CHARACTER FORMULA FOR WEIL REPRESENTATIONS IN TERMS OF FROBENIUS TRACES

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ABSTRACT. It is known that the étale cohomology of a potentially good abelian variety over a local field K is determined by its Euler factors over the extensions of K. We extend this to all abelian varieties, show that it is enough to take extensions where A is semistable, and give a uniform version over p-adic fields where the extensions are the same for all abelian varieties of a given dimension. The results are explicit, and apply to a wide class of Weil-Deligne representations.

Ramified l-adic local Galois representations tend to be unwelcome. We give an explicit character formula which reconstructs such a representation from its Frobenius traces over extensions where it is unramified. We also record some consequences for étale cohomology of abelian varieties and other l-adic representations. The paper is a number-theoretic counterpart of [13], which concerns representations of finite groups.

Our setting is as follows. Let K be a non-archimedean local field, with residue field $\mathbb{F}_K$, separable closure K^s and algebraic closure $\bar{K}$. We write K^{nr} for the maximal unramified extension of K in K^s, $I_K = I_{K^s/K}$ for the absolute inertia group and $\mathrm{Frob}_K \in G_K = \mathrm{Gal}(K^s/K)$ for an arithmetic Frobenius element. Recall that the Weil group W_K is generated by I_K and Frob_K; its topology is the usual profinite one on I_K and discrete on $W_K/I_K \cong \mathbb{Z}$. A Weil representation over K is a continuous finite-dimensional complex representation of W_K. We refer the reader to [23] for the background on Weil and Weil-Deligne representations.

We begin with an analogue of the character formula [13, Cor. 9] for Weil representations:

Theorem 1. *Let F/K be a finite Galois extension of local fields and ρ a semisimple Weil representation over K that factors through $\mathrm{Gal}(F^{nr}/K)$. Let $\{\rho_i\}_{i\in\Lambda}$ be a set of irreducible representations of $\mathrm{Gal}(F/K)$, with exactly one in each set of unramified twists[1]. Write $I < \mathrm{Gal}(F/K)$ for the inertia group and $m_i = \langle \mathrm{Res}_I \rho_i, \mathrm{Res}_I \rho_i \rangle$. Then*

   (i) *There are unramified representations Ψ_i such that $\rho \cong \bigoplus_{i\in\Lambda} \rho_i \otimes \Psi_i$.*

   (ii) *Fix $i \in \Lambda$. For every $d > 0$, $\mathrm{Tr}\,\Psi_i(\mathrm{Frob}_K^{dm_i}) = \frac{1}{|I| m_i} \sum_L \overline{\mathrm{Tr}\,\rho_i(\mathrm{Frob}_L)} \, \mathrm{Tr}\,\rho(\mathrm{Frob}_L)$, where L ranges over extensions of K in F^{nr} of ramification degree $|I|$ and residue degree dm_i. Note that F^{nr}/L is unramified, so $\rho(\mathrm{Frob}_L), \rho_i(\mathrm{Frob}_L)$ are well-defined.*

  (iii) *The twist $\rho_i \otimes \Psi_i$ is uniquely determined by* (ii).

  (iv) *Concretely, suppose $\dim \Psi_i \leq N$. There is a unique $0 \leq n \leq N$ and $\lambda_1, ..., \lambda_n \in \mathbb{C}^\times$ such that $\sum_k \lambda_k^d = \mathrm{Tr}\,\Psi_i(\mathrm{Frob}_K^{dm_i})$ for $d = 1, ..., N$. Then $\Psi_i(\mathrm{Frob}_K)$ has eigenvalues $\sqrt[m_i]{\lambda_1}, ..., \sqrt[m_i]{\lambda_n}$ for some choice of the roots, and $\rho_i \otimes \Psi_i$ is independent of this choice.*

Let ℓ be a prime different from the residue characteristic of K. As an application, we deduce that the ℓ-adic representation $\rho_\ell = H^1_{\text{ét}}(A_{\bar{K}}, \mathbb{Q}_\ell)$ of an abelian variety A/K is

