

POLYNOMIAL DYNAMICAL SYSTEMS AND DIFFERENTIATION OF GENUS 4 HYPERELLIPTIC FUNCTIONS

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ABSTRACT. We give an explicit solution to the problem of differentiation of hyperelliptic functions in genus 4 case. We describe explicitly the polynomial Lie algebras and polynomial dynamical systems connected to this problem.

1. INTRODUCTION

Let $g \in \mathbb{N}$. We denote the coordinates in the complex space \mathbb{C}^g by $t = (t_1, t_3, \dots, t_{2g-1})$. For a meromorphic function f on \mathbb{C}^g , a vector $\omega \in \mathbb{C}^g$ is a *period* if $f(t + \omega) = f(t)$ for all $t \in \mathbb{C}^g$. If a meromorphic function f has $2g$ independent periods in \mathbb{C}^g , then f is called an *Abelian function*. Thus, an Abelian function is a meromorphic function on the complex torus \mathbb{C}^g/Γ , where Γ is the lattice formed by the periods. See [1].

A plane nonsingular algebraic curve of genus g determines a lattice Γ as the set of periods of its holomorphic differentials. The torus \mathbb{C}^g/Γ is the *Jacobian variety* of the curve.

In [2] the problem of differentiation of the field of Abelian functions on Jacobian varieties of genus g curves is considered. In this work we consider a special case of this problem. Namely, we consider the model of the *universal hyperelliptic curve of genus g*

$$\mathcal{V}_\lambda = \{(x, y) \in \mathbb{C}^2 : y^2 = x^{2g+1} + \lambda_4 x^{2g-1} + \lambda_6 x^{2g-2} + \dots + \lambda_{4g} x + \lambda_{4g+2}\}.$$

Each curve is defined by specialization of parameters $\lambda = (\lambda_4, \lambda_6, \dots, \lambda_{4g}, \lambda_{4g+2}) \in \mathbb{C}^{2g}$.

The indices of all the coordinates $t = (t_1, t_3, \dots, t_{2g-1}) \in \mathbb{C}^g$ and of the parameters $\lambda = (\lambda_4, \lambda_6, \dots, \lambda_{4g}, \lambda_{4g+2}) \in \mathbb{C}^{2g}$ determine their weights. Namely, $\text{wt } t_k = -k$ and $\text{wt } \lambda_k = k$. For suitable weights of the other variables, all the equations in this paper are of homogeneous weight.

Let $\mathcal{B} \subset \mathbb{C}^{2g}$ be the subspace of parameters such that the curve \mathcal{V}_λ is nonsingular for $\lambda \in \mathcal{B}$. Then we have $\mathcal{B} = \mathbb{C}^{2g} \setminus \Sigma$, where Σ is the *discriminant hypersurface* of the universal curve. In Section 2 we describe the polynomial vector fields in \mathcal{B} tangent to Σ .

For each $\lambda \in \mathcal{B}$ the set of periods of holomorphic differentials on the curve \mathcal{V}_λ generates a lattice Γ_λ of rank $2g$ in \mathbb{C}^g . A *hyperelliptic function of genus g* is a meromorphic function on $\mathbb{C}^g \times \mathcal{B}$ such that, for each $\lambda \in \mathcal{B}$, its restriction to $\mathbb{C}^g \times \lambda$ is an Abelian function with lattice of periods Γ_λ . Thus, a hyperelliptic function is a function defined on an open dense subset of the total space \mathcal{U} of the fiber bundle $\pi: \mathcal{U} \rightarrow \mathcal{B}$ with fiber over $\lambda \in \mathcal{B}$ the Jacobian variety $\mathcal{J}_\lambda = \mathbb{C}^g/\Gamma_\lambda$ of the curve \mathcal{V}_λ . The universal bundle of Jacobians of hyperelliptic curves \mathcal{U} is introduced in [3]. We denote by \mathcal{F} the field of hyperelliptic functions of genus g .

The *problem of differentiation of hyperelliptic functions* is a genus g analogue of a genus 1 result of [4], see also §1.2 in [2]. We consider this problem in the form:

Problem 1.1 (Problem 1.1 in [5]).

- (1) Find the $3g$ generators of the \mathcal{F} -module $\text{Der } \mathcal{F}$ of derivations of the field \mathcal{F} .
- (2) Describe the structure of Lie algebra $\text{Der } \mathcal{F}$ (i.e. find the commutation relations).

A general approach to the solution of this problem is given in [2]. An overview with examples in the cases of genus $g = 1$ elliptic and $g = 2$ hyperelliptic curves is given in [6].

We use the theory of hyperelliptic Kleinian functions (see [7], [8], [9], [10], and [11] for elliptic functions). Take coordinates (t, λ) in $\mathbb{C}^g \times \mathcal{B} \subset \mathbb{C}^{3g}$. Let $\sigma(t, \lambda)$ be the hyperelliptic sigma function. We denote $\partial_k = \frac{\partial}{\partial t_k}$. We use the notation

$$\zeta_k = \partial_k \ln \sigma(t, \lambda), \quad \wp_{k_1, \dots, k_n} = -\partial_{k_1} \cdots \partial_{k_n} \ln \sigma(t, \lambda),$$

where $n \geq 2$ and $k_s \in \{1, 3, \dots, 2g - 1\}$. The functions \wp_{k_1, \dots, k_n} give examples of hyperelliptic functions.

Consider the diagram

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\varphi} & \mathbb{C}^{3g} \\ \downarrow \pi & & \downarrow \rho \\ \mathcal{B} & \longrightarrow & \mathbb{C}^{2g} \end{array} \quad (1)$$

The fiber bundle $\pi : \mathcal{U} \rightarrow \mathcal{B}$ and the embedding $\mathcal{B} \subset \mathbb{C}^{2g}$ are described above. In Section 3 we describe the maps φ and ρ following [5]. By Dubrovin–Novikov theorem [3], the space \mathcal{U} is birationally isomorphic to the complex linear space \mathbb{C}^{3g} . We construct such an isomorphism φ explicitly. We use a fundamental result from the theory of hyperelliptic Abelian functions (see [10], Chapter 5): Any hyperelliptic function can be represented as a rational function in $\wp_{1,k}$ and $\wp_{1,1,k}$, where $k \in \{1, 3, \dots, 2g - 1\}$. Theorem 3.1 in Section 3 gives a set of relations between the derivatives of these functions. We use it to introduce a set of generators in \mathcal{F} . The map φ will be determined by this set of generators. The map ρ will be a polynomial map that makes the diagram (1) commutative.

We denote the ring of polynomials in $\lambda \in \mathbb{C}^{2g}$ by \mathcal{P} . For the polynomial map $\rho : \mathbb{C}^{3g} \rightarrow \mathbb{C}^{2g}$ we call a vector field \mathcal{D} in \mathbb{C}^{3g} *projectable* if there exists a vector field L in \mathbb{C}^{2g} such that $\mathcal{D}(\rho^*f) = \rho^*L(f)$ for any $f \in \mathcal{P}$. The vector field L is the *pushforward* of \mathcal{D} . A corollary of this definition is that for a projectable vector field \mathcal{D} we have $\mathcal{D}(\rho^*\mathcal{P}) \subset \rho^*\mathcal{P}$.

We consider the problem:

Problem 1.2. *Find $3g$ polynomial vector fields in \mathbb{C}^{3g} projectable for $\rho : \mathbb{C}^{3g} \rightarrow \mathbb{C}^{2g}$ and independent at any point in $\rho^{-1}(\mathcal{B})$. Construct their polynomial Lie algebra.*

There is a direct relation (2) between Problems 1.1 and 1.2. In Section 4 we give vector fields that give part of both solutions for general genus g . The pushforwards of these solutions are the polynomial vector fields described in Section 2. In Section 5 we give polynomial dynamical systems that arise from this relation.

In the case of genus $g = 2$ the solution to Problem 1.2 and the corresponding homogeneous polynomial dynamical systems in \mathbb{C}^6 are given in Section 4 of [12]. An explicit solution to Problem 1.1 is given in Theorems B.3 and B.6 of [12]. We recall these results in Section 6.

In the case of genus $g = 3$ an explicit solution to Problem 1.1 is given in [5]. The proof is based on a solution to Problem 1.2. We give these results in Section 7. The corresponding polynomial dynamical systems in \mathbb{C}^9 are presented in Section 8 of [13].

The case of genus $g = 4$ is considered in Sections 8, 9, 10, and 11 of this work. We find the solution to Problem 1.1 in Theorem 8.2 and the solution to Problem 1.2 in Theorem 9.1. Some of the results of these Sections were first obtained in [14]. Here we complete the descriptions of the Lie algebras, see Section 10. Our approach is based on a construction of the generators of $\text{Der } \mathcal{F}$ based on a result from [15].

In all the cases that we consider the generators \mathcal{L}_k of $\text{Der } \mathcal{F}$, that solve Problem 1.1, and the polynomial vector fields \mathcal{D}_k in \mathbb{C}^{3g} , that solve Problem 1.2, are related as

$$\mathcal{L}_k(\varphi^*b_{i,j}) = \varphi^*\mathcal{D}_k(b_{i,j}) \quad (2)$$

for coordinate functions $b_{i,j} \in \mathbb{C}^{3g}$. We note that these solutions give explicit solutions of the Problem considered in [2] for hyperelliptic curves \mathcal{V}_λ of genus 2, 3, and 4.

2. POLYNOMIAL VECTOR FIELDS TANGENT TO THE DISCRIMINANT HYPERSURFACE
OF THE UNIVERSAL CURVE

Let us define the *polynomial Lie algebra of vector fields tangent to the discriminant hypersurface* Σ in \mathbb{C}^{2g} . We denote it by $\mathcal{L}_{\mathcal{B}}$. Here \mathbb{C}^{2g} is the complex linear space with coordinates (λ) , \mathcal{P} is the ring of polynomials in (λ) and Σ is defined in Section 1.

For the polynomial Lie algebra $\mathcal{L}_{\mathcal{B}}$, the generators $\{L_0, L_2, L_4, \dots, L_{4g-2}\}$ are the vector fields

$$L_{2k} = \sum_{s=2}^{2g+1} v_{2k+2, 2s-2}(\lambda) \frac{\partial}{\partial \lambda_{2k}}, \quad v_{2k+2, 2s-2}(\lambda) \in \mathcal{P}.$$

By [16], the structure of a polynomial Lie algebra as a \mathcal{P} -module with generators $1, L_0, L_2, L_4, \dots, L_{4g-2}$ is determined by the polynomial matrices $V(\lambda) = (v_{2i, 2j}(\lambda))$, where $i, j = 1, \dots, 2g$, and $C(\lambda) = (c_{2i, 2j}^{2k}(\lambda))$, where $i, j, k = 0, \dots, 2g - 1$, such that

$$[L_{2i}, L_{2j}] = \sum_{k=0}^{2g-1} c_{2i, 2j}^{2k}(\lambda) L_{2k}, \quad [L_{2i}, \lambda_{2j+4}] = v_{2i+2, 2j+2}(\lambda), \quad [\lambda_{2i+4}, \lambda_{2j+4}] = 0. \quad (3)$$

In the case of the Lie algebra $\mathcal{L}_{\mathcal{B}}$, explicit expressions for the matrix $V(\lambda)$ can be found in Section 4.1 of [17]. The elements of this matrix are given by the following formulas. For convenience, we assume that $\lambda_s = 0$ for all $s \notin \{0, 4, 6, \dots, 4g, 4g + 2\}$ and $\lambda_0 = 1$. Let $k, m \in \{1, 2, \dots, 2g\}$. If $k \leq m$, then we set

$$v_{2k, 2m}(\lambda) = \sum_{s=0}^{k-1} 2(k+m-2s)\lambda_{2s}\lambda_{2(k+m-s)} - \frac{2k(2g-m+1)}{2g+1}\lambda_{2k}\lambda_{2m}, \quad (4)$$

and if $k > m$, then we set $v_{2k, 2m}(\lambda) = v_{2m, 2k}(\lambda)$.

The vector field L_0 is the Euler vector field; namely, since $\text{wt } \lambda_{2k} = 2k$, we have

$$[L_0, \lambda_{2k}] = 2k\lambda_{2k}, \quad [L_0, L_{2k}] = 2kL_{2k}.$$

This determines the weights of the vector fields L_k , namely, $\text{wt } L_{2k} = 2k$.

Lemma 2.1 (Lemma 4.3 in [15]).

$$[L_2, L_{2k}] = 2(k-1)L_{2k+2} + \frac{4(2g-k)}{(2g+1)}(\lambda_{2k+2}L_0 - \lambda_4L_{2k-2}).$$

This Lemma determines the polynomials $c_{2i, 2j}^{2k}(\lambda)$ in (3). For general i the polynomials $c_{2i, 2j}^{2k}(\lambda)$ in (3) are described in Theorem 2.5 of [18]. We can derive them directly from the explicit expressions (4).

