# ASYMPTOTIC SLOPES AND STRONG SEMISTABILITY ON SURFACES

### MITRA KOLEY AND A. J. PARAMESWARAN

ABSTRACT. In this article we study asymptotic slopes of strongly semistable vector bundles on a smooth projective surface. A connection between asymptotic slopes and strong restriction theorem of a strongly semistable vector bundle is shown. We also give an equivalent criterion of strong semistability of a vector bundle in terms of its asymptotic slopes under some assumptions on the surface and on the bundle.

#### 1. Introduction

Let X be a smooth/normal projective variety over an algebraically closed field  $\mathbb{K}$  and H be an ample line bundle on X. Let E be a vector bundle on X. A subbundle  $0 \neq F \subset E$  of rank k, is said to be maximal, if  $\deg F$  is maximal among all subbundles of rank k. Maximal subbundles of vector bundles over a smooth projective curve have been studied by many authors. Maximal line subbundles of a rank two bundle on a smooth projective curve have been studied in [9]. For higher rank vector bundles again on curves, maximal subbundles are studied in [12] and in many subsequent papers.

In [13], the second author and Subramanian studied the behavior of maximal subbundles of a vector bundle on a smooth projective curve after finite pull backs. We briefly discuss their results here. Let C be a smooth projective curve defined over an algebraically closed field  $\mathbb{K}$  of arbitrary characteristic. Let E be a vector bundle of rank r over C. For each  $1 \leq k < r$ , the slope of maximal subbundle is denoted by  $e_k(E)$ ,

$$e_k(E) := \operatorname{Max}\{\frac{\operatorname{deg}(W)}{k} \mid W \subset E \text{ is a subbundle of rank } k\}.$$

Define the asymptotic k-spectrum  $\mathcal{AS}_k(E)$  and the asymptotic k-slope  $\nu_k(E)$  as follows:

$$\mathcal{AS}_k(E) := \{ \frac{e_k(f^*(E))}{\deg f} \}.$$

$$\nu_k(E) := \text{Limsup } \frac{e_k(f^*(E))}{\deg f} = \text{Limsup } \mathcal{AS}_k(E).$$

where the supremum is taken over all finite morphisms  $f: D \to C$ . One of their main result is the following:

**Theorem 1.1** ([13, Theorem 4.1]). Let C be a smooth projective curve defined over an algebraically closed field  $\mathbb{K}$  of arbitrary characteristic and E be a vector bundle on C. Then E is strongly semistable if and only if  $\nu_k(E) = \mu(E)$  for some k. Moreover if  $\nu_k(E) = \mu(E)$  for some k, then  $\nu_j(E) = \mu(E)$  for all j.

<sup>2010</sup> Mathematics Subject Classification. 20C08, 20F55.

Key words and phrases. Asymptotic slopes; strong semistability.

Moreover in that paper [13], the authors gave an explicit formula of  $\nu_k(E)$  for an arbitrary vector bundle E in terms of degrees and ranks of the strong Harder-Narasimhan factors of E.

Here in this article we study asymptotic k-spectrum of vector bundles defined over a smooth projective surface and wanted to understand whether similar results still hold for smooth projective surfaces. First we show that even if one defines asymptotic k-slope of a vector bundle similar to curve case; one can not expect an analogue of Theorem 1.1. This is because such an analogues result for rank 2 vector bundles yields strong restriction theorem (Theorem 3.4). However we show that an analogues theorem for discriminant zero strongly semistable bundles with some more additional conditions on the bundle and on the underlying surface (see section 4 for more details).

The paper is organized as follows: in Section 2, we recollect relevant definitions and some useful facts about strong semistability and discriminants. In Section 3 we prove that an analogue theorem to Theorem 1.1 for rank 2 vector bundles implies strong restriction theorem (Theorem 3.4). Section 4 is devoted on the study of asymptotic k-spectrum of vector bundles of arbitrary rank with zero discriminants.

#### 2. Preliminaries

In this section we recall some definitions and some useful facts of relevant topics which we need in later sections. Here in general the underlying space is always assumed to be a normal projective surface.

Let X be a smooth/normal projective surface over an algebraically closed field  $\mathbb{K}$ . Let H be an ample line bundle on X and E be a torsion free sheaf of rank r defined over X. Then the slope of E with respect to H is defined by

$$\mu(E) = \frac{c_1(E) \cdot H}{r}.$$

A torsion free sheaf E on X is called semistable (resp. stable) if for every nonzero subsheaf W of E,  $\mu(W) \leq \mu(E)$  (resp.  $\mu(W) < \mu(E)$ ); equivalently for every torsion free quotient sheaf Q of E, we have  $\mu(Q) \geq \mu(E)$  (resp.  $\mu(Q) > \mu(E)$ ).

When  $\mathbb{K}$  is a field of characteristic p > 0, let  $F_X : X \to X$  denote the absolute Frobenius morphism. Then a vector bundle E over X is called strongly semistable (resp. strongly stable) if for all  $n \geq 0$ , the Frobenius pull back  $F_X^{n*}(E)$  of E is semistable (resp. stable).

Given a torsion free sheaf E, there exists a unique increasing filtration of torsion free sheaves (known as the Harder-Narasimhan filtration)

$$E_{\bullet}: \{0 = F_0 \subset F_1 \subset \cdots \subset F_l = E\}$$

such that for each i,  $F_i/F_{i-1}$  is a semistable torsion free sheaf with slope  $\mu_i$  satisfying  $\mu_i = \mu(\frac{F_i}{F_{i-1}}) > \mu_{i+1} = \mu(\frac{F_{i+1}}{F_i})$ . The torsion free sheaves  $\{F_i/F_{i-1}: 1 \leq i \leq l\}$  are called the Harder-Narasimhan factors

The torsion free sheaves  $\{F_i/F_{i-1}: 1 \leq i \leq l\}$  are called the Harder-Narasimhan factors of the bundle E. The factor  $F_1$  is called the maximal destabilizing subsheaf of E. It's slope  $\mu(F_1)$  is denoted by  $\mu_{max}(E)$ . The factor  $E/F_{l-1}$  is called the minimal destabilizing quotient of E and it's slope  $\mu(E/F_{l-1})$  is denoted by  $\mu_{min}(E)$ .

If K is a field of characteristic p > 0, then a filtration

$$E_{\bullet}: \{0 = F_0 \subset F_1 \subset \cdots \subset F_l = E\}$$

is called the *strong Harder-Narasimhan* filtration of E, if it is the Harder-Narasimhan filtration of E and for each i, the factor  $F_i/F_{i-1}$  is a strongly semistable sheaf. When X is smooth, by a theorem of Langer ([10]), for any torsion free sheaf E, there exists an  $n_0 \in \mathbb{N}$  such that for all  $F_X^{n_0*}(E)$  has strong Harder-Narasimhan filtration. Now we recall the following useful lemma.

**Lemma 2.1.** ([7, Lemma 1.3.3]) Let F and G be torsion free sheaves such that  $\mu_{min}(F) > \mu_{max}(G)$  then Hom(F, G) = 0.

Discriminant of a sheaf is an important invariant. Let E be a torsion free sheaf of rank r then the discriminant  $\Delta(E)$  of E is defined by

$$\Delta(E) = (r-1)c_1(E)^2 - 2rc_2(E),$$

where for each i,  $c_i(E)$  denotes the i-th Chern class of E. In particular, if E is a vector bundle of rank 2, then  $\Delta(E) = c_1(E)^2 - 4c_2(E)$ . In the following Proposition we write down a few facts about discriminant which we will use in the later sections. Let X be a smooth surface.

**Proposition 2.2.** (1) Let E be a torsion free sheaf and L be a line bundle on X, then

$$\Delta(E) = \Delta(E^*)$$
 and  $\Delta(E \otimes L) = \Delta(E)$ .

- (2) If E is a strongly semistable torsion free sheaf on X, then  $\Delta(E) \leq 0$ .
- (3) Let E be a vector bundle of rank r. If  $\Delta(E) = 0$ , then  $\Delta(S^n(E)) = 0$ , for all n > 0.

*Proof.* (1) is an easy computation and left to the reader.

- (2) When X is defined over a field of characteristic 0, then it follows from Theorem 3.4.1 of [7]. When X is defined over a field of prime characteristic p > 0, the proposition follows from Theorem 0.1 of [10].
- (3) This part may be known to experts. Since we are unable to find a reference, we include a proof for the convenience of reader.

By Lemma 10.1 of [2],

$$c_1(S^n(E)) = \binom{n+r-1}{r}c_1(E) = P_r(n)c_1(E).$$

$$c_2(S^n(E)) = P_{r+1}(n)[c_2(E) - \frac{r-1}{2r}c_1(E)^2] + \frac{1}{2}[P_r^2(n) - \frac{n}{r}P_r(n)]c_1(E)^2,$$

where  $P_r(n) = \binom{n+r-1}{r}$ . That is

$$c_2(S^n(E)) = -\frac{P_{r+1}(n)\Delta(E)}{2r} + \frac{1}{2}[P_r^2(n) - \frac{n}{r}P_r(n)]c_1(E)^2.$$

Since  $\Delta(E) = 0$ , in our case

$$c_2(S^n(E)) = \frac{1}{2}[P_r^2(n) - \frac{n}{r}P_r(n)]c_1(E)^2.$$

Note that  $\operatorname{rank}(S^n(E)) = \binom{n+r-1}{r-1}$ .

$$\Delta(S^{n}(E)) = (\operatorname{rk} S^{n}(E) - 1)c_{1}^{2}(S^{n}(E)) - 2\operatorname{rk}(S^{n}(E))c_{2}(S^{n}(E))$$

$$= \left[\binom{n+r-1}{r-1} - 1\right]\binom{n+r-1}{r}^{2}c_{1}^{2}(E) - \left(\binom{n+r-1}{r-1}\right)\left[\binom{n+r-1}{r}^{2} - \frac{n}{r}\binom{n+r-1}{r}\right]c_{1}^{2}(E)$$

$$= \left[\frac{n}{r}\binom{n+r-1}{r}\binom{n+r-1}{r-1} - \binom{n+r-1}{r}^{2}\right]c_{1}^{2}(E)$$

$$= 0.$$

**Lemma 2.3.** Let X be a smooth surface and H be an ample line bundle on X. Let  $0 \to V \to E \to Q \to 0$  be an exact sequence of torsion free sheaves such that  $\mu(V) = \mu(Q)$ . If E is strongly semistable with  $\Delta(E) = 0$ , then  $\Delta(Q) = \Delta(V) = 0$ .