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[1]i.e. $\rho_i \neq \rho_j \otimes (1\text{-dim})$ unramified, and each irreducible of $\mathrm{Gal}(F/K)$ is $\rho_i \otimes (1\text{-dim unramified})$ for some i

determined by the traces $\operatorname{Tr} \rho_\ell^{I_L}(\operatorname{Frob}_L)$ over the fields L where A is semistable. (Here, as usual, $\rho_\ell^{I_L}$ is the maximal subrepresentation unramified over L.) In fact, when K has characteristic 0, we can choose a universal finite list of such fields to control all abelian varieties of a fixed dimension.

Theorem 2. *Let $g \geq 0$, and suppose $\operatorname{char} K = 0$. There is a finite Galois extension K_g/K such that*

  (i) *Every g-dimensional abelian variety A/K has semistable reduction over K_g.*
  (ii) *The G_K-representation $\rho_\ell = H^1_{\text{ét}}(A_{\bar{K}}, \mathbb{Q}_\ell)$ is uniquely determined by the traces $\operatorname{Tr} \rho_\ell^{I_L}(\operatorname{Frob}_L)$ over subfields $K \subset L \subset K_g$ for which K_g/L is unramified.*
 (iii) *When $g \geq 2$, for every smooth projective curve X/K of genus g the G_K-representation $H^1_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_\ell)$ is uniquely determined by point counts $\#\bar{\mathcal{X}}_L(\mathbb{F}_L)$ with L as in (ii). Here $\mathcal{X}_L$ is the minimal regular model of X/L with special fibre $\bar{\mathcal{X}}_L/\mathbb{F}_L$.*

We caution the reader that the result is false when $\operatorname{char} K > 0$, see Remark 11.

We now return to the case of arbitrary characteristic. Theorem 1 allows one to explicitly reconstruct local Galois representations from the restrictions to subgroups where they become semistable or unramified (extending [12]). For example, suppose X/K is a curve with 'very bad' reduction. Then we can identify the Galois representation $\rho_l = H^1_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_\ell)$, provided we know a Galois extension F/K where X becomes semistable, and we can count points on the reduction of X in extensions of K where X is semistable. We restore the decomposition of ρ_l into irreducibles via a weighted average of Frobenius traces as in Theorem 1 (ii), which can be computed as point counts on the reduced curves. We give a numerical example at the end of this paper, which illustrates this in detail. This style of argument has been used for specific Galois groups (see [12, §§3-4], [11, Thm 7.3], [5], [17, App. A]), and Theorem 1 gives a universal character formula which applies for arbitrary groups.

Here are some theoretical consequences of Theorem 1 for general representations. The following corollary is essentially due to Saito [20, Lemma 1(1)]; see also Remark 9.

Corollary 3. *Every semisimple Weil representation ρ over K is uniquely determined by the traces $\operatorname{Tr} \rho(\operatorname{Frob}_L)$, where L varies over finite separable extensions of K over which ρ is unramified.*

Notation. Recall that a Frobenius semisimple Weil-Deligne representation $\rho = (\rho_{\text{Weil}}, N)$ over K can be decomposed as $\rho = \bigoplus_{n \geq 1} \rho_n \otimes \operatorname{Sp}_n$, where ρ_n are semisimple Weil representations (i.e. have $N = 0$), and Sp_n is the n-dimensional special representation [23, 4.1.4–4.1.5]. In particular, $\rho^{N=0} := \ker N = \bigoplus \rho_n$.

We say that ρ is *weight-monodromy compatible* if the eigenvalues of Frob_K on ρ_i are Weil numbers of absolute value $q^{\frac{n-1}{2}}$, where $q = |\mathbb{F}_K|$. We use the same term for ℓ-adic representations when their associated Weil-Deligne representation is weight-monodromy compatible.

