

H^2 -Optimal Reduction of Positive Networks using Riemannian Augmented Lagrangian Method

Sota Misawa and Kazuhiro Sato

Abstract—In this study, we formulate the model reduction problem of a stable and positive network system as a constrained Riemannian optimization problem with the H^2 -error objective function of the original and reduced network systems. We improve the reduction performance of the clustering-based method, which is one of the most known methods for model reduction of positive network systems, by using the output of the clustering-based method as the initial point for the proposed method. The proposed method reduces the dimension of the network system while preserving the properties of stability, positivity, and interconnection structure by applying the Riemannian augmented Lagrangian method (RALM) and deriving the Riemannian gradient of the Lagrangian. To check the efficiency of our method, we conduct a numerical experiment and compare it with the clustering-based method in the sense of H^2 -error and H^∞ -error.

Index Terms—Positive network, structure-preserving model reduction, Riemannian optimization

I. INTRODUCTION

Model reduction is a crucial step in designing controllers for large-scale network systems, and thus some reduction methods have been proposed such as the balanced truncation (BT) method [1] and the application of the iterative rational Krylov algorithm (IRKA) [2]–[6]. However, the BT and IRKA do not preserve the original interconnection structure in spite of the importance for controlling and monitoring [7]–[10]. To resolve this issue, the interconnection structure preserving model reduction methods for network systems have been proposed for a few decades. For example, [11] proposed to preserve the scale-free property of networks by formulating the interconnection constraints as the eigenvector centrality. In this method, the reduced network remains a flow network if the initial network is a flow network. Furthermore, [12] introduced network reduction methods which preserve the interconnection structure of subsystems.

Moreover, model reduction methods of positive network systems whose outputs are always nonnegative under nonnegative inputs are important, because the systems are often found in real world applications such as pharmacokinetics, metabolism, epidemiology, ecology, and logistics [13], [14]. In a positivity-preserving manner, some model reduction methods have been proposed in [15]–[17]. For instance, the clustering-based method [16], [17] is one of the most known methods which preserves the positivity. Furthermore, in [16], [17], the theoretical bounds of the H^2 -error between the original and

reduced systems are provided. Nonetheless, the clustering-based method does not guarantee the H^2 -optimality between the original and reduced network systems.

The H^2 optimal model reduction methods have been proposed in [18]–[21] based on Riemannian optimization [22]. In particular, in [21], the set of stable matrices was endowed with the geometry of a Riemannian manifold, as summarized in Section II. D in this paper. That is, the Riemannian optimization algorithm in [21] always produces a reduced asymptotically stable system at each iteration. However, the above methods are not appropriate for network systems, because the resulting reduced systems do not preserve the original interconnection structure. That is, it is difficult to physically interpret the resulting reduced model.

For the H^2 optimal network system reduction, [23] proposed the H^2 optimal reduction method for linear consensus networks consisting of diffusively coupled single-integrators, which uses the clustering-based model reduction method as an initial network of the algorithm and aims to minimize the H^2 error by selecting suitable edge weights of the reduced network. Moreover, the method preserves the interconnection structure of the original network system. However, since this method is based on matrix inequalities, it is difficult to simply extend to general positive networks to preserve the positive property.

Therefore, to reduce large-scale asymptotically stable positive network systems, we formulate a novel H^2 optimization problem as a Riemannian optimization problem with constraints. The introduction of the constraints is for preserving the positivity and interconnection structure of the original network system, and is the major difference with the existing problem formulations in [18]–[21]. To define the constraints, we use the result of a clustering-based model reduction method. That is, the problem formulation in this paper can be regarded as a generalization of that in [21]. The main contribution is to develop the Riemannian augmented Lagrangian method (RALM) [24] for solving the problem. That is, the RALM preserves not only the stability and positivity but also the interconnection structure of the original system. To this end, we derive the Riemannian gradients, that are different from those of the objective function in [21], of the Lagrangian composed of the objective function and a penalty term.

The remainder of this paper is organized as follows. In Section II, we describe the preliminary knowledge about asymptotically stable positive network systems, clustering-based network reduction, and the Riemannian manifold of stable matrices. In Section III, we formulate a novel H^2 optimal model reduction problem for preserving the positivity and

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interconnection structure of the original network system as a Riemannian optimization problem with constraints. In Section IV, we propose the RALM algorithm by deriving the gradients of the Lagrangian. In Section V, to illustrate the effectiveness of our method, we conduct a numerical experiment on a stable positive network and compare it with the clustering-based method from the viewpoints of H^2 -error and H^∞ -error. Finally, our conclusions are presented in Section VI.

