

# PBW DEGENERATIONS, QUIVER GRASSMANNIANS, AND TORIC VARIETIES

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*Dedicated to the memory of Ernest Borisovich Vinberg*

ABSTRACT. We present a review on the recently discovered link between the Lie theory, the theory of quiver Grassmannians, and various degenerations of flag varieties. Our starting point is the induced Poincaré–Birkhoff–Witt filtration on the highest weight representations and the corresponding PBW degenerate flag varieties.

## 1. INTRODUCTION

The celebrated Poincaré–Birkhoff–Witt theorem claims that there exists a filtration on the universal enveloping of a Lie algebra such that the associated graded algebra is isomorphic to the symmetric algebra. The PBW filtration on the universal enveloping algebra of a nilpotent subalgebra of a simple Lie algebra induces a filtration on the representation space of a highest weight module. The natural problem is to study this filtration and the corresponding graded space. Quite unexpectedly, the problem turned out to be related to numerous representation-theoretic, algebro-geometric, and combinatorial questions. Our goal is to give an overview of the whole story and to describe various links between different parts of the picture. The main objects of study are monomial bases, convex polytopes, flag and Schubert varieties, their degenerations, quiver Grassmannians, and toric varieties.

The paper is organized as follows. In Section 2 we collect representation-theoretic results of algebraic nature. Section 3 is devoted to the geometric representation theory. In Section 4 we discuss combinatorics emerging from the cellular decomposition of the PBW degenerate flag varieties. In Section 5 we describe the link between the Lie theory and the theory of quiver Grassmannians. Finally, Section 6 treats toric degenerations.

Throughout the paper we work over the field of complex numbers.

## 2. REPRESENTATION THEORY: ALGEBRA

Let  $\mathfrak{g}$  be a simple Lie algebra with the set  $R^+$  of positive roots. Let  $\alpha_i, \omega_i, i = 1, \dots, n-1$  be the simple roots and the fundamental weights. Let  $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$  be the Cartan decomposition. For  $\alpha \in R^+$ , we

denote by  $e_\alpha \in \mathfrak{n}^+$  and  $f_\alpha \in \mathfrak{n}^-$  the corresponding Chevalley generators. We denote by  $P^+$  the set of dominant integral weights.

Consider the PBW filtration on the universal enveloping algebra  $U(\mathfrak{n}^-)$ :

$$U(\mathfrak{n}^-)_s = \text{span}\{x_1 \cdots x_l : x_i \in \mathfrak{n}^-, l \leq s\},$$

for example,  $U(\mathfrak{n}^-)_0 = \mathbb{C} \cdot 1$ .

For a dominant integral weight  $\lambda = m_1\omega_1 + \cdots + m_{n-1}\omega_{n-1}$ , let  $V_\lambda$  be the corresponding irreducible highest weight  $\mathfrak{g}$ -module with a highest weight vector  $v_\lambda$ . Since  $V_\lambda = U(\mathfrak{n}^-)v_\lambda$ , we have an increasing filtration  $(V_\lambda)_s$  on  $V_\lambda$ ,

$$(V_\lambda)_s = U(\mathfrak{n}^-)_s v_\lambda.$$

We call this filtration the PBW filtration and study the associated graded space  $V_\lambda^a = \text{gr}V_\lambda$ .

Let us consider an example for fundamental weights in type  $A$ . Let  $V_{\omega_1}$  be the vector representation of  $\mathfrak{sl}_n$  with a basis  $w_1, \dots, w_n$  and consider  $V_{\omega_k} \simeq \Lambda^k V_{\omega_1}$  for  $k = 1, \dots, n-1$ . Then  $(V_{\omega_k})_s$  is spanned by the wedge products  $w_{i_1} \wedge \cdots \wedge w_{i_k}$  such that the number of indices  $a$  with  $i_a > k$  is at most  $s$ .

The following holds true [57]:

- (1) The action of  $U(\mathfrak{n}^-)$  on  $V_\lambda$  induces the structure of an  $S(\mathfrak{n}^-)$ -module on  $V_\lambda^a$  and  $V_\lambda^a = S(\mathfrak{n}^-)v_\lambda$ .
- (2) The action of  $U(\mathfrak{n}^+)$  on  $V_\lambda$  induces the structure of a  $U(\mathfrak{n}^+)$ -module on  $V_\lambda^a$ .

Our aims are to describe  $V_\lambda^a$  as an  $S(\mathfrak{n}^-)$ -module and to find a basis of  $V_\lambda^a$ . We present the answer in type  $A$ . For similar results in other types, see [16, 58, 59, 75, 76, 98].

