

# THE VERY EFFECTIVE COVERS OF KO AND KGL OVER DEDEKIND SCHEMES

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ABSTRACT. We answer a question of Hoyois–Jelisiejew–Nardin–Yakerson regarding framed models of motivic connective  $K$ -theory spectra over Dedekind schemes. That is, we show that the framed suspension spectrum of the presheaf of groupoids of vector bundles (respectively non-degenerate symmetric bilinear bundles) is the effective cover of KGL (respectively very effective cover of KO). One consequence is that, over any scheme, we obtain a spectral sequence from Spitzweck’s motivic cohomology to homotopy algebraic  $K$ -theory; it is strongly convergent under mild assumptions.

## 1. STATEMENT OF RESULTS

Let  $S$  be a scheme. The category  $\mathcal{P}_\Sigma(\mathrm{Cor}^{\mathrm{fr}}(S))$  of presheaves with framed transfers [5, §2.3] is a motivic analog of the classical category of  $\mathcal{E}_\infty$ -monoids. We have the *framed suspension spectrum* functor

$$\Sigma_{\mathrm{fr}}^\infty : \mathcal{P}_\Sigma(\mathrm{Cor}^{\mathrm{fr}}(S)) \rightarrow \mathcal{SH}(S)$$

which was constructed in [6, Theorem 18]. By analogy with the classical situation, one might expect that many interesting motivic spectra can be obtained as framed suspension spectra. This is indeed the case; see [8, §1.1] for a summary.

This note concerns the following examples of the above idea. One has framed presheaves [8, §6]

$$\mathrm{Vect}, \mathrm{Bil} \in \mathcal{P}_\Sigma(\mathrm{Cor}^{\mathrm{fr}}(S))$$

where  $\mathrm{Vect}(X)$  is the groupoid of vector bundles on  $X$  and  $\mathrm{Bil}(X)$  is the groupoid of vector bundles with a non-degenerate symmetric bilinear form. There exist Bott elements

$$\beta \in \pi_{2,1}\Sigma_{\mathrm{fr}}^\infty \mathrm{Vect} \quad \text{and} \quad \tilde{\beta} \in \pi_{8,4}\Sigma_{\mathrm{fr}}^\infty \mathrm{Bil}$$

and canonical equivalences [7, Proposition 5.1] [8, Proposition 6.7]

$$(\Sigma_{\mathrm{fr}}^\infty \mathrm{Vect})[\beta^{-1}] \simeq \mathrm{KGL} \quad \text{and} \quad (\Sigma_{\mathrm{fr}}^\infty \mathrm{Bil})[\tilde{\beta}^{-1}] \simeq \mathrm{KO}.$$

Here KGL is the motivic spectrum representing homotopy algebraic  $K$ -theory and KO is the motivic spectrum representing homotopy hermitian  $K$ -theory.<sup>1</sup> Again by comparison with the classical situation, this suggests that  $\Sigma_{\mathrm{fr}}^\infty \mathrm{Vect}$  and  $\Sigma_{\mathrm{fr}}^\infty \mathrm{Bil}$  should be motivic analogs of *connective*  $K$ -theory spectra. Another way of producing “connective” versions is by passing to (very) effective covers [12, 11]. It was proved in [8, 7] that these two notions of connective motivic  $K$ -theory spectra coincide, provided that  $S$  is regular over a field.

Our main result is to extend this comparison to more general base schemes. We denote by  $H\mathbb{Z}$  Spitzweck’s motivic cohomology spectrum [11] and by  $HW$  the periodic Witt cohomology spectrum [3, Definition 4.6].