Example 2.2. *In the case of genus $g = 2$ we have*

$$\begin{aligned} [L_2, L_4] &= \frac{8}{5}\lambda_6L_0 - \frac{8}{5}\lambda_4L_2 + 2L_6, & [L_2, L_6] &= \frac{4}{5}\lambda_8L_0 - \frac{4}{5}\lambda_4L_4, \\ [L_4, L_6] &= -2\lambda_{10}L_0 + \frac{6}{5}\lambda_8L_2 - \frac{6}{5}\lambda_6L_4 + 2\lambda_4L_6. \end{aligned}$$

Example 2.3. *In the case of genus $g = 3$ the expressions for $c_{2i, 2j}^{2k}$ are given in Lemma 4.3 of [5]. In particular, we have:*

$$\begin{aligned} [L_2, L_4] &= 2L_6 + \frac{16}{7}(\lambda_6L_0 - \lambda_4L_2), & [L_2, L_6] &= 4L_8 + \frac{12}{7}(\lambda_8L_0 - \lambda_4L_4), & (5) \\ [L_2, L_8] &= 6L_{10} + \frac{8}{7}(\lambda_{10}L_0 - \lambda_4L_6). \end{aligned}$$

In the case of genus $g = 4$ the expressions for $c_{2i, 2j}^{2k}$ are given in Example 8.1.

3. POLYNOMIAL MAP RELATED TO
THE UNIVERSAL BUNDLE OF JACOBIANS OF HYPERELLIPTIC CURVES

Theorem 3.1 ([8], §3 in [9]). *For $i, k \in \{1, 3, \dots, 2g-1\}$ we have the relations*

$$\begin{aligned} \wp_{1,1,1,i} &= 6\wp_{1,1}\wp_{1,i} + 6\wp_{1,i+2} - 2\wp_{3,i} + 2\lambda_4\delta_{i,1}, \\ \wp_{1,1,i}\wp_{1,1,k} &= 4(\wp_{1,1}\wp_{1,i}\wp_{1,k} + \wp_{1,k}\wp_{1,i+2} + \wp_{1,i}\wp_{1,k+2} + \wp_{k+2,i+2}) - \\ &\quad - 2(\wp_{1,i}\wp_{3,k} + \wp_{1,k}\wp_{3,i} + \wp_{k,i+4} + \wp_{i,k+4}) + \\ &\quad + 2\lambda_4(\delta_{i,1}\wp_{1,k} + \delta_{k,1}\wp_{1,i}) + 2\lambda_{i+k+4}(2\delta_{i,k} + \delta_{k,i-2} + \delta_{i,k-2}). \end{aligned}$$

Corollary 3.2 (Corollary 5.2 in [5]). *Consider the map $\tilde{\varphi} : \mathcal{U} \dashrightarrow \mathbb{C}^{\frac{g(g+9)}{2}}$ with the coordinates (b, p, λ) in $\mathbb{C}^{\frac{g(g+9)}{2}}$. Here $b = (b_{i,j}) \in \mathbb{C}^{3g}$ with $i \in \{1, 2, 3\}$, $j \in \{1, 3, \dots, 2g-1\}$, $p = (p_{k,l}) \in \mathbb{C}^{\frac{g(g-1)}{2}}$ with $k, l \in \{3, 5, \dots, 2g-1\}$ for $k \leq l$, and $\lambda = (\lambda_s) \in \mathbb{C}^{2g}$ with $s \in \{4, 6, \dots, 4g, 4g+2\}$. Set*

$$\tilde{\varphi} : (t, \lambda) \mapsto (b_{1,j}, b_{2,j}, b_{3,j}, p_{k,l}, \lambda_s) = (\wp_{1,j}(t, \lambda), \wp_{1,1,j}(t, \lambda), \wp_{1,1,1,j}(t, \lambda), 2\wp_{k,l}(t, \lambda), \lambda_s).$$

We denote $p_{l,k} = p_{k,l}$ for $k, l \in \{3, 5, \dots, 2g-1\}$. Then the image of $\tilde{\varphi}$ lies in $\mathcal{S} \subset \mathbb{C}^{\frac{g(g+9)}{2}}$, where \mathcal{S} is determined by the set of $g(g+3)/2$ equations

$$\begin{aligned} b_{3,1} &= 6b_{1,1}^2 + 4b_{1,3} + 2\lambda_4, \\ b_{3,k} &= 6b_{1,1}b_{1,k} + 6b_{1,k+2} - p_{3,k}, \\ b_{2,1}^2 &= 4b_{1,1}^3 + 4b_{1,1}b_{1,3} - 4b_{1,5} + 2p_{3,3} + 4\lambda_4b_{1,1} + 4\lambda_6, \\ b_{2,1}b_{2,k} &= 4b_{1,1}^2b_{1,k} + 2b_{1,3}b_{1,k} + 4b_{1,1}b_{1,k+2} - 2b_{1,k+4} - \\ &\quad - b_{1,1}p_{3,k} + 2p_{3,k+2} - p_{5,k} + 2\lambda_4b_{1,k} + 2\lambda_8\delta_{3,k}, \\ b_{2,j}b_{2,k} &= 4b_{1,1}b_{1,j}b_{1,k} + 4b_{1,k}b_{1,j+2} + 4b_{1,j}b_{1,k+2} + 2p_{k+2,j+2} - \\ &\quad - b_{1,j}p_{3,k} - b_{1,k}p_{3,j} - p_{k,j+4} - p_{j,k+4} + 2\lambda_{j+k+4}(2\delta_{j,k} + \delta_{k,j-2} + \delta_{j,k-2}), \end{aligned}$$

for $j, k \in \{3, \dots, 2g-1\}$ and any variable equal to zero if the index is out of range.

Theorem 3.3 (Theorem 5.3 in [5]). *The projection $\pi_1 : \mathbb{C}^{\frac{g(g+9)}{2}} \rightarrow \mathbb{C}^{3g}$ on the first $3g$ coordinates gives the isomorphism $\mathcal{S} \simeq \mathbb{C}^{3g}$. Therefore, the coordinates $(b_{i,j})$ uniformize \mathcal{S} .*

We have $\tilde{\varphi} : \mathcal{U} \dashrightarrow \mathcal{S} \simeq \mathbb{C}^{3g}$ and we denote by φ the composition $\pi_1 \circ \tilde{\varphi} : \mathcal{U} \dashrightarrow \mathbb{C}^{3g}$.

We obtain the diagram

$$\begin{array}{ccc} & \mathbb{C}^{\frac{g(g+9)}{2}} & \xlongequal{\quad} \mathbb{C}^{3g} \times \mathbb{C}^{\frac{g(g-1)}{2}} \times \mathbb{C}^{2g} \\ & \uparrow \tilde{\varphi} & \swarrow \pi_1 \\ \mathcal{U} & \xrightarrow{\varphi} \mathcal{S} \simeq \mathbb{C}^{3g} & \\ \downarrow \pi & \downarrow \rho & \swarrow \pi_3 \\ \mathcal{B} & \xrightarrow{\quad} \mathbb{C}^{2g} & \end{array} \tag{6}$$

Corollary 3.4 (Corollary 5.5 in [5]). *The projection $\pi_3 : \mathbb{C}^{\frac{g(g+9)}{2}} \rightarrow \mathbb{C}^{2g}$ on the last $2g$ coordinates in Corollary 3.2 restricted to $\mathcal{S} \simeq \mathbb{C}^{3g}$ gives the polynomial map $\rho : \mathbb{C}^{3g} \rightarrow \mathbb{C}^{2g}$.*

Corollary 3.5 (Corollary 5.4 in [5]). *The projection $\pi_2 : \mathbb{C}^{\frac{g(g+9)}{2}} \rightarrow \mathbb{C}^{\frac{g(g-1)}{2}}$ on the middle $g(g-1)/2$ coordinates in Corollary 3.2 restricted to $\mathcal{S} \simeq \mathbb{C}^{3g}$ gives a polynomial map $\mathbb{C}^{3g} \rightarrow \mathbb{C}^{\frac{g(g-1)}{2}}$.*

The proof of these Corollaries in [5] allows to obtain recursive expressions for the polynomials λ_s and $p_{k,l}$ in $b_{i,j}$. We will describe the polynomial maps $\rho : \mathbb{C}^{3g} \rightarrow \mathbb{C}^{2g}$ and $\mathbb{C}^{3g} \rightarrow \mathbb{C}^{\frac{g(g-1)}{2}}$ explicitly for any g in our next work.

4. EXPLICIT DESCRIPTION OF POLYNOMIAL VECTOR FIELDS PROJECTABLE FOR ρ

Now let us describe explicitly some of the polynomial vector fields \mathcal{D}_k projectable for $\rho: \mathbb{C}^{3g} \rightarrow \mathbb{C}^{2g}$. We construct them using the relation

$$\mathcal{L}_k(\varphi^*b_{i,j}) = \varphi^*\mathcal{D}_k(b_{i,j}) \quad (7)$$

for $\mathcal{L}_k \in \text{Der } \mathcal{F}$.

Remark 4.1 (Section 2 in [5]). *The operators $\mathcal{L}_{2k-1} = \partial_{2k-1}$ for $k \in \{1, 2, 3, \dots, g\}$ belong to the Lie algebra of derivations of \mathcal{F} . Their pushforwards for π are zero.*

Lemma 4.2 (Lemma 6.2 in [5]). *For the polynomial vector field*

$$\mathcal{D}_1 = \sum_j b_{2,j} \frac{\partial}{\partial b_{1,j}} + b_{3,j} \frac{\partial}{\partial b_{2,j}} + 4(2b_{1,1}b_{2,j} + b_{2,1}b_{1,j} + b_{2,j+2}) \frac{\partial}{\partial b_{3,j}}$$

where $b_{2,2g+1} = 0$ we have $\mathcal{L}_1(\varphi^*b_{i,j}) = \varphi^*\mathcal{D}_1(b_{i,j})$ for all $i \in \{1, 2, 3\}, j \in \{1, 3, \dots, 2g-1\}$.

Lemma 4.3 (cf. Lemma 6.3 in [5]). *Set $p_{s,1} = 2b_{1,s}$. For $s \in \{3, 5, \dots, 2g-1\}$ for the polynomial vector fields*

$$\mathcal{D}_s = \frac{1}{2} \sum_{k=1}^g \left(\mathcal{D}_1(p_{s,2k-1}) \frac{\partial}{\partial b_{1,2k-1}} + \mathcal{D}_1(\mathcal{D}_1(p_{s,2k-1})) \frac{\partial}{\partial b_{2,2k-1}} + \mathcal{D}_1(\mathcal{D}_1(\mathcal{D}_1(p_{s,2k-1}))) \frac{\partial}{\partial b_{3,2k-1}} \right)$$

we have $\mathcal{L}_s(\varphi^*b_{i,j}) = \varphi^*\mathcal{D}_s(b_{i,j})$ for all $i \in \{1, 2, 3\}, j \in \{1, 3, \dots, 2g-1\}$.

Corollary 4.4. *The polynomial vector fields \mathcal{D}_s for $s \in \{1, 3, \dots, 2g-1\}$ are projectable for the polynomial map $\rho: \mathbb{C}^{3g} \rightarrow \mathbb{C}^{2g}$. Their pushforwards are zero.*

Theorem 4.5 (Theorem 6.1 in [13]). *The operators*

$$\begin{aligned} \mathcal{L}_0 &= L_0 - \sum_{s=1}^g (2s-1)t_{2s-1}\partial_{2s-1}, \\ \mathcal{L}_2 &= L_2 - \zeta_1\partial_1 - \sum_{s=1}^{g-1} (2s-1)t_{2s-1}\partial_{2s+1} + \frac{4}{2g+1}\lambda_4 \sum_{s=1}^{g-1} (g-s)t_{2s+1}\partial_{2s-1}, \\ \mathcal{L}_4 &= L_4 - \zeta_3\partial_1 - \zeta_1\partial_3 - \sum_{s=1}^{g-2} (2s-1)t_{2s-1}\partial_{2s+3} - \\ &\quad - \lambda_4 \sum_{s=1}^{g-1} (2s-1)t_{2s+1}\partial_{2s+1} + \frac{6}{2g+1}\lambda_6 \sum_{s=1}^{g-1} (g-s)t_{2s+1}\partial_{2s-1} \end{aligned}$$

belong to the Lie algebra of derivations of \mathcal{F} .

Lemma 4.6 (Equation (22) in [5]). *For the polynomial vector field*

$$\mathcal{D}_0 = \sum_j (j+1)b_{1,j} \frac{\partial}{\partial b_{1,j}} + (j+2)b_{2,j} \frac{\partial}{\partial b_{2,j}} + (j+3)b_{3,j} \frac{\partial}{\partial b_{3,j}}$$

we have $\mathcal{L}_0(\varphi^*b_{i,j}) = \varphi^*\mathcal{D}_0(b_{i,j})$ for all $i \in \{1, 2, 3\}, j \in \{1, 3, \dots, 2g-1\}$.

Corollary 4.7. *The polynomial vector field \mathcal{D}_0 is projectable for the polynomial map ρ with pushforward L_0 .*

Problem 4.8. *Describe explicitly the polynomial vector fields \mathcal{D}_2 and \mathcal{D}_4 such that $\mathcal{L}_k(\varphi^*b_{i,j}) = \varphi^*\mathcal{D}_k(b_{i,j})$ for $k = 2, 4$ and $i \in \{1, 2, 3\}, j \in \{1, 3, \dots, 2g-1\}$.*

We will give a solution of this problem based on results of [13] and [2] in our next work.