The positive roots in type  $A_{n-1}$  are of the form  $\alpha_{i,j} = \alpha_i + \cdots + \alpha_j$  with  $1 \leq i \leq j \leq n-1$ . Recall that a Dyck path is a sequence  $\mathbf{p} = (\beta(0), \beta(1), \dots, \beta(k))$  of positive roots of  $\mathfrak{sl}_n$  satisfying the following conditions: if  $k = 0$ , then  $\mathbf{p}$  is of the form  $\mathbf{p} = (\alpha_i)$  for some simple root  $\alpha_i$ , and if  $k \geq 1$ , then the first and last elements are simple roots, and if  $\beta(s) = \alpha_{p,q}$ , then  $\beta(s+1) = \alpha_{p,q+1}$  or  $\beta(s+1) = \alpha_{p+1,q}$ .

Here is an example of a path for  $\mathfrak{sl}_6$ :

$$(\alpha_2, \alpha_2 + \alpha_3, \alpha_2 + \alpha_3 + \alpha_4, \alpha_3 + \alpha_4, \alpha_4, \alpha_4 + \alpha_5, \alpha_5).$$

For a multiexponent  $\mathbf{s} = \{s_\beta\}_{\beta > 0}$ ,  $s_\beta \in \mathbb{Z}_{\geq 0}$ , let  $f^{\mathbf{s}} = \prod_{\beta \in R^+} f_\beta^{s_\beta} \in S(\mathfrak{n}^-)$ . For an integral dominant  $\mathfrak{sl}_n$ -weight  $\lambda = \sum_{i=1}^{n-1} m_i \omega_i$ , let  $S(\lambda)$  be the set of all multiexponents  $\mathbf{s} = (s_\beta)_{\beta \in R^+} \in \mathbb{Z}_{\geq 0}^{R^+}$  such that for all Dyck paths  $\mathbf{p} = (\beta(0), \dots, \beta(k))$ ,

$$(1) \quad s_{\beta(0)} + s_{\beta(1)} + \cdots + s_{\beta(k)} \leq m_i + m_{i+1} + \cdots + m_j,$$

where  $\beta(0) = \alpha_i$  and  $\beta(k) = \alpha_j$ .

The polytopes in  $\mathbb{R}_{\geq 0}^{R^+}$  defined by inequalities (1) are referred to as the FFLV polytopes. For their combinatorial properties and connection

to the Gelfand–Tsetlin polytopes [74], see [5, 66, 46, 43]. The following theorem holds true [57].

**Theorem 2.1.** *The vectors  $f^{\mathbf{s}}v_\lambda$ ,  $\mathbf{s} \in S(\lambda)$ , form a basis of  $V_\lambda^a$ . In addition,  $S(\lambda) + S(\mu) = S(\lambda + \mu)$ .*

We note that Theorem 2.1 implies that the elements  $f^{\mathbf{s}}v_\lambda$ ,  $\mathbf{s} \in S(\lambda)$  form a basis of the classical representation  $V_\lambda$  provided an order of factors is fixed in each monomial  $f^{\mathbf{s}}$  (see [120]).

Let us describe the Lie algebra  $\mathfrak{g}^a$  acting on  $V_\lambda^a$ . As a vector space,  $\mathfrak{g}^a$  is isomorphic to  $\mathfrak{g}$ . The Borel  $\mathfrak{b} \subset \mathfrak{g}^a$  is a subalgebra, the nilpotent subalgebra  $\mathfrak{n}^- \subset \mathfrak{g}^a$  is an abelian ideal, and  $\mathfrak{b}$  acts on the space  $\mathfrak{n}^-$  as on the quotient  $\mathfrak{g}/\mathfrak{b}$ . Then for any  $\lambda \in P^+$  the structure of the  $\mathfrak{g}$ -module on  $V_\lambda$  induces the structures of  $\mathfrak{g}^a$  module on  $V_\lambda^a$ .

Note that  $V_\lambda^a = S(\mathfrak{n}^-)v_\lambda$  is a cyclic  $S(\mathfrak{n}^-)$ -module, so we can write  $V_\lambda^a \simeq S(\mathfrak{n}^-)/I(\lambda)$ , for some ideal  $I(\lambda) \subset S(\mathfrak{n}^-)$ .

