A GENERALIZED SECOND MAIN THEOREM FOR CLOSED SUBSCHEMES

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Abstract. Let $Y_1, \ldots, Y_q$ be closed subschemes which are located in ℓ-subgeneral position with index κ in a complex projective variety X of dimension n. Let A be an ample Cartier divisor on X. We obtain that if a holomorphic curve $f : \mathbb{C} \to X$ is Zariski-dense, then for every $\epsilon > 0$,

$$\sum_{j=1}^{q} \epsilon_{Y_j}(A) m_f(r, Y_j) \leq_{exc} \left(\frac{(\ell - n + \kappa)(n+1)}{\kappa} + \epsilon \right) T_{f,A}(r).$$

This generalizes the second main theorems for general position case due to Heier-Levin [AM J. Math. 143(2021), no. 1, 213-226] and subgeneral position case due to He-Ru [J. Number Theory 229(2021), 125-141]. In particular, whenever all the Y_j are reduced to Cartier divisors, we also give a second main theorem with the distributive constant. The corresponding Schmidt's subspace theorem for closed subschemes in Diophantine approximation is also given.

1. Introduction and main results

In 1933, H. Cartan [1] established the important second main theorem for linearly nondegenerate holomorphic mappings from $\mathbb{C}^m$ into $\mathbb{P}^N(\mathbb{C})$ intersecting hyperplanes in general position. He also conjectured that it should be true for hyperplanes in subgeneral position. In 1983, Nochka [8] confirmed the Cartan's conjecture by introducing the Nochka weight and Nochka constant. In 2009, Ru [11] extended the Cartan's second main theorem to the case of Cartier divisors in complex projective variety as follows (For more background of Nevanlinna theory, we refer to [12]). Here we use notations from Nevanlinna theory which can be found in Section 2.

Theorem 1.1. [11] *Let X be a complex projective variety of dimension n and $D_1, \ldots, D_q$ be effective Cartier divisors on X, located in general position. Suppose that there exists an ample Cartier divisor A on X and positive integers d_j such that $D_j \sim d_j A$ for $j = 1, \ldots, q$. Let $f : \mathbb{C} \to X$ be a*

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holomorphic map with Zariski dense image. Then, for every $\epsilon > 0$,

$$\sum_{j=1}^{q} \frac{1}{d_j} m_f(r, D_j) \leq_{exc} (n+1+\epsilon) T_{f,A}(r),$$

where $\leq_{exc}$ means the inequality holds for all $r \in \mathbb{R}^{\geq 0}$ outside a set of finite Lebesgue measure.

In 2017, Ru and Wang [13] considered the case of closed subschemes and gave the second main theorem as follows.

Theorem 1.2. [13] *Let X be a projective variety. Let $Y_1, \ldots, Y_q$ be closed subschemes of X such that at most ℓ of the closed subschemes meet at any point $x \in X$. Let A be a big Cartier divisor on X. Let $f : \mathbb{C} \to X$ be a holomorphic curve with Zariski-dense image. Let*

$$\beta_{A,Y_j} = \lim_{N \to \infty} \frac{\sum_{m=1}^{\infty} h^0(\tilde{X}_j, N\pi_j^* A - mE_i)}{N h^0(X, NA)}, \ j = 0, \ldots, q,$$

where $\pi_j : \tilde{X}_j \to X$ is the blowing-up of X along Y_j, with associated exceptional divisor E_j. Then, for every $\epsilon > 0$,

$$\sum_{j=1}^{q} m_f(r, Y_j) \leq_{exc} \ell(\max_{1 \leq j \leq q} \{\beta_{A,Y_j}^{-1}\} + \epsilon) T_{f,A}(r),$$

where $\leq_{exc}$ means the inequality holds for all $r \in \mathbb{R}^{\geq 0}$ outside a set of finite Lebesgue measure.