For every proper smooth variety X/K and $0 \leq i \leq 2 \dim X$, $H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_\ell)(\frac{i}{2})$ is conjectured to be weight-monodromy compatible [8], and this is often known (see [21, §1] for a summary). Because a weight-monodromy compatible representation is determined by $\rho^{N=0}$, from Corollary 3 we get

Corollary 4. *Frobenius-semisimple Weil-Deligne representations $\rho = (\rho_{\text{Weil}}, N)$ over K satisfying the weight-monodromy compatibility are uniquely determined by the traces $\text{Tr}\,\rho^{N=0}(\text{Frob}_L)$, where L varies over finite separable extensions of K over which ρ_{Weil} is unramified.*

Corollary 5. *Let X/K be a proper smooth variety and $0 \leq i \leq 2\dim X$. If the G_K-representation $\rho_\ell = H^i_{\acute{e}t}(X_{\bar{K}}, \mathbb{Q}_\ell)(\frac{i}{2})$ is Frobenius semisimple and weight-monodromy compatible, then it is uniquely determined by $\text{Tr}\,\rho_\ell^{I_L}(\text{Frob}_L)$ for finite separable extensions L/K for which I_L acts unipotently. In particular, if these traces are independent of ℓ, then so is the Weil-Deligne representation associated to $H^i_{\acute{e}t}(X_{\bar{K}}, \mathbb{Q}_\ell)$.*

For semistable abelian varieties these traces are known to be independent of ℓ [15, IX, Thm 4.3], and we get the well-known independence of ℓ for Weil-Deligne representations of general abelian varieties (see [6, Ex. 8.10], [14, Rem. 2.4.6] and [1, Prop. 2.8.1] for a proof):

Corollary 6. *For an abelian variety A/K, the Weil–Deligne representation associated to $H^1_{\acute{e}t}(A_{\bar{K}}, \mathbb{Q}_\ell)$ is independent of ℓ.*

Finally, we also have a version of Theorem 2 for semisimple ℓ-adic representations:

Theorem 7. *Suppose* char $K = 0$. *Let $n \geq 0$ and $\mathcal{F}/\mathbb{Q}_\ell$ a finite extension. There is a finite Galois extension $K_{\mathcal{F},n}/K$ such that all continuous semisimple representations $\rho_\ell : G_K \to \text{GL}_n(\mathcal{F})$ are*

   (1) *unramified over subfields $K \subset L \subset K_{\mathcal{F},n}$ for which $K_{\mathcal{F},n}/L$ is unramified;*
   (2) *uniquely determined by the traces $\text{Tr}\,\rho_\ell(\text{Frob}_L)$ for L as in (1).*

Moreover, any finite extension $E/K_{\mathcal{F},n}$, with E Galois over K, has this property.

Again, because a Frobenius semisimple weight-monodromy compatible representation is determined by its maximal semisimple subrepresentation, we deduce

Corollary 8. *Suppose* char $K = 0$. *All Frobenius semisimple weight-monodromy compatible representations $\rho_\ell : G_K \to \text{GL}_n(\mathcal{F})$ are uniquely determined by the traces $\text{Tr}\,\rho_\ell^{I_L}(\text{Frob}_L)$ with L and $K_{\mathcal{F},n}$ (or E) as in the theorem.*

Remark 9. Our strategy of proving that Weil representations are determined by their Frobenius traces over extensions where they are unramified is known, and goes back at least to Chebotarev. He used the fact that after adjoining enough roots of unity, a finite extension of number fields has abundance of cyclotomic extensions inside it to reduce his density theorem to Hecke's cyclotomic case (see e.g. [19]). Similarly, Corollary 3 relies on the abudance of unramified extensions inside F^{nr}/K for a finite Galois extension F/K. What is new is Theorem 1, which makes this explicit in practice.