II. PRELIMINARIES

A. Notation

For a Riemannian manifold M , the tangent space at $x \in M$ is denoted by $T_x M$. We remark $\langle \cdot, \cdot \rangle_x^{(M)} : T_x M \times T_x M \rightarrow \mathbb{R}$ is an inner product at $x \in M$. For a vector $v \in \mathbb{R}^m$, $\|v\|_2$ denotes the usual Euclidean norm. The L^2 space on \mathbb{R}^m for $m \in \mathbb{Z}_{>0}$ is denoted by $L^2(\mathbb{R}^m)$ with the norm $\|f\|_{L^2} := \sqrt{\int_0^\infty \|f(t)\|_2^2 dt}$, where $f : [0, \infty) \rightarrow \mathbb{R}^m$ is a measurable function. The H^2 and H^∞ norms of a linear system whose transfer function is G are defined by $\|G\|_{H^2} = \sqrt{\frac{1}{2\pi} \int_{-\infty}^\infty \|G(i\omega)\|_F^2 d\omega}$ and $\|G\|_{H^\infty} = \sup_{\omega \in \mathbb{R}} \sigma_{\max}(G(i\omega))$, respectively, where $\|\cdot\|_F$ is the Frobenius norm and $\sigma_{\max}(G(i\omega))$ denotes the maximum singular value of $G(i\omega)$.

B. Asymptotically Stable Positive Network System

In this paper, we consider

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) \end{cases} \quad (1)$$

as the original large-scale network system with the state $x(t) \in \mathbb{R}^n$, input $u(t) \in \mathbb{R}^m$, output $y(t) \in \mathbb{R}^p$, and coefficient matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, and $C \in \mathbb{R}^{p \times n}$.

We assume the following:

- 1) The original network system (1) is asymptotically stable. That is, the real parts of all the eigenvalues of the matrix A are negative. In this case, we call A a stable matrix. The matrix A is Metzler. That is, every off-diagonal entry of A is nonnegative.
- 2) The matrices B and C are nonnegative.

Assumptions 2) and 3) mean that the original network system (1) is essentially positive, as shown in [13]. That is, not only the output $y(t)$ but also the state $x(t)$ is nonnegative under the nonnegative input $u(t)$ and initial state $x(0)$. In fact, the solution to system (1) is given by $x(t) = \exp(At)x(0) + \int_0^t \exp(A(t-\tau))Bu(\tau)d\tau$. If A is Metzler, $\exp(At)$ for any $t \in \mathbb{R}$ is nonnegative. Thus, Assumptions 2) and 3) imply that $x(t)$ and $y(t)$ are nonnegative under the nonnegative input $u(t)$ and initial state $x(0)$.

We denote the original network graph by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$.

C. Network Reduction Based on Clustering

For fixed $r (< n)$, we reduce the original system (1) to a r -dimensional system

$$\begin{cases} \dot{x}_r(t) = A_r x_r(t) + B_r u(t) \\ y_r(t) = C_r x_r(t), \end{cases} \quad (2)$$

where $x_r(t) \in \mathbb{R}^r$, $y_r(t) \in \mathbb{R}^p$, and $(A_r, B_r, C_r) \in \mathbb{R}^{r \times r} \times \mathbb{R}^{r \times m} \times \mathbb{R}^{p \times r}$. We define $\mathcal{G}_r = (\mathcal{V}_r, \mathcal{E}_r)$ as the reduced network graph associated to system (2).

As shown in [15], we can obtain a reduced model by aggregating clusters $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_r$, that are illustrated in Fig. 1, into the nodes of \mathcal{G}_r as follows.

$$\begin{cases} \Pi^\top \Pi \dot{x}_r(t) = \Pi^\top A \Pi x_r(t) + \Pi^\top B u(t), \\ y_r(t) = C \Pi x_r(t), \end{cases} \quad (3)$$

where $\Pi \in \mathbb{R}^{n \times r}$ is the characteristic matrix and

$$(\Pi)_{ij} = \begin{cases} 1, & \text{if } i \in \mathcal{C}_j, \\ 0, & \text{otherwise.} \end{cases}$$

In this case, reduced system (3) has the same interconnection structure with the original network system (1). That is, if there is a directed path from $i \in \mathcal{V}_r$ to $j \in \mathcal{V}_r$, there is a directed path from a node of $\pi^{-1}(i)$ to a node of $\pi^{-1}(j)$ in \mathcal{G} . Here, $\pi : \mathcal{V} \rightarrow \mathcal{V}_r$ is the associated map to the characteristic matrix Π of the clustering $\{\mathcal{C}_1, \dots, \mathcal{C}_r\}$. We remark that $\Pi^\top \Pi$ is a diagonal matrix whose diagonal elements are the number of nodes in each cluster.

D. Riemannian Manifold of Stable Matrices

We will explain that the space of stable matrices \mathbb{S}_r can be regarded as a Riemannian manifold.

As shown in Proposition 1 of [25], for any stable matrix $A_r \in \mathbb{R}^{r \times r}$, there exists a $(J_r, R_r, Q_r) \in \mathbb{S}_r$ satisfying $A_r = (J_r - R_r)Q_r$. Conversely, for any $(J_r, R_r, Q_r) \in \mathbb{S}_r$, $(J_r - R_r)Q_r$ is stable. Here, $\mathbb{S}_r := \text{Skew}(r) \times \text{Sym}_+(r) \times \text{Sym}_+(r)$, $\text{Skew}(r)$ is the set of all skew symmetric matrices, and $\text{Sym}_+(r)$ is the set of all symmetric positive definite matrices.