*Proof.* Let rank E = n. By definition  $\Delta(E) = (n-1)c_1(E)^2 - 2nc_2(E)$ . Let rank V = k and rank Q = l. Then n = k+l. Then  $\Delta(E) = (k+l-1)c_1(E)^2 - 2(k+l)c_2(E)$ . Therefore we have,

$$kl\Delta(E) = kl(k+l-1)c_1(E)^2 - 2kl(k+l)c_2(E)$$

$$= kl(k+l-1)(c_1(V) + c_1(Q))^2 - 2kl(k+l)(c_1(V)c_1(Q) - c_2(V) - c_2(Q))$$

$$= (k+l)l[(k-1)c_1(V) - 2kc_2(V)] + (k+l)k[(l-1)c_1(Q) - 2lc_2(Q)]$$

$$+ [l^2c_1(V)^2 + k^2c_1(Q)^2 - 2klc_1(V)c_1(Q)]$$

$$= (k+l)l\Delta(V) + (k+l)k\Delta(Q) + (lc_1(V) - kc_1(Q))^2.$$

Hence

$$\frac{\Delta(E)}{k+l} = \frac{\Delta(V)}{k} + \frac{\Delta(Q)}{l} + \frac{1}{kl(k+l)} (lc_1(V) - kc_1(Q))^2.$$

$$\leq \frac{\Delta(V)}{k} + \frac{\Delta(Q)}{l} + \frac{kl}{(k+l)H^2} (\mu(V) - \mu(Q))^2$$

$$= \frac{\Delta(V)}{k} + \frac{\Delta(Q)}{l}.$$

The middle inequality follows from Hodge index theorem and the last equality follows because  $\mu(V) = \mu(Q)$ . Since E is strongly semistable and  $\mu(V) = \mu(Q) = \mu(E)$ , it follows that V, Q are also strongly semistable. Hence  $\Delta(V) \leq 0$  and  $\Delta(Q) \leq 0$  by Proposition 2.2 (2). Since  $\Delta(E) = 0$ ,  $\Delta(Q) = 0$  and  $\Delta(V) = 0$ .

Now in next proposition we will see that any torsion free strong semistable sheaf with zero discriminant is a vector bundle. This fact might be known to experts but we include its prof for the sake of completeness.

**Proposition 2.4.** Let (X, H) be a smooth polarized surface. Let E be a torsion free strongly semistable sheaf with  $\Delta(E) = 0$ . Then E is a vector bundle.

*Proof.* Let  $E^{**}$  denote the reflexive closure of E. Since E is strongly semistable, so is  $E^{**}$ . Consider the exact sequence

$$0 \to E \to E^{**} \to E^{**}/E \to 0.$$

Since E is torsion free, it is locally free in codimension  $\geq 2$ . Hence Supp $(E^{**}/E)$  is a finite set of points. Therefore

$$c_1(E^{**}/E) = 0$$
 and  $c_2(E^{**}/E) \le 0$ .

Note that  $c_2(E^{**}/E) = 0$  if and only if  $\text{Supp}(E^{**}/E)$  is empty, this is the case precisely when  $E = E^{**}$ . Hence

$$c_1(E^{**}) = c_1(E)$$
 and  $c_2(E^{**}) = c_2(E) + c_2(E^{**}/E)$ .

Now

$$\Delta(E^{**}) = (r-1)c_1^2(E^{**}) - 2rc_2(E^{**})$$

$$= (r-1)c_1^2(E) - 2rc_2(E) - 2rc_2(E^{**}/E)$$

$$= \Delta(E) - 2rc_2(E^{**}/E).$$

Now since  $E^{**}$  is strongly semistable,  $\Delta(E^{**}) \leq 0$ . Hence  $\Delta(E) = 0$  implies  $c_2(E^{**}/E) = 0$ , i.e.  $E = E^{**}$ . Hence E is reflexive. Now the proposition follows from the fact that reflexive sheaves on smooth surfaces are vector bundles.

Next we observe that Theorem 3.1 of [11] which is proved stable bundles can be extended for the semistable case also.

**Proposition 2.5.** Let (X, H) be a smooth polarized surface with H an ample line bundle. Let E be a vector bundle of rank  $r \geq 2$  with  $\Delta(E) = 0$ . Assume that E is strongly semistable. Let  $C \in |H|$  be any smooth effective divisor, then  $E|_C$  is also strongly semistable.

Proof. Let E be a strongly semistable bundle with  $\Delta(E) = 0$ . If E is strongly stable we are done by Theorem 3.1 of [11]. If not, there exists  $e \geq 0$  and an exact sequence  $0 \to V \to F_X^{e*}E \to Q \to 0$ , such that  $\mu(V) = \mu(F_X^{e*}E) = \mu(Q)$ . Hence V and Q are also strongly semistable with  $\Delta(V) = \Delta(Q) = 0$  by Lemma 2.3. One also notes that by Proposition 2.4 V and Q are also bundles. Since rank of V and Q are smaller than rank of E, by induction on rank,  $V|_C$  and  $Q|_C$  are strongly semistable for all smooth effective divisor  $C \in |H|$ . Since V, Q are bundles, the following is an exact sequence of bundles.

$$0 \to V|_C \to F_X^{e*}E|_C \to Q|_C \to 0.$$

Now one can see that strong semistablity of  $V|_C$  and  $Q|_C$  implies  $F_C^{e*}E|_C$  is strongly semistable since  $\mu(V|_C) = \mu(F_X^{e*}E|_C) = \mu(Q|_C)$  and hence  $E|_C$  is strongly semistable for all smooth effective divisor  $C \in |H|$ .

# 3. Asymptotic slopes and strong restriction

In this section we show that one can not expect an analogue of Theorem 1.1 for arbitrary semistable vector bundles on smooth projective surfaces. Recall that Theorem 1.1 states that, a vector bundle E on a smooth projective curve C is strongly semistable if and only if  $\nu_K(E) = \mu(E)$  for all k.

First we define asymptotic spectrum on surfaces. Let X be a smooth projective surface over an algebraically closed field  $\mathbb{K}$  of characteristic p > 0. Let H be an ample line bundle on X.

**Definition 3.1.** Let E be a vector bundle of rank  $r \geq 2$  on X.

For each  $1 \le k < r$ , we denote the slope of maximal subsheaf of rank k by  $e_k(E)$ , and

$$e_k(E) := \operatorname{Max} \{ \frac{\operatorname{deg}(W)}{k} \mid W \subset E \text{ is a subsheaf of rank } k \}$$

Define the asymptotic k-spectrum  $\mathcal{AS}_k(E)$  and the asymptotic k-slope  $\nu_k(E)$  as follows: Let  $f: \tilde{X} \to X$  be a finite morphism with  $\tilde{X}$  normal.

$$\mathcal{AS}_k(E) := \{ \frac{e_k(f^*(E))}{\deg f} \}$$

$$\nu_k(E) := Limsup \frac{e_k(f^*(E))}{\deg f} = Limsup \mathcal{AS}_k(E).$$

where the supremum is taken over all finite morphisms  $f: \tilde{X} \to X$  with normal  $\tilde{X}$ .

Consider  $X = \mathbb{P}^2$  and  $E = T\mathbb{P}^2$ , the tangent bundle of  $\mathbb{P}^2$ . It is known that  $T\mathbb{P}^2$  is a strongly semistable bundle. With respect to the very ample line bundle  $\mathcal{O}_X(1)$ ,  $\mu(T\mathbb{P}^2) = 3/2$ . In the following example we see that if we consider only composite of Frobenius morphisms  $F^n : \mathbb{P}^2 \to \mathbb{P}^2$ , and the sequence  $\{\frac{e_1(F^{n*}(E))}{\deg F^n} : n > 0\}$ , then Limsup  $\frac{e_1(F^{n*}(E))}{\deg F^n} < 3/2 = \mu(T\mathbb{P}^2)$ .

**Example 3.2.** Consider  $X = \mathbb{P}^2$  defined over a field of characteristic p > 0. Let  $E = T\mathbb{P}^2$ , the tangent bundle of  $\mathbb{P}^2$ . On a line  $l \simeq \mathbb{P}^1$ , it is known that  $T\mathbb{P}^2|_l \simeq \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ . Hence  $F^*T\mathbb{P}^2|_l \simeq \mathcal{O}_{\mathbb{P}^1}(2p) \oplus \mathcal{O}_{\mathbb{P}^1}(p)$ . Consider the following exact sequence of sheaves:

$$0 \to K \to F^*T\mathbb{P}^2 \to Q \to 0.$$

Now restricts this to any line  $l \simeq \mathbb{P}^1$ , and using  $F^*T\mathbb{P}^2|_l \simeq \mathcal{O}_{\mathbb{P}^1}(2p) \oplus \mathcal{O}_{\mathbb{P}^1}(p)$ , we have (3.1)  $0 \to K|_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(2p) \oplus \mathcal{O}_{\mathbb{P}^1}(p) \to Q|_{\mathbb{P}^1} \to 0$ .