As mentioned above, Corollary 3 is due to Saito [20, Lemma 1(1)]. His assumption $n(\sigma) \geq 0$ is slightly stronger but the proof only uses $n(\sigma) > 0$, which is equivalent to that in Corollary 3. The proof of [20, Lemma 1(1)] does not explain why m is a function of the traces, and the field cut out by σ is not a finite extension of K. However, both of these issues can be circumvented with an argument similar to the proof of Corollary 3 below, by comparing two representations instead of taking one.

PROOF OF THEOREM 1

(i) This is standard; see e.g. [12, Lemma 2.3] and its proof (taking F as in the theorem).

(ii) By linearity in ρ, it suffices to prove this when $\rho = \rho_i \otimes \Psi$, with Ψ unramified 1-dimensional. Because the formula is also invariant under twisting by unramified 1-dimensional characters, we may assume $\Psi = \mathbf{1}$. In particular Ψ factors through $\mathrm{Gal}(F/K)$, and this case is proved in [13, Cor 9 (iii)]. Note that ranging over L is the same as ranging over $g \in \mathrm{Frob}_K^{dm_i} I_{F^{nr}/K}$ under the correspondence $L \mapsto \mathrm{Frob}_L$ and $g \mapsto g$-invariants of F^{nr}.

(iii),(iv) This is essentially [13, Cor 9 (iv)]. It is stated there for finite groups, but the proof is the same: (iii) and (iv) follow formally from (ii) plus the fact that ρ_i is invariant under twisting by unramified characters of order m_i, which is [13, Lemma 5]. $\square$

PROOF OF COROLLARY 3

Let ρ_1, ρ_2 be two non-isomorphic semisimple Weil representations over K. They have finite inertia images, and so there is a finite Galois extension F/K such that both ρ_1 and ρ_2 factor through $\mathrm{Gal}(F^{nr}/K)$. By Theorem 1, ρ_1 and ρ_2 can be distinguished from one another by their Frobenius traces over extensions L/K inside F^{nr} over which they are unramified.

PROOF OF THEOREM 7

Fix n and $\mathcal{F}$ as in the theorem. First consider continuous representations $\tau : G_K \to \mathrm{GL}_n(\mathcal{F})$ with finite image. After taking a $\tau(G_K)$-invariant $O_{\mathcal{F}}$-lattice, we may assume that τ lands in $\mathrm{GL}_n(O_{\mathcal{F}})$. Recall that $\mathrm{GL}_n(O_{\mathcal{F}})$ has a compact open subgroup U with no elements of finite order (e.g. using exp and log as in [22, Appendix]). Therefore $|\tau(G_K)| \leq (\mathrm{GL}_n(O_{\mathcal{F}}) : U)$, and τ factors thorough a Galois extension of K of at most this degree. As $\mathrm{char}\, K = 0$, there are only finitely many separable extensions of K of a given degree by a theorem of Krasner [18], and so there is a finite Galois extension F/K, such that every τ with finite image factors through $\mathrm{Gal}(F/K)$. It follows that every continuous semisimple representation $\rho : G_K \to \mathrm{GL}_n(\mathcal{F})$ factors through $\mathrm{Gal}(F^{nr}/K)$.

Now apply Theorem 1. Let $K_{\mathcal{F},n}$ be the compositum of all L/K in F^{nr} with ramification degree $e_{L/K} = e_{F/K}$ and residue degree over K bounded by n^3. By the theorem (parts (ii)-(iv) and using that $dm_i \leq n^3$ in (iv)), Frobenius traces over these fields determine $\rho_{\mathbb{C}} : W_K \to \mathrm{GL}_n(\mathbb{C})$ for some chosen embedding $\mathcal{F} \hookrightarrow \mathbb{C}$. Since $W_K \subset G_K$ is dense and $\rho_{\mathbb{C}}$ is continuous, it is determined on all of G_K as well. Therefore ρ itself is uniquely determined as a representation to $\mathrm{GL}_n(\mathcal{F})$. The same is true for any finite extension E of $K_{\mathcal{F},n}$, Galois over K. $\square$

PROOF OF THEOREM 2

We will prove a slightly stronger statement:

Theorem 10. *Suppose* $\mathrm{char}\, K = 0$. *Fix* $g, c \in \mathbb{N}$ *and a prime* ℓ *different from the residue characteristic of* K. *There is a finite Galois extension* $K_{\ell,g,c}/K$ *such that*

(i) *Every* g-*dimensional abelian variety* A/K *has semistable reduction over* $K_{\ell,g,c}$.