The Euclidean space $\mathbb{R}^{k \times l}$ is a Riemannian manifold with the metric $\langle \xi, \eta \rangle_P^{(\mathbb{R}^{k \times l})} := \text{tr}(\xi^\top \eta)$. For $\text{Skew}(r)$, which is an Euclidean embedded submanifold endowed with the restricted metric $\langle \xi, \eta \rangle_P^{(\mathbb{R}^{r \times r})}|_{\text{Skew}(r)}$ and also linear subspace of $\mathbb{R}^{r \times r}$, the Riemannian gradient of $f : \text{Skew}(r) \rightarrow \mathbb{R}$ is calculated from the Euclidean gradient $\nabla \bar{f}(x)$ easily using the orthogonal projection onto the tangent space; $\text{grad } f(x) = \text{skew}(\nabla \bar{f}(x)) := \frac{1}{2}(\nabla \bar{f}(x) - \nabla \bar{f}(x)^\top)$. For more detail discussion, see [26, Chapter 3] and [22]. The Riemannian metric of $\text{Sym}_+(r)$ is defined as $\langle \xi, \eta \rangle_P^{(\text{Sym}_+(r))} := \text{tr}(P^{-1} \xi P^{-1} \eta)$, for any $P \in \text{Sym}_+(r)$ and $\xi, \eta \in T_P \text{Sym}_+(r)$ [27, Chapter XII]. The Riemannian manifold $\text{Sym}_+(r)$ with this metric has a closed form of the exponential map $\text{Exp}_P(\xi) = P \exp(P^{-1} \xi)$, where $\exp(P)$ is a matrix exponential, and the Riemannian gradient on $\text{Sym}_+(r)$ is calculated as $\text{grad } f(P) = P \text{sym}(\nabla \bar{f}(P)) P$ by letting $\text{sym}(S) := \frac{1}{2}(S + S^\top)$ [19].

The set \mathbb{S}_r is a product of the Riemannian manifolds $\text{Skew}(r)$ and $\text{Sym}_+(r)$. Thus, \mathbb{S}_r is a Riemannian manifold with the canonically induced metric.

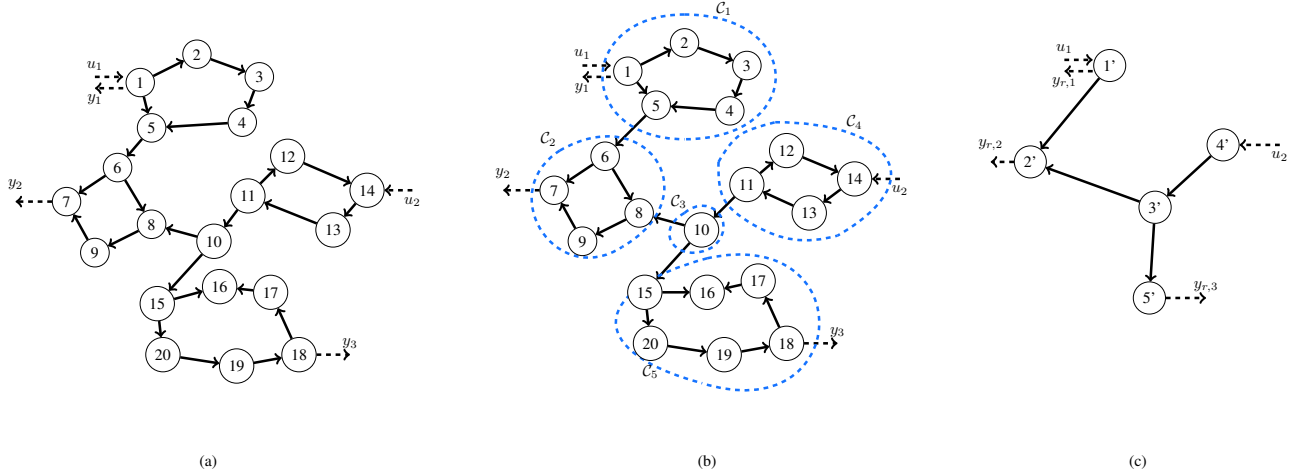


Fig. 1. An example of clustering-based network reduction. The original network (a) is clustered into (b) and the reduced network with the clusters is (c). The input and output structure after reduction is the same as the original network.