**Theorem 1.1.** *Let  $S$  be a scheme.*

(1) *Suppose that  $f_1(H\mathbb{Z}) = 0 \in \mathcal{SH}(S)$ . The canonical map*

$$\Sigma_{\mathrm{fr}}^\infty \mathrm{Vect} \rightarrow f_0 \mathrm{KGL} \in \mathcal{SH}(S)$$

*is an equivalence.*

(2) *Suppose in addition that  $1/2 \in S$  and  $HW_{\geq 2} = 0 \in \mathcal{SH}(S)$ . The canonical map*

$$\Sigma_{\mathrm{fr}}^\infty \mathrm{Bil} \rightarrow \tilde{f}_0 \mathrm{KO} \in \mathcal{SH}(S)$$

*is an equivalence.*

*These assumptions are satisfied if  $S$  is essentially smooth over a Dedekind scheme (containing  $1/2$  in case (2)).*

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<sup>1</sup>As a notational convention for this introduction, whenever we mention KO we shall assume that  $1/2 \in S$ .

*Remark 1.2.* That the assumptions are satisfied for Dedekind schemes is proved in [4, Proposition B.4] for (1) and in [3, Lemma 3.8] for (2). They in fact hold for all schemes; this will be recorded elsewhere.

*Example 1.3.* Bott periodicity implies formally that  $f_n \text{KGL} \simeq \Sigma^{2n,n} f_0 \text{KGL}$  and  $s_n(\text{KGL}) \simeq \Sigma^{2n,n} f_0(\text{KGL})/\beta$ . Theorem 1.1(1) implies that  $f_0(\text{KGL})/\beta \simeq H\mathbb{Z}$  (see Lemma 2.1). Hence in this situation the slice filtration for KGL yields a convergent spectral sequence, with  $E_2$ -page given by (Spitzweck's) motivic cohomology.

**Notation.** We use notation for standard motivic categories and spectra, as in [3] and [8].

## 2. PROOFS

As a warm-up, we treat the case of KGL. Recall that the functor  $\Sigma_{\text{fr}}^\infty$  inverts group-completion. The Bott element lifts to  $\beta : (\mathbb{P}^1, \infty) \rightarrow \text{Vect}^{\text{gp}}$  [7, §5]. We also have the rank map  $\text{Vect}^{\text{gp}} \rightarrow \mathbb{Z} \in \mathcal{P}_\Sigma(\text{Cor}^{\text{fr}}(S))$ . The composite

$$(\mathbb{P}^1, \infty) \wedge \text{Vect}^{\text{gp}} \xrightarrow{\beta} \text{Vect}^{\text{gp}} \wedge \text{Vect}^{\text{gp}} \xrightarrow{m} \text{Vect}^{\text{gp}} \rightarrow \mathbb{Z}$$

is null-homotopic after motivic localization, since  $\mathbb{Z}$  is motivically local and truncated and  $(\mathbb{P}^1, \infty) \xrightarrow{\text{mot}} S^1 \wedge \mathbb{G}_m$ .

**Lemma 2.1.** *The induced map*

$$(\Sigma_{\text{fr}}^\infty \text{Vect})/\beta \rightarrow \Sigma_{\text{fr}}^\infty \mathbb{Z} \simeq H\mathbb{Z}$$

*is an equivalence.*

*Proof.* The equivalence  $\Sigma_{\text{fr}}^\infty \mathbb{Z} \simeq H\mathbb{Z}$  is [6, Theorem 21]. Since all terms are stable under base change [8, proof of Lemma 7.5] [6, Lemma 16], we may assume that  $S = \text{Spec}(\mathbb{Z})$ . Using [4, Proposition B.3] we further reduce to the case where  $S$  is the spectrum of a perfect field. In this case  $\Sigma_{\text{fr}}^\infty \text{Vect} \simeq f_0 \text{KGL}$  and so  $(\Sigma_{\text{fr}}^\infty \text{Vect})/\beta \simeq s_0 \text{KGL} \simeq H\mathbb{Z}$  (see e.g. [1, Proposition 2.7]).  $\square$

*Proof of Theorem 1.1(1).* Note first that if  $U \subset S$  is an open subscheme, and any of the assumptions of Theorem 1.1 holds for  $S$ , it also holds for  $U$ . On the other hand, if one of the conclusions holds for all  $U$  in an open cover, it holds for  $S$ . It follows that we may assume that  $S$  is qcqs, e.g. affine.