If the map  $\mathcal{O}_{\mathbb{P}^1}(2p) \to Q|_{\mathbb{P}^1}$  induced from (3.1) is a zero map, then the map  $\mathcal{O}_{\mathbb{P}^1}(p) \to Q|_{\mathbb{P}^1}$  induced from (3.1) is surjective, hence  $\mathcal{O}_{\mathbb{P}^1}(p) \simeq Q|_{\mathbb{P}^1}$ . Therefore  $\mu(Q|_{\mathbb{P}^1}) = p < 3p/2 = \mu(F^*T\mathbb{P}^2)|_{\mathbb{P}^1}$ , which contradicts that  $F^*T\mathbb{P}^2$  is a semistable bundle.

Hence the induced map  $\mathcal{O}_{\mathbb{P}^1}(2p) \to Q|_{\mathbb{P}^1}$  from (3.1), is nonzero, hence it is an injective map of sheaves, hence  $\mu(Q)|_{\mathbb{P}^1} \geq 2p$ . Hence with respect to  $\mathcal{O}_X(p), \mu(Q) \geq 2p^2$ . Similar calculation for  $F^n : \mathbb{P}^2 \to \mathbb{P}^2$  shows that if Q is any quotient of  $F^{n*}T\mathbb{P}^2$ , then with respect to  $\mathcal{O}_X(p^n), \mu(Q) \geq 2p^{2n} > \mu(F^{n*}T\mathbb{P}^2)$ . Hence  $Limsup \frac{e_1(F^{n*}(E))}{\deg F^n} < 3/2 = \mu(T\mathbb{P}^2)$ .

Before stating our main result of the section we prove a useful lemma.

**Lemma 3.3.** Let E be a strongly semistable vector bundle on a smooth polarized surface (X, H) over an algebraically closed field of characteristic p > 0. For any k, if  $\nu_k(E) = \mu(E)$ , then  $\nu_k(F_X^*E) = \mu(F_X^*E)$ .

Proof. Since  $\nu_k(E) = \mu(E)$ , there is a sequence of finite coverings  $f_n: X_n \to X$  and a subbundles  $F_n$  of  $f_n^*E$  rank k such that  $\frac{\mu(F_n)}{\deg f_n} \to \mu(E)$ . Hence  $\frac{\mu(F_{X_n}^*F_n)}{\deg f_n} \to \mu(F_X^*E)$ . Since  $F_X^*E$  is also strongly semistable, then  $\nu_k(F_X^*E) \leq \mu(F_X^*E)$ . By definition,  $\lim_{n \to \infty} \frac{\mu(F_{X_n}^*F_n)}{\deg f_n} \leq \nu_k(F_X^*E)$ . Hence the lemma.

Now we are ready to prove our main result of this section.

**Theorem 3.4.** Let (X, H) be a smooth polarized surface over an algebraically closed field of characteristic p > 0 with H very ample line bundle. Suppose E be a rank 2 strongly semistable vector bundle. If  $\nu_1(E) = \mu(E)$ , then  $E|_C$  is strongly semistable for a general smooth  $C \in |H|$ .

*Proof.* Let X be a smooth surface with a fixed very ample polarization H. Let E be a rank 2 strongly semistable vector bundle on X. Let  $C \in |H|$  be a general smooth curve such that  $F_C^{n*}E|_C$  is not semistable for some  $n \geq 0$ . If n > 1, then replacing E by  $F_X^{n-1*}(E)$ , and using Lemma 3.3 we can assume that  $F_C^*E|_C$  is not semistable. Let

$$0 \to F_1 \to F_C^* E|_C \to F_2 \to 0$$

be the Harder-Narasimhan filtration of  $F_C^*E|_C$ .

Choose  $0 < \epsilon < \mu(F_1) - \mu(F_C^*E|_C)$ . Since  $\nu_1(E) = \mu(E)$ , there exists an  $\tilde{X} \xrightarrow{f} X$  such that  $\tilde{X}$  is normal, f is finite and an exact sequence

$$0 \to L_1 \to f^*E \to L_2 \to 0$$

with  $L_i$  line bundles and  $\frac{\mu(L_2)-\mu(f^*E)}{\deg f} < \epsilon/p$ . Now consider the curve  $D := f^{-1}C$  in  $\tilde{X}$ . Since C is general, by [15] D is geometrically unibranched. Hence D is irreducible. Let  $\pi: \tilde{D} \to D_{red}$  be the normalization of  $D_{red}$ . Let  $\psi$  denote the composite morphism  $\tilde{D} \to D_{red} \to D \to C$ . Note that  $\psi$  is also finite.

On  $\tilde{D}$  we have

$$0 \rightarrow \psi^* F_1 \rightarrow \psi^* F_C^* E|_C$$

and  $\deg \psi^* F_1 - \deg \psi^* F_C^* E|_C = \deg \psi(\deg F_1 - \deg F_C^* E|_C) > 0.$ 

Hence  $\psi^* F_C^* E|_C$  is not semistable and  $E' = \psi^* F_1$  is a destabilizing subbundle of  $\psi^* F_C^* E|_C$ .

Let  $\tau$  denote the morphism  $\tilde{D} \to D_{red} \to D$ . On  $\tilde{D}$ , we also have an exact sequence

$$0 \to F_{\tilde{D}}^*(\tau^*(L_1|_D)) \to F_{\tilde{D}}^*(\tau^*(f^*E|_D)) \to F_{\tilde{D}}^*\tau^*(L_2|_D)) \to 0$$

Now

$$\frac{1}{\deg \psi}(\mu(E') - \mu(F_{\tilde{D}}^*(\tau^*L_2|_D))) = \frac{\mu(E') - \mu(\psi^*F_C^*E|_C) + \mu(\psi^*F_C^*E|_C) - \mu(F_{\tilde{D}}^*\tau^*L_2|_D)}{\deg \psi}$$

$$= \frac{\mu(E') - \mu(\psi^*F_C^*E|_C) + \mu(F_{\tilde{D}}^*\tau^*f^*E|_C) - \mu(F_{\tilde{D}}^*\tau^*L_2|_D)}{\deg \psi}$$

$$= \mu(F_1) - \mu(F_C^*E|_C) - \frac{p \deg \tau[\mu(f^*E) - \mu(L_2)]}{\deg \psi}$$

$$> \mu(F_1) - \mu(F_C^*E|_C) - \epsilon$$

$$> 0$$

where the third equality follows from Proposition 6 of [8]. Hence  $\operatorname{Hom}(E', F_{\tilde{D}}^*\tau^*L_2|_D) = 0$ . Therefore  $0 \to E' \to \tau^*F_D^*L_1|_D$  and  $\mu(E') < \mu(\tau^*F_D^*L_1|_D) < \mu(\tau^*F_D^*(f^*E)|_D)$ , contradiction. Hence  $F_C^*E|_C$  is semistable for a general smooth  $C \in |H|$ .

Remark 3.5. (1) Now suppose E is a rank 2 strongly semistable vector bundle such that  $\nu_1(E) = \mu(E)$ , then  $E|_C$  is strongly semistable for a general smooth  $C \in |H|$ ; in particular  $E|_C$  is semistable for a general smooth  $C \in |H|$ . But this not true in general (see Example 3.2). Hence we can not expect  $\nu_1(E) = \mu(E)$  for arbitrary rank 2 strongly semistable bundle.

In general, if E is a strongly semistable vector bundle on a polarized variety (X, H), then whether  $E|_C$  is strongly semistable for a very general hypersurface C is an open question. However by Theorem 3.1 of [11], it is known that if E is a strongly stable vector bundle with  $\Delta(E) = 0$ , then  $E|_C$  is also strongly stable for all smooth curve  $C \in |H|$ . In the next section we study asymptotic slope for strongly semistable vector bundles with zero discriminant (i.e.  $\Delta(E) = 0$ ).

(2) One might expect to get a generalization of Theorem 3.4 for a vector bundle of arbitrary rank. But at present we are unable to extend the Theorem 3.4 for vector bundles of arbitrary rank.

## 4. Asymptotic slopes and strong semistability

Here in this section we prove analogue of Theorem 1.1 for strongly semistable vector bundles E of arbitrary rank with zero discriminants i.e.  $\Delta(E) = 0$  and  $c_1(E) = H$ . In order to do this we appeal to the Kodaira type vanishing theorem in characteristic p > 0, for this we assume some additional condition (which will be clear from the following) on the polarized surface and on the vector bundle.

Let X be a smooth projective surface over an algebraically closed field  $\mathbb{K}$  of characteristic p>0. Let H be an ample line bundle. Let E be a strongly semistable vector bundle of rank  $r\geq 2$  on X with respect to the polarization H with  $\Delta(E)=0$ . Let  $\pi$  denote the natural morphism  $\operatorname{Gr}(k,E)\to X$ . We assume that X,E,H admit a lifting  $\overline{X},\overline{E}$  and  $\overline{H}$  respectively to  $W_2(\mathbb{K})$ . Then  $\operatorname{Gr}(k,\overline{E})$  is a lifting of  $\operatorname{Gr}(k,E)$  to  $W_2(\mathbb{K})$ . Indeed since  $\overline{E}$  is a lifting, then  $\overline{E}\times_{\overline{X}}X\simeq E$  and there is a natural injection  $X\hookrightarrow \overline{X}$ . Hence

$$\operatorname{Gr}(k, \overline{E}) \times_{\overline{X}} X = \operatorname{Gr}(k, \overline{E}|_X) = \operatorname{Gr}(k, E).$$

Hence  $Gr(k, \overline{E})$  is a lifting of Gr(k, E).