III. PROBLEM FORMULATION

A. Initial Reduced Network

To define the constraints for the interconnection structure, we calculate the reduced model (2) on $\mathcal{G}_r = (\mathcal{V}_r, \mathcal{E}_r)$ by using the clustering-based algorithm as follows;

$$\begin{cases} A_r^{(0)} := (\Pi^\top \Pi)^{-1} \Pi^\top A \Pi - \alpha I_r, \\ B_r^{(0)} := (\Pi^\top \Pi)^{-1} \Pi^\top B, \\ C_r^{(0)} := C \Pi \end{cases} \quad (4)$$

where $\alpha \geq 0$ is a sufficiently large constant such that $A_r^{(0)}$ is stable. Note that $(\Pi^\top \Pi)^{-1} \Pi^\top A \Pi$ is not stable in general, even if A is stable. For example, consider

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 6 & -2 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 10 & -5 \end{bmatrix}, \quad \Pi = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix},$$

where A is a stable and Metzler matrix. Then, $(\Pi^\top \Pi)^{-1} \Pi^\top A \Pi = \begin{bmatrix} \frac{1}{3} & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$, which is Metzler, but is not stable. It is also notable that the non-diagonal nonzero entries of A_r correspond to the edges of \mathcal{G}_r . Also, the nonzero entries of B_r and C_r imply the input and output position, respectively.

To preserve the interconnection structure, every entry of the feasible solution (A_r, B_r, C_r) should be zero if the corresponding entry of the initial matrix $(A_r^{(0)}, B_r^{(0)}, C_r^{(0)})$ is zero and should be nonnegative if the corresponding entry is nonnegative. For convenience, $z(X)$ denotes the indices sets of the zero-entries of the matrix X and $\text{nn}(X)$ denotes the indices sets of the nonnegative-entries of the matrix X for $X = A_r^{(0)}, B_r^{(0)}, C_r^{(0)}$. It is notable that $z(A_r^{(0)})$ and $\text{nn}(A_r^{(0)})$ do not contain the diagonal indices of $A_r^{(0)}$.

B. H^2 Optimization Problem with constraints

We consider an H^2 optimal model reduction problem using the transfer functions G of (1) and G_r of (2) to reconstruct a novel reduced model (A_r, B_r, C_r) of preserving the positivity and interconnection structure better than a given reduced model $(A_r^{(0)}, B_r^{(0)}, C_r^{(0)})$ in the sense of the H^2 norm. This is because $\sup_{t \geq 0} \|y(t) - y_r(t)\|_2 \leq \|G - G_r\|_{H^2}$ assuming that $\|u\|_{L^2} \leq 1$, as explained in [2], [21]. This inequality indicates that the maximum output error norm can be expected to become almost zero when $\|G - G_r\|_{H^2}^2 = 2F(A_r, B_r, C_r) + \|G\|_{H^2}^2$ is sufficiently small, where

$$\begin{aligned} F(A_r, B_r, C_r) &:= \frac{1}{2} \text{tr}(C_r S C_r^\top - 2C_r X^\top C^\top) \\ &= \frac{1}{2} \text{tr}(B_r^\top S B_r + 2B_r^\top Y B_r). \end{aligned} \quad (5)$$

Here, (X, Y, S, T) are the solutions to the following Sylvester equations

$$AX + XA_r^\top + BB_r^\top = 0, \quad (6)$$

$$A^\top Y + YA_r - C^\top C_r = 0, \quad (7)$$

$$A_r S + SA_r^\top + B_r B_r^\top = 0, \quad (8)$$

$$A_r^\top T + TA_r + C_r^\top C_r = 0. \quad (9)$$

Therefore, we formulate the following optimization problem.

$$\begin{aligned} &\underset{\substack{(J_r, R_r, Q_r), B_r, C_r \\ \in \mathcal{M}_r}}{\text{minimize}} && F(A_r, B_r, C_r) \\ &\text{subject to} && A_r := (J_r - R_r)Q_r, \\ & && g_{ij}^{(A_r)}(A_r) \leq 0, \forall (i, j) \in \text{nn}(A_r^{(0)}), \\ & && g_{ij}^{(B_r)}(B_r) \leq 0, \forall (i, j) \in \text{nn}(B_r^{(0)}), \\ & && g_{ij}^{(C_r)}(C_r) \leq 0, \forall (i, j) \in \text{nn}(C_r^{(0)}), \\ & && g_{ij}^{(A_r)}(A_r) = 0, \forall (i, j) \in z(A_r^{(0)}), \\ & && g_{ij}^{(B_r)}(B_r) = 0, \forall (i, j) \in z(B_r^{(0)}), \\ & && g_{ij}^{(C_r)}(C_r) = 0, \forall (i, j) \in z(C_r^{(0)}), \end{aligned} \quad (10)$$

where $M_r := \mathbb{S}_r \times \mathbb{R}^{r \times m} \times \mathbb{R}^{p \times r}$ and $g_{ij}^{(X)}(X) := -(X)_{ij}$ for $X = A_r, B_r, C_r$ and (i, j) is an index of X .

The problem (10) is an H^2 optimal model reduction problem with nonnegativity and interconnection structure-preserving constraints. The reduced system (2) corresponding to a feasible solution to (10) is always an asymptotically stable, positive, and has the same interconnection structure with the original system (1).