The following theorem holds in types *A* and *C* [57]:

**Theorem 2.2.**  $I(\lambda) = S(\mathfrak{n}^-)(U(\mathfrak{n}^+) \circ \text{span}\{f_\alpha^{(\lambda, \alpha^\vee)+1}, \alpha \in R^+\})$ .

This theorem should be understood as a commutative analogue of the well-known description of  $V_\lambda$  as the quotient

$$V_\lambda \simeq U(\mathfrak{n}^-)/\langle f_\alpha^{(\lambda, \alpha^\vee)+1}, \alpha \in R^+ \rangle$$

(see, for example, [73, 85]).

The proof of the theorems above is based on the following claim available in types *A* and *C* [57, 58, 59].

**Theorem 2.3.** *Let  $\lambda, \mu \in P^+$ . Then*

$$V_{\lambda+\mu}^a \simeq U(\mathfrak{g}^a)(v_\lambda \otimes v_\mu) \hookrightarrow V_\lambda^a \otimes V_\mu^a$$

as  $\mathfrak{g}^a$ -modules.

The algebraic and representation theoretic properties of the PBW filtration and the  $\mathfrak{g}^a$  action in more general settings are considered in [9, 10, 16, 19, 50, 51, 55, 69, 47, 64, 75, 76, 98, 106, 107].

### 3. REPRESENTATION THEORY: GEOMETRY

Let  $G$  be a simple simply-connected Lie group with the Lie algebra  $\mathfrak{g}$ . Let  $B \subset G$  be a Borel subgroup with the Lie algebra  $\mathfrak{b}$ . Each space  $V_\lambda$ ,  $\lambda \in P^+$  is equipped with the natural structure of a  $G$ -module. Therefore  $G$  acts on the projectivization  $\mathbb{P}(V_\lambda)$ . The (generalized) flag variety  $\mathcal{F}_\lambda \hookrightarrow \mathbb{P}(V_\lambda)$  is defined as the  $G$ -orbit of the line  $\mathbb{C}v_\lambda$  (see [72, 93]). Each variety  $\mathcal{F}_\lambda$  is isomorphic to the quotient of  $G$  by the parabolic subgroup leaving the point  $\mathbb{C}v_\lambda \in \mathbb{P}(V_\lambda)$  invariant. In particular, for  $G = \text{SL}_n$  and a fundamental weight  $\lambda = \omega_k$  the flag variety  $\mathcal{F}_\lambda$  is isomorphic to the Grassmannian  $\text{Gr}_k(n)$ . For a regular weight  $\lambda$ , the flag variety  $\mathcal{F}_\lambda$  sits inside  $\prod_{k=1}^{n-1} \text{Gr}_k(n)$  and consists of chains of embedded subspaces.

In what follows, we mostly consider the case  $G = \mathrm{SL}_n$  and regular  $\lambda$ , the general type  $A$  case can be treated similarly (see [52, 53, 55, 56]). We denote the complete type  $A_{n-1}$  flag variety by  $\mathcal{F}_n$  (it is known to be independent of a regular weight  $\lambda$ ). The variety  $\mathcal{F}_n$  admits Plücker embedding into the product of projective spaces  $\prod_{k=1}^{n-1} \mathbb{P}(\Lambda^k(\mathbb{C}^n))$ . The homogeneous coordinate ring (also known as the Plücker algebra) is a quotient of the polynomial algebra in Plücker variables  $X_I$ ,  $I \subset [n]$  by the quadratic Plücker ideal.

Recall the Lie algebra  $\mathfrak{g}^a$  acting on  $V_\lambda^a$ . We now describe the corresponding Lie group  $G^a$ . Let  $M = \dim \mathfrak{n}$  and let  $\mathbb{G}_a$  be the additive group of the field  $\mathbb{C}$ . The Lie group  $G^a$  is a semidirect product  $\mathbb{G}_a^M \rtimes B$  of the normal subgroup  $\mathbb{G}_a^M$  and the Borel subgroup  $B$ . The action by conjugation of  $B$  on  $\mathbb{G}_a^M$  is induced from the  $B$ -action on  $(\mathfrak{n}^-)^a \simeq \mathfrak{g}/\mathfrak{b}$ .

We now define the degenerate flag varieties  $\mathcal{F}_\lambda^a$  [52]. Let  $[v_\lambda] \in \mathbb{P}(V_\lambda^a)$  be the line  $\mathbb{C}v_\lambda$ .