5. POLYNOMIAL DYNAMICAL SYSTEMS RELATED TO DIFFERENTIATIONS
OF HYPERELLIPTIC FUNCTIONS

In Section 4 the generators \mathcal{L}_k of $\text{Der } \mathcal{F}$ and the polynomial vector fields \mathcal{D}_k in \mathbb{C}^{3g} are related by (7). In this Section we give the graded homogeneous polynomial dynamical systems in \mathbb{C}^{3g} determined by these vector fields. We follow the approach of [12], where such a description is given in the case of genus $g = 1$ and $g = 2$. See also Section 8 of [13]. By definition, the dynamical system S_k corresponding to the vector field \mathcal{D}_k is given by

$$\frac{\partial}{\partial \tau_k} b_{i,j} = \mathcal{D}_k(b_{i,j}).$$

The dynamical system S_0 corresponding to the Euler vector field \mathcal{D}_0 is given by

$$\frac{\partial}{\partial \tau_0} b_{i,j} = (i+j)b_{i,j}, \quad \text{for } i \in \{1, 2, 3\}, \quad j \in \{1, 3, \dots, 2g-1\}.$$

The dynamical system S_1 corresponding to the vector field \mathcal{D}_1 for $j \in \{1, 3, \dots, 2g-1\}$ is given by

$$\frac{\partial}{\partial \tau_1} b_{i,j} = b_{i+1,j}, \quad \text{for } i \in \{1, 2\}, \quad \frac{\partial}{\partial \tau_1} b_{3,j} = 4(2b_{1,1}b_{2,j} + b_{2,1}b_{1,j} + b_{2,j+2}),$$

where $b_{2,2g+1} = 0$.

The dynamical systems S_s for $s \in \{3, 5, \dots, 2g-1\}$ corresponding to vector fields \mathcal{D}_s for $j \in \{1, 3, \dots, 2g-1\}$ are given by

$$\frac{\partial}{\partial \tau_s} b_{1,j} = \frac{1}{2} \mathcal{D}_1(p_{s,j}), \quad \frac{\partial}{\partial \tau_s} b_{2,j} = \frac{1}{2} \mathcal{D}_1(\mathcal{D}_1(p_{s,j})), \quad \frac{\partial}{\partial \tau_s} b_{3,j} = \frac{1}{2} \mathcal{D}_1(\mathcal{D}_1(\mathcal{D}_1(p_{s,j}))),$$

where $p_{s,1} = 2b_{1,s}$.

6. LIE ALGEBRA OF DERIVATIONS OF HYPERELLIPTIC FUNCTIONS: GENUS 2 CASE

The following results were obtained in [12]. We give them in the notation of this work using the explicit formulas from Section 4.

Theorem 6.1 (Theorem B.3 and Theorem B.6 in [12]). *In the case of genus $g = 2$ the following vector fields give a solution to Problem 1.1*

$$\begin{aligned} \mathcal{L}_1 &= \partial_1, \quad \mathcal{L}_0 = L_0 - t_1 \partial_1 - 3t_3 \partial_3, & \mathcal{L}_4 &= L_4 - \zeta_3 \partial_1 - \zeta_1 \partial_3 - \lambda_4 t_3 \partial_3 + \frac{6}{5} \lambda_6 t_3 \partial_1, \\ \mathcal{L}_3 &= \partial_3, \quad \mathcal{L}_2 = L_2 - \zeta_1 \partial_1 - t_1 \partial_3 + \frac{4}{5} \lambda_4 t_3 \partial_1, & \mathcal{L}_6 &= L_6 - \zeta_3 \partial_3 + \frac{3}{5} \lambda_8 t_3 \partial_1. \end{aligned}$$

For $m, l \in \{1, 2\}$, $m \leq l$, and $k \in \{1, 2, 3, 4, 6\}$, the commutation relations are

$$\begin{aligned} [\mathcal{L}_0, \mathcal{L}_k] &= k\mathcal{L}_k, & [\mathcal{L}_1, \mathcal{L}_3] &= 0, \\ [\mathcal{L}_1, \mathcal{L}_2] &= \wp_{1,1} \mathcal{L}_1 - \mathcal{L}_3, & [\mathcal{L}_3, \mathcal{L}_2] &= \left(\wp_{1,3} + \frac{4}{5} \lambda_4 \right) \mathcal{L}_1, \\ [\mathcal{L}_1, \mathcal{L}_4] &= \wp_{1,3} \mathcal{L}_1 + \wp_{1,1} \mathcal{L}_3, & [\mathcal{L}_3, \mathcal{L}_4] &= \left(\wp_{3,3} + \frac{6}{5} \lambda_6 \right) \mathcal{L}_1 + (\wp_{1,3} - \lambda_4) \mathcal{L}_3, \\ [\mathcal{L}_1, \mathcal{L}_6] &= \wp_{1,3} \mathcal{L}_3, & [\mathcal{L}_3, \mathcal{L}_6] &= \frac{3}{5} \lambda_8 \mathcal{L}_1 + \wp_{3,3} \mathcal{L}_3, \\ [\mathcal{L}_{2m}, \mathcal{L}_{2l+2}] &= \sum_{s=0}^3 c_{2m,2l+2}^{2s}(\lambda) \mathcal{L}_{2s} - \frac{1}{2} \wp_{2m-1,2l-1,3} \mathcal{L}_1 + \frac{1}{2} \wp_{1,2m-1,2l-1} \mathcal{L}_3. \end{aligned}$$

The expressions for the coordinates (λ) and (p) are (see eq. (4.3)-(4.6) and (4.8) in [12]):

$$\begin{aligned}\lambda_4 &= -3b_{1,1}^2 + \frac{1}{2}b_{3,1} - 2b_{1,3}, \\ \lambda_6 &= 2b_{1,1}^3 + \frac{1}{4}b_{2,1}^2 - \frac{1}{2}b_{1,1}b_{3,1} - 2b_{1,1}b_{1,3} + \frac{1}{2}b_{3,3}, \\ \lambda_8 &= (4b_{1,1}^2 + b_{1,3})b_{1,3} - \frac{1}{2}(b_{3,1}b_{1,3} - b_{2,1}b_{2,3} + b_{1,1}b_{3,3}), \\ \lambda_{10} &= 2b_{1,1}b_{1,3}^2 + \frac{1}{4}b_{2,3}^2 - \frac{1}{2}b_{1,3}b_{3,3}, & p_{3,3} &= 6b_{1,1}b_{1,3} - b_{3,3}.\end{aligned}$$

Theorem 6.2 (Section 4 in [12]). *In the case of genus $g = 2$ the homogeneous polynomial vector fields $\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4, \mathcal{D}_6$ give a solution to Problem 1.2. Here the vector fields $\mathcal{D}_1, \mathcal{D}_3$ and \mathcal{D}_0 are determined by Lemmas 4.2, 4.3 and 4.6 for $g = 2$. Set $p_{1,k} = 2b_{1,k}$ and $q_{i,j,k} = \mathcal{D}_i(p_{j,k})$ for $i, j, k \in \{1, 3\}$. The polynomial vector fields $\mathcal{D}_2, \mathcal{D}_4, \mathcal{D}_6$ are determined by the conditions*

$$\begin{aligned}\mathcal{D}_2(b_{1,1}) &= \frac{8}{5}\lambda_4 + 2b_{1,1}^2 + 4b_{1,3}, & (8) \\ \mathcal{D}_4(b_{1,1}) &= \frac{2}{5}\lambda_6 - 2b_{1,1}b_{1,3} + b_{3,3}, \\ \mathcal{D}_6(b_{1,1}) &= \frac{1}{5}\lambda_8 + \frac{1}{2}(b_{3,1}b_{1,3} - b_{1,1}b_{3,3}) - b_{1,3}^2, \\ \mathcal{D}_2(b_{1,3}) &= -\frac{4}{5}\lambda_4 b_{1,1} + 2b_{1,1}b_{1,3}, \\ \mathcal{D}_4(b_{1,3}) &= -\frac{6}{5}\lambda_6 b_{1,1} + 2\lambda_4 b_{1,3} - 4b_{1,1}^2 b_{1,3} + b_{3,1}b_{1,3} - \frac{1}{2}b_{2,1}b_{2,3}, \\ \mathcal{D}_6(b_{1,3}) &= -\frac{8}{5}\lambda_8 b_{1,1} + 2\lambda_6 b_{1,3} - 2b_{1,1}b_{1,3}^2 - b_{2,3}^2 + b_{1,3}b_{3,3}.\end{aligned}$$

and the relations

$$[\mathcal{D}_1, \mathcal{D}_2] = b_{1,1}\mathcal{D}_1 - \mathcal{D}_3, \quad [\mathcal{D}_1, \mathcal{D}_4] = b_{1,3}\mathcal{D}_1 + b_{1,1}\mathcal{D}_3, \quad [\mathcal{D}_1, \mathcal{D}_6] = b_{1,3}\mathcal{D}_3. \quad (9)$$

These relations determine the polynomials $\mathcal{D}_k(b_{i,j})$ for $k \in \{2, 4, 6\}$, $i \in \{2, 3\}$, $j \in \{1, 3\}$. For $m, l \in \{1, 2\}$, $m \leq l$, and $k \in \{1, 2, 3, 4, 6\}$, the commutation relations are (9) and

$$\begin{aligned}[\mathcal{D}_0, \mathcal{D}_k] &= k\mathcal{D}_k & [\mathcal{D}_1, \mathcal{D}_3] &= 0, \\ [\mathcal{D}_3, \mathcal{D}_2] &= \left(b_{1,3} + \frac{4}{5}\lambda_4\right)\mathcal{D}_1, & [\mathcal{D}_3, \mathcal{D}_4] &= \left(\frac{1}{2}p_{3,3} + \frac{6}{5}\lambda_6\right)\mathcal{D}_1 + (b_{1,3} - \lambda_4)\mathcal{D}_3, \\ [\mathcal{D}_3, \mathcal{D}_6] &= \frac{3}{5}\lambda_8\mathcal{D}_1 + \frac{1}{2}p_{3,3}\mathcal{D}_3, \\ [\mathcal{D}_{2m}, \mathcal{D}_{2l+2}] &= \sum_{k=0}^3 c_{2m,2l+2}^{2k}(\lambda)\mathcal{D}_{2k} - \frac{1}{4}q_{2m-1,2l-1,3}\mathcal{D}_1 + \frac{1}{4}q_{1,2m-1,2l-1}\mathcal{D}_3.\end{aligned}$$

Proof. This Theorem follows from Theorem 4.6 in [12] and the expressions in the proof of this Theorem. Note that this expressions are based on the relation (7) for \mathcal{L}_k given in Theorem 6.1. See also Theorems 4.7 and B.3 in [12] and Section 8 in [5]. \square

Let us note that the result of Theorem 6.2 is equivalent to the corresponding result in Example 15 in [2]. Some misprints in [2] were corrected in [12]. In terms of the work [2], the polynomial vector fields \mathcal{D}_k determine the action of the operators \mathcal{L}_k on \mathcal{F} . The latter is determined by the values $\mathcal{L}_k(\wp_{1,1}) \in \mathcal{F}$, that correspond to $\mathcal{D}_k(b_{1,1})$, given by (8).