Moreover, the optimization problem (10) can be regarded as a Riemannian optimization problem with the constraints by introducing the Riemannian metric

$$\begin{aligned} \langle \eta, \xi \rangle_x^{(M_r)} := & \langle \xi, \eta \rangle_{J_r}^{(\mathbb{R}^{r \times r})} |_{\text{Skew}(r)} + \langle \xi, \eta \rangle_{R_r}^{(\text{Sym}_+(r))} \\ & + \langle \xi, \eta \rangle_{Q_r}^{(\text{Sym}_+(r))} + \langle \xi, \eta \rangle_{B_r}^{(\mathbb{R}^{r \times m})} + \langle \xi, \eta \rangle_{C_r}^{(\mathbb{R}^{p \times r})} \end{aligned} \quad (11)$$

into the set M_r , where $x = (J_r, R_r, Q_r, B_r, C_r)$, and $\xi = (\xi_1, \xi_2, \xi_3, \xi_4, \xi_5)$, $\eta = (\eta_1, \eta_2, \eta_3, \eta_4, \eta_5) \in T_x M_r$. Therefore, we can develop an algorithm for solving (10) based on Riemannian optimization [22].

Remark 1. According to Theorem 4.C.2 in [28], if the matrix A has a dominant diagonal that is negative, A is stable. Using this fact, we can formulate another optimization problem by adding the inequality constraints to enforce the strict diagonally dominance. However, this is just a sufficient condition for A to be stable unlike our formulation in (10). That is, our formulation is useful to decrease the objective function value compared with the addition of the inequality constraints, because the search space of A_r is wider.

IV. PROPOSED METHOD

Because the optimization problem (10) is a Riemannian optimization problem with constraints, we develop an algorithm for solving (10) based on RALM proposed in [24].

The Lagrangian function of (10) for RALM is as follows:

$$\begin{aligned} \mathcal{L}_\rho((J_r, R_r, Q_r, B_r, C_r), \lambda, \gamma) = & F(A_r, B_r, C_r) \\ & + \frac{\rho}{2} \sum_{V \in \{A_r, B_r, C_r\}} \left(\sum_{(i,j) \in \text{nn}(V^{(0)})} \max \left\{ 0, \frac{\lambda_{ij}^{(V)}}{\rho} + g_{ij}^{(V)}(V) \right\} \right. \\ & \left. + \sum_{(i,j) \in \text{z}(V^{(0)})} \left(g_{ij}^{(V)}(V) + \frac{\gamma_{ij}^{(V)}}{\rho} \right)^2 \right), \end{aligned}$$

where $\rho > 0$ is a penalty parameter and $\gamma_{ij}^{A_r} \in \mathbb{R}^{r \times r}$, $\gamma_{ij}^{B_r} \in \mathbb{R}^{r \times m}$, $\gamma_{ij}^{C_r} \in \mathbb{R}^{p \times r}$, $\lambda_{ij}^{A_r} \in \mathbb{R}_{\geq 0}^{r \times r}$, $\lambda_{ij}^{B_r} \in \mathbb{R}_{\geq 0}^{r \times m}$, and $\lambda_{ij}^{C_r} \in \mathbb{R}_{\geq 0}^{p \times r}$ are the hyper parameters of RALM.

The algorithm is shown in Algorithm 1. Here, in Algorithm 1, dist is the distance function on the Riemannian manifold M_r equipped with the Riemannian metric (11). That is,

$$\begin{aligned} \text{dist}^{(M_r)}(x, y)^2 = & \|J_r - J'_r\|_F^2 + \left\| \log R_r^{-1/2} R'_r R_r^{-1/2} \right\|_F^2 \\ & + \left\| \log Q_r^{-1/2} Q'_r Q_r^{-1/2} \right\|_F^2 + \|B_r - B'_r\|_F^2 + \|C_r - C'_r\|_F^2 \end{aligned}$$

where $x = (J_r, R_r, Q_r, B_r, C_r)$, $y = (J'_r, R'_r, Q'_r, B'_r, C'_r) \in M_r$ and $\xi = (\xi_1, \xi_2, \xi_3, \xi_4, \xi_5)$, $\eta = (\eta_1, \eta_2, \eta_3, \eta_4, \eta_5) \in T_x M_r$. For the details of step 9 in Algorithm 1, see [22].

To solve the subproblem of step 3 in Algorithm 1 by using a Riemannian line search method [22], we calculate the

Euclidean gradients of \mathcal{L}_ρ . As shown in [29], the Euclidean gradients \bar{F} with respect to A_r, B_r and C_r are calculated as

$$\begin{aligned} \nabla_{A_r} \bar{F} = TS + Y^T X, \quad \nabla_{B_r} \bar{F} = TB_r + Y^T B, \\ \nabla_{C_r} \bar{F} = C_r S - CX, \end{aligned}$$

respectively, where (X, Y, S, T) are the solutions to (6)-(9). Besides, it is easily seen that