**Definition 3.1.** The variety  $\mathcal{F}_\lambda^a \hookrightarrow \mathbb{P}(V_\lambda^a)$  is the closure of the  $G^a$ -orbit of  $[v_\lambda]$ ,

$$\mathcal{F}_\lambda^a = \overline{G^a[v_\lambda]} = \overline{\mathbb{G}_a^M[v_\lambda]} \hookrightarrow \mathbb{P}(V_\lambda^a).$$

We note that the orbit  $G[v_\lambda] \hookrightarrow \mathbb{P}(V_\lambda)$  coincides with its closure, but the orbit  $G^a[v_\lambda]$  does not; in fact,  $\mathcal{F}_\lambda^a$  is the so-called  $\mathbb{G}_a^M$ -variety, see [81, 6, 7]. Theorem 2.3 implies that in types  $A$  and  $C$  the varieties  $\mathcal{F}_\lambda^a$  depend only on the regularity class of  $\lambda$ , i.e.,  $\mathcal{F}_\lambda^a$  is isomorphic to  $\mathcal{F}_\mu^a$  if and only if the sets of fundamental weights showing up in  $\lambda$  and  $\mu$  coincide (see [97] for the study of a similar question for Schubert varieties).

In types  $A$  and  $C$ , we have rather explicit description of the degenerate flag varieties [53, 61]. In particular, for  $\mathfrak{g} = \mathfrak{sl}_n$  one has  $\mathcal{F}_{\omega_k}^a \simeq \mathrm{Gr}_k(n)$ . To describe the PBW degenerate flag varieties in type  $A$ , we introduce the following notation: let  $W$  be an  $n$ -dimensional vector space with a basis  $w_1, \dots, w_n$ . Let us denote by  $\mathrm{pr}_k : W \rightarrow W$  the projection along  $w_k$ . We denote the regular PBW degenerate flag variety by  $\mathcal{F}_n^a$ . The following theorem holds [52, 53] (we use the shorthand notation  $[n] = \{1, \dots, n\}$ ).

**Theorem 3.2.** *One has*

$$\mathcal{F}_n^a \simeq \{(V_1, \dots, V_{n-1}) : V_k \in \mathrm{Gr}_k(W), k \in [n]; \mathrm{pr}_{k+1} V_k \subset V_{k+1}, k \in [n-1]\}.$$

Using this description, one proves the following theorem [34, 35, 32] (see also [95]).

**Theorem 3.3.** *The variety  $\mathcal{F}_n^a$  is isomorphic to a Schubert variety for the group  $\mathrm{SL}_{2n-1}$ .*

The symplectic PBW degenerations are described in [61] (see also [14]).

For a partition  $\lambda = (\lambda_1 \geq \dots \geq \lambda_{n-1} \geq 0)$ , we denote by  $Y_\lambda$  the corresponding Young diagram. Recall that the classical  $\mathrm{SL}_n$  flag variety admits an embedding to the product of Grassmannians. The corresponding homogeneous coordinate ring (the Plücker algebra) is generated by the Plücker variables  $X_I$ ,  $I \subset [n]$  and is known to be isomorphic to the direct sum  $\bigoplus_{\lambda \in P^+} V_\lambda^*$  (see [72]). There is a one-to-one bijection between the Plücker variables and columns filled with numbers from  $[n]$  (the numbers increase from top to bottom). Then the semistandard Young tableaux provide a basis of the homogeneous coordinate ring of  $\mathrm{SL}_n/B$  (one takes the product of Plücker variables, corresponding to the columns of a tableau). Similar result holds true in the PBW degenerate situation.

We denote by  $\mu_j$  the length of the  $j$ th column of a diagram.

**Definition 3.4.** A semistandard PBW-tableau of shape  $\lambda$  is a filling  $T_{i,j}$  of the Young diagram  $Y_\lambda$  with numbers  $1, \dots, n$ . The number  $T_{i,j} \in \{1, \dots, n\}$  is attached to the box  $(i, j)$ . The filling  $T_{i,j}$  has to satisfy the following properties:

- (1) if  $T_{i,j} \leq \mu_j$ , then  $T_{i,j} = i$ ;
- (2) if  $i_1 < i_2$  and  $T_{i_1,j} \neq i_1$ , then  $T_{i_1,j} > T_{i_2,j}$ ;
- (3) for any  $j > 1$  and any  $i$ , there exists  $i_1 \geq i$  such that  $T_{i_1,j-1} \geq T_{i,j}$ .

One can show that the number of shape  $\lambda$  semistandard PBW-tableaux is equal to  $\dim V_\lambda$ . Moreover, the following theorem holds [52] (see also [78, 60]).