7. LIE ALGEBRA OF DERIVATIONS OF HYPERELLIPTIC FUNCTIONS: GENUS 3 CASE

Theorem 7.1 (Theorem 10.1 and Corollary 10.2 in [5]). *In the case of genus $g = 3$ the following vector fields give a solution to Problem 1.1*

$$\begin{aligned}
\mathcal{L}_1 &= \partial_1, & \mathcal{L}_3 &= \partial_3, & \mathcal{L}_5 &= \partial_5, \\
\mathcal{L}_0 &= L_0 - t_1 \partial_1 - 3t_3 \partial_3 - 5t_5 \partial_5, \\
\mathcal{L}_2 &= L_2 - \left(\zeta_1 - \frac{8}{7} \lambda_4 t_3 \right) \partial_1 - \left(t_1 - \frac{4}{7} \lambda_4 t_5 \right) \partial_3 - 3t_3 \partial_5, \\
\mathcal{L}_4 &= L_4 - \left(\zeta_3 - \frac{12}{7} \lambda_6 t_3 \right) \partial_1 - \left(\zeta_1 + \lambda_4 t_3 - \frac{6}{7} \lambda_6 t_5 \right) \partial_3 - (t_1 + 3\lambda_4 t_5) \partial_5, \\
\mathcal{L}_6 &= L_6 - \left(\zeta_5 - \frac{9}{7} \lambda_8 t_3 \right) \partial_1 - \left(\zeta_3 - \frac{8}{7} \lambda_8 t_5 \right) \partial_3 - (\zeta_1 + \lambda_4 t_3 + 2\lambda_6 t_5) \partial_5, \\
\mathcal{L}_8 &= L_8 + \left(\frac{6}{7} \lambda_{10} t_3 - \lambda_{12} t_5 \right) \partial_1 - \left(\zeta_5 - \frac{10}{7} \lambda_{10} t_5 \right) \partial_3 - (\zeta_3 + \lambda_8 t_5) \partial_5, \\
\mathcal{L}_{10} &= L_{10} + \left(\frac{3}{7} \lambda_{12} t_3 - 2\lambda_{14} t_5 \right) \partial_1 + \frac{5}{7} \lambda_{12} t_5 \partial_3 - \zeta_5 \partial_5,
\end{aligned}$$

The commutation relations are

$$[\mathcal{L}_0, \mathcal{L}_k] = k\mathcal{L}_k, \quad k = 1, 2, 3, 4, 5, 6, 8, 10, \quad [\mathcal{L}_1, \mathcal{L}_3] = 0, \quad [\mathcal{L}_1, \mathcal{L}_5] = 0, \quad [\mathcal{L}_3, \mathcal{L}_5] = 0,$$

$$\begin{aligned}
\begin{pmatrix} [\mathcal{L}_1, \mathcal{L}_2] \\ [\mathcal{L}_1, \mathcal{L}_4] \\ [\mathcal{L}_1, \mathcal{L}_6] \\ [\mathcal{L}_1, \mathcal{L}_8] \\ [\mathcal{L}_1, \mathcal{L}_{10}] \end{pmatrix} &= \begin{pmatrix} \wp_{1,1} & -1 & 0 \\ \wp_{1,3} & \wp_{1,1} & -1 \\ \wp_{1,5} & \wp_{1,3} & \wp_{1,1} \\ 0 & \wp_{1,5} & \wp_{1,3} \\ 0 & 0 & \wp_{1,5} \end{pmatrix} \begin{pmatrix} \mathcal{L}_1 \\ \mathcal{L}_3 \\ \mathcal{L}_5 \end{pmatrix}, \\
\begin{pmatrix} [\mathcal{L}_3, \mathcal{L}_2] \\ [\mathcal{L}_3, \mathcal{L}_4] \\ [\mathcal{L}_3, \mathcal{L}_6] \\ [\mathcal{L}_3, \mathcal{L}_8] \\ [\mathcal{L}_3, \mathcal{L}_{10}] \end{pmatrix} &= \frac{3}{7} \begin{pmatrix} 5\lambda_4 \\ 4\lambda_6 \\ 3\lambda_8 \\ 2\lambda_{10} \\ \lambda_{12} \end{pmatrix} \mathcal{L}_1 + \begin{pmatrix} \wp_{1,3} - \lambda_4 & 0 & -3 \\ \wp_{3,3} & \wp_{1,3} - \lambda_4 & 0 \\ \wp_{3,5} & \wp_{3,3} & \wp_{1,3} - \lambda_4 \\ 0 & \wp_{3,5} & \wp_{3,3} \\ 0 & 0 & \wp_{3,5} \end{pmatrix} \begin{pmatrix} \mathcal{L}_1 \\ \mathcal{L}_3 \\ \mathcal{L}_5 \end{pmatrix}, \\
\begin{pmatrix} [\mathcal{L}_5, \mathcal{L}_2] \\ [\mathcal{L}_5, \mathcal{L}_4] \\ [\mathcal{L}_5, \mathcal{L}_6] \\ [\mathcal{L}_5, \mathcal{L}_8] \\ [\mathcal{L}_5, \mathcal{L}_{10}] \end{pmatrix} &= \frac{2}{7} \begin{pmatrix} 2\lambda_4 \\ 3\lambda_6 \\ 4\lambda_8 \\ 5\lambda_{10} \\ 6\lambda_{12} \end{pmatrix} \mathcal{L}_3 - \begin{pmatrix} 0 \\ 3\lambda_4 \\ 2\lambda_6 \\ \lambda_8 \\ 0 \end{pmatrix} \mathcal{L}_5 + \begin{pmatrix} \wp_{1,5} & 0 & 0 \\ \wp_{3,5} & \wp_{1,5} & 0 \\ \wp_{5,5} & \wp_{3,5} & \wp_{1,5} \\ -\lambda_{12} & \wp_{5,5} & \wp_{3,5} \\ -2\lambda_{14} & -\lambda_{12} & \wp_{5,5} \end{pmatrix} \begin{pmatrix} \mathcal{L}_1 \\ \mathcal{L}_3 \\ \mathcal{L}_5 \end{pmatrix}, \\
\begin{pmatrix} [\mathcal{L}_2, \mathcal{L}_4] \\ [\mathcal{L}_2, \mathcal{L}_6] \\ [\mathcal{L}_2, \mathcal{L}_8] \\ [\mathcal{L}_2, \mathcal{L}_{10}] \end{pmatrix} &= \begin{pmatrix} \sum_{k=0}^5 c_{2,4}^{2k}(\lambda) \mathcal{L}_{2k} \\ \sum_{k=0}^5 c_{2,6}^{2k}(\lambda) \mathcal{L}_{2k} \\ \sum_{k=0}^5 c_{2,8}^{2k}(\lambda) \mathcal{L}_{2k} \\ \sum_{k=0}^5 c_{2,10}^{2k}(\lambda) \mathcal{L}_{2k} \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -\wp_{1,1,3} & \wp_{1,1,1} & 0 \\ -\wp_{1,3,3} - \wp_{1,1,5} & \wp_{1,1,3} & \wp_{1,1,1} \\ -2\wp_{1,3,5} & \wp_{1,1,5} & \wp_{1,1,3} \\ -\wp_{1,5,5} & 0 & \wp_{1,1,5} \end{pmatrix} \begin{pmatrix} \mathcal{L}_1 \\ \mathcal{L}_3 \\ \mathcal{L}_5 \end{pmatrix}, \quad (10) \\
\begin{pmatrix} [\mathcal{L}_4, \mathcal{L}_6] \\ [\mathcal{L}_4, \mathcal{L}_8] \\ [\mathcal{L}_4, \mathcal{L}_{10}] \\ [\mathcal{L}_6, \mathcal{L}_8] \\ [\mathcal{L}_6, \mathcal{L}_{10}] \\ [\mathcal{L}_8, \mathcal{L}_{10}] \end{pmatrix} &= \begin{pmatrix} \sum_{k=0}^5 c_{4,6}^{2k}(\lambda) \mathcal{L}_{2k} \\ \sum_{k=0}^5 c_{4,8}^{2k}(\lambda) \mathcal{L}_{2k} \\ \sum_{k=0}^5 c_{4,10}^{2k}(\lambda) \mathcal{L}_{2k} \\ \sum_{k=0}^5 c_{6,8}^{2k}(\lambda) \mathcal{L}_{2k} \\ \sum_{k=0}^5 c_{6,10}^{2k}(\lambda) \mathcal{L}_{2k} \\ \sum_{k=0}^5 c_{8,10}^{2k}(\lambda) \mathcal{L}_{2k} \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -\wp_{3,3,3} & \wp_{1,3,3} - 2\wp_{1,1,5} & 2\wp_{1,1,3} \\ -2\wp_{3,3,5} & 0 & 2\wp_{1,3,3} \\ -\wp_{3,5,5} & -\wp_{1,5,5} & 2\wp_{1,3,5} \\ -2\wp_{3,5,5} & 2\wp_{1,5,5} - \wp_{3,3,5} & \wp_{3,3,3} \\ -\wp_{5,5,5} & -\wp_{3,5,5} & \wp_{3,3,5} + \wp_{1,5,5} \\ 0 & -\wp_{5,5,5} & \wp_{3,5,5} \end{pmatrix} \begin{pmatrix} \mathcal{L}_1 \\ \mathcal{L}_3 \\ \mathcal{L}_5 \end{pmatrix}.
\end{aligned}$$

The expressions for the coordinates (λ) and (p) are (see Section 9 in [5]):

$$\begin{aligned}\lambda_4 &= -3b_{1,1}^2 + \frac{1}{2}b_{3,1} - 2b_{1,3}, & \lambda_6 &= 2b_{1,1}^3 + \frac{1}{4}b_{2,1}^2 - \frac{1}{2}b_{1,1}b_{3,1} - 2b_{1,1}b_{1,3} + \frac{1}{2}b_{3,3} - 2b_{1,5}, \\ \lambda_8 &= 4b_{1,1}^2b_{1,3} - \frac{1}{2}(b_{3,1}b_{1,3} - b_{2,1}b_{2,3} + b_{1,1}b_{3,3}) + b_{1,3}^2 - 2b_{1,1}b_{1,5} + \frac{1}{2}b_{3,5}, \\ \lambda_{10} &= 2b_{1,1}b_{1,3}^2 + \frac{1}{4}b_{2,3}^2 - \frac{1}{2}b_{1,3}b_{3,3} - \frac{1}{2}(b_{3,1}b_{1,5} - b_{2,1}b_{2,5} + b_{1,1}b_{3,5}) + (4b_{1,1}^2 + 2b_{1,3})b_{1,5}, \\ \lambda_{12} &= 4b_{1,1}b_{1,3}b_{1,5} - \frac{1}{2}(b_{3,3}b_{1,5} - b_{2,3}b_{2,5} + b_{1,3}b_{3,5}) + b_{1,5}^2, \\ \lambda_{14} &= 2b_{1,1}b_{1,5}^2 + \frac{1}{4}b_{2,5}^2 - \frac{1}{2}b_{1,5}b_{3,5}, & p_{3,3} &= 6b_{1,1}b_{1,3} - b_{3,3} + 6b_{1,5}, \\ p_{3,5} &= 6b_{1,1}b_{1,5} - b_{3,5}, & p_{5,5} &= (b_{3,1}b_{1,5} - b_{2,1}b_{2,5} + b_{1,1}b_{3,5}) - 2(4b_{1,1}^2 + b_{1,3})b_{1,5}.\end{aligned}$$

Theorem 7.2. *In the case of genus $g = 3$ the homogeneous polynomial vector fields \mathcal{D}_s for $s \in \{0, 1, 2, 3, 4, 5, 6, 8, 10\}$ give a solution to Problem 1.2. The vector fields $\mathcal{D}_1, \mathcal{D}_3, \mathcal{D}_5$ and \mathcal{D}_0 are determined by Lemmas 4.2, 4.3 and 4.6 for $g = 3$. We set $p_{1,k} = 2b_{1,k}$ and $q_{i,j,k} = \mathcal{D}_i(p_{j,k})$ for $i, j, k \in \{1, 3, 5\}$. The polynomial vector fields \mathcal{D}_2 and \mathcal{D}_4 are determined by the conditions*

$$\begin{aligned}\mathcal{D}_2(b_{1,1}) &= \frac{12}{7}\lambda_4 + 2b_{1,1}^2 + 4b_{1,3}, & \mathcal{D}_4(b_{1,1}) &= \frac{4}{7}\lambda_6 - 2b_{1,1}b_{1,3} + b_{3,3} + 2b_{1,5}, \\ \mathcal{D}_2(b_{1,3}) &= -\frac{8}{7}\lambda_4b_{1,1} + 2b_{1,1}b_{1,3} + 6b_{1,5}, \\ \mathcal{D}_4(b_{1,3}) &= -\frac{12}{7}\lambda_6b_{1,1} - 10b_{1,1}^2b_{1,3} + 2b_{3,1}b_{1,3} - 4b_{1,3}^2 - \frac{1}{2}b_{2,1}b_{2,3} + 2b_{1,1}b_{1,5}, \\ \mathcal{D}_2(b_{1,5}) &= -\frac{4}{7}\lambda_4b_{1,3} + 2b_{1,1}b_{1,5}, \\ \mathcal{D}_4(b_{1,5}) &= -\frac{6}{7}\lambda_6b_{1,3} - 16b_{1,1}^2b_{1,5} + 3b_{3,1}b_{1,5} - 8b_{1,3}b_{1,5} - \frac{1}{2}b_{2,1}b_{2,5},\end{aligned}$$

and the relations

$$\begin{pmatrix} [\mathcal{D}_1, \mathcal{D}_2] \\ [\mathcal{D}_1, \mathcal{D}_4] \end{pmatrix} = \begin{pmatrix} b_{1,1} & -1 & 0 \\ b_{1,3} & b_{1,1} & -1 \end{pmatrix} \begin{pmatrix} \mathcal{D}_1 \\ \mathcal{D}_3 \\ \mathcal{D}_5 \end{pmatrix}. \quad (11)$$

The polynomial vector fields $\mathcal{D}_6, \mathcal{D}_8$ and \mathcal{D}_{10} are determined by the relations

$$\begin{aligned}\mathcal{D}_6 &= \frac{1}{4}(2[\mathcal{D}_2, \mathcal{D}_4] + b_{2,3}\mathcal{D}_1 - b_{2,1}\mathcal{D}_3) - \frac{8}{7}(\lambda_6\mathcal{D}_0 - \lambda_4\mathcal{D}_2), & (12) \\ \mathcal{D}_8 &= \frac{1}{8}\left(2[\mathcal{D}_2, \mathcal{D}_6] + \left(\frac{1}{2}\mathcal{D}_1(p_{3,3}) + b_{2,5}\right)\mathcal{D}_1 - b_{2,3}\mathcal{D}_3 - b_{2,1}\mathcal{D}_5\right) - \frac{3}{7}(\lambda_8\mathcal{D}_0 - \lambda_4\mathcal{D}_4), \\ \mathcal{D}_{10} &= \frac{1}{12}(2[\mathcal{D}_2, \mathcal{D}_8] + \mathcal{D}_1(p_{3,5})\mathcal{D}_1 - b_{2,5}\mathcal{D}_3 - b_{2,3}\mathcal{D}_5) - \frac{4}{21}(\lambda_{10}\mathcal{D}_0 - \lambda_4\mathcal{D}_6).\end{aligned}$$

The commutation relations are obtained from the relations in Theorem 7.1 by the correspondence

$$\mathcal{L}_k = \varphi^*\mathcal{D}_k, \quad \varphi^*p_{i,j} = 2\wp_{i,j}, \quad \varphi^*q_{i,j,k} = 2\wp_{i,j,k}.$$

Proof. This Theorem follows from Section 9 in [5]. Note that the relations (11) determine the polynomials $\mathcal{D}_k(b_{i,j})$ for $k \in \{2, 4\}$, $i \in \{2, 3\}$, $j \in \{1, 3, 5\}$. For \mathcal{L}_k given in Theorem 7.1 the relation (7) holds. Cf. eq. (5), (10), and (12). \square

The polynomial dynamical systems in \mathbb{C}^9 determined by the vector fields \mathcal{D}_k are presented in Section 8 of [13].