$$\begin{aligned} \sum_{(i,j) \in \text{nn}(A_r^{(0)})} \nabla_{A_r} \bar{g}_{ij}^{(A_r)} &= (I_r - \mathbf{1}(r, r)) \odot \chi_{\text{nn}(A_r^{(0)})}, \\ \sum_{(i,j) \in \text{nn}(B_r^{(0)})} \nabla_{B_r} \bar{g}_{ij}^{(B_r)} &= -\chi_{\text{nn}(B_r^{(0)})}, \\ \sum_{(i,j) \in \text{nn}(C_r^{(0)})} \nabla_{C_r} \bar{g}_{ij}^{(C_r)} &= -\chi_{\text{nn}(C_r^{(0)})}, \\ \sum_{(i,j) \in \text{z}(A_r^{(0)})} \nabla_{A_r} \bar{g}_{ij}^{(A_r)} &= (I_r - \mathbf{1}(r, r)) \odot \chi_{\text{z}(A_r^{(0)})}, \\ \sum_{(i,j) \in \text{z}(B_r^{(0)})} \nabla_{B_r} \bar{g}_{ij}^{(B_r)} &= -\chi_{\text{z}(B_r^{(0)})}, \\ \sum_{(i,j) \in \text{z}(C_r^{(0)})} \nabla_{C_r} \bar{g}_{ij}^{(C_r)} &= -\chi_{\text{z}(C_r^{(0)})}, \end{aligned}$$

where \odot denotes element-wise product and $\mathbf{1}(k, l)$ is a $k \times l$ matrix whose entries are all 1. Here, $\chi_{\text{nn}(V)}$ and $\chi_{\text{z}(V)}$ are matrices of the same shape as (V) , being defined as

$$\begin{aligned} (\chi_{\text{nn}(V)})_{i,j} &= \begin{cases} 1, & \text{if } (i, j) \in \text{nn}(V), \\ 0, & \text{otherwise,} \end{cases} \\ (\chi_{\text{z}(V)})_{i,j} &= \begin{cases} 1, & \text{if } (i, j) \in \text{z}(V), \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

for $V \in \{A_r^{(0)}, B_r^{(0)}, C_r^{(0)}\}$. Then, the Euclidean gradient of the Lagrangian is written as

$$\begin{aligned} \nabla_{A_r} \bar{\mathcal{L}}_\rho((J_r, R_r, Q_r, B_r, C_r), \lambda, \gamma) &= \nabla_{A_r} \bar{F}(A_r, B_r, C_r) \\ &+ \left(U_i^{(A_r)} \odot \chi_{\text{nn}(A_r^{(0)})} + U_e^{(A_r)} \odot \chi_{\text{z}(A_r^{(0)})} \right) \\ &\odot (I_r - \mathbf{1}(r, r)), \\ \nabla_{B_r} \bar{\mathcal{L}}_\rho((J_r, R_r, Q_r, B_r, C_r), \lambda, \gamma) &= \nabla_{B_r} \bar{F}(A_r, B_r, C_r) \\ &- \left(U_i^{(B_r)} \odot \chi_{\text{nn}(B_r^{(0)})} + U_e^{(B_r)} \odot \chi_{\text{z}(B_r^{(0)})} \right), \\ \nabla_{C_r} \bar{\mathcal{L}}_\rho((J_r, R_r, Q_r, B_r, C_r), \lambda, \gamma) &= \nabla_{C_r} \bar{F}(A_r, B_r, C_r) \\ &- \left(U_i^{(C_r)} \odot \chi_{\text{nn}(C_r^{(0)})} + U_e^{(C_r)} \odot \chi_{\text{z}(C_r^{(0)})} \right), \end{aligned}$$

and

$$\begin{aligned} U_i^{(V)} &:= \left(\max \left\{ 0, \frac{\lambda_{ij}^{(V)}}{\rho} + g_{ij}^{(V)}(V) \right\} \right)_{(i,j) \in \mathcal{I}_V} \\ U_e^{(V)} &:= \left(\frac{\lambda_{ij}^{(V)}}{\rho} + g_{ij}^{(V)}(V) \right)_{(i,j) \in \mathcal{I}_V} \end{aligned}$$

are matrices for each $V \in \{A_r, B_r, C_r\}$. Using the chain rule, we obtain the Euclidean gradients $\nabla_{J_r} \tilde{\mathcal{L}}_\rho = (\nabla_{A_r} \tilde{\mathcal{L}}_\rho) Q_r$, $\nabla_{R_r} \tilde{\mathcal{L}}_\rho = -(\nabla_{A_r} \tilde{\mathcal{L}}_\rho) Q_r$, and $\nabla_{Q_r} \tilde{\mathcal{L}}_\rho = -(J_r + R_r)(\nabla_{A_r} \tilde{\mathcal{L}}_\rho)$.

Finally, we calculate the Riemannian gradients from the Euclidean gradients in the same manner described in Section II-D. The Riemannian gradients are used to solve the subproblem of step 3 in Algorithm 1.

Algorithm 1 Riemannian augmented Lagrangian method (RALM).

Require: The function F in (5) and the constraints $\{g_{ij}^{(V)}\}_{(i,j) \in \mathcal{I}_V, V \in \{A_r, B_r, C_r\}}$ in (10) on Riemannian manifold M_r .