**Theorem 3.5.** *The homogeneous coordinate ring of  $\mathcal{F}_n^a$  (also known as the PBW degenerate Plücker algebra) is isomorphic to the direct sum of dual PBW degenerate modules  $(V_\lambda^a)^*$ ,  $\lambda \in P^+$ . The ideal of relations is quadratic and is generated by degenerate Plücker relations. The PBW semistandard tableaux parametrize a basis in the coordinate ring.*

Certain infinite-dimensional analogues of the results described above are obtained in [62, 108]. However, this direction has not been seriously pursued so far.

#### 4. TOPOLOGY AND COMBINATORICS

In this section we describe a cellular decomposition of the type  $A$  complete PBW degenerate flag varieties  $\mathcal{F}_n^a$  (see [53, 14, 61] for a more general picture).

Let us fix an  $n$ -dimensional vector space  $W$  with a basis  $w_1, \dots, w_n$ . Let  $\mathbf{I} = (I_1, \dots, I_{n-1})$  be a collection of subsets of the set  $[n]$  such that  $|I_k| = k$ . We denote by  $p_{\mathbf{I}} \in \prod_{k=1}^{n-1} \mathrm{Gr}_k(W)$  a point in the product of Grassmann varieties such that the  $k$ th component is equal to the linear span of  $w_i$  with  $i \in I_k$ . Theorem 3.2 implies that  $p_{\mathbf{I}} \in \mathcal{F}_n^a$  if and only if

$$(2) \quad I_k \subset I_{k+1} \cup \{k+1\} \quad \text{for all } k = 1, \dots, n-2.$$

The following theorem is proved in [53].

**Theorem 4.1.** *The  $G^a$  orbits of the points  $p_{\mathbf{I}}$  provide a cellular decomposition of  $\mathcal{F}_n^a$ .*

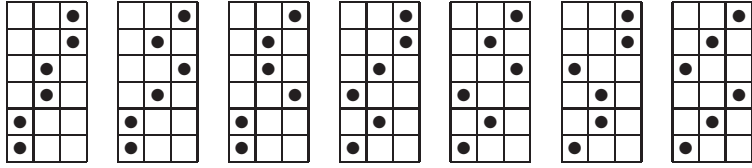
A natural problem is to compute the Euler characteristic and Poincaré polynomial of  $\mathcal{F}_n^a$ . The answer is given in terms of the normalized median Genocchi numbers and the Dellac configurations.

The normalized median Genocchi numbers  $h_n$ ,  $n = 0, 1, 2, \dots$  form a sequence which starts with 1, 1, 2, 7, 38, 295 [104]. The earliest definition was given by Dellac in [36] (see also [12, 13, 17, 39, 40, 41, 54, 79, 92, 121, 122]). Consider a rectangle with  $n$  columns and  $2n$  rows. It contains  $n \times 2n$  boxes labeled by pairs  $(l, j)$ , where  $l = 1, \dots, n$  is the number of a column and  $j = 1, \dots, 2n$  is the number of a row. A Dellac configuration  $D$  is a subset of boxes, subject to the following conditions:

- (1) each column contains exactly two boxes from  $D$ ,
- (2) each row contains exactly one box from  $D$ ,
- (3) if the  $(l, j)$ th box is in  $D$ , then  $l \leq j \leq n + l$ .

Let  $DC_n$  be the set of such configurations. Then the number of elements in  $DC_n$  is equal to  $h_n$ .

We list all Dellac's configurations for  $n = 3$ .



The importance of the median Genocchi numbers comes from the following theorem [53].

**Theorem 4.2.** *The number of collections  $\mathbf{I}$  subject to conditions (2) is equal to the normalized median Genocchi number  $h_n$ . The Euler characteristic of  $\mathcal{F}_n^a$  is equal to  $h_n$ .*

An explicit formula for the numbers  $h_n$  is available (see [29]), namely

$$(3) \quad h_n = \sum_{f_0, \dots, f_n \geq 0} \prod_{k=1}^n \binom{1+f_{k-1}}{f_k} \prod_{k=0}^{n-1} \binom{1+f_{k+1}}{f_k}$$

with  $f_0 = f_n = 0$ .