(ii) *The* G_K-*representation* $\rho_\ell = H^1_{\text{ét}}(A_{\bar{K}}, \mathbb{Q}_\ell)$ *is uniquely determined by* $\mathrm{Tr}\, \rho_\ell^{I_L}(\mathrm{Frob}_L)$ *for subfields* $K \subset L \subset K_{\ell,g,c}$ *with* $|\mathbb{F}_L| > c$ *and* $K_{\ell,g,c}/L$ *unramified.*

(iii) *When $g \geq 2$, for every smooth projective curve X/K of genus g, the G_K-representation $H^1_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_\ell)$ is uniquely determined by point counts $\#\bar{\mathcal{X}}_L(\mathbb{F}_L)$ with L as in (ii). Here $\mathcal{X}_L$ is the minimal regular model of X/L with special fibre $\bar{\mathcal{X}}_L/\mathbb{F}_L$.*

Proof. (i), (ii) Suppose $c = 1$. Recall that $H^1_{\text{ét}}(A_{\bar{K}}, \mathbb{Q}_\ell)$ is Frobenius semisimple and weight-monodromy compatible [15, IX]. For L/K finite, I_L acts unipotently on it if and only if A/L is semistable [15, IX, 3.5/3.8], [2, 7.4.6]. By Theorem 7 and Corollary 8 for $n = 2g$ and $\mathcal{F} = \mathbb{Q}_\ell$, any choice of $K_{\ell,g,1}$ that contains $K_{\mathcal{F},n}$ satisfies (i) and (ii).

Now let c be arbitrary, and choose a prime $d \neq \ell$ so that $[\mathbb{Q}_\ell(\zeta_d) : \mathbb{Q}_\ell] > 4g^2$ and $|\mathbb{F}_K|^d > c$. (All but finitely many primes satisfy these conditions.) Let K' be the unramified extension of K of degree d. Construct $K'_{\ell,g,1}$ by the $c = 1$ case over K', enlarging it if necessary to contain $K_{\mathcal{F},n}$. We claim that $K_{\ell,g,c} = K'_{\ell,g,1}$ satisfies (i) and (ii). By construction, (i) holds.

Suppose A/K is an abelian variety. Frobenius traces over subfields $K' \subset L \subset K'_{\ell,g,1}$ as in (ii) determine the restriction of ρ_ℓ to K' uniquely by the properties of $K_{\mathcal{F},n}$. Such L have $|\mathbb{F}_L| \geq |\mathbb{F}_{K'}| = |\mathbb{F}_K|^d > c$, so it suffices to show that this restriction determines ρ.

By Theorem 7 and Corollary 8 again, it suffices to reconstruct Frobenius traces over subfields $K \subset L \subset K_{\mathcal{F},n}$ with $K_{\mathcal{F},n}/L$ unramified. If L contains K' then we are done, as we have already reconstructed the restriction of ρ_ℓ to K'. Suppose $L \not\supset K'$, and let $\alpha_1, ..., \alpha_m$ ($m \leq 2g$) be the eigenvalues of $\rho^{I_L}(\text{Frob}_L)$. As LK'/L has degree d (as d is prime), we know $\alpha_1^d, ..., \alpha_m^d$, the Frobenius eigenvalues over LK' from the restriction to K'.