Input: Initial point $x_0 = (J^{(0)}, R^{(0)}, Q^{(0)}, B^{(0)}, C^{(0)}) \in M_r$; initial hyper parameters $\sigma = (\sigma_{A_r}, \sigma_{B_r}, \sigma_{C_r}) \in \mathbb{R}^{r \times r} \times \mathbb{R}^{r \times m} \times \mathbb{R}^{p \times r}$, $\rho \in \mathbb{R}_{\geq 0}$, $\lambda = (\lambda_{A_r}, \lambda_{B_r}, \lambda_{C_r}) \in \mathbb{R}_{> 0}^{r \times r} \times \mathbb{R}_{> 0}^{r \times m} \times \mathbb{R}_{> 0}^{p \times r}$, and $\gamma = (\gamma_{A_r}, \gamma_{B_r}, \gamma_{C_r}) \in \mathbb{R}^{r \times r} \times \mathbb{R}^{r \times m} \times \mathbb{R}^{p \times r}$; initial accuracy tolerance $\varepsilon > 0$; minimum tolerance $\varepsilon_{\min} > 0$ s.t. $\varepsilon_{\min} < \varepsilon$; constants $\theta_\rho > 1, \theta_\varepsilon \in (0, 1); \theta_\sigma \in (0, 1)$; boundary vectors $\lambda^{\min} = (\lambda_{A_r}^{\min}, \lambda_{B_r}^{\min}, \lambda_{C_r}^{\min}), \lambda^{\max} = (\lambda_{A_r}^{\max}, \lambda_{B_r}^{\max}, \lambda_{C_r}^{\max}) \in \mathbb{R}_{\geq 0}^{r \times r} \times \mathbb{R}_{\geq 0}^{r \times m} \times \mathbb{R}_{\geq 0}^{p \times r}$ s.t. $\lambda_V^{\min} \leq \lambda_V \leq \lambda_V^{\max}$, and $\gamma^{\min} = (\gamma_{A_r}^{\min}, \gamma_{B_r}^{\min}, \gamma_{C_r}^{\min}), \gamma^{\max} = (\gamma_{A_r}^{\max}, \gamma_{B_r}^{\max}, \gamma_{C_r}^{\max}) \in \mathbb{R}^{r \times r} \times \mathbb{R}^{r \times m} \times \mathbb{R}^{p \times r}$ s.t. $\gamma_V^{\min} \leq \gamma_V \leq \gamma_V^{\max}$ for $V \in \{A_r, B_r, C_r\}$; minimum step size $d_{\min} > 0$.

- 1: **for** $k = 0, 1, \dots$ **do**
 - 2: Solve the subproblem below using Riemannian line search methods within a tolerance ε :
 - 3:
$$\min_{x \in M_r} \mathcal{L}_\rho(x, \lambda, \gamma)$$
 - 4: and set the solution as $x_{k+1} = (J^{(k+1)}, R^{(k+1)}, Q^{(k+1)}, B^{(k+1)}, C^{(k+1)})$
 - 5: Set $A^{(k+1)} = (J^{(k+1)} - R^{(k+1)})Q^{(k+1)}$
 - 6: **if** $\text{dist}^{(M_r)}(x_k, x_{k+1}) < d_{\min}$ and $\varepsilon \leq \varepsilon_{\min}$ **then**
 - 7: **return** x_{k+1}
 - 8: **end if**
 - 9: Update the hyper parameters $\lambda, \gamma, \sigma, \rho$, and accuracy tolerance ε .
 - 10: **end for**
-

Remark 2. For the time complexity of the proposed method, the bottleneck of the algorithm is to calculate the objective function F and its gradients because F internally requires the solutions for large-scale Sylvester equations (6) and (7). However, in many applications, these Sylvester equations have the sparse-dense structure. That is,

- 1) the original matrix A is large-scale, but is sparse.
- 2) the reduced matrix A_r is small-scale, but is dense.

In this situation, we can use an efficient algorithm whose computational complexity is greatly smaller than $O(n^3)$ for solving (6) and (7) such as the method proposed in Section 3 in [30].

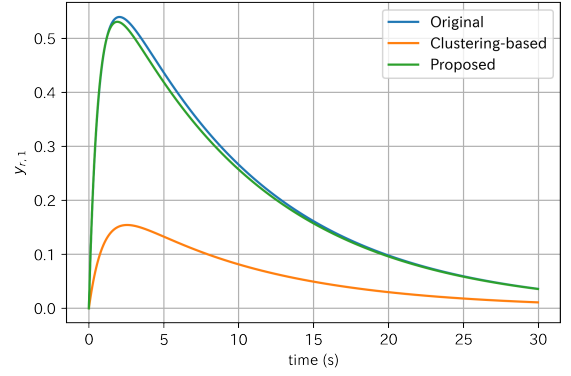


Fig. 2: The output $y_{r,1}(t)$ for each original system (blue), reduced model by clustering-based method (orange), and reduced model by proposed method (green)