8. LIE ALGEBRA OF DERIVATIONS OF HYPERELLIPTIC FUNCTIONS: GENUS 4 CASE

Example 8.1. *In the case of genus $g = 4$ Lemma 2.1 implies*

$$\begin{pmatrix} [L_2, L_4] \\ [L_2, L_6] \\ [L_2, L_8] \\ [L_2, L_{10}] \\ [L_2, L_{12}] \\ [L_2, L_{14}] \end{pmatrix} = \frac{1}{9} \begin{pmatrix} 24\lambda_6 & -24\lambda_4 & 0 & 18 & 0 & 0 & 0 & 0 \\ 20\lambda_8 & 0 & -20\lambda_4 & 0 & 36 & 0 & 0 & 0 \\ 16\lambda_{10} & 0 & 0 & -16\lambda_4 & 0 & 54 & 0 & 0 \\ 12\lambda_{12} & 0 & 0 & 0 & -12\lambda_4 & 0 & 72 & 0 \\ 8\lambda_{14} & 0 & 0 & 0 & 0 & -8\lambda_4 & 0 & 90 \\ 4\lambda_{10} & 0 & 0 & 0 & 0 & 0 & -4\lambda_4 & 0 \end{pmatrix} \tilde{L}, \quad (13)$$

where $\tilde{L} = (L_0 \ L_2 \ L_4 \ L_6 \ L_8 \ L_{10} \ L_{12} \ L_{14})^\top$.

We denote the matrix $(c_{2,2j}^{2k}(\lambda))$ by $\mathcal{C}_2(\lambda)$. The right hand side of (13) is $\mathcal{C}_2(\lambda)\tilde{L}$.

Theorem 8.2 (cf. Corollary 6.2 in [15]). *In the case of genus $g = 4$ the following vector fields give a solution to Problem 1.1*

$$\begin{aligned} \mathcal{L}_1 &= \partial_1, & \mathcal{L}_3 &= \partial_3, & \mathcal{L}_5 &= \partial_5, & \mathcal{L}_7 &= \partial_7, \\ \mathcal{L}_0 &= L_0 - t_1\partial_1 - 3t_3\partial_3 - 5t_5\partial_5 - 7t_7\partial_7, \\ \mathcal{L}_2 &= L_2 - \zeta_1\partial_1 + \frac{4}{3}\lambda_4t_3\partial_1 - \left(t_1 - \frac{8}{9}\lambda_4t_5\right)\partial_3 - \left(3t_3 - \frac{4}{9}\lambda_4t_7\right)\partial_5 - 5t_5\partial_7, \\ \mathcal{L}_4 &= L_4 - \zeta_3\partial_1 - \zeta_1\partial_3 + \\ &\quad + 2\lambda_6t_3\partial_1 - \left(\lambda_4t_3 - \frac{4}{3}\lambda_6t_5\right)\partial_3 - \left(t_1 + 3\lambda_4t_5 - \frac{2}{3}\lambda_6t_7\right)\partial_5 - (3t_3 + 5\lambda_4t_7)\partial_7, \\ \mathcal{L}_6 &= L_6 - \zeta_5\partial_1 - \zeta_3\partial_3 - \zeta_1\partial_5 + \\ &\quad + \frac{5}{3}\lambda_8t_3\partial_1 + \frac{16}{9}\lambda_8t_5\partial_3 - \left(\lambda_4t_3 + 2\lambda_6t_5 - \frac{8}{9}\lambda_8t_7\right)\partial_5 - (t_1 + 3\lambda_4t_5 + 4\lambda_6t_7)\partial_7, \\ \mathcal{L}_8 &= L_8 - \zeta_7\partial_1 - \zeta_5\partial_3 - \zeta_3\partial_5 - \zeta_1\partial_7 + \left(\frac{4}{3}\lambda_{10}t_3 - \lambda_{12}t_5\right)\partial_1 + \\ &\quad + \frac{20}{9}\lambda_{10}t_5\partial_3 - \left(\lambda_8t_5 - \frac{10}{9}\lambda_{10}t_7\right)\partial_5 - (\lambda_4t_3 + 2\lambda_6t_5 + 3\lambda_8t_7)\partial_7, \\ \mathcal{L}_{10} &= L_{10} - \zeta_7\partial_3 - \zeta_5\partial_5 - \zeta_3\partial_7 + \\ &\quad + (\lambda_{12}t_3 - 2\lambda_{14}t_5 - \lambda_{16}t_7)\partial_1 + \frac{5}{3}\lambda_{12}t_5\partial_3 + \frac{4}{3}\lambda_{12}t_7\partial_5 - (\lambda_8t_5 + 2\lambda_{10}t_7)\partial_7, \\ \mathcal{L}_{12} &= L_{12} - \zeta_7\partial_5 - \zeta_5\partial_7 + \\ &\quad + \left(\frac{2}{3}\lambda_{14}t_3 - 3\lambda_{16}t_5 - 2\lambda_{18}t_7\right)\partial_1 + \left(\frac{10}{9}\lambda_{14}t_5 - \lambda_{16}t_7\right)\partial_3 + \frac{14}{9}\lambda_{14}t_7\partial_5 - \lambda_{12}t_7\partial_7, \\ \mathcal{L}_{14} &= L_{14} - \zeta_7\partial_7 + \left(\frac{1}{3}\lambda_{16}t_3 - 4\lambda_{18}t_5\right)\partial_1 + \left(\frac{5}{9}\lambda_{16}t_5 - 2\lambda_{18}t_7\right)\partial_3 + \frac{7}{9}\lambda_{16}t_7\partial_5. \end{aligned}$$

We denote this Lie algebra by \mathcal{L} . The commutation relations are given in Lemma 8.3, Lemma 8.4, Lemma 8.5, and Corollary 10.5.

Lemma 8.3. *For the commutators in the Lie algebra \mathcal{L} we have the relations:*

$$\begin{aligned} [\mathcal{L}_0, \mathcal{L}_k] &= k\mathcal{L}_k, & k &= 0, 1, 2, 3, 4, 5, 6, 7, 8, 10, 12, 14; \\ [\mathcal{L}_k, \mathcal{L}_m] &= 0, & k, m &= 1, 3, 5, 7. \end{aligned}$$

Proof. We use the explicit expressions for \mathcal{L}_k and the fact that \mathcal{L}_0 is the Euler vector field, thus $\mathcal{L}_0\zeta_k = k\zeta_k$. \square

Lemma 8.4. *For the commutators in the Lie algebra \mathcal{L} we have the relations:*

$$\begin{aligned}
\begin{pmatrix} [\mathcal{L}_1, \mathcal{L}_2] \\ [\mathcal{L}_1, \mathcal{L}_4] \\ [\mathcal{L}_1, \mathcal{L}_6] \\ [\mathcal{L}_1, \mathcal{L}_8] \\ [\mathcal{L}_1, \mathcal{L}_{10}] \\ [\mathcal{L}_1, \mathcal{L}_{12}] \\ [\mathcal{L}_1, \mathcal{L}_{14}] \end{pmatrix} &= \begin{pmatrix} \wp_{1,1} & -1 & 0 & 0 \\ \wp_{1,3} & \wp_{1,1} & -1 & 0 \\ \wp_{1,5} & \wp_{1,3} & \wp_{1,1} & -1 \\ \wp_{1,7} & \wp_{1,5} & \wp_{1,3} & \wp_{1,1} \\ 0 & \wp_{1,7} & \wp_{1,5} & \wp_{1,3} \\ 0 & 0 & \wp_{1,7} & \wp_{1,5} \\ 0 & 0 & 0 & \wp_{1,7} \end{pmatrix} \begin{pmatrix} \mathcal{L}_1 \\ \mathcal{L}_3 \\ \mathcal{L}_5 \\ \mathcal{L}_7 \end{pmatrix}, \\
\begin{pmatrix} [\mathcal{L}_3, \mathcal{L}_2] \\ [\mathcal{L}_3, \mathcal{L}_4] \\ [\mathcal{L}_3, \mathcal{L}_6] \\ [\mathcal{L}_3, \mathcal{L}_8] \\ [\mathcal{L}_3, \mathcal{L}_{10}] \\ [\mathcal{L}_3, \mathcal{L}_{12}] \\ [\mathcal{L}_3, \mathcal{L}_{14}] \end{pmatrix} &= \begin{pmatrix} \wp_{1,3} - \lambda_4 & 0 & -3 & 0 \\ \wp_{3,3} & \wp_{1,3} - \lambda_4 & 0 & -3 \\ \wp_{3,5} & \wp_{3,3} & \wp_{1,3} - \lambda_4 & 0 \\ \wp_{3,7} & \wp_{3,5} & \wp_{3,3} & \wp_{1,3} - \lambda_4 \\ 0 & \wp_{3,7} & \wp_{3,5} & \wp_{3,3} \\ 0 & 0 & \wp_{3,7} & \wp_{3,5} \\ 0 & 0 & 0 & \wp_{3,7} \end{pmatrix} \begin{pmatrix} \mathcal{L}_1 \\ \mathcal{L}_3 \\ \mathcal{L}_5 \\ \mathcal{L}_7 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 7\lambda_4 \\ 6\lambda_6 \\ 5\lambda_8 \\ 4\lambda_{10} \\ 3\lambda_{12} \\ 2\lambda_{14} \\ \lambda_{16} \end{pmatrix} \mathcal{L}_1, \\
\begin{pmatrix} [\mathcal{L}_5, \mathcal{L}_2] \\ [\mathcal{L}_5, \mathcal{L}_4] \\ [\mathcal{L}_5, \mathcal{L}_6] \\ [\mathcal{L}_5, \mathcal{L}_8] \\ [\mathcal{L}_5, \mathcal{L}_{10}] \\ [\mathcal{L}_5, \mathcal{L}_{12}] \\ [\mathcal{L}_5, \mathcal{L}_{14}] \end{pmatrix} &= \begin{pmatrix} \wp_{1,5} & 0 & 0 & -5 \\ \wp_{3,5} & \wp_{1,5} & -3\lambda_4 & 0 \\ \wp_{5,5} & \wp_{3,5} & \wp_{1,5} - 2\lambda_6 & -3\lambda_4 \\ \wp_{5,7} - \lambda_{12} & \wp_{5,5} & \wp_{3,5} - \lambda_8 & \wp_{1,5} - 2\lambda_6 \\ -2\lambda_{14} & \wp_{5,7} - \lambda_{12} & \wp_{5,5} & \wp_{3,5} - \lambda_8 \\ -3\lambda_{16} & -2\lambda_{14} & \wp_{5,7} & \wp_{5,5} \\ -4\lambda_{18} & -3\lambda_{16} & 0 & \wp_{5,7} \end{pmatrix} \begin{pmatrix} \mathcal{L}_1 \\ \mathcal{L}_3 \\ \mathcal{L}_5 \\ \mathcal{L}_7 \end{pmatrix} + \frac{4}{9} \begin{pmatrix} 2\lambda_4 \\ 3\lambda_6 \\ 4\lambda_8 \\ 5\lambda_{10} \\ 6\lambda_{12} \\ 7\lambda_{14} \\ 8\lambda_{16} \end{pmatrix} \mathcal{L}_3, \\
\begin{pmatrix} [\mathcal{L}_7, \mathcal{L}_2] \\ [\mathcal{L}_7, \mathcal{L}_4] \\ [\mathcal{L}_7, \mathcal{L}_6] \\ [\mathcal{L}_7, \mathcal{L}_8] \\ [\mathcal{L}_7, \mathcal{L}_{10}] \\ [\mathcal{L}_7, \mathcal{L}_{12}] \\ [\mathcal{L}_7, \mathcal{L}_{14}] \end{pmatrix} &= \begin{pmatrix} \wp_{1,7} & 0 & 0 & 0 \\ \wp_{3,7} & \wp_{1,7} & 0 & -5\lambda_4 \\ \wp_{5,7} & \wp_{3,7} & \wp_{1,7} & -4\lambda_6 \\ \wp_{7,7} & \wp_{5,7} & \wp_{3,7} & \wp_{1,7} - 3\lambda_8 \\ -\lambda_{16} & \wp_{7,7} & \wp_{5,7} & \wp_{3,7} - 2\lambda_{10} \\ -2\lambda_{18} & -\lambda_{16} & \wp_{7,7} & \wp_{5,7} - \lambda_{12} \\ 0 & -2\lambda_{18} & -\lambda_{16} & \wp_{7,7} \end{pmatrix} \begin{pmatrix} \mathcal{L}_1 \\ \mathcal{L}_3 \\ \mathcal{L}_5 \\ \mathcal{L}_7 \end{pmatrix} + \frac{2}{9} \begin{pmatrix} 2\lambda_4 \\ 3\lambda_6 \\ 4\lambda_8 \\ 5\lambda_{10} \\ 6\lambda_{12} \\ 7\lambda_{14} \\ 8\lambda_{16} \end{pmatrix} \mathcal{L}_5.
\end{aligned}$$

The proof follows from the explicit expressions for \mathcal{L}_k .