In order to compute the Poincaré polynomial of  $\mathcal{F}_n^a$ , we define a length  $l(D)$  of a Dellac configuration  $D$  as the number of pairs  $(l_1, j_1)$ ,  $(l_2, j_2)$  such that the boxes  $(l_1, j_1)$  and  $(l_2, j_2)$  are both in  $D$  and  $l_1 < l_2$ ,  $j_1 > j_2$ . This definition resembles the definition of the length of a permutation. We note that in the classical case the complex dimension of the cell attached to a permutation  $\sigma$  in a flag variety is equal to the number of pairs  $j_1 < j_2$  such that  $\sigma(j_1) > \sigma(j_2)$ , which equals to the length of  $\sigma$ . One has [53]:

**Theorem 4.3.** *The complex dimension of the cell in  $\mathcal{F}_n^a$  containing a point  $p_{\mathbf{1}}$  is equal to  $l(D)$ . Thus the Poincaré polynomial  $h_n(q) = P_{\mathcal{F}_n^a}(q)$  is given by  $h_n(q) = \sum_{D \in DC_n} q^{l(D)}$ .*

The first four polynomials  $h_n(q)$  are as follows:

$$\begin{aligned} h_1(q) &= 1, & h_2(q) &= 1 + q, \\ h_3(q) &= 1 + 2q + 3q^2 + q^3, \\ h_4(q) &= 1 + 3q + 7q^2 + 10q^3 + 10q^4 + 6q^5 + q^6. \end{aligned}$$

The following (fermionic type) formula for the polynomials  $h_n(q)$  is obtained in [29] using the geometry of quiver Grassmannians:

$$(4) \quad h_n(q) = \sum_{f_1, \dots, f_{n-1} \geq 0} q^{\sum_{k=1}^{n-1} (k-f_k)(1-f_k+f_{k+1})} \prod_{k=1}^n \binom{1+f_{k-1}}{f_k}_q \prod_{k=0}^{n-1} \binom{1+f_{k+1}}{f_k}_q$$

(we assume  $f_0 = f_n = 0$ ). The formula is given in terms of the  $q$ -binomial coefficients

$$\binom{m}{n}_q = \frac{m_q!}{n_q!(m-n)_q!}, \quad m_q! = \prod_{i=1}^m \frac{1-q^i}{1-q}.$$

## 5. QUIVER GRASSMANNIANS

Theorem 3.2 provides a link between the PBW degenerate flag varieties and quiver Grassmannians. Let  $Q$  be a quiver with the set of vertices  $Q_0$  and the set of arrows  $Q_1$ . For two vectors  $\mathbf{e}, \mathbf{d} \in \mathbb{Z}^{Q_0}$ , we denote by  $\langle \mathbf{e}, \mathbf{d} \rangle$  the value of the Euler form of the quiver. For a  $Q$  module  $M$  and a dimension vector  $\mathbf{e} \in \mathbb{Z}_{\geq 0}^{Q_0}$ , we denote by  $\text{Gr}_{\mathbf{e}}(M)$  the quiver Grassmannian consisting of  $\mathbf{e}$ -dimensional subrepresentations of  $M$ . For more details on the quiver representation theory, see [8, 22, 23, 113]. The general theory of quiver Grassmannians can be found in [25] (see also [1, 2, 24, 26, 84, 96, 109, 110]).

Now let  $Q$  be an equioriented type  $A_{n-1}$  quiver. We label the vertices by the numbers from 1 to  $n$ . Then the set  $Q_1$  consists of arrows  $i \rightarrow i+1$ ,  $i \in [n-1]$ . The indecomposable representations of  $Q$  are labeled by pairs  $1 \leq i \leq j \leq n$ ; the representation  $U_{i,j}$  is supported on vertices from  $i$  to  $j$  and is one-dimensional at each vertex. The projective indecomposable representations are given by  $P_k = U_{k,n}$  and the injective indecomposables are  $I_k = U_{1,k}$ . In particular, the path algebra  $A$  of  $Q$  is isomorphic to the direct sum  $\bigoplus_{k=1}^{n-1} P_k$  and the dual  $A^*$  is the direct sum  $\bigoplus_{k=1}^{n-1} I_k$  of all indecomposable injectives.

By the very definition, the classical complete flag variety  $\text{SL}_n/B$  is isomorphic to the quiver Grassmannian  $\text{Gr}_{\dim A}(P_1^{\oplus n})$ . The following observation was made in [29]:

$$(5) \quad \mathcal{F}_n^a \simeq \text{Gr}_{\dim A}(A \oplus A^*).$$



The realization (5) provides additional tools for the study of algebro-geometric and combinatorial properties of the degenerate flag varieties (see [29, 30, 31, 32]). In particular, one recovers and generalizes [31, 33] the Bott–Samelson type construction for the resolution of singularities of  $\mathcal{F}_n^a$  [56] (see also [88, 112] for further generalizations). The resolution is constructed as a quiver Grassmannian for a larger quiver attached to  $Q$ .