Now we state our main result of this section.

**Theorem 4.1.** Let E be a strongly semistable vector bundle of rank  $r \geq 2$  with  $\Delta(E) = 0$  on a smooth polarized surface (X, H) such that X, E, H admit a liftings  $\overline{X}$ ,  $\overline{E}$ ,  $\overline{H}$  respectively to  $W_2(\mathbb{K})$ . Moreover if  $c_1(E) = H$ , then E is strongly semistable if and only if  $\nu_k(E) = \mu(E)$  for all k.

*Proof.* Here we prove the if direction and give an outline of the only if direction.

Suppose  $\nu_k(E) = \mu(E)$  for all k. We will show that E is strongly semistable. If not, there exists a finite morphism  $f: \tilde{X} \to X$  and a subsheaf  $W \to f^*E$  of rank k such that  $\mu(W) > \mu(f^*E)$ . Hence  $\nu_k(E) > \mu(E)$ , which is a contradiction.

Now we give an outline of the proof of only if direction. To prove the only if direction of the theorem we construct smooth surfaces  $f_n: \tilde{X}_n \to X$  and subbundles  $V_n \subset f_n^*(E)$  such that  $\frac{\mu(V_n)}{\deg f_n}$  converges to  $\mu(E)$ .

To find such surfaces, we consider Gr(k, E) and let  $\pi : Gr(k, E) \to X$  denote the natural morphism. Let  $\mathcal{N} = \{m^2 : m \in \mathbb{N} \text{ and } m \text{ is divisible by } rH^2\}$ . For any  $n \in \mathcal{N}$ , define  $\mathcal{L}_n := -nk\frac{\mu(E)}{H^2}H + n^{1/2}H$ . Note that  $\mathcal{L}_n$  is a genuine line bundle for all  $n \in \mathcal{N}$ . We show

that  $\mathcal{O}_{Gr(k,E)}(n) \otimes \pi^* \mathcal{L}_n$  is very ample line bundle for all large  $n \in \mathcal{N}$ . Then cutting down by appropriate k(r-k) sections of  $\mathcal{O}_{Gr(k,E)}(n) \otimes \pi^* \mathcal{L}_n$  we get desired surfaces.

The strategy to show very ampleness of the line bundles  $\mathcal{O}_{Gr(k,)}(n) \otimes \pi^* \mathcal{L}_n$  is make use of Kodaira vanishing theorem in characteristic p and Lemma 2.3 of [1]. Let  $C \in |H|$  be any smooth curve, then  $E|_C$  is also strongly semistable by Proposition 2.5. Also  $\mu(\mathcal{L}_n|_C) = -nk\mu(E) + n^{1/2}H \cdot H$ , hence for large  $n \in \mathcal{N}$ ,  $\mu(\mathcal{L}_n) > 2g - nk\mu(E)$  where g is the genus of C, hence by Lemma 2.3 of [1],  $\mathcal{O}_{Gr(k,E|_C)}(n) \otimes \pi^*\mathcal{L}_n|_C$  is very ample on  $\pi^{-1}(C) = Gr(k, E|_C)$ . We use the very ampleness of  $\mathcal{O}_{Gr(k,E|_C)}(n) \otimes \pi^*\mathcal{L}_n$  to get very ampleness of  $\mathcal{O}_{Gr(k,E)}(n) \otimes \pi^*\mathcal{L}_n$ .

In order to complete the proof of Theorem 4.1 we need several results (up to Proposition 4.10). The following two lemmas might be known to experts, but here we include a proof for the convenience to the readers.

**Lemma 4.2.** Let X be a smooth projective surface over a field of characteristic p > 0. Let H be an ample line bundle and V be a strongly semistable vector bundle of rank r with  $\Delta(V) = 0$ . Then the line bundle  $\mathcal{O}(1)$  on Gr(k, V) is ample if and only if det(V) is ample.

*Proof.* First we prove the lemma for k = 1. Then  $Gr(1, V) = \mathbb{P}(V)$ . Note that  $Sym^r V \otimes det(V)^*$  is also strongly semistable and  $\Delta(Sym^r V \otimes det(V)^*) = \Delta(Sym^r V) = 0$ . Also

$$c_1(\operatorname{Sym}^r V \otimes \det(V)^*) = \binom{2r-1}{r} c_1(V) - \operatorname{rk}(\operatorname{Sym}^r(E)) c_1(\det(V))$$
$$= \binom{2r-1}{r} c_1(V) - \binom{2r-1}{r} c_1(V)$$
$$= 0.$$

Hence  $c_2(\operatorname{Sym}^r V \otimes \det(V)^*) = 0$ . Therefore by Proposition 5.1 of [11],  $\operatorname{Sym}^r V \otimes \det(V)^*$  is nef. Now since  $\det(V)$  is ample  $\operatorname{Sym}^r V = \operatorname{Sym}^r V \otimes \det(V)^* \otimes \det(V)$  is also ample. Hence V is ample by Proposition 2.4 of [5].

Now we prove the lemma for k > 1. Note that Gr(k, V) embeds in  $\mathbb{P}(\Lambda^k V)$  by Plücker embedding and  $\mathcal{O}_{\mathbb{P}(\Lambda^k V)}(1)$  pulls back to  $\mathcal{O}_{Gr(k,V)}(1)$ . Hence in order to show  $\mathcal{O}_{Gr(k,V)}(1)$  ample it is enough to show that  $\mathcal{O}_{\mathbb{P}(\Lambda^k V)}(1)$  is ample. Now as V is strongly semistable, then  $\Lambda^k V$  is so. Also  $\Delta(\Lambda^k V) = 0$ , since  $\Delta(V) = 0$  (by Lemma 2.3). Then by above paragraph  $\mathcal{O}_{\mathbb{P}(\Lambda^k V)}(1)$  is ample. Hence the lemma.

**Lemma 4.3.** Let E be vector bundle on X, then

$$c_1(E^{\otimes n}) = nr^{n-1}c_1(E).$$

*Proof.* The proof follows from the repeated application of the following formula: given two vector bundles  $V_1$  and  $V_2$  of rank  $r_1$  and  $r_2$  respectively,

$$c_1(V_1 \otimes V_2) = r_2 c_1(V_1) + r_1 c_1(V_2).$$

The following Theorem plays a crucial role in proving very ampleness of  $\mathcal{O}_{Gr(k,E}(n)\otimes\pi^*\mathcal{L}_n$ .

**Theorem 4.4.** With the same hypothesis as in Theorem 4.1, fix  $m \ge 1$ . For any smooth curve  $C \in |mH|$ , the cohomology module

$$H^1(Gr(k, E), \mathcal{O}(n) \otimes \pi^* \mathcal{L}_n(-C)) = 0$$

for large  $n \in \mathcal{N}$ , where  $\mathcal{L}_n = -\frac{nk}{r}H + n^{1/2}H$ .

*Proof.* In the proof of this theorem we will make use to Kodaira type vanishing theorem in characteristic p > 0. We prove the theorem in two steps.

Let L be an ample line bundle on X. Then in the first step we will show that  $\mathcal{O}_{Gr(k,E)}(n) \otimes \frac{-nk}{r} \pi^* H \otimes \pi^* L$  is ample for large  $n \in \mathcal{N}$ . In the next step using first step and Kodaira type vanishing theorem along with Serre duality we will conclude the theorem.

**Step1**: We have the following commutative diagram

$$\operatorname{Gr}(k,E) \xrightarrow{i} \mathbb{P}(\Lambda^{k}(E))$$

$$\downarrow^{\pi_{1}}_{X}$$

such that  $\mathcal{O}_{\mathrm{Gr}(k,E)}(n) \otimes \frac{-nk}{r} \pi^* H \otimes \pi^* L = i^* (\mathcal{O}_{\mathbb{P}(\Lambda^k(E))}(n) \otimes \frac{-nk}{r} \pi_1^* H \otimes \pi_1^* L)$ . Hence in order to show  $\mathcal{O}_{\mathrm{Gr}(k,E)}(n) \otimes \frac{-nk}{r} \pi^* H \otimes \pi^* L$  is ample on  $\mathrm{Gr}(k,E)$  it is sufficient to show that  $\mathcal{O}_{\mathbb{P}(\Lambda^k(E))}(n) \otimes \frac{-nk}{r} \pi_1^* H \otimes \pi_1^* L$  is ample on  $\mathbb{P}(\Lambda^k E)$ .

Using Lemma 4.3, one can check that  $\det(E^{\otimes nk} \otimes \frac{-nk}{r} H \otimes L)$  is ample. Since  $\Delta(E) = 0$ , hence  $\Delta(E^{\otimes nk}) = 0$ , and thus by Proposition 2.2(1) we have

$$\Delta(E^{\otimes nk} \otimes \frac{-nk}{r} H \otimes L) = 0.$$

Hence by Lemma 4.2,  $E^{\otimes nk} \otimes \frac{-nk}{r} H \otimes L$  is ample. Since quotient of an ample bundle is ample, it follows that  $\operatorname{Sym}^n(\Lambda^k(E)) \otimes \frac{-nk}{r} H \otimes L$  is also ample.