Lemma 8.5. *For the commutators in the Lie algebra \mathcal{L} we have the relations*

$$\begin{pmatrix} [\mathcal{L}_2, \mathcal{L}_4] \\ [\mathcal{L}_2, \mathcal{L}_6] \\ [\mathcal{L}_2, \mathcal{L}_8] \\ [\mathcal{L}_2, \mathcal{L}_{10}] \\ [\mathcal{L}_2, \mathcal{L}_{12}] \\ [\mathcal{L}_2, \mathcal{L}_{14}] \end{pmatrix} = \mathcal{C}_2(\lambda) \begin{pmatrix} \mathcal{L}_0 \\ \mathcal{L}_2 \\ \mathcal{L}_4 \\ \mathcal{L}_6 \\ \mathcal{L}_8 \\ \mathcal{L}_{10} \\ \mathcal{L}_{12} \\ \mathcal{L}_{14} \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -\wp_{1,1,3} & \wp_{1,1,1} & 0 & 0 \\ -\wp_{1,3,3} - \wp_{1,1,5} & \wp_{1,1,3} & \wp_{1,1,1} & 0 \\ -2\wp_{1,3,5} - \wp_{1,1,7} & \wp_{1,1,5} & \wp_{1,1,3} & \wp_{1,1,1} \\ -2\wp_{1,3,7} - \wp_{1,5,5} & \wp_{1,1,7} & \wp_{1,1,5} & \wp_{1,1,3} \\ -2\wp_{1,5,7} & 0 & \wp_{1,1,7} & \wp_{1,1,5} \\ -\wp_{1,7,7} & 0 & 0 & \wp_{1,1,7} \end{pmatrix} \begin{pmatrix} \mathcal{L}_1 \\ \mathcal{L}_3 \\ \mathcal{L}_5 \\ \mathcal{L}_7 \end{pmatrix},$$

where the polynomial matrix $\mathcal{C}_2(\lambda) = (c_{2,2j}^{2k}(\lambda))$ is given in (13).

Proof. From Theorem 6.3 in [15] we obtain the expressions for $\mathcal{L}_{2k}\zeta_s$, where $s = 1, 3, 5, 7$ and $k = 1, 2, 3, 4, 5, 6, 7$. Provided this, the proof follows from the explicit expressions for \mathcal{L}_k . \square

9. GENERATORS IN THE POLYNOMIAL LIE ALGEBRA IN \mathbb{C}^{12}

In the case of genus $g = 4$ the expressions for the coordinates (λ) and (p) in $b_{i,j}$ obtained in Corollaries 3.4 and 3.5 are the following:

$$\begin{aligned}
\lambda_4 &= -3b_{1,1}^2 + \frac{1}{2}b_{3,1} - 2b_{1,3}, & \lambda_6 &= 2b_{1,1}^3 + \frac{1}{4}b_{2,1}^2 - \frac{1}{2}b_{1,1}b_{3,1} - 2b_{1,1}b_{1,3} + \frac{1}{2}b_{3,3} - 2b_{1,5}, \\
\lambda_8 &= 4b_{1,1}^2b_{1,3} + b_{1,3}b_{1,3} - 2b_{1,1}b_{1,5} + \frac{1}{2}b_{3,5} - 2b_{1,7} - \frac{1}{2}(b_{3,1}b_{1,3} - b_{2,1}b_{2,3} + b_{1,1}b_{3,3}), & (14) \\
\lambda_{10} &= 2(b_{1,1}b_{1,3}^2 + b_{1,5}b_{1,3} - b_{1,1}b_{1,7} + 2b_{1,1}^2b_{1,5}) + \frac{1}{4}b_{2,3}^2 - \frac{1}{2}(b_{1,3}b_{3,3} - b_{3,7} + b_{3,1}b_{1,5} - b_{2,1}b_{2,5} + b_{1,1}b_{3,5}), \\
\lambda_{12} &= 4b_{1,1}(b_{1,3}b_{1,5} + b_{1,1}b_{1,7}) + b_{1,5}^2 + 2b_{1,3}b_{1,7} - \\
&\quad - \frac{1}{2}(b_{3,3}b_{1,5} - b_{2,3}b_{2,5} + b_{1,3}b_{3,5} + b_{3,1}b_{1,7} - b_{2,1}b_{2,7} + b_{1,1}b_{3,7}), \\
\lambda_{14} &= 2b_{1,1}b_{1,5}^2 + \frac{1}{4}b_{2,5}^2 - \frac{1}{2}b_{1,5}b_{3,5} + 4b_{1,1}b_{1,3}b_{1,7} + 2b_{1,5}b_{1,7} - \frac{1}{2}(b_{3,3}b_{1,7} - b_{2,3}b_{2,7} + b_{1,3}b_{3,7}), \\
\lambda_{16} &= 4b_{1,1}b_{1,5}b_{1,7} + b_{1,7}^2 - \frac{1}{2}(b_{3,5}b_{1,7} - b_{2,5}b_{2,7} + b_{1,5}b_{3,7}), & \lambda_{18} &= 2b_{1,1}b_{1,7}^2 + \frac{1}{4}b_{2,7}^2 - \frac{1}{2}b_{1,7}b_{3,7}, \\
p_{3,3} &= 6b_{1,1}b_{1,3} - b_{3,3} + 6b_{1,5}, & p_{3,5} &= 6b_{1,1}b_{1,5} - b_{3,5} + 6b_{1,7}, & p_{3,7} &= 6b_{1,1}b_{1,7} - b_{3,7}, \\
p_{5,5} &= b_{3,1}b_{1,5} - b_{2,1}b_{2,5} + b_{1,1}b_{3,5} - 2(4b_{1,1}^2 + b_{1,3})b_{1,5} + 10b_{1,1}b_{1,7} - 2b_{3,7}, \\
p_{5,7} &= b_{3,1}b_{1,7} - b_{2,1}b_{2,7} + b_{1,1}b_{3,7} - 2(4b_{1,1}^2 + b_{1,3})b_{1,7}, \\
p_{7,7} &= b_{3,3}b_{1,7} - b_{2,3}b_{2,7} + b_{1,3}b_{3,7} - 2(4b_{1,1}b_{1,3} + b_{1,5})b_{1,7}.
\end{aligned}$$

Theorem 9.1. *In the case of genus $g = 4$ the homogeneous polynomial vector fields \mathcal{D}_s for $s \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 10, 12, 14\}$ give a solution to Problem 1.2. The vector fields $\mathcal{D}_1, \mathcal{D}_3, \mathcal{D}_5, \mathcal{D}_7$, and \mathcal{D}_0 are determined by Lemmas 4.2, 4.3 and 4.6 for $g = 4$. We set $p'_{i,j} = \mathcal{D}_1(p_{i,j})$ and $p''_{i,j} = \mathcal{D}_1(\mathcal{D}_1(p_{i,j}))$ for $i, j \in \{3, 5, 7\}$. The polynomial vector fields \mathcal{D}_2 and \mathcal{D}_4 are determined by the conditions*

$$\begin{aligned}
\mathcal{D}_2(b_{1,1}) &= -b_{1,1}^2 + \frac{1}{2}b_{3,1} + 2b_{1,3} + \frac{7}{9}\lambda_4, & (15) \\
\mathcal{D}_2(b_{1,3}) &= -b_{1,1}b_{1,3} + \frac{1}{2}b_{3,3} + 3b_{1,5} - \frac{4}{3}\lambda_4b_{1,1} + \frac{1}{2}p_{3,3}, \\
\mathcal{D}_2(b_{1,5}) &= -b_{1,1}b_{1,5} + \frac{1}{2}b_{3,5} + 5b_{1,7} - \frac{8}{9}\lambda_4b_{1,3} + \frac{1}{2}p_{3,5}, \\
\mathcal{D}_2(b_{1,7}) &= -b_{1,1}b_{1,7} + \frac{1}{2}b_{3,7} - \frac{4}{9}\lambda_4b_{1,5} + \frac{1}{2}p_{3,7}, \\
\mathcal{D}_4(b_{1,1}) &= -2b_{1,1}b_{1,3} + b_{3,3} + 2b_{1,5} + \frac{2}{3}\lambda_6, \\
\mathcal{D}_4(b_{1,3}) &= -b_{1,3}^2 + 3b_{1,7} + \lambda_4b_{1,3} - 2\lambda_6b_{1,1} - \lambda_8 - \frac{1}{2}b_{1,1}p_{3,3} + \frac{1}{2}p''_{3,3} + \frac{1}{2}p_{3,5}, \\
\mathcal{D}_4(b_{1,5}) &= -b_{1,3}b_{1,5} + 3\lambda_4b_{1,5} - \frac{4}{3}\lambda_6b_{1,3} - \frac{1}{2}b_{1,1}p_{3,5} + \frac{1}{2}p''_{3,5} + \frac{1}{2}p_{5,5}, \\
\mathcal{D}_4(b_{1,7}) &= -b_{1,3}b_{1,7} + 5\lambda_4b_{1,7} - \frac{2}{3}\lambda_6b_{1,5} - \frac{1}{2}b_{1,1}p_{3,7} + \frac{1}{2}p''_{3,7} + \frac{1}{2}p_{5,7},
\end{aligned}$$

and the relations

$$\begin{pmatrix} [\mathcal{D}_1, \mathcal{D}_2] \\ [\mathcal{D}_1, \mathcal{D}_4] \end{pmatrix} = \begin{pmatrix} b_{1,1} & -1 & 0 \\ b_{1,3} & b_{1,1} & -1 \end{pmatrix} \begin{pmatrix} \mathcal{D}_1 \\ \mathcal{D}_3 \\ \mathcal{D}_5 \end{pmatrix}. \quad (16)$$

The polynomial vector fields $\mathcal{D}_6, \mathcal{D}_8$ and \mathcal{D}_{10} are determined by the relations

$$\begin{aligned}\mathcal{D}_6 &= \frac{1}{4} (2[\mathcal{D}_2, \mathcal{D}_4] + b_{2,3}\mathcal{D}_1 - b_{2,1}\mathcal{D}_3) - \frac{4}{3} (\lambda_6\mathcal{D}_0 - \lambda_4\mathcal{D}_2), \\ \mathcal{D}_8 &= \frac{1}{8} \left(2[\mathcal{D}_2, \mathcal{D}_6] + \left(\frac{1}{2}p'_{3,3} + b_{2,5} \right) \mathcal{D}_1 - b_{2,3}\mathcal{D}_3 - b_{2,1}\mathcal{D}_5 \right) - \frac{5}{9} (\lambda_8\mathcal{D}_0 - \lambda_4\mathcal{D}_4), \\ \mathcal{D}_{10} &= \frac{1}{12} (2[\mathcal{D}_2, \mathcal{D}_8] + (p'_{3,5} + b_{2,7}) \mathcal{D}_1 - b_{2,5}\mathcal{D}_3 - b_{2,3}\mathcal{D}_5 - b_{2,1}\mathcal{D}_7) - \frac{8}{27} (\lambda_{10}\mathcal{D}_0 - \lambda_4\mathcal{D}_6), \\ \mathcal{D}_{12} &= \frac{1}{16} \left(2[\mathcal{D}_2, \mathcal{D}_{10}] + \left(p'_{3,7} + \frac{1}{2}p'_{5,5} \right) \mathcal{D}_1 - b_{2,7}\mathcal{D}_3 - b_{2,5}\mathcal{D}_5 - b_{2,3}\mathcal{D}_7 \right) - \frac{1}{6} (\lambda_{12}\mathcal{D}_0 - \lambda_4\mathcal{D}_8), \\ \mathcal{D}_{14} &= \frac{1}{20} (2[\mathcal{D}_2, \mathcal{D}_{12}] + p'_{5,7}\mathcal{D}_1 - b_{2,7}\mathcal{D}_3 - b_{2,5}\mathcal{D}_5 - b_{2,3}\mathcal{D}_7) - \frac{4}{45} (\lambda_{14}\mathcal{D}_0 - \lambda_4\mathcal{D}_{10}).\end{aligned}\tag{17}$$

We denote by \mathcal{D} the polynomial Lie algebra generated by the polynomial vector fields \mathcal{D}_s for $s \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 10, 12, 14\}$. The commutation relations for \mathcal{D} are given in Lemma 10.1, Lemma 10.2, Lemma 10.3, and Lemma 10.4.