Since the degenerate flag varieties have many nice properties, it is natural to study the quiver Grassmannians  $\text{Gr}_{\dim P}(P \oplus I)$  for arbitrary projective representation  $P$  and an injective representation  $I$  and a Dynkin quiver  $Q$  (the so-called well-behaved quiver Grassmannians). We summarize the main properties of these quiver Grassmannians in the following theorem (see [29, 30]).

**Theorem 5.1.** *Let  $P$  and  $I$  be a projective and an injective representations of a Dynkin quiver  $Q$ . Then the quiver Grassmannian  $X = \text{Gr}_{\dim P}(P \oplus I)$  has the following properties:*

- (1)  $\dim X = \langle \dim P, \dim I \rangle$ ,
- (2)  $X$  is irreducible and normal,
- (3)  $X$  is locally a complete intersection,
- (4) there exists an algebraic group  $G \subset \text{Aut}(P \oplus I)$  acting on  $X$  with finitely many orbits.

For a dimension vector  $\mathbf{d} \in \mathbb{Z}_{\geq 0}^{Q_0}$ , let  $R_{\mathbf{d}}$  be the variety of  $Q$ -representations of dimension  $\mathbf{d}$ . The group  $GL_{\mathbf{d}} = \prod_{i \in Q_0} GL_{d_i}$  acts on  $R_{\mathbf{d}}$  by base change and the orbits are parameterized by the isoclasses of  $\mathbf{d}$ -dimensional representations of  $Q$ . The closure of orbits induces the degeneration order on the set of isoclasses. Fixing a dimension vector  $\mathbf{e}$ , we obtain a family  $\text{Gr}_{\mathbf{e}}(\mathbf{d})$  of  $\mathbf{e}$ -dimensional quiver Grassmannians over the representation space  $R_{\mathbf{d}}$  (the so-called universal quiver Grassmannian). Let us denote the projection map  $\text{Gr}_{\mathbf{e}}(\mathbf{d}) \rightarrow R_{\mathbf{d}}$  by  $p_{\mathbf{e}, \mathbf{d}}$ .

We are interested in the case when  $Q$  is the equioriented type  $A_{n-1}$  quiver,  $\mathbf{d} = (n, \dots, n)$  and  $\mathbf{e} = (1, 2, \dots, n-1)$ . Then both the classical and the PBW degenerate flag varieties are isomorphic to the fibers of  $p_{\mathbf{e}, \mathbf{d}}$ . It is thus natural to ask about the properties of the whole family. The  $GL_{\mathbf{d}}$  orbits on  $R_{\mathbf{d}}$  are parametrized by the tuples  $\mathbf{r}$  of ranks  $r_{i,j}$  of the compositions of the maps between the  $i$ th and  $j$ th vertices. We define three rank tuples  $\mathbf{r}^0$ ,  $\mathbf{r}^1$ , and  $\mathbf{r}^2$  by

$$r_{i,j}^0 = n + 1, \quad r_{i,j}^1 = n + 1 - (j - i), \quad r_{i,j}^2 = n - (j - i).$$

Then the corresponding representations of  $Q$  are given by  $M^0 = P_1^{\oplus n}$ ,  $M^1 = A \oplus A^*$ , and

$$M^2 = \bigoplus_{k=1}^{n-1} P_k \oplus \bigoplus_{k=1}^{n-2} I_k \oplus S,$$



where  $S$  is the direct sum of all simple modules of  $Q$ . One has  $\mathrm{SL}_n/B \simeq \mathrm{Gr}_e(M^0)$ ,  $\mathcal{F}_n^a \simeq \mathrm{Gr}_e(M^1)$ . In [27] we prove the following theorem:

**Theorem 5.2.** (a) *The quiver Grassmannian  $p_{\mathbf{e},\mathbf{d}}^{-1}(M^2)$  is of expected dimension  $n(n-1)/2$ . It is reducible and the number of irreducible components is equal to the  $n$ th Catalan number.*  
 (b) *The flat irreducible locus of  $\mathrm{Gr}_e(\mathbf{d})$  consists of the fibers  $p_{\mathbf{e},\mathbf{d}}^{-1}(M)$  such that  $M$  degenerates to  $M^1$ .*  
 (c) *The flat locus of  $\mathrm{Gr}_e(\mathbf{d})$  consists of the fibers  $p_{\mathbf{e},\mathbf{d}}^{-1}(M)$  such that  $M$  degenerates to  $M^2$ .*

The case of partial flag varieties is considered in [28].