Now consider the following commutative diagram

$$\mathbb{P}(\Lambda^{k}(E)) \xrightarrow{i} \mathbb{P}(\operatorname{Sym}^{n}(\Lambda^{k}(E))) \longleftarrow \mathbb{P}(\operatorname{Sym}^{n}(\Lambda^{k}(E)) \otimes \frac{-nk}{r}H \otimes L)$$

$$\downarrow^{\tilde{\pi}_{1}} \qquad \downarrow^{\tilde{\pi}_{1}}$$

such that

$$\mathcal{O}_{\mathbb{P}(\Lambda^{k}(E))}(n) \otimes \frac{-nk}{r} \pi_{1}^{*}H \otimes \pi_{1}^{*}L = i^{*}[\mathcal{O}_{\mathbb{P}(\operatorname{Sym}^{n}(\Lambda^{k}(E)))}(1) \otimes \frac{-nk}{r} \tilde{\pi}_{1}^{*}H \otimes \tilde{\pi}_{1}^{*}L]$$
$$= i^{*}[\mathcal{O}_{\mathbb{P}(\operatorname{Sym}^{n}(\Lambda^{k}(E)))}(1)] \otimes \frac{-nk}{r} \pi_{1}^{*}H \otimes \pi_{1}^{*}L.$$

Since  $\mathcal{O}_{\mathbb{P}(\operatorname{Sym}^n(\Lambda^k(E))\otimes \frac{-nk}{r}H\otimes L)}(1)$  is ample on  $\mathbb{P}(\operatorname{Sym}^n(\Lambda^k(E))\otimes \frac{-nk}{r}H\otimes L)$ ,  $\mathcal{O}_{\mathbb{P}(\operatorname{Sym}^n(\Lambda^k(E)))}(1)\otimes \frac{-nk}{r}\pi_1^*H\otimes \tilde{\pi}_1^*L$  is ample on  $\mathbb{P}(\operatorname{Sym}^n(\Lambda^k(E)))$ . Hence  $\mathcal{O}_{\mathbb{P}(\Lambda^k(E))}(n)\otimes \frac{-nk}{r}\pi_1^*H\otimes \pi_1^*L$  is ample on  $\mathbb{P}(\Lambda^k(E))$ . Thus  $\mathcal{O}_{\operatorname{Gr}(k,E)}(n)\otimes \frac{-nk}{r}\pi^*H\otimes \pi^*L$  is ample on  $\operatorname{Gr}(k,E)$ .

**Step2**: Let  $C \in |mH|$  be any smooth curve. Consider the short exact sequence

$$0 \to \mathcal{O}(-\pi^{-1}C) \to \mathcal{O}_{Gr(k,E)} \to \mathcal{O}_{\pi^{-1}C} \to 0.$$

which gives long exact sequence in homology modules,

$$0 \to H^0(Gr(k, E), \mathcal{O}(n) \otimes \pi^*(\mathcal{L}_n(-C))) \to H^0(Gr(k, E), \mathcal{O}(n) \otimes \pi^*\mathcal{L}_n) \to$$
$$H^0(Gr(k, E|_C), \mathcal{O}(n) \otimes \mathcal{L}_n|_{\pi^{-1}C}) \to H^1(Gr(k, E), \mathcal{O}(n) \otimes \pi^*\mathcal{L}_n(-C)) \to \cdots$$

Now we claim that  $H^1(\operatorname{Gr}(k,E),\mathcal{O}_n\otimes\pi^*(\mathcal{L}_n(-C)))=0$  for all large enough  $n\in\mathcal{N}$ . Note that  $\mathcal{O}_{\operatorname{Gr}(k,E)}(n)\otimes\pi^*(\mathcal{L}_n(-C))=\mathcal{O}_{\operatorname{Gr}(k,E)}(n)\otimes\pi^*\frac{-nk}{r}H\otimes\pi^*[n^{1/2}H\otimes\mathcal{O}(-C)].$ Now we can choose  $n\in\mathcal{N}$  and  $n\gg 0$  such that,  $n^{1/2}H\otimes\mathcal{O}(-C)$  is ample. Take  $L_n=n^{1/2}H\otimes\mathcal{O}(-C)$ . Hence by Step 1,  $\mathcal{O}_{\operatorname{Gr}(k,E)}(n)\otimes\pi^*(\mathcal{L}_n(-C))$  is ample for all  $n\gg 0$  with  $n\in\mathcal{N}$ . Again we can choose  $n\in\mathcal{N}$  sufficiently large such that  $\mathcal{O}_{\operatorname{Gr}(k,E)}(n)\otimes\pi^*(\mathcal{L}_n(-C))\otimes K^*_{\operatorname{Gr}(k,E)}$  is also ample. This can be seen as follows: since  $\Delta(E)=0$  and  $\det(E)$  is ample, by Lemma 4.2,  $\mathcal{O}_{\operatorname{Gr}(k,E)}(1)$  is ample on  $\operatorname{Gr}(k,E)$ . Hence for  $K^*_{\operatorname{Gr}(k,E)}$ , there exists s divisible by r(H.H) such that  $\mathcal{O}_{\operatorname{Gr}(k,E)}(s)\otimes K^*_{\operatorname{Gr}(k,E)}$  is ample. We can choose  $n\in\mathcal{N}$  large such that  $(n^{1/2}-\frac{sk}{r})H(-C)$  is ample. Hence

$$\mathcal{O}_{Gr(k,E)}(n) \otimes \pi^*(\mathcal{L}_n(-C)) \otimes K^*_{Gr(k,E)}$$

$$= \mathcal{O}_{\mathrm{Gr}(k,E)}(n-s) \otimes \pi^* \frac{-(n-s)k}{r} H \otimes \pi^*((n^{1/2} - \frac{sk}{r})H(-C)) \otimes \mathcal{O}_{\mathrm{Gr}(k,E)}(s) \otimes K^*_{\mathrm{Gr}(k,E)}(s)$$

Now both the line bundles  $\mathcal{O}_{Gr(k,E)}(n-s) \otimes \pi^* \frac{-(n-s)k}{r} H \otimes \pi^*((n^{1/2} - \frac{sk}{r})H(-C)),$   $\mathcal{O}_{Gr(k,E)}(s) \otimes K^*_{Gr(k,E)}$  are ample, hence  $\mathcal{O}_{Gr(k,E)}(n) \otimes \pi^*(\mathcal{L}_n(-C)) \otimes K^*_{Gr(k,E)}$  is also ample for large n. Then Kodaira type vanishing theorem says [4],  $H^i(Gr(k,E),\mathcal{O}(-n) \otimes (\pi^*(\mathcal{L}_n(-C)))^* \otimes K_{Gr(k,E)}) = 0$  for all large enough  $n \in \mathcal{N}$  and for all  $i < \dim Gr(k,E)$ . Hence by Serre duality the claim follows.

Next we prove that  $\mathcal{O}_{Gr(k,E)}(n) \otimes \pi^* \mathcal{L}_n$  is very ample, for all large  $n \in \mathcal{N}$ . But first we show a useful lemma.

**Lemma 4.5.** Let X be any projective variety and H be a very ample divisor on X. Let  $x \in X$ , and  $\tau : Bl_x(X) \to X$  denote the blow up of X at x with E as exceptional divisor. Then  $2\tau^*H - E$  is very ample on  $Bl_x(X)$ .

*Proof.* Since H is very ample, with respect to a fixed embedding X can be realized as a closed subvariety of  $\mathbb{P}^N$  for some  $N \in \mathbb{N}$ . Hence  $Bl_x(X) \subseteq Bl_x(\mathbb{P}^N)$ . Let  $\tilde{E}$  denote the exceptional divisor of  $\tilde{\tau}: Bl_x(\mathbb{P}^N) \to \mathbb{P}^N$ . Now in order to show that  $2\tau^*H - E$  is very ample on  $Bl_x(X)$ , it is enough to show that  $2\mathcal{O}_{\mathbb{P}^N}(1) - \tilde{E}$  is very ample on  $Bl_x(\mathbb{P}^N)$ .

Note that  $\mathcal{O}_{\mathbb{P}^N}(1) - \tilde{E}$  gives the projection morphism from  $Bl_x(\mathbb{P}^N) \to \mathbb{P}^{N-1}$  and  $\mathcal{O}_{\mathbb{P}^N}(1)$  gives the morphism from  $Bl_x(\mathbb{P}^N) \to \mathbb{P}^N$ . Hence together  $2\tilde{\tau}^*\mathcal{O}_{\mathbb{P}^N}(1) - \tilde{E}$  gives a morphism  $Bl_x(\mathbb{P}^N) \to \mathbb{P}^N \times \mathbb{P}^{N-1}$  which is the natural morphism of  $Bl_x(\mathbb{P}^N) \hookrightarrow \mathbb{P}^N \times \mathbb{P}^{N-1}$ . Hence  $2\tilde{\tau}^*\mathcal{O}_{\mathbb{P}^N}(1) - \tilde{E}$  is very ample on  $Bl_x(\mathbb{P}^N)$ .

**Theorem 4.6.** The line bundle  $\mathcal{O}_{Gr(k,E)}(n) \otimes \pi^* \mathcal{L}_n$  is very ample, for all large  $n \in \mathcal{N}$ , where  $\mathcal{L}_n = -\frac{nk}{r}H + n^{1/2}H$ .

*Proof.* In order to show that the line bundle  $\mathcal{O}_{Gr(k,E)}(n) \otimes \pi^* \mathcal{L}_n$  is very ample, we need to show that the line bundle separates points and separates tangent vectors.