Note that α_1 generates an extension of $\mathbb{Q}_l$ of degree $\leq 2g$. Now, α_1^d has a unique dth root that generates an extension of $\mathbb{Q}_\ell$ of degree $\leq 2g$, namely α_1. Indeed, if $\alpha_1 \zeta_d^i$ also generates such an extension for $i \neq 0 \mod d$, then

$$[\mathbb{Q}_\ell(\alpha_1, \zeta_d) : \mathbb{Q}_\ell] = [\mathbb{Q}_\ell(\alpha_1, \alpha_1 \zeta_d^i) : \mathbb{Q}_\ell] \leq (2g)^2 = 4g^2,$$

contradicting $[\mathbb{Q}_\ell(\zeta_d) : \mathbb{Q}_\ell] > 4g^2$. Therefore α_1 is indeed determined by α_1^d, and similarly for all the other α_i, as required.

(iii) We claim that any $K_{\ell,g,1}$ that satisfies (i) and (ii) also satisfies (iii), provided $c > 16g^2$ (which we may clearly assume).

Let A be the Jacobian of X. Recall that there is a canonical isomorphism

$$H^1_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_l) \cong H^1_{\text{ét}}(A_{\bar{K}}, \mathbb{Q}_l).$$

By (ii), this representation, say ρ_ℓ, is determined by the traces $\text{Tr}\, \rho_\ell^{I_L}(\text{Frob}_L)$ for L as in (ii). By (i), over these fields A is semistable, hence so is X as $g \geq 2$ [9, Thm. 1.2]. Therefore

$$H^1_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_l)^{I_L} \cong H^1_{\text{ét}}((\bar{\mathcal{X}}_L)_{\bar{\mathbb{F}}_L}, \mathbb{Q}_l)$$

as $\text{Gal}(\bar{\mathbb{F}}_L/\mathbb{F}_L)$-modules, see e.g. [11, Thm. B.1]. By the Grothendieck-Lefschetz trace formula [7, p. 86, Thm 3.1],

$$\#\bar{\mathcal{X}}_L(\mathbb{F}_L) = t_0 - t_1 + t_2,$$

where t_i is the trace of the Frobenius automorphism on $H^i_{\text{ét}}((\bar{\mathcal{X}}_L)_{\bar{\mathbb{F}}_L}, \mathbb{Q}_l)$. In particular, $t_1 = \text{Tr}\, \rho_\ell^{I_L}(\text{Frob}_L)$ by the above. Note that $t_0 = 1$ as $\bar{\mathcal{X}}_L$ is connected, and t_2 is a multiple of $q = |\mathbb{F}_L|$ as $H^2_{\text{ét}}((\bar{\mathcal{X}}_L)_{\bar{\mathbb{F}}_L}, \mathbb{Q}_l)$ is the permutation module on the irreducible components of $\bar{\mathcal{X}}_L$ twisted by $\mathbb{Q}_l(1)$. Now, $|t_1| \leq 2g\sqrt{q}$ because all eigenvalues of $\rho_\ell^{I_L}(\text{Frob}_L)$ have absolute

values 1 or $\sqrt{q}$ and $\dim \rho_\ell^{I_L} \le 2g$. Because $q \ge c > 16g^2$, we have $2g\sqrt{q} < q/2$, and so

$$t_1 \bmod q \;=\; (t_0 + t_2 - \#\bar{\mathcal{X}}_L(\mathbb{F}_L)) \bmod q \;=\; (1 - \#\bar{\mathcal{X}}_L(\mathbb{F}_L)) \bmod q$$

recovers t_1 uniquely. This shows that $\#\bar{\mathcal{X}}_L(\mathbb{F}_L)$ determines $t_1 = \operatorname{Tr} \rho_\ell^{I_L}(\operatorname{Frob}_L)$, and the collection of these for varying L determines $H^1_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_l)$.   □

An example

We end with a numerical example of how Theorem 1 works in practice.