## 6. TORIC DEGENERATIONS

As explained in Section 5, the degeneration of the classical flag variety into the PBW degenerate flag variety can be considered within a family of quiver Grassmannians over the representation space of the quiver. In particular, the study of other degenerations (intermediate and deeper ones) leads to the new and interesting results and examples. Yet another direction is to make a connection between the PBW degeneration and toric degenerations [20] of flag varieties (the latter attracted a lot of attention in the last two decades, see [3, 4, 11, 15, 18, 44, 45, 63, 80, 94]). One of the most famous examples is a flat degeneration of  $\mathcal{F}_n$  into the toric variety with the Newton polytope being the Gelfand–Tsetlin polytope [74, 119, 89, 77]. We are able to prove the following theorem.

**Theorem 6.1.** *The complete flag variety  $\mathcal{F}_n = \mathrm{SL}_n/B$  admits a flat degeneration to the toric variety corresponding to a FFLV polytope with regular highest weight. This degeneration factors through the PBW degeneration.*

The GT and FFLV polytopes are identified with the order and chain polytopes of a certain poset (see [5, 118, 46, 100, 102]). Several proofs of Theorem 6.1 are available. Essentially, there are three different approaches:

- (1) via the representation space  $V_\lambda$ ,
- (2) via the Gröbner theory for the Plücker ideal,
- (3) via the SAGBI theory for the Plücker algebra.

The first approach is utilized in [60, 47, 42]. The approach is similar to the PBW degeneration construction: instead of attaching degree one to each Chevalley generator, one uses a weight system, attaching weight  $a_{i,j}$  to the generators  $f_{\alpha_{i,j}}$  for all positive roots. For certain weight systems, one gets a filtration on the universal enveloping algebra, which leads to a filtered (and then graded) representation space and degenerate flag variety. Here comes the theorem (see [47, 60]).

**Theorem 6.2.** *Consider the PBW filtration with the weight system  $a_{i,j} = (j - i + 1)(n - j)$ . Then*

- (1) *in the associated graded space the nonzero monomials in  $f_\alpha$  form a basis,*
- (2) *the associated graded space is acted upon by the symmetric algebra  $S(\mathfrak{n}^-)$  and the degenerate flag variety is a  $\mathbb{G}_a^{n(n-1)/2}$  variety,*
- (3) *the corresponding degenerate flag variety is toric with the Newton polytope being the FFLV polytope.*

Instead of working with the representation space, one may start with the algebraic variety  $\mathcal{F}_n$  from the very beginning. As an intermediate step one considers the theory of Newton–Okounkov (NO) bodies [105, 87]. The connection between the NO bodies and toric degenerations is used in many papers, see, e.g., [4, 48, 80, 86]. The following holds true.

**Theorem 6.3.** *The toric variety attached to the FFLV polytope can be constructed as a Newton–Okounkov body for certain valuations. The valuations are obtained via Lie theory [60] or geometrically [71, 70, 90, 91].*

Recall the Plücker coordinates  $X_I$ , the quadratic Plücker ideal defining the flag variety  $\mathcal{F}_n$  inside the product of the projectivized fundamental representations and the Plücker algebra (the quotient by the Plücker ideal). There are two general constructions leading to the degenerations of algebraic varieties: Gröbner theory for the defining ideals [103, 82] (see also [116, 117, 42] for the tropical version) and the SAGBI (subalgebra analogues of the Gröbner bases for ideals) theory [111] (see also [21, 82, 83]). The former construction works with the defining ideals, attaching certain degrees to the variables, and the latter deals with the quotient algebras, using certain monomial orders. In our setting the following claims hold (see [42, 99]).

**Theorem 6.4.** *There exists a maximal cone in the Gröbner fan of the Plücker ideal such that a general point corresponds to the monomial ideal defined by the PBW semistandard tableaux. There exists a monomial order on the set of Plücker variables such that the monomials in Plücker variables corresponding to the PBW semistandard tableaux form a SAGBI basis of the Plücker algebra.*

Let us close with the remark that it would be very interesting to construct and study toric degenerations for affine flag varieties [93] and semiinfinite flag varieties [49, 67]. The first steps in this direction were made in [114, 115]. From the representation theory point of view, this would lead to new constructions of bases and character formulas for the integrable representations of affine algebras and global Weyl and Demazure modules for current algebras [37, 38, 65, 101].

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