Proof. First we note that the polynomial vector fields \mathcal{D}_s are determined uniquely by the Theorem. The polynomial vector fields $\mathcal{D}_1, \mathcal{D}_3, \mathcal{D}_5, \mathcal{D}_7$, and \mathcal{D}_0 are determined by Lemmas 4.2, 4.3 and 4.6. Relations (15) determine the polynomials $\mathcal{D}_2(b_{1,j})$ and $\mathcal{D}_4(b_{1,j})$ for $j \in \{1, 3, 5, 7\}$. Relations (16) determine the polynomials $\mathcal{D}_2(b_{i,j})$ and $\mathcal{D}_4(b_{i,j})$ for $i \in \{2, 3\}$, $j \in \{1, 3, 5, 7\}$. Relations (17) determine the polynomial vector fields \mathcal{D}_{2k} for $k \in \{3, 4, 5, 6, 7\}$ given the polynomial vector fields \mathcal{D}_s for $s \in \{1, 3, 5, 7\}$ and \mathcal{D}_{2m} for $m < k$.

Thus we have a system of 12 polynomial vector fields that are explicitly defined in the coordinates (b) . By Corollary 4.4, the polynomial vector fields \mathcal{D}_s for $s \in \{1, 3, 5, 7\}$ are projectable for the polynomial map $\rho: \mathbb{C}^{12} \rightarrow \mathbb{C}^8$. Their pushforwards are zero. By Corollary 4.7, the polynomial vector field \mathcal{D}_0 is projectable for the polynomial map ρ with pushforward L_0 . It is a direct calculation to check that the polynomial vector fields \mathcal{D}_{2k} for $k \in \{1, 2, 3, 4, 5, 6, 7\}$ are projectable for the polynomial map ρ with pushforwards L_{2k} . This proves the claim of the Theorem that the vector fields \mathcal{D}_s give a solution to Problem 1.2.

The commutation relations given in Section 10 in Lemmas 10.1, 10.2, 10.3, and 10.4 are proved by direct computation. \square

Theorem 9.2. *For the polynomial vector fields \mathcal{D}_k for $k \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 10, 12, 14\}$, determined by Theorem 9.1, we have*

$$\mathcal{L}_k(\varphi^*b_{i,j}) = \varphi^*\mathcal{D}_k(b_{i,j}),\tag{18}$$

where the vector fields \mathcal{L}_k are given in Theorem 8.2.

Proof. For the polynomial vector fields $\mathcal{D}_1, \mathcal{D}_3, \mathcal{D}_5, \mathcal{D}_7$, and \mathcal{D}_0 the claim of the Theorem is given by Lemmas 4.2, 4.3, and 4.6. For $k \in \{1, 2, 3, 4, 5, 6, 7\}$ define the vector fields $\tilde{\mathcal{D}}_{2s}$ by the relation (18). The vector fields $\tilde{\mathcal{D}}_{2s}$ are linear combinations of the fields \mathcal{D}_k for $k \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 10, 12, 14\}$ with coefficients rational functions in (b) . The pushforward of $\tilde{\mathcal{D}}_{2s}$ is L_{2s} , which is equal to the pushforward of \mathcal{D}_{2s} . Therefore we have

$$\tilde{\mathcal{D}}_{2s} = \mathcal{D}_{2s} + \alpha_{2s}^1(b)\mathcal{D}_1 + \alpha_{2s}^3(b)\mathcal{D}_3 + \alpha_{2s}^5(b)\mathcal{D}_5 + \alpha_{2s}^7(b)\mathcal{D}_7$$

for some rational functions $\alpha_{2s}^j(b)$. Comparing results of Lemma 8.4 and Lemma 10.2, we obtain $\mathcal{D}_i(\alpha_{2s}^j(b)) = 0$ for $i \in \{1, 3, 5, 7\}$, thus $\alpha_{2s}^j(b)$ is a rational function in (λ) . We have $\text{wt } \alpha_{2s}^j(b) = 2s - j$, where j is odd, while $\text{wt } \lambda_m$ is even for all m (see (14)), thus we obtain $\alpha_{2s}^j(b) = 0$ and $\tilde{\mathcal{D}}_{2s} = \mathcal{D}_{2s}$. \square

10. COMMUTATION RELATIONS IN THE POLYNOMIAL LIE ALGEBRA IN \mathbb{C}^{12}

Lemma 10.1. *The relations hold for the commutators in the polynomial Lie algebra \mathcal{D} :*

$$\begin{aligned} [\mathcal{D}_0, \mathcal{D}_k] &= k\mathcal{D}_k, & k &= 0, 1, 2, 3, 4, 5, 6, 7, 8, 10, 12, 14; \\ [\mathcal{D}_k, \mathcal{D}_m] &= 0, & k, m &= 1, 3, 5, 7. \end{aligned}$$

Lemma 10.2. *The relations hold for the commutators in the polynomial Lie algebra \mathcal{D} :*

$$\begin{aligned} \begin{pmatrix} [\mathcal{D}_1, \mathcal{D}_2] \\ [\mathcal{D}_1, \mathcal{D}_4] \\ [\mathcal{D}_1, \mathcal{D}_6] \\ [\mathcal{D}_1, \mathcal{D}_8] \\ [\mathcal{D}_1, \mathcal{D}_{10}] \\ [\mathcal{D}_1, \mathcal{D}_{12}] \\ [\mathcal{D}_1, \mathcal{D}_{14}] \end{pmatrix} &= \begin{pmatrix} b_{1,1} & -1 & 0 & 0 \\ b_{1,3} & b_{1,1} & -1 & 0 \\ b_{1,5} & b_{1,3} & b_{1,1} & -1 \\ b_{1,7} & b_{1,5} & b_{1,3} & b_{1,1} \\ 0 & b_{1,7} & b_{1,5} & b_{1,3} \\ 0 & 0 & b_{1,7} & b_{1,5} \\ 0 & 0 & 0 & b_{1,7} \end{pmatrix} \begin{pmatrix} \mathcal{D}_1 \\ \mathcal{D}_3 \\ \mathcal{D}_5 \\ \mathcal{D}_7 \end{pmatrix}, & (19) \\ \begin{pmatrix} [\mathcal{D}_3, \mathcal{D}_2] \\ [\mathcal{D}_3, \mathcal{D}_4] \\ [\mathcal{D}_3, \mathcal{D}_6] \\ [\mathcal{D}_3, \mathcal{D}_8] \\ [\mathcal{D}_3, \mathcal{D}_{10}] \\ [\mathcal{D}_3, \mathcal{D}_{12}] \\ [\mathcal{D}_3, \mathcal{D}_{14}] \end{pmatrix} &= \frac{1}{2} \begin{pmatrix} 2b_{1,3} - 2\lambda_4 & 0 & -6 & 0 \\ p_{3,3} & 2b_{1,3} - 2\lambda_4 & 0 & -6 \\ p_{3,5} & p_{3,3} & 2b_{1,3} - 2\lambda_4 & 0 \\ p_{3,7} & p_{3,5} & p_{3,3} & 2b_{1,3} - 2\lambda_4 \\ 0 & p_{3,7} & p_{3,5} & p_{3,3} \\ 0 & 0 & p_{3,7} & p_{3,5} \\ 0 & 0 & 0 & p_{3,7} \end{pmatrix} \begin{pmatrix} \mathcal{D}_1 \\ \mathcal{D}_3 \\ \mathcal{D}_5 \\ \mathcal{D}_7 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 7\lambda_4 \\ 6\lambda_6 \\ 5\lambda_8 \\ 4\lambda_{10} \\ 3\lambda_{12} \\ 2\lambda_{14} \\ \lambda_{16} \end{pmatrix} \mathcal{D}_1, \\ \begin{pmatrix} [\mathcal{D}_5, \mathcal{D}_2] \\ [\mathcal{D}_5, \mathcal{D}_4] \\ [\mathcal{D}_5, \mathcal{D}_6] \\ [\mathcal{D}_5, \mathcal{D}_8] \\ [\mathcal{D}_5, \mathcal{D}_{10}] \\ [\mathcal{D}_5, \mathcal{D}_{12}] \\ [\mathcal{D}_5, \mathcal{D}_{14}] \end{pmatrix} &= \frac{1}{2} \begin{pmatrix} 2b_{1,5} & 0 & 0 & -10 \\ p_{3,5} & 2b_{1,5} & -6\lambda_4 & 0 \\ p_{5,5} & p_{3,5} & 2b_{1,5} - 4\lambda_6 & -6\lambda_4 \\ p_{5,7} - 2\lambda_{12} & p_{5,5} & p_{3,5} - 2\lambda_8 & 2b_{1,5} - 4\lambda_6 \\ -4\lambda_{14} & p_{5,7} - 2\lambda_{12} & p_{5,5} & p_{3,5} - 2\lambda_8 \\ -6\lambda_{16} & -4\lambda_{14} & p_{5,7} & p_{5,5} \\ -8\lambda_{18} & -6\lambda_{16} & 0 & p_{5,7} \end{pmatrix} \begin{pmatrix} \mathcal{D}_1 \\ \mathcal{D}_3 \\ \mathcal{D}_5 \\ \mathcal{D}_7 \end{pmatrix} + \frac{4}{9} \begin{pmatrix} 2\lambda_4 \\ 3\lambda_6 \\ 4\lambda_8 \\ 5\lambda_{10} \\ 6\lambda_{12} \\ 7\lambda_{14} \\ 8\lambda_{16} \end{pmatrix} \mathcal{D}_3, \\ \begin{pmatrix} [\mathcal{D}_7, \mathcal{D}_2] \\ [\mathcal{D}_7, \mathcal{D}_4] \\ [\mathcal{D}_7, \mathcal{D}_6] \\ [\mathcal{D}_7, \mathcal{D}_8] \\ [\mathcal{D}_7, \mathcal{D}_{10}] \\ [\mathcal{D}_7, \mathcal{D}_{12}] \\ [\mathcal{D}_7, \mathcal{D}_{14}] \end{pmatrix} &= \frac{1}{2} \begin{pmatrix} 2b_{1,7} & 0 & 0 & 0 \\ p_{3,7} & 2b_{1,7} & 0 & -10\lambda_4 \\ p_{5,7} & p_{3,7} & 2b_{1,7} & -8\lambda_6 \\ p_{7,7} & p_{5,7} & p_{3,7} & 2b_{1,7} - 6\lambda_8 \\ -2\lambda_{16} & p_{7,7} & p_{5,7} & p_{3,7} - 4\lambda_{10} \\ -4\lambda_{18} & -2\lambda_{16} & p_{7,7} & p_{5,7} - 2\lambda_{12} \\ 0 & -4\lambda_{18} & -2\lambda_{16} & p_{7,7} \end{pmatrix} \begin{pmatrix} \mathcal{D}_1 \\ \mathcal{D}_3 \\ \mathcal{D}_5 \\ \mathcal{D}_7 \end{pmatrix} + \frac{2}{9} \begin{pmatrix} 2\lambda_4 \\ 3\lambda_6 \\ 4\lambda_8 \\ 5\lambda_{10} \\ 6\lambda_{12} \\ 7\lambda_{14} \\ 8\lambda_{16} \end{pmatrix} \mathcal{D}_5. \end{aligned}$$

Lemma 10.3. *The relations hold for the commutators in the polynomial Lie algebra \mathcal{D} :*

$$\begin{pmatrix} [\mathcal{D}_2, \mathcal{D}_4] \\ [\mathcal{D}_2, \mathcal{D}_6] \\ [\mathcal{D}_2, \mathcal{D}_8] \\ [\mathcal{D}_2, \mathcal{D}_{10}] \\ [\mathcal{D}_2, \mathcal{D}_{12}] \\ [\mathcal{D}_2, \mathcal{D}_{14}] \end{pmatrix} = \mathcal{C}_2(\lambda) \begin{pmatrix} \mathcal{D}_0 \\ \mathcal{D}_2 \\ \mathcal{D}_4 \\ \mathcal{D}_6 \\ \mathcal{D}_8 \\ \mathcal{D}_{10} \\ \mathcal{D}_{12} \\ \mathcal{D}_{14} \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -b_{2,3} & b_{2,1} & 0 & 0 \\ -\frac{1}{2}p'_{3,3} - b_{2,5} & b_{2,3} & b_{2,1} & 0 \\ -p'_{3,5} - b_{2,7} & b_{2,5} & b_{2,3} & b_{2,1} \\ -p'_{3,7} - \frac{1}{2}p'_{5,5} & b_{2,7} & b_{2,5} & b_{2,3} \\ -p'_{5,7} & 0 & b_{2,7} & b_{2,5} \\ -\frac{1}{2}p'_{7,7} & 0 & 0 & b_{2,7} \end{pmatrix} \begin{pmatrix} \mathcal{D}_1 \\ \mathcal{D}_3 \\ \mathcal{D}_5 \\ \mathcal{D}_7 \end{pmatrix},$$

where the polynomial matrix $\mathcal{C}_2(\lambda) = (c_{2,2j}^{2k}(\lambda))$ is given in (13).