Since  $f_1(H\mathbb{Z}) = 0$  we find (using Lemma 2.1) that

$$\beta : \Sigma_{\text{fr}}^\infty \text{Vect} \rightarrow \Sigma^{-2,-1} \Sigma_{\text{fr}}^\infty \text{Vect}$$

induces an equivalence on  $f_i$  for  $i \geq 0$ . It follows that in the directed system

$$\Sigma_{\text{fr}}^\infty \text{Vect} \xrightarrow{\beta} \Sigma^{-2,-1} \Sigma_{\text{fr}}^\infty \text{Vect} \xrightarrow{\beta} \Sigma^{-4,-2} \Sigma_{\text{fr}}^\infty \text{Vect} \xrightarrow{\beta} \dots$$

all maps induce an equivalence on  $f_0$ . Since the colimit is KGL,  $f_0$  commutes with colimits (here we use that  $X$  is qcqs, via [4, Proposition A.3(2)]) and  $\Sigma_{\text{fr}}^\infty \text{Vect}$  is effective (like any framed suspension spectrum), the result follows.  $\square$

The proof for KO is an elaboration on these ideas. From now on we assume that  $1/2 \in S$ . Recall from [3, Definition 2.6, Lemma 2.7] the motivic spectrum

$$\underline{k}^M \simeq (H\mathbb{Z}/2)/\tau \in \mathcal{SH}(S).$$

For the time being, assume  $S$  is Dedekind. Taking framed loops we obtain

$$\underline{k}_1^M := \Omega_{\text{fr}}^\infty \Sigma^{1,1} \underline{k}^M \in \mathcal{P}_\Sigma(\text{Cor}^{\text{fr}}(S)).$$

**Lemma 2.2.** *Let  $S$  be a Dedekind scheme,  $1/2 \in S$ .*

- (1) *We have  $\underline{k}_1^M \simeq a_{\text{Nis}} \tau_{\leq 0} \mathbb{G}_m/2$ , where  $\mathbb{G}_m \in \mathcal{P}_\Sigma(\text{Cor}^{\text{fr}}(S))$  denotes the sheaf  $\mathcal{O}^\times$  with its usual structure of transfers [9, Example 2.4].*
- (2) *If  $f : S' \rightarrow S$  is a morphism of Dedekind schemes then  $f^* \underline{k}_1^M \xrightarrow{\text{mot}} \underline{k}_1^M \in \mathcal{P}_\Sigma(\text{Cor}^{\text{fr}}(S'))$ .*
- (3) *The canonical map  $\Sigma_{\text{fr}}^\infty \underline{k}_1^M \rightarrow \Sigma^{1,1} \underline{k}^M \in \mathcal{SH}(S)$  is an equivalence.*

For this and some of the following arguments, it will be helpful to recall that we have an embedding of  $\mathcal{Spc}^{\text{fr}}(S)^{\text{gp}}$  into the stable category of spectral presheaves on  $\text{Cor}^{\text{fr}}(S)$ . In particular, many fiber sequences in  $\mathcal{Spc}^{\text{fr}}(S)$  are cofiber sequences.

*Proof.* (1) Clear by construction since  $H_{\acute{e}t}^1(X, \mu_2) \simeq \mathcal{O}^\times(X)/2$  for  $X$  affine.

(2) By (1) we have a cofiber sequence  $\Sigma\mu_2 \rightarrow a_{\text{Nis}}\mathbb{G}_m/2 \rightarrow \underline{k}_1^M \in \mathcal{P}_\Sigma(\text{Cor}^{\text{fr}}(S))$ . Since pullback of framed presheaves preserves cofiber sequences and commutes with forgetting transfers up to motivic equivalence [6, Lemma 16] we reduce to the same assertion about  $\mathbb{G}_m, \mu_2$ , viewed as presheaves without transfers. Since they are representable, the assertion is clear.