Let $K = \mathbb{Q}_7$ and let $F/\mathbb{Q}_7$ be a wildly ramified Galois extension of degree 21 with (non-abelian) Galois group $G = C_7 \rtimes C_3$, residue degree 3 and ramification degree 7. The character table of G is as follows, after fixing a choice of a cube root of unity ζ_3 and $\sqrt{-7}$ in $\mathbb{C}$:

	1	τ	τ^{-1}	σ	σ^{-1}
$\mathbf{1}$	1	1	1	1	1
χ	1	ζ_3^2	ζ_3	1	1
$\bar{\chi}$	1	ζ_3	ζ_3^2	1	1
ρ_1	3	0	0	$\frac{-1-\sqrt{-7}}{2}$	$\frac{-1+\sqrt{-7}}{2}$
ρ_2	3	0	0	$\frac{-1+\sqrt{-7}}{2}$	$\frac{-1-\sqrt{-7}}{2}$

TABLE 1. Character table of $G = C_7 \rtimes C_3$

Consider a genus 3 curve X whose Jacobian has bad reduction over $\mathbb{Q}_7$, but X acquires good reduction over F. Our goal is to determine the Galois representation $\rho = H^1_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} \mathbb{C}$. By the Néron-Ogg-Shafarevich criterion (or just the proper smooth base change theorem), I_K acts on V through $I_{F/\mathbb{Q}_7} = C_7$. By Theorem 1(i), and the fact that the characteristic polynomials of inertia elements have rational coefficients by [22, Thm. 2],

$$\rho \cong (\rho_1 \otimes \Psi_1) \oplus (\rho_2 \otimes \Psi_2),$$

for some unramified characters Ψ_1, Ψ_2.

Theorem 1 lets us determine the Ψ_i as follows. Let $\tilde{F}$ be the degree 7 unramified extension of F. For each element $g \in I_{F/\mathbb{Q}_7} = C_7$ let $\tilde{g} \in \operatorname{Gal}(\tilde{F}/\mathbb{Q}_7)$ be the unique element that projects to $g \in \operatorname{Gal}(F/\mathbb{Q}_7)$ and acts as $\operatorname{Frob}^3_{\mathbb{Q}_7}$ on the residue field. Write $F_g = \tilde{F}^{\langle \tilde{g} \rangle}$ for the corresponding degree 21 extensions of $\mathbb{Q}_7$. Note that $\operatorname{Jac} X$ has good reduction over F_g, since $\tilde{F}/F_g$ is unramified.

By the theorem,

$$\Psi_i(\operatorname{Frob}^3_{\mathbb{Q}_7}) = \frac{1}{|C_7|} \frac{1}{\langle \operatorname{Res}_{C_7} \rho_i, \operatorname{Res}_{C_7} \rho_i \rangle} \sum_{g \in I_{F/\mathbb{Q}_7}} \overline{\operatorname{Tr} \rho_i(g)} \operatorname{Tr} \rho(\operatorname{Frob}_{F_g}) =$$

$$= \frac{1}{21} \sum_{g \in I_{F/\mathbb{Q}_7}} \overline{\operatorname{Tr} \rho_i(g)} \cdot (7^3 + 1 - \#\overline{\mathcal{X}_{F_g}}(\mathbb{F}_{7^3})),$$

where $\mathcal{X}_{F_g}$ is a regular model of X over F_g, by the Grothendieck-Lefschetz trace formula (as in proof of Theorem 10 (iii)). Moreover, $\Psi_i(\operatorname{Frob}_{\mathbb{Q}_7})$ can be taken to be any cube root of this value, as the discrepancy vanishes when taking the tensor product with ρ_i.

As a particular example, let $F/\mathbb{Q}_7$ be the splitting field of

$$f(x) = x^7 + 21x^6 + 7,$$

taken from [16]. Let α be a root of f in F; it is a uniformiser of F. Let $\zeta \in F$ be a primitive 18th root of unity. One checks that the roots α_j of f have expansions

$$\alpha_j = \alpha + j\zeta\alpha^2 + O(\alpha^3), \qquad j = 0, ..., 6.$$

The generators of G are then determined by

$$\sigma \colon \alpha_j \mapsto \alpha_{j+1} \text{ (order 7)}, \qquad \tau \colon \alpha_j \mapsto \alpha_{2j} \text{ (order 3)},$$

with indices taken modulo 7.

