \cdots \tau_{i_k} \rangle. \tag{5.2}$$

We call $\langle \tau_{i_1} \cdots \tau_{i_k} \rangle_\infty$ *the stabilized generalized BGW correlators. For example,*

$$\langle \tau_1 \rangle_\infty = c_1, \quad \langle \tau_2 \rangle_\infty = c_2, \quad \langle \tau_1^2 \rangle = c_1, \quad \langle \tau_3 \rangle_\infty = c_3,$$
$$\langle \tau_2 \tau_1 \rangle_\infty = 2c_2, \quad \langle \tau_1^3 \rangle_\infty = 2c_1, \quad \langle \tau_4 \rangle_\infty = c_4, \quad \langle \tau_3 \tau_1 \rangle_\infty = 3c_3,$$
$$\langle \tau_2^2 \rangle_\infty = 4c_3 - 2c_1(c_1+1), \quad \langle \tau_2 \tau_1^2 \rangle_\infty = 6c_2, \quad \langle \tau_1^4 \rangle_\infty = 6c_1.$$

The partition function of these stabilized correlators and its relation to the KP hierarchy deserves a further study.

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