(3) Using [4, Proposition B.3], (2) and [3, Theorem 4.4] we may assume that  $S$  is the spectrum of a perfect field. In this case  $\Sigma_{\text{fr}}^\infty \Omega_{\text{fr}}^\infty \simeq \tilde{f}_0$  [5, Theorem 3.5.14(i)], so we need only prove that  $\Sigma^{1,1}\underline{k}_1^M$  is very effective. But this is clear since we have the cofiber sequence  $\Sigma^{1,0}H\mathbb{Z}/2 \xrightarrow{\tau} \Sigma^{1,1}H\mathbb{Z}/2 \rightarrow \Sigma^{1,1}\underline{k}_1^M$  and  $H\mathbb{Z}/2$  is very effective.  $\square$

**Construction 2.3.** The assignment  $V \mapsto (V \oplus V^*, \varphi_V)$  sending a vector bundle to its associated (hyperbolic) symmetric bilinear bundle upgrades to a morphism

$$\text{Vect} \rightarrow \text{Bil} \in \mathcal{P}_\Sigma(\text{Cor}^{\text{fr}}(S))^{BC_2},$$

where  $\text{Vect}$  carries the  $C_2$ -action coming from passing to dual bundles, and  $\text{Bil}$  carries the trivial  $C_2$ -action.

*Proof.* Since the presheaves are 1-truncated, all the required coherence data can be written down by hand.  $\square$

**Lemma 2.4.** *Let  $S$  be a Dedekind scheme containing  $1/2$ .*

(1) *The map*

$$(\text{Vect}^{\text{sp}})_{hC_2} \rightarrow \text{Bil}^{\text{sp}}$$

*induces an isomorphism on  $a_{\text{Nis}}\pi_i$  for  $i = 1, 2$ .*

(2) *The homotopy orbits spectral sequence yields*

$$a_{\text{Nis}}\pi_0(\text{Vect}^{\text{sp}})_{hC_2} \simeq \mathbb{Z},$$

*an exact sequence*

$$0 \rightarrow \underline{k}_1^M \rightarrow a_{\text{Nis}}\pi_1(\text{Vect}^{\text{sp}})_{hC_2} \rightarrow \mathbb{Z}/2 \rightarrow 0$$

*and a map*

$$a_{\text{Nis}}\pi_2(\text{Vect}^{\text{sp}})_{hC_2} \rightarrow \mathbb{Z}/2,$$

*all as presheaves with framed transfers.*

*Proof.* (1) This follows from the cofiber sequence  $K_{hC_2} \rightarrow \text{GW} \rightarrow L$  [10, Theorem 7.6] using that  $a_{\text{Nis}}\pi_i L = 0$  unless  $i \equiv 0 \pmod{4}$ .

(2) The homotopy orbit spectral sequence just arises from the Postnikov filtration of  $\text{Vect}^{\text{sp}}$  and the formation of homotopy orbits and hence is compatible with transfers. Its  $E_2$  page takes the form

$$H_i(C_2, a_{\text{Nis}}\pi_j \text{Vect}^{\text{sp}}) \Rightarrow a_{\text{Nis}}\pi_{i+j}(\text{Vect}^{\text{sp}})_{hC_2}.$$

The form of the differentials of the spectral sequence implies that the terms  $H_i(C_2, a_{\text{Nis}}\pi_j \text{Vect}^{\text{sp}})$  are permanent cycles for  $i \leq 1$ , and survive to  $E_\infty$  for  $(i, j) = (0, 0)$  and  $(i, j) = (1, 1)$ . One has  $a_{\text{Nis}}\pi_0 \text{Vect}^{\text{sp}} = \mathbb{Z}$  with the trivial action and  $a_{\text{Nis}}\pi_1 \text{Vect}^{\text{sp}} = \mathbb{G}_m$  [13, Lemma III.1.4] with the inversion action. This already yields the first assertion. A straightforward computation shows that

$$H_*(C_2, \mathbb{Z}) = \mathbb{Z}, \mathbb{Z}/2, 0, \mathbb{Z}/2, \dots$$

and

$$H_*(C_2, \mathbb{G}_m) = \underline{k}_1^M, \mu_2, \underline{k}_1^M, \dots$$

Since  $H_2(C_2, \mathbb{Z}) = 0$ , no differential can hit the  $(i, j) = (0, 1)$  spot either, yielding the second assertion. Moreover this implies that  $H_1(C_2, \mathbb{G}_m) = \mu_2$  is the bottom of the filtration of  $\pi_2$ . It follows that there is a map  $a_{\text{Nis}}\pi_2(\text{Vect}^{\text{sp}})_{hC_2} \rightarrow A$ , where  $A$  is a quotient of  $\mu_2$ . To prove that  $A = \mu_2$  it suffices to check this on sections over a field, in which case we can use the hermitian motivic spectral sequence of [2].  $\square$