First we will show that  $\mathcal{O}_{Gr(k,E)}(n) \otimes \pi^* \mathcal{L}_n$  separates points. Take two points  $z_1, z_2 \in Gr(k, E)$ . Let  $\pi(z_1) = x, \pi(z_2) = y$ . By Theorem 3.1 of [3], one can choose a smooth curve  $C \in |mH|$  that contains x, y. Consider the short exact sequence of sheaves:

 $0 \to \mathcal{O}_{Gr(k,E)}(n) \otimes \pi^* \mathcal{L}_n(-C) \to \mathcal{O}_{Gr(k,E)}(n) \otimes \pi^* \mathcal{L}_n \to \mathcal{O}_{Gr(k,E|_C)}(n) \otimes \pi^* \mathcal{L}_n|_{Gr(k,E|_C)} \to 0$ which yields the following long exact sequence in homology:

$$\cdots \to H^0(Gr(k, E), \mathcal{O}(n) \otimes \pi^* \mathcal{L}_n) \to H^0(Gr(k, E|_C), \mathcal{O}(n) \otimes \pi^* \mathcal{L}_n)$$
$$\to H^1(Gr(k, E), \mathcal{O}(n) \otimes \pi^* \mathcal{L}_n(-C)) \to \cdots$$

By Theorem 4.4,

$$H^1(Gr(k, E), \mathcal{O}(n) \otimes \pi^* \mathcal{L}_n(-C)) = 0.$$

Hence the morphism

$$\cdots \to H^0(Gr(k, E), \mathcal{O}(n) \otimes \pi^* \mathcal{L}_n) \to H^0(Gr(k, E|_C), \mathcal{O}(n) \otimes \pi^* \mathcal{L}_n)$$

is surjective. By Lemma 2.3 of [1],  $\mathcal{O}_{Gr(k,E|_C)}(n) \otimes \pi^* \mathcal{L}_n$  separates points on  $Gr(k,E|_C)$ . Hence there exists a section  $\sigma$  in  $\mathcal{O}_{Gr(k,E|_C)}(n) \otimes \pi^* \mathcal{L}_n$  such that  $\sigma(z_1) = 0$  and  $\sigma(z_2) \neq 0$ . Then choose a lift  $\tilde{\sigma} \in H^0(Gr(k,E), \mathcal{O}(n) \otimes \pi^* \mathcal{L}_n)$  of  $\sigma$  which also has the property that  $\tilde{\sigma}(z_1) = 0$  and  $\tilde{\sigma}(z_2) \neq 0$ . Hence  $\mathcal{O}_{Gr(k,E)}(n) \otimes \pi^* \mathcal{L}_n$  separates points.

Now we will show that  $\mathcal{O}_{Gr(k,E)}(n) \otimes \pi^* \mathcal{L}_n$  separates tangent vectors of Gr(k,E).

Let  $\tilde{\pi}: T_z(Gr(k, E)) \to T_x(X)$  be the natural morphism induced by  $\pi: Gr(k, E) \to X$  with kernel  $T_z(Gr(k, E|_x))$ . Let  $v \in T_z(Gr(k, E))$ .

Case 1: Suppose that  $\tilde{\pi}(v) = 0 \in T_x(X)$ . Then  $v \in T_z(Gr(k, E|_x))$ . Choose a smooth curve  $C \in |H|$  such that  $x \in C$ . Then  $v \in T_z(Gr(k, E|_x)) \subseteq T_z(Gr(k, E|_C))$ . Since by Lemma 2.3 of [1], on  $Gr(k, E|_C)$ ,  $\mathcal{O}_{Gr(k, E|_C)}(n) \otimes \pi^* \mathcal{L}_n$  separates tangent vectors, there exists a section  $\sigma \in H^0(Gr(k, E|_C, \mathcal{O}(n) \otimes \pi^* \mathcal{L}_n))$  such that  $\sigma(z) = 0$  and  $v \notin T_z(\operatorname{div}(\sigma))$ . Since the natural morphism

$$H^0(Gr(k, E), \mathcal{O}(n) \otimes \pi^* \mathcal{L}_n) \to H^0(Gr(k, E|_C), \mathcal{O}(n) \otimes \pi^* \mathcal{L}_n)$$

is surjective, there exists a lift  $\tilde{\sigma} \in H^0(Gr(k, E), \mathcal{O}(n) \otimes \pi^* \mathcal{L}_n)$  of  $\sigma$ . Note that  $\tilde{\sigma}(z) = 0$  and also  $v \notin T_z(\operatorname{div}(\tilde{\sigma}))$ . Hence in this case  $\mathcal{O}_{Gr(k,E)}(n) \otimes \pi^* \mathcal{L}_n$  separates tangent vectors.

Case 2: Suppose that  $\tilde{\pi}(v) = w \neq 0$ . We will show that there exists a smooth  $D \in |mH|$ , with  $\frac{m}{2}H$  ample such that  $w \in T_x(D)$ . Hence  $v \in T_z(Gr(k, E|_D))$ , and then the rest of the arguments follow from case 1.

To show such smooth  $D \in |mH|$  with  $w \in T_x(D)$  exists let us consider the blow up morphism  $\tau : Bl_x(X) \to X$  where  $Bl_x(X)$  denotes the blow up of X at x with E as the exceptional divisor. One can note that  $E = \mathbb{P}(T_x(X))$ . Let y represent w in E. Also one can assume that  $\frac{m}{2}H$  is very ample and hence by Lemma 4.5,  $\tau^*mH - E$  is very ample on  $Bl_x(X)$ . Let C be a smooth curve in  $\tau^*mH - E$  passing through y (such smooth curve exists by the proof of Theorem 3.1 of [3]). Since  $(\tau^*mH - E).E = 1$ ,  $\tau(C) \in |mH|$  is a smooth curve in X and  $w \in T_x(\tau(C))$ . Hence the proof.

Now since  $\mathcal{O}_{Gr(k,E)}(n) \otimes \pi^* \mathcal{L}_n$  is very ample for all large  $n \in \mathcal{N}$ , we can choose k(r-k) sections such that they cut down Gr(k,E) into a smooth surface  $\tilde{X}_n$ . Next we show that  $\tilde{X}_n$  is finite over X for all large  $n \in \mathcal{N}$ . First we prove a general proposition concerning a general hyperplane section of a flat family is flat.

**Proposition 4.7.** Let  $f: X \to Y$  be a flat family obtained by a morphism f from a smooth variety X to a smooth surface Y. Let  $X \subset \mathbb{P}^N$  be an embedding obtained by an very ample divisor  $\mathcal{L}$  such that the general fiber is not a linear subspace of  $\mathbb{P}^N$ . Then for a general hyperplane  $H, X \cap H \to Y$  is also a flat family induced by f.

Proof. For each  $y \in Y$ , let  $X_y$  denote the fibre. Consider the incidence variety  $S = \{(y, H) : X_y \subset H\} \subset Y \times \check{\mathbb{P}}^N$ . Let  $p_1$  and  $p_2$  denote the projections from S to Y and to  $\check{\mathbb{P}}^N$  respectively. Then the fibre of  $p_1$  over a point  $y \in Y$  denoted by  $S_y = \{H \in \check{\mathbb{P}}^N : X_y \subset H\}$ . Note that  $S_y$  is a linear subspace of  $\check{\mathbb{P}}^N$  of dimension  $N - \dim \langle X_y \rangle - 1$ , where  $\langle X_y \rangle$  denotes the smallest linear subspace containing  $X_y$  (i.e. the linear span of  $X_y$ ). Since

for each  $y \in Y$ ,  $\dim S_y \leq N - \dim X_y - 1$  then  $\dim S \leq N - \dim X_y - 1 + \dim Y$ . The image  $p_2(S)$  is a proper closed subset of  $\check{\mathbb{P}}^N$  unless  $\dim X_y \leq 1$ . Hence for the case when each  $y \in Y$ ,  $\dim X_y \geq 2$ , there will be a general hyperplane H such that it contains no fibre  $X_y$ , in other words, for each y,  $\dim X_y \cap H = \dim X_y - 1$ . By Bertini's theorem, shrinking the open set if required, we may assume  $X \cap H$  is also smooth, hence in this case  $X \cap H \to Y$  is a morphism induced from f between smooth varieties with equidimensional fibres, hence  $X \cap H \to Y$  is a flat family. Now we consider the case when  $\dim X_y = 1$ , for some g i.e. g is a curve. Consider g is a flat family in either a finite set of points or a curve g in g. Hence g is not a linear subspace of g in g is either a finite set of points or a curve g in g. Hence g is a proper closed subset of g in g is a flat family.

Next we give an example which shows that the hypothesis that the general fiber is not a linear subspace of  $\mathbb{P}^N$  is necessary.

**Example 4.8.** Consider  $X = SL_3/B$  which embeds in  $\mathbb{P}^8$ ,  $Y = \mathbb{P}^2$ . Let  $f: X \to Y$  denote the morphism which sends each full flag to its linear subspace. One notes that each fibre is a linear space  $\mathbb{P}^1$  in  $\mathbb{P}^8$ . It is known that for a general hyperplane H,  $X \cap H \to Y$  is birational. If  $X \cap H \to Y$  is a flat family. it would be a finite map and hence an isomorphism, which is not true as we show next that there is no section from  $\mathbb{P}^2 \to SL_3/B$ . Let  $D_1$  and  $D_2$  be the divisors in  $SL_3/B$ . They are pull backs of lines from  $\mathbb{P}^2$  and  $\mathbb{P}^2$  dual. Since the square of a line in  $\mathbb{P}^2$  is a point, we get  $D_1^2 = L_1$  and  $D_2^2 = L_2$ , where  $L_1$  and  $L_2$  are fibres. The very ample divisor H on  $SL_3/B$  is  $D_1 + D_2$ . If we intersect these divisor  $D_1 \cdot D_2 = L_1 + L_2$ . Hence by squaring we get:

$$(D_1 + D_2)^2 = D_1^2 + D_2^2 + 2D_1 \cdot D_2 = 3D_1 \cdot D_2.$$

Assume there is a map f from  $\mathbb{P}^2$  to  $SL_3/B$  and  $f^*(D_1)=a$  and  $f^*(D_2)=b$ . Then  $f^*(H)=f^*(D_1+D_2)=a+b$ . Hence  $f^*(H^2)=f^*(D_1+D_2)^2=(a+b)^2=a^2+b^2+2ab$ . Hence we get:  $a^2+b^2+2ab=3ab$ . subtracting 4ab from both sides we obtain  $(a-b)^2=-ab$ , which is true only when a=0 and b=0.