V. EXPERIMENT

A. Experimental Conditions

We conducted a numerical experiment to verify the effectiveness of the proposed method in the sense of H^2 -error and H^∞ -error. We used the network shown in Fig. 1 with the random positive weight sampled independently from the uniform distribution on $[0, 1]$. Then, the coefficient matrices are

$$A = -\mathcal{L} - 0.1I_n,$$

$$B_{ij} = \begin{cases} 1 & \text{if } (i, j) = (1, 1) \text{ or } (14, 2) \\ 0 & \text{otherwise.} \end{cases}, \quad (12)$$

$$C_{ij} = \begin{cases} 1 & \text{if } (i, j) = (1, 1) \text{ or } (2, 7) \text{ or } (3, 18) \\ 0 & \text{otherwise.} \end{cases},$$

where \mathcal{L} is a loopy Laplacian of the random weighted network and the second term of A is for numerical stability. We used Riemannian conjugate gradient descent method [22] with the Riemannian gradients obtained in Section III as the subsolver in Algorithm 1. For each iteration, we used the after-100-iteration output of the Riemannian conjugate gradient descent method as its solution. The parameters for Algorithm 1 are shown in Table I.

We obtained the initial iterative point $(J_r^{(0)}, R_r^{(0)}, Q_r^{(0)}, B_r^{(0)}, C_r^{(0)})$ by the following way.

- 1) Calculate $A_r^{(0)}, B_r^{(0)}$, and $C_r^{(0)}$ using (4).
- 2) Solve the Lyapunov equation $A_r^{(0)\top} Q_r^{(0)} + Q_r^{(0)} A_r^{(0)} = -I_r$ for $Q_r^{(0)}$.
- 3) Calculate $J_r^{(0)} = \frac{1}{2}(A_r^{(0)}(Q_r^{(0)})^{-1} - (Q_r^{(0)})^{-1}A_r^{(0)\top})$ and $R_r^{(0)} = -\frac{1}{2}(A_r^{(0)}(Q_r^{(0)})^{-1} + (Q_r^{(0)})^{-1}A_r^{(0)\top})$.

Here, $(J_r^{(0)}, R_r^{(0)}, Q_r^{(0)}) \in \mathbb{S}_r$, as shown in [21].

We define the relative H^2 and H^∞ errors as

$$\text{Err}_2(G_r) := \frac{\|G - G_r\|_{H^2}}{\|G\|_{H^2}}, \quad \text{Err}_\infty(G_r) := \frac{\|G - G_r\|_{H^\infty}}{\|G\|_{H^\infty}}.$$

B. Result

The H^2 -error of clustering-based method was $\text{Err}_2(G_r) = 75.7\%$ and its H^∞ -error was $\text{Err}_\infty(G_r) = 70.24\%$. On

TABLE I
RALM PARAMETERS USED IN EXPERIMENT

| parameter | $\sigma^{(0)}$ | ρ_0 | $\lambda^{(0)}$ | λ^{\min} | λ^{\max} | $\gamma^{(0)}$ | γ^{\min} | γ^{\max} | ε_0 | ε_{\min} | θ_ρ | θ_ε | θ_σ | d_{\min} |
|-----------|----------------|----------|-----------------|------------------|------------------|----------------|-----------------|-----------------|-----------------|----------------------|---------------|----------------------|-----------------|------------|
| value | 0 | 10 | 1.0 | 0 | 10 | 0 | -1.5 | 1.5 | 1.0 | 10^{-16} | 1.01 | 0.95 | 0.9 | 10^{-8} |

the other hand, the H^2 -error of proposed method was $\text{Err}_2(G_r) = 3.14\%$ and its H^∞ -error was $\text{Err}_\infty(G_r) = 4.67\%$. It is easily seen that the proposed method improves the clustering-based method in the sense of not only H^2 -error but also H^∞ -error. Moreover, in Fig. 2, we illustrate the example outputs corresponding to the input

$$u(t) = \begin{pmatrix} \exp(-0.1t) |\cos(100\pi t)| \\ \exp(-0.1t) |\sin(200\pi t)| \end{pmatrix}.$$

VI. CONCLUSION

We formulated the model reduction problem of an asymptotically stable and positive network system as a constrained Riemannian optimization problem with H^2 -error of the original and reduced network systems as the objective function. Our method reduces the dimension of the network system while preserving the properties of stability, positivity, and interconnection structure by applying RALM and deriving the Riemannian gradients of the Lagrangian. We proposed to use the initial point of the clustering-based method. We conducted a numerical experiment and compare it with the clustering-based method in the sense of H^2 -error and H^∞ -error and verified that our method improved the reduction performance of the clustering-based method.

We note that our proposed algorithm can be easily extended to the case of a positive network system with multi-dimensional subsystems. Moreover, instead of an initial model generated by the clustering method as explained in Section II-C, we can use other arbitrary reduction methods which preserve stability, positivity, and interconnection structure, as the initial model of our proposed algorithm.

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