We have  $a_{\text{Nis}}\pi_0 \text{Bil}^{\text{sp}} \simeq \underline{GW}$ . Thus we can form the following filtration of  $\text{Bil}^{\text{sp}}$  refining the Postnikov filtration

$$\text{Bil}^{\text{sp}} \leftarrow F_1 \text{Bil}^{\text{sp}} \leftarrow F_2 \text{Bil}^{\text{sp}} \leftarrow F_3 \text{Bil}^{\text{sp}} \leftarrow F_4 \text{Bil}^{\text{sp}} \in \mathcal{P}_\Sigma(\text{Cor}^{\text{fr}}(S))$$

with subquotients given Nisnevich-locally by

$$(2.1) \quad \underline{GW}, \Sigma\mathbb{Z}/2, \Sigma\underline{k}_1^M, \Sigma^2\mathbb{Z}/2.$$

Recall also the framed presheaf  $\text{Alt} \in \mathcal{P}_\Sigma(\text{Cor}^{\text{fr}}(S))$  sending a scheme to the groupoid of vector bundles with a non-degenerate alternating form. Tensoring with the canonical alternating (virtual) form  $H(1) - h$

on  $H\mathbb{P}^1$  (where  $H(1)$  is the tautological rank 2 alternating form on  $H\mathbb{P}^1$ , and  $h$  is the standard alternating form on a trivial vector bundle of rank 2) yields maps

$$\sigma_1 : H\mathbb{P}^1 \wedge \text{Alt}^{\text{gp}} \rightarrow \text{Bil}^{\text{gp}} \quad \text{and} \quad \sigma_2 : H\mathbb{P}^1 \wedge \text{Bil}^{\text{gp}} \rightarrow \text{Alt}^{\text{gp}};$$

by construction we have  $\tilde{\beta} = \sigma_1\sigma_2$  (recall that  $H\mathbb{P}^1 \xrightarrow{\text{mot}} S^{4,2}$ ).

**Lemma 2.5.** *Let  $S$  be a Dedekind scheme,  $1/2 \in S$ .*

(1) *The composite*

$$H\mathbb{P}^1 \wedge \text{Alt}^{\text{gp}} \xrightarrow{\sigma_1} \text{Bil}^{\text{gp}} \rightarrow \text{Bil}^{\text{gp}}/F_4\text{Bil}^{\text{gp}}$$

*is motivically null. The induced map*

$$\Sigma_{\text{fr}}^{\infty} \text{cof}(\sigma_1) \rightarrow \Sigma_{\text{fr}}^{\infty} \text{Bil}^{\text{gp}}/F_4\text{Bil}^{\text{gp}}$$

*is an equivalence.*

(2) *The composite*

$$H\mathbb{P}^1 \wedge \text{Bil}^{\text{gp}} \xrightarrow{\sigma_2} \text{Alt}^{\text{gp}} \xrightarrow{rk/2} \mathbb{Z}$$

*is motivically null. The induced map*

$$\Sigma_{\text{fr}}^{\infty} \text{cof}(\sigma_2) \rightarrow \Sigma_{\text{fr}}^{\infty} \mathbb{Z}$$