**Theorem 4.9.** For all large  $n \in \mathcal{N}$ , the morphism  $\tilde{X}_n \to X$  induced by  $\pi$  is a finite morphism.

*Proof.* By Proposition 4.7, a general section of  $\mathcal{O}_{Gr(k,E)}(n) \otimes \pi^* \mathcal{L}_n$  cuts each fibre of  $\pi: Gr(K,E) \to X$  into a variety of dimension exactly one less. Using Proposition 4.7 repeatedly one see that the morphism  $\tilde{X}_n \to X$  is quasi finite. Since the morphism is also proper, it is a finite map by Zariski's main theorem.

Since  $\tilde{X}_n$  is complete intersection, then  $\deg \mathcal{O}_{Gr(k,E)}(1)|_{\tilde{X}_n}$  with respect to  $\tilde{X}_n.\pi^*H$ , can be calculated as the cup product of the cycle classes of the corresponding divisors with the class of  $\mathcal{O}_{Gr(k,E)}(1)$ .

Let  $D = \tilde{X}_n \cdot \pi^* H$ .

On Gr(k, E), we have the universal exact sequence:

$$0 \to \mathcal{S}(E) \to \pi^*E \to \mathcal{Q}(E) \to 0.$$

Hence

$$0 \to \mathcal{S}(E)|_{\tilde{X}_n} \to \pi^* E|_{\tilde{X}_n} \to \mathcal{Q}(E)|_{\tilde{X}_n} \to 0.$$

**Proposition 4.10.** Let  $f_n: \tilde{X}_n \to X$  denote the morphism induced by  $\pi$ . Then

$$\lim_{n \to \infty} \frac{\mu(f_n^* E) - \mu(\mathcal{S}(E)|_{\tilde{X}_n})}{\deg f_n} = 0,$$

where  $\mu$  is taken with respect to D.

Proof.

$$\deg \mathcal{Q}(E)|_{\tilde{X}} = [\mathcal{O}(n) \otimes \pi^* \mathcal{L}_n]^{k(r-k)} \cdot [\pi^* H] \cdot [\mathcal{O}(1)]$$

$$= ([\mathcal{O}(n)] + [\pi^* \mathcal{L}_n])^{k(r-k)} \cdot [\pi^* H] \cdot [\mathcal{O}(1)]$$

$$= ([\mathcal{O}(n)]^{k(r-k)} + k(r-k)[\mathcal{O}(n)]^{k(r-k)-1} \cdot [\pi^* \mathcal{L}_n]) \cdot [\pi^* H] \cdot [\mathcal{O}(1)]$$

$$= n^{k(r-k)}[\mathcal{O}(1)]^{k(r-k+1)} \cdot [\pi^* H] + k(r-k)n^{k(r-k)-1}[\mathcal{O}(1)]^{k(r-k)} \cdot [\pi^* \mathcal{L}_n] \cdot [\pi^* H]$$

$$= n^{k(r-k)}[\mathcal{O}(1)]^{k(r-k)+1} \cdot [\pi^* H] + k(r-k)n^{k(r-k)-1}([\mathcal{O}(1)]^{k(r-k)} \cdot \mathcal{F}) \deg \mathcal{L}_n$$

the last equality follows from the fact that

$$[\mathcal{O}(1)]^{k(r-k)} \cdot [\pi^* \mathcal{L}_n] \cdot [\pi^* H] = ([\mathcal{O}(1)]^{k(r-k)} \cdot \mathcal{F}) \deg \mathcal{L}_n,$$

where  $\mathcal{F}$  denotes any fiber of  $\pi: Gr(k, E) \to X$ .

$$\operatorname{deg} \pi^* E|_{\tilde{X}} = [\mathcal{O}(n) \otimes \pi^* \mathcal{L}_n]^{k(r-k)} \cdot [\pi^* H] \cdot [\pi^* \det E] 
= n^{k(r-k)} [\mathcal{O}(1)]^{k(r-k)} \cdot [\pi^* H] \cdot [\pi^* \det E] + [\pi^* \mathcal{L}_n] \cdot [\pi^* H] \cdot [\pi^* \det E] 
= n^{k(r-k)} ([\mathcal{O}(1)] \cdot \mathcal{F}) \operatorname{deg} E + 0$$

the last equality follows from the fact that

$$[\pi^* \mathcal{L}_n] \cdot [\pi^* H] \cdot [\pi^* \det E] = 0.$$

Need to find  $[\mathcal{O}_{Gr(k,E)}(1)]^{k(r-k)+1} \cdot [\pi^*H]$ . Note that if  $C \in |H|$  be any smooth curve, then

$$[\mathcal{O}_{Gr(k,E)}(1)]^{k(r-k)+1} \cdot [\pi^* H] = [\mathcal{O}_{Gr(k,E|_C)}(1)]^{k(r-k)+1}.$$

Hence by Lemma 2.3 of [13]

$$[\mathcal{O}_{Gr(k,E)}(1)]^{k(r-k)+1} \cdot [\pi^*H] = (k(r-k)+1)k\mu(E)([\mathcal{O}_{Gr(k,E)}(1)]^{k(r-k)} \cdot \mathcal{F})$$

Hence

$$\begin{split} \deg \mathcal{S}(E)|_{\tilde{X}_{n}} &= \deg \pi^{*}E|_{\tilde{X}} - \deg \mathcal{O}(1)|_{\tilde{X}} \\ &= n^{k(r-k)}([\mathcal{O}(1)]^{k(r-k)} \cdot \mathcal{F}) \deg E - n^{k(r-k)}[\mathcal{O}(1)]^{k(n-k)+1} \cdot [\pi^{*}H] \\ &- (k(r-k)n^{k(r-k)-1}([\mathcal{O}(1)]^{k(r-k)} \cdot \mathcal{F}) \deg \mathcal{L}_{n} \\ &= n^{k(r-k)}([\mathcal{O}(1)]^{k(r-k)} \cdot \mathcal{F}) \deg E - n^{k(r-k)}(k(r-k)+1)k\mu(E)([\mathcal{O}_{\mathrm{Gr}(k,E)}(1)]^{k(r-k)} \cdot \mathcal{F}) \\ &- k(r-k)n^{k(r-k)-1}([\mathcal{O}(1)]^{k(r-k)} \cdot \mathcal{F}) \deg \mathcal{L}_{n} \\ &= n^{k(r-k)}([\mathcal{O}(1)]^{k(r-k)} \cdot \mathcal{F})r\mu(E) - n^{k(r-k)}(k(r-k)+1)k\mu(E)([\mathcal{O}_{\mathrm{Gr}(k,E)}(1)]^{k(r-k)} \cdot \mathcal{F}) \\ &- k(r-k)n^{k(r-k)-1}([\mathcal{O}(1)]^{k(r-k)} \cdot \mathcal{F}) \deg \mathcal{L}_{n} \\ &= (r-k)n^{k(r-k)}([\mathcal{O}(1)]^{k(r-k)} \cdot \mathcal{F})\mu(E) - n^{k(r-k)}k(r-k)k\mu(E)([\mathcal{O}_{\mathrm{Gr}(k,E)}(1)]^{k(r-k)} \cdot \mathcal{F}) \\ &- k(r-k)n^{k(r-k)-1}([\mathcal{O}(1)]^{k(r-k)} \cdot \mathcal{F})(-nk\mu(E) + n^{1/2}H \cdot H) \\ &= (r-k)n^{k(r-k)}([\mathcal{O}(1)]^{k(r-k)} \cdot \mathcal{F})\mu(E) - k(r-k)n^{k(r-k)-1}n^{1/2}([\mathcal{O}(1)]^{k(r-k)} \cdot \mathcal{F})H \cdot H. \end{split}$$

Hence

$$\mu(\mathcal{S}(E)_{\tilde{X}_n}) = \frac{\deg \mathcal{S}(E)|_{\tilde{X}_n}}{r - k} \\ = n^{k(r-k)} ([\mathcal{O}(1)]^{k(r-k)} \cdot \mathcal{F}) \mu(E) - k n^{k(r-k)-1} n^{1/2} ([\mathcal{O}(1)]^{k(r-k)} \cdot \mathcal{F}) H \cdot H.$$

Note that the degree of  $f_n$  is equal to the cardinality of a general fiber of  $f_n$  which equals to  $[\mathcal{O}_{Gr(k,E_x)}(n)]^{k(r-k)} = n^{k(r-k)}([\mathcal{O}_{Gr(k,E)}(1)]^{k(r-k)} \cdot \mathcal{F}).$ 

Hence

$$\frac{\mu(\mathcal{S}(E)_{\tilde{X}_n})}{\deg f_n} = \frac{n^{k(r-k)}([\mathcal{O}(1)]^{k(r-k)} \cdot \mathcal{F})\mu(E) - kn^{k(r-k)-1}n^{1/2}([\mathcal{O}(1)]^{k(r-k)} \cdot \mathcal{F})H \cdot H}{n^{k(r-k)}([\mathcal{O}_{Gr(k,E)}(1)]^{k(r-k)} \cdot \mathcal{F})}$$

$$= \mu(E) - \frac{kn^{1/2}H \cdot H}{n}$$

Therefore

$$\lim_{n \to \infty} \frac{f_n^* E - \mu(\mathcal{S}(E)|_{\tilde{X}_n})}{\deg f_n} = \lim_{n \to \infty} \frac{\deg f_n \mu(E) - \mu(\mathcal{S}(E)|_{\tilde{X}_n})}{\deg f_n}$$

$$= \lim_{n \to \infty} \left[ \mu(E) - \mu(E) + \frac{kn^{1/2} H \cdot H}{n} \right]$$

$$= \lim_{n \to \infty} \frac{kn^{1/2} H \cdot H}{n}$$

$$= 0.$$

Completion of the proof of Theorem 4.1. Now we complete the proof of the only if direction of theorem. Suppose E is strongly semistable. Then for any finite morphism  $f: \tilde{X} \to X$ ,  $f^*(E)$  is semistable. Hence for all k, if W is a subsheaf of  $f^*E$ ,  $\mu(W) \le$ 

 $\mu(f^*E)$ . Therefore  $\nu_k(E) \leq \mu(E)$  for all k. Now as before one can construct  $\tilde{X}_n$  and by Proposition 4.10, the theorem follows.