Lemma 10.4. Set $p_{1,k} = 2b_{1,k}$ and $q_{i,j,k} = \mathcal{D}_i(p_{j,k})$ for $i, j, k \in \{1, 3, 5, 7\}$. The relations for $m, l \in 2, 3, 4, 5, 6$, $m \leq l$ hold for the commutators in the polynomial Lie algebra \mathcal{D} :

$$[\mathcal{D}_{2m}, \mathcal{D}_{2l+2}] = \sum_{k=0}^7 c_{2m,2l+2}^{2k}(\lambda) \mathcal{D}_{2k} + (A_{2m,2l+2}) (\mathcal{D}_1 \ \mathcal{D}_3 \ \mathcal{D}_5 \ \mathcal{D}_7)^\top,$$

where

$$\begin{pmatrix} A_{4,6} \\ A_{4,8} \\ A_{4,10} \\ A_{4,12} \\ A_{4,14} \\ A_{6,8} \\ A_{6,10} \\ A_{6,12} \\ A_{6,14} \\ A_{8,10} \\ A_{8,12} \\ A_{8,14} \\ A_{10,12} \\ A_{10,14} \\ A_{12,14} \end{pmatrix} = \frac{1}{4} \begin{pmatrix} -q_{3,3,3} & q_{1,3,3} - 2q_{1,1,5} & 2q_{1,1,3} & 0 \\ -2q_{3,3,5} & -2q_{1,1,7} & 2q_{1,3,3} & 2q_{1,1,3} \\ -2q_{3,3,7} - q_{3,5,5} & -q_{1,5,5} & 2q_{1,3,5} & 2q_{1,3,3} \\ -2q_{3,5,7} & -2q_{1,5,7} & 2q_{1,3,7} & 2q_{1,3,5} \\ -q_{3,7,7} & -q_{1,7,7} & 0 & 2q_{1,3,7} \\ q_{3,3,7} - 2q_{3,5,5} & 2q_{1,5,5} - q_{3,3,5} - 2q_{1,3,7} & q_{3,3,3} - 2q_{1,1,7} & q_{1,3,3} + 2q_{1,1,5} \\ -q_{5,5,5} - 2q_{3,5,7} & 2q_{1,5,7} - q_{3,5,5} - q_{3,3,7} & q_{3,3,5} + q_{1,5,5} - 2q_{1,3,7} & q_{3,3,3} + 2q_{1,3,5} \\ -2q_{5,5,7} & -2q_{3,5,7} & q_{3,3,7} & q_{3,3,5} + 2q_{1,5,5} \\ -q_{5,7,7} & -q_{3,7,7} & -q_{1,7,7} & q_{3,3,7} + 2q_{1,5,7} \\ -2q_{3,7,7} - q_{5,5,7} & -q_{5,5,5} + 2q_{1,7,7} & 2q_{1,5,7} + q_{3,5,5} - 2q_{3,3,7} & 2q_{3,3,5} - q_{1,5,5} \\ -2q_{5,7,7} & -2q_{5,5,7} & 2q_{1,7,7} & 2q_{3,5,5} \\ -q_{7,7,7} & -q_{5,7,7} & -q_{3,7,7} & 2q_{3,5,7} + q_{1,7,7} \\ 0 & -2q_{5,7,7} & 2q_{3,7,7} - q_{5,5,7} & q_{5,5,5} \\ 0 & -q_{7,7,7} & -q_{5,7,7} & q_{5,5,7} + q_{3,7,7} \\ 0 & 0 & -q_{7,7,7} & q_{5,7,7} \end{pmatrix}.$$

Corollary 10.5 (From Lemma 10.4 and Theorem 9.2). The relations for $m, l \in 2, 3, 4, 5, 6$, $m \leq l$ hold for the commutators in the polynomial Lie algebra \mathcal{L} :

$$[\mathcal{L}_{2m}, \mathcal{L}_{2l+2}] = \sum_{k=0}^7 c_{2m,2l+2}^{2k}(\lambda) \mathcal{L}_{2k} + (A_{2m,2l+2}) (\mathcal{L}_1 \ \mathcal{L}_3 \ \mathcal{L}_5 \ \mathcal{L}_7)^\top,$$

where

$$\begin{pmatrix} \mathcal{A}_{4,6} \\ \mathcal{A}_{4,8} \\ \mathcal{A}_{4,10} \\ \mathcal{A}_{4,12} \\ \mathcal{A}_{4,14} \\ \mathcal{A}_{6,8} \\ \mathcal{A}_{6,10} \\ \mathcal{A}_{6,12} \\ \mathcal{A}_{6,14} \\ \mathcal{A}_{8,10} \\ \mathcal{A}_{8,12} \\ \mathcal{A}_{8,14} \\ \mathcal{A}_{10,12} \\ \mathcal{A}_{10,14} \\ \mathcal{A}_{12,14} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -\wp_{3,3,3} & \wp_{1,3,3} - 2\wp_{1,1,5} & 2\wp_{1,1,3} & 0 \\ -2\wp_{3,3,5} & -2\wp_{1,1,7} & 2\wp_{1,3,3} & 2\wp_{1,1,3} \\ -2\wp_{3,3,7} - \wp_{3,5,5} & -\wp_{1,5,5} & 2\wp_{1,3,5} & 2\wp_{1,3,3} \\ -2\wp_{3,5,7} & -2\wp_{1,5,7} & 2\wp_{1,3,7} & 2\wp_{1,3,5} \\ -\wp_{3,7,7} & -\wp_{1,7,7} & 0 & 2\wp_{1,3,7} \\ \wp_{3,3,7} - 2\wp_{3,5,5} & 2\wp_{1,5,5} - \wp_{3,3,5} - 2\wp_{1,3,7} & \wp_{3,3,3} - 2\wp_{1,1,7} & \wp_{1,3,3} + 2\wp_{1,1,5} \\ -\wp_{5,5,5} - 2\wp_{3,5,7} & 2\wp_{1,5,7} - \wp_{3,5,5} - \wp_{3,3,7} & \wp_{3,3,5} + \wp_{1,5,5} - 2\wp_{1,3,7} & \wp_{3,3,3} + 2\wp_{1,3,5} \\ -2\wp_{5,5,7} & -2\wp_{3,5,7} & \wp_{3,3,7} & \wp_{3,3,5} + 2\wp_{1,5,5} \\ -\wp_{5,7,7} & -\wp_{3,7,7} & -\wp_{1,7,7} & \wp_{3,3,7} + 2\wp_{1,5,7} \\ -2\wp_{3,7,7} - \wp_{5,5,7} & -\wp_{5,5,5} + 2\wp_{1,7,7} & 2\wp_{1,5,7} + \wp_{3,5,5} - 2\wp_{3,3,7} & 2\wp_{3,3,5} - \wp_{1,5,5} \\ -2\wp_{5,7,7} & -2\wp_{5,5,7} & 2\wp_{1,7,7} & 2\wp_{3,5,5} \\ -\wp_{7,7,7} & -\wp_{5,7,7} & -\wp_{3,7,7} & 2\wp_{3,5,7} + \wp_{1,7,7} \\ 0 & -2\wp_{5,7,7} & 2\wp_{3,7,7} - \wp_{5,5,7} & \wp_{5,5,5} \\ 0 & -\wp_{7,7,7} & -\wp_{5,7,7} & \wp_{5,5,7} + \wp_{3,7,7} \\ 0 & 0 & -\wp_{7,7,7} & \wp_{5,7,7} \end{pmatrix}.$$

11. POLYNOMIAL DYNAMICAL SYSTEMS IN \mathbb{C}^{12} RELATED TO DIFFERENTIATIONS
OF GENUS 4 HYPERELLIPTIC FUNCTIONS

As we have noted throughout the work, the generators \mathcal{L}_k of $\text{Der } \mathcal{F}$ and the polynomial vector fields \mathcal{D}_k in \mathbb{C}^{3g} are related by (7). The polynomial vector fields $\mathcal{D}_0, \mathcal{D}_1$, and \mathcal{D}_s for $s \in \{3, 5, \dots, 2g-1\}$ are given in Section 4. Section 5 gives the graded homogeneous polynomial dynamical systems S_0, S_1 , and S_s for $s \in \{3, 5, \dots, 2g-1\}$ in \mathbb{C}^{3g} determined by these vector fields.

In this Section in the case of genus $g = 4$ we determine the remaining graded homogeneous polynomial dynamical systems S_{2k} for $k \in \{1, 2, 3, 4, 5, 6, 7\}$. By definition, the dynamical system S_{2k} corresponding to the vector field \mathcal{D}_{2k} is given by

$$\frac{\partial}{\partial \tau_{2k}} b_{i,j} = \mathcal{D}_{2k}(b_{i,j}).$$

It has been noted in [2] in the case of genus $g = 2$ that to determine such systems it is sufficient to determine the polynomials $\mathcal{D}_{2k}(b_{1,1})$. Indeed, we have the relation

$$\mathcal{D}_{2k}(\lambda_m) = L_{2k}(\lambda_m)$$

for $m \in \{4, 6, 8, 10, 12, 14, 16, 18\}$ that determines the action of \mathcal{D}_{2k} on the coordinates (λ) . The relations (14) imply

$$\begin{aligned} b_{1,3} &= -\frac{3}{2}b_{1,1}^2 + \frac{1}{4}b_{3,1} - \frac{1}{2}\lambda_4, \\ b_{1,5} &= b_{1,1}^3 + \frac{1}{8}b_{2,1}^2 - \frac{1}{4}b_{1,1}b_{3,1} - b_{1,1}b_{1,3} + \frac{1}{4}b_{3,3} - \frac{1}{2}\lambda_6, \\ b_{1,7} &= 2b_{1,1}^2b_{1,3} + \frac{1}{2}b_{1,3}b_{1,3} - b_{1,1}b_{1,5} + \frac{1}{4}b_{3,5} - \frac{1}{4}(b_{3,1}b_{1,3} - b_{2,1}b_{2,3} + b_{1,1}b_{3,3}) - \frac{1}{2}\lambda_8. \end{aligned}$$

Corollary 4.4 implies $\mathcal{D}_1(\lambda_m) = 0$ and Lemma 4.2 implies

$$b_{2,s} = \mathcal{D}_1(b_{1,s}), \quad b_{3,s} = \mathcal{D}_1(\mathcal{D}_1(b_{1,s})), \quad \text{for } s \in \{1, 3, 5, 7\}.$$

Finally, the relation (19) determines $\mathcal{D}_{2k}(\mathcal{D}_1^{(n)}(b_{1,1}))$ provided $\mathcal{D}_{2k}(b_{1,1})$ is given. To conclude, we give explicitly the values $\mathcal{D}_{2k}(b_{1,1})$ for $k \in \{1, 2, 3, 4, 5, 6, 7\}$. By (15) we have

$$\begin{aligned} \mathcal{D}_2(b_{1,1}) &= -b_{1,1}^2 + 2b_{1,3} + \frac{1}{2}b_{3,1} + \frac{7}{9}\lambda_4, \\ \mathcal{D}_4(b_{1,1}) &= b_{3,3} - 2b_{1,1}b_{1,3} + 2b_{1,5} + \frac{2}{3}\lambda_6. \end{aligned}$$

The remaining values are

$$\begin{aligned} \mathcal{D}_6(b_{1,1}) &= -b_{1,3}^2 + 2b_{1,7} - 2b_{1,1}b_{1,5} + \frac{1}{2}b_{3,1}b_{1,3} - \frac{1}{2}b_{3,3}b_{1,1} + \frac{3}{2}b_{3,5} + \frac{5}{9}\lambda_8, \\ \mathcal{D}_8(b_{1,1}) &= b_{1,5}b_{3,1} - b_{1,1}b_{3,5} - 2b_{1,3}b_{1,5} - 2b_{1,1}b_{1,7} + 2b_{3,7} + \frac{4}{9}\lambda_{10}, \\ \mathcal{D}_{10}(b_{1,1}) &= -b_{1,5}^2 - 2b_{1,3}b_{1,7} + \frac{1}{2}b_{3,3}b_{1,5} - \frac{1}{2}b_{3,5}b_{1,3} + \frac{3}{2}b_{3,1}b_{1,7} - \frac{3}{2}b_{3,7}b_{1,1} + \frac{1}{3}\lambda_{12}, \\ \mathcal{D}_{12}(b_{1,1}) &= b_{1,7}b_{3,3} - b_{1,3}b_{3,7} - 2b_{1,5}b_{1,7} + \frac{2}{9}\lambda_{14}, \\ \mathcal{D}_{14}(b_{1,1}) &= -b_{1,7}^2 + \frac{1}{2}b_{3,5}b_{1,7} - \frac{1}{2}b_{3,7}b_{1,5} + \frac{1}{9}\lambda_{16}. \end{aligned}$$

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