Now take $X : y^2 = f(x)$. It acquires good reduction over $\mathbb{Q}_7(\alpha)$, and using Magma [3] we find that

$$\#\overline{\mathcal{X}}_{F_{\mathrm{id}}}(\mathbb{F}_{7^3}) = 7^3 + 1 = 344,$$

$$\#\overline{\mathcal{X}}_{F_\sigma}(\mathbb{F}_{7^3}) = \#\overline{\mathcal{X}}_{F_{\sigma^2}}(\mathbb{F}_{7^3}) = \#\overline{\mathcal{X}}_{F_{\sigma^4}}(\mathbb{F}_{7^3}) = 7^3 - 7^2 + 1 = 295,$$

$$\#\overline{\mathcal{X}}_{F_{\sigma^3}}(\mathbb{F}_{7^3}) = \#\overline{\mathcal{X}}_{F_{\sigma^5}}(\mathbb{F}_{7^3}) = \#\overline{\mathcal{X}}_{F_{\sigma^6}}(\mathbb{F}_{7^3}) = 7^3 + 7^2 + 1 = 393,$$

which leads to $\Psi_1(\mathrm{Frob}_{\mathbb{Q}_7}) = \sqrt{-7}$ and $\Psi_2(\mathrm{Frob}_{\mathbb{Q}_7}) = -\sqrt{-7}$.

Conversely, let $X' : y^2 = x^7 + 420x^6 - 245x^3 + 1225x^2 - 833x + 189$. (The right-hand side is the minimal polynomial of $-\alpha^2 - \alpha$.) This curves acquires good reduction over the same fields as X, but has the opposite point counts (that is, 295 and 393 are interchanged), so that here $\Psi_1(\mathrm{Frob}_{\mathbb{Q}_7}) = -\sqrt{-7}$ and $\Psi_2(\mathrm{Frob}_{\mathbb{Q}_7}) = \sqrt{-7}$.

What makes the example interesting is that the two curves have the same point counts over all extensions of $\mathbb{Q}_7$ of degree at most 20. In that sense our result here is best possible.

Remark 11. The assumption char $K = 0$ in Theorems 2, 7, 10 and Corollary 8 is necessary, as we will now explain.

First, suppose M/K is any ramified separable extension of prime degree $p > 2$. It is obtained by adjoining the root of some Eisenstein polynomial $f(x)$. Consider the hyperelliptic Jacobian of dimension $g = (p-1)/2$

$$A = \mathrm{Jac}\, C, \qquad C : y^2 = f(x).$$

In the terminology of [10], C has one proper cluster of size p (because I_K acts transitively on roots of $f(x)$ and $\deg f = p$ is prime). By [10, Thm 1.20],

$$\rho_l = H^1_{\text{ét}}(A_{\bar{K}}, \mathbb{Q}_\ell) \cong \gamma \otimes (\pi \ominus \mathbf{1}),$$

as an I_K-representation, where γ is some tame character and π is the permutation character of I_K on the roots of f.

Now, suppose char $K = p$. By [18], there are extensions M/K as above with arbitrary large valuation of the discriminant. Therefore, by the conductor-discriminant formula and the definition of the conductor, for every $n \geq 1$ there is a separable polynomial $f(x) \in K[x]$ of degree p such that the higher ramification group $G_n \lhd I_K$ acts non-trivially on the roots of $f(x)$.

Let K_g/K be any finite Galois extension. Then G_n acts trivially on K_g for some n. However, the above construction produces an abelian variety of dimension g for which $\rho_l^{I_{K_g}} \subset \rho_l^{G_n} = 0$. Therefore A has non-semistable reduction over K_g and $\mathrm{Tr}\, \rho_\ell^{I_L}(\mathrm{Frob}_L) = 0$ for every subfield $K \subset L \subset K_g$. In other words, Theorems 2, 7, 10 and Corollary 8 fail over function fields.

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