**Remark 4.11.** One might hope to get a similar result of Theorem 4.12, without the assumption  $c_1(E) = H$ , without even the lifting assumptions on the surface and the bundle. But at present we have no idea how to avoid Kodaira vanishing theorem.

However Theorem 4.1 has the following corollary.

Corollary 4.12. Let E be a strongly semistable vector bundle of rank  $r \geq 2$  with  $\Delta(E) = 0$  on a smooth polarized surface (X, H) such that X, E, H admit liftings  $\overline{X}, \overline{E}, \overline{H}$  respectively to  $W_2(\mathbb{K})$ . Suppose E is also strongly semistable with respect to  $c_1(E) + mH$  for all  $m \gg 0$ . Then  $\nu_k(E) = \mu(E)$  for all k, where slope is taken with respect to  $c_1(E) + mH$  for some large m.

Before going to the proof, we first prove a useful lemma.

**Lemma 4.13.** Let E be a vector bundle on a smooth polarized surface X. Let  $\mathcal{L}$  be a line bundle on X. Then for any k,

$$\nu_k(E) = \mu(E)$$
 if and only if  $\nu_k(E \otimes \mathcal{L}) = \mu(E \otimes \mathcal{L})$ 

*Proof.* Note that if there exists a sequence of finite coverings  $f_n: X_n \to X$  and subbundles  $F_n$  of  $f_n^*E$  of rank k, then for each n,  $F_n \otimes f_n^*\mathcal{L}$  is also a subbundle of  $f_n^*(E \otimes \mathcal{L})$  of rank k. Similarly if there exists a sequence of finite coverings  $f_n: X_n \to X$  and subbundles  $G_n$  of  $f_n^*(E \otimes \mathcal{L})$  of rank k, then for each n,  $G_n \otimes f_n^*\mathcal{L}^{-1}$  is also a subbundle of  $f_n^*E$ . We also have

$$\lim_{n\to\infty} \frac{\mu(F_n)}{\deg f_n} + \deg \mathcal{L} = \lim_{n\to\infty} \frac{\mu(F_n \otimes f_n^*\mathcal{L})}{\deg f_n}.$$

Thus

$$\nu_k(E \otimes \mathcal{L}) = \nu_k(E) + \deg \mathcal{L}.$$

Hence the lemma follows.

Completion of the proof of Corollary 4.12. First note that E is strongly semistable if and only if for any line bundle  $\mathcal{L}$ ,  $E \otimes \mathcal{L}$  is so and we also have that  $\Delta(E) = \Delta(E \otimes \mathcal{L})$ . Also by Lemma 4.13,  $\nu_1(E) = \mu(E)$  if and only if  $\nu_1(E \otimes \mathcal{L}) = \mu(E \otimes \mathcal{L})$ . Also  $c_1(E \otimes nH) = c_1(E) + rnc_1(H)$ . Hence the corollary follows from Theorem 4.1.

We conclude this section with the following remarks where we give criterion, when the hypothesis "E is strongly semistable with respect to  $c_1(E) + mH$  for all  $m \gg 0$ " of Corollary 4.12 holds.

- **Remark 4.14.** (1) When E is a vector bundle with  $c_1(E) = c_2(E) = 0$ , then it satisfies all the hypothesis of Theorem 4.12. Hence in this case given an ample line bundle H, E is strongly semistable with respect to H if and only if  $\nu_k(E) = \mu(E)$  for all k.
  - (2) If E is a strongly semistable with respect to  $c_1(E)$ , then E is strongly semistable with respect to  $c_1(E) + mH$  for all m, as  $\mu_{c_1(E)+mH}(\_) = \mu_{c_1(E)}(\_) + \mu_{mH}(\_)$ . Next suppose that E is not strongly semistable with respect to  $c_1(E)$  Consider  $\lim_{n\to\infty} \frac{\mu_H(F_X^{n*}E)-\mu_H(W_n)}{p^n}$  where  $W_n$  denote a maximal subsheaf of  $F_X^{n*}E$ . Suppose the limit is nonzero say  $\delta > 0$ . Since E is not strongly semistable with respect to  $c_1(E)$ , then there exists  $n_0$  such that  $F_X^{n*}$  is not semistable for all  $n \geq n_0$ . By [10],

it is known that there exists  $n_0 + k$  such that if V is the maximal destabilizing subsheaf  $F_X^{n_0+k^*}(E)$  then  $F_X^*(V)$  is the maximal destabilizing subsheaf of  $F_X^{n_0+k+1^*}(E)$ . Let  $V_{n_0}, \ldots, V_{n_K}$  are maximal destabilizing subsheaves of  $F_X^{n_0*}(E), \ldots, F_X^{n_0+k^*}(E)$ . Let  $\epsilon_i = \frac{\mu_H(V_{n_0+i}) - \mu_H(F_X^{n_0+i^*}(E))}{p^{n_0+i}}$ . Now choose m such that  $m\delta > \epsilon_i$  for all i. Let V be a subsheaf of  $F_X^{n_0*}(E)$ , then

$$\frac{\mu_{c_1(E)+mH}(F_X^{n*}(E)) - \mu_{c_1(E)+mH}(V)}{p^n} = \frac{\mu_{c_1(E)}(F_X^{n*}(E)) - \mu_{c_1(E)}(V) + \mu_{mH}(F_X^{n*}(E)) - \mu_{mH}(V)}{p^n}$$

$$\geq -\max\{\epsilon_i : i\} + m\delta$$

$$> 0.$$

Hence whenever  $\delta > 0$ , E is strongly semistable with respect to  $c_1(E) + mH$  for all  $m \gg 0$ .

## References

- [1] Indranil Biswas and A. J. Parameswaran. A criterion for virtual global generation. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), 5(1):39–53, 2006.
- [2] F. A. Bogomolov. Holomorphic tensors and vector bundles on projective manifolds. *Izv. Akad. Nauk SSSR Ser. Mat.*, 42(6):1227–1287, 1439, 1978.
- [3] Steven Diaz and David Harbater. Strong Bertini Theorems. Trans. Amer. Math. Soc., 324(1):73–86, 1991.
- [4] Hélène Esnault and Eckart Viehweg. Lectures on vanishing theorems, volume 20 of DMV Seminar. Birkhäuser Verlag, Basel, 1992.
- [5] Robin Hartshorne. Ample vector bundles. Inst. Hautes Études Sci. Publ. Math., (29):63-94, 1966.
- [6] Robin Hartshorne. Algebraic geometry. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.
- [7] Daniel Huybrechts and Manfred Lehn. The geometry of moduli spaces of sheaves. Aspects of Mathematics, E31. Friedr. Vieweg & Sohn, Braunschweig, 1997.
- [8] Steven L. Kleiman. Toward a numerical theory of ampleness. Ann. of Math. (2), 84:293–344, 1966.
- [9] H. Lange and M. S. Narasimhan. Maximal subbundles of rank two vector bundles on curves. Math. Ann., 266(1):55–72, 1983.
- [10] Adrian Langer. Semistable sheaves in positive characteristic. Ann. of Math. (2), 159(1):251–276, 2004.
- [11] Adrian Langer. On the S-fundamental group scheme. Ann. Inst. Fourier (Grenoble), 61(5):2077–2119 (2012), 2011.
- [12] Shigeru Mukai and Fumio Sakai. Maximal subbundles of vector bundles on a curve. *Manuscripta Math.*, 52(1-3):251–256, 1985.
- [13] A. J. Parameswaran and S. Subramanian. On the spectrum of asymptotic slopes. In *Teichmüller theory and moduli problem*, volume 10 of *Ramanujan Math. Soc. Lect. Notes Ser.*, pages 519–528. Ramanujan Math. Soc., Mysore, 2010.
- [14] S. Ramanan and A. Ramanathan. Some remarks on the instability flag. *Tohoku Math. J.* (2), 36(2):269–291, 1984.
- [15] Bin Zhang. Théorèmes du type Bertini en caractéristique positive. Arch. Math. (Basel), 64(3):209–215, 1995.

STAT-MATH UNIT, INDIAN STATISTICAL INSTITUTE, 203 B.T. ROAD, KOLKATA, INDIA 700035 *Email address*: mitra.koley@gmail.com

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai, India 400005

Email address: param@math.tifr.res.in