*is an equivalence.*

*Proof.* (1) Write  $C$  for the cofiber computed in the category of spectral presheaves on  $\text{Cor}^{\text{fr}}(S)$ . Then  $C$  admits a finite filtration, with subquotients corresponding to those in (2.1). Since each of those is the infinite loop space of a motivic spectrum, it follows that  $C$  is in fact motivically local. Consequently  $C$  corresponds to  $\text{Bil}^{\text{gp}}/F_4\text{Bil}^{\text{gp}}$  under the embedding into spectral presheaves. These contortions tell us that there are fiber sequences

$$F_{i+1}\text{Bil}^{\text{gp}}/F_4\text{Bil}^{\text{gp}} \rightarrow F_i\text{Bil}^{\text{gp}}/F_4\text{Bil}^{\text{gp}} \rightarrow F_i\text{Bil}^{\text{gp}}/F_{i+1}\text{Bil}^{\text{gp}}$$

for  $i < 4$ . Hence to prove that the composite is null, it suffices to prove that there are no maps from  $\Sigma^{4,2}\text{Alt}^{\text{gp}}$  into the motivic localizations of the subquotients of the filtration given in (2.1). These motivic localizations are  $\underline{GW}$ ,  $L_{\text{Nis}}K(\mathbb{Z}/2, 1)$ ,  $L_{\text{Nis}}K(k_1^M, 1)$  and  $L_{\text{Nis}}K(\mathbb{Z}/2, 2)$  (since they are motivically equivalent to the subquotients, and motivically local because they are infinite loop spaces of the motivic spectra  $H\mathbb{Z}$ ,  $\Sigma \underline{k}^M$ ,  $\Sigma^{2,1}\underline{k}^M$ ,  $\Sigma^2 \underline{k}^M$ ). It suffices to prove that  $\Omega^{4,2}$  of these subquotients vanishes, which is clear. Next we claim that  $\Sigma_{\text{fr}}^{\infty} \text{Bil}^{\text{gp}}/F_4\text{Bil}^{\text{gp}}$  is stable under base change (among Dedekind schemes containing  $1/2$ ). Indeed the defining fiber sequences of  $F_4\text{Bil}^{\text{gp}}$  are also cofiber sequences, and so  $\Sigma_{\text{fr}}^{\infty} \text{Bil}^{\text{gp}}/F_4\text{Bil}^{\text{gp}}$  is obtained by iterated extension from spectra stable under base change (see Lemma 2.2(2) for  $\underline{k}_1^M$ , [8, proof of Lemma 7.5] for  $\text{Bil}$  and  $\text{Alt}$ , and [6, Lemma 16] for  $\mathbb{Z}/2$ ). To prove that the induced map is an equivalence we thus reduce as before to  $S = \text{Spec}(k)$ ,  $k$  a perfect field of characteristic  $\neq 2$ . In this case the result is a straightforward consequence of the hermitian motivic filtration of [2].

(2) The proof is essentially the same as for (1), but easier.  $\square$

We now arrive at the main result.

**Theorem 2.6.** *Let  $S$  be a scheme containing  $1/2$  such that*

$$f_1(H\mathbb{Z}) = 0 = HW_{\geq 2} \in \mathcal{SH}(S).$$

*The canonical maps*

$$\Sigma_{\text{fr}}^{\infty} \text{Bil} \rightarrow \tilde{f}_0 \text{KO} \quad \text{and} \quad \Sigma_{\text{fr}}^{\infty} \text{Alt} \rightarrow \tilde{f}_0 \Sigma^{4,2} \text{KO}$$

*are equivalences.*

*Proof.* As before we may assume that  $S$  is qcqs.

We know that  $\text{KO}$  is the colimit of

$$\Sigma_{\text{fr}}^{\infty} \text{Bil} \xrightarrow{\sigma_2} \Sigma^{-4,-2} \Sigma_{\text{fr}}^{\infty} \text{Alt} \xrightarrow{\sigma_1} \Sigma^{-8,-4} \text{Bil} \xrightarrow{\sigma_2} \dots$$

It is hence enough to prove that  $\sigma_1 : \Sigma^{-8n,-4n} \Sigma_{\text{fr}}^{\infty} \text{Bil} \rightarrow \Sigma^{-8n-4,-4n-2} \Sigma_{\text{fr}}^{\infty} \text{Alt}$  induces an equivalence on  $\tilde{f}_0$  for every  $n \geq 0$ , and similarly for  $\sigma_2$ . (Here we use that  $S$  is qcqs, so that  $\tilde{f}_0$  preserves filtered colimits.) Given a cofiber sequence  $A \rightarrow B \rightarrow C$ , in order to prove that  $\tilde{f}_0 A \simeq \tilde{f}_0 B$ , it suffices to show that  $\text{Map}(X, C) = *$  for every  $X \in \mathcal{SH}(S)^{\text{veff}}$ , i.e. that  $C \in \mathcal{SH}(S)^{\text{veff}\perp}$ .

Over  $\mathbb{Z}[1/2]$ , the cofiber of  $\sigma_1$  has a finite filtration, with subquotients

$$\Sigma^{-4,-2} \Sigma_{\text{fr}}^{\infty} \underline{GW}, \Sigma^{-3,-2} \Sigma_{\text{fr}}^{\infty} \mathbb{Z}/2, \Sigma^{-3,-2} \Sigma_{\text{fr}}^{\infty} \underline{k}_1^M, \Sigma^{-2,-2} \Sigma_{\text{fr}}^{\infty} \mathbb{Z}/2,$$

and the cofiber of  $\sigma_2$  is  $\Sigma^{-4,-2}\Sigma_{\text{fr}}^\infty\mathbb{Z}$ . Using [6, Corollary 22], [8, Theorem 7.3] and Lemma 2.2(3), we can identify the list of cofibers as

$$\Sigma^{-4,-2}H\tilde{\mathbb{Z}}, \Sigma^{-3,-2}H\mathbb{Z}/2, \Sigma^{-2,-1}\underline{k}^M, \Sigma^{-2,-2}H\mathbb{Z}/2, \Sigma^{-4,-2}H\mathbb{Z}.$$

These spectra are stable under arbitrary base change (essentially by definition), and hence for arbitrary  $S$  the cofibers of  $\sigma_1, \sigma_2$  are obtained as finite extensions, with cofibers in the above list. To conclude the proof, it will thus suffice to show that all spectra in the above list are in  $\mathcal{SH}(S)^{\text{veff}\perp}$ .

Note that if  $E \in \mathcal{SH}(S)$  then  $E \in \mathcal{SH}(S)^{\text{veff}\perp}$  if and only if  $\Omega^\infty E \simeq *$ . In particular this holds if  $f_0 E = 0$ . This holds for  $\Sigma^{m,n}H\mathbb{Z}$  as soon as  $n < 0$ , by assumption. Hence it also holds for  $\Sigma^{m,n}H\mathbb{Z}/2$  in the same case ( $f_0$  being a stable functor) and for

$$\Sigma^{m,n}\underline{k}^M \simeq \text{cof}(\Sigma^{m,n-1}H\mathbb{Z}/2 \xrightarrow{\tau} \Sigma^{m,n}H\mathbb{Z}/2).$$

The only spectrum left in our list is  $\Sigma^{-4,-2}H\tilde{\mathbb{Z}}$ . Using [3, Definition 4.1] we see now that  $\Omega^\infty\Sigma^{-4,-2}H\tilde{\mathbb{Z}} \simeq \Omega^\infty\Sigma^{-4,-2}\underline{K}^W$ , so we may treat the latter spectrum. We have  $\underline{K}^W/\eta \simeq \underline{k}^M$  [3, Lemma 3.9], whence  $\eta : \Sigma^{-4-n,-2-n}\underline{K}^W \rightarrow \Sigma^{-5-n,-3-n}\underline{K}^W$  induces an equivalence on  $\Omega^\infty$ . Since  $\Omega^\infty$  commutes with filtered colimits, we see that  $\Sigma^{-4,-2}\underline{K}^W \in \mathcal{SH}(S)^{\text{veff}\perp}$  if and only if  $\Sigma^{-4,-2}\underline{K}^W[\eta^{-1}] \in \mathcal{SH}(S)^{\text{veff}\perp}$ . This latter spectrum is the same as  $\Sigma^{-2}HW$  [3, Lemma 3.9], and

$$\tilde{f}_0(\Sigma^{-2}HW) \simeq \tilde{f}_0((\Sigma^{-2}HW)_{\geq 0}) \simeq \tilde{f}_0(\Sigma^{-2}(HW)_{\geq 2}) = 0$$

by assumption. □

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