When the closed subschemes $Y_j = y_j$ are distinct points in X and $\dim X = n$, then one may take $\ell = 1$ in Theorem 1.2, Mckinnon and Roth [7] have shown that the Seshadri constants $\epsilon_{y_j}(A)$ and β_{A,y_j} have the following relation:

$$\beta_{A,y_j} \geq \frac{n}{n+1} \epsilon_{y_j}(A).$$

In order to improve the condition $D_j \sim d_j A$ in Theorem 1.1, Heier and Levin [4] used the notion of Seshadri constant $\epsilon_{D_j}(L)$ (see Definition 2.1), and obtained the following generalization of the second main Theorem for closed subschemes in general position.

Theorem 1.3. [4, Theorem 1.8] *Let X be a complex projective variety of dimension n. Let $Y_0, \ldots, Y_q$ be closed subschemes of X in general position, $f : \mathbb{C} \to X$ be a holomorphic map with Zariski dense image. Let A be an ample Cartier divisor on X. Then, for every $\epsilon > 0$,*

$$\sum_{j=0}^{q} \epsilon_{Y_j}(A) m_f(r, Y_j) \leq_{exc} (n+1+\epsilon) T_{f,A}(r),$$

where $\leq_{exc}$ means the inequality holds for all $r \in \mathbb{R}^{\geq 0}$ outside a set of finite Lebesgue measure.

In 2021, He and Ru [3] considered the ℓ-subgeneral position case and generalized Theorem 1.3 by using somewhat different methods. In this paper, we continue to generalize them to the case of the index κ of ℓ-subgeneral position. The concept of the index κ of ℓ-subgeneral position by Ji-Yan-Yu [5] is generalized from Cartier divisors to closed subschemes.

Definition 1.4. Let X be a projective variety of dimension n which is defined over an arbitrary field k with characteristic zero and let $Y_1, \dots, Y_q$ be q closed subschemes of X. Let ℓ and κ be two positive integers such that $\ell \geq \dim X \geq \kappa$.

(a). The closed subschemes $\{Y_1, \dots, Y_q\}$ are said to be in general position in X if for any subset $I \subset \{1, \dots, q\}$ with $\sharp I \leq \dim X + 1$,

$$\mathit{codim}\left(\bigcap_{i\in I} Y_i \cap X\right) \geq \sharp I.$$

(b). The closed subschemes $\{Y_1, \dots, Y_q\}$ are said to be in ℓ-subgeneral position in X if for any subset $I \subset \{1, \dots, q\}$ with $\sharp I \leq \ell + 1$,

$$\dim\left(\bigcap_{i\in I} Y_i \cap X\right) \leq \ell - \sharp I.$$

(c). The closed subschemes $Y_1, \dots, Y_q$ are said to be in ℓ-subgeneral position with index κ if $Y_1, \dots, Y_q$ are in ℓ-subgeneral position and for any subset $J \subset \{1, \dots, q\}$ with $\sharp I \leq \kappa$,

$$\mathit{codim}\left(\bigcap_{i\in I} Y_i \cap X\right) \geq \sharp I.$$

Theorem 1.5. *Let $f : \mathbb{C} \to X$ be a meromorphic map with Zariski dense image, where X is a complex projective variety of dimension n. Let $Y_1, \dots, Y_q$ be closed subschemes which is located in ℓ-subgeneral position with index κ in X, and $\ell \geq n$. Let A be an ample Cartier divisor on X. Then, for any $\epsilon > 0$,*

$$\sum_{j=1}^{q} \epsilon_{Y_j}(A) m_f(r, Y_j) \leq_{exc} \left(\frac{(\ell - n + \kappa)(n+1)}{\kappa} + \epsilon\right) T_{f,A}(r),$$

where $\leq_{exc}$ means the inequality holds for all $r \in \mathbb{R}^{\geq 0}$ outside a set of finite Lebesgue measure.

To prepare for the proof of Theorem 1.5, we need firstly to obtain a second main theorem for Cartier divisors with distributive constant, which is also an interesting topic independently. The concept of distributive constant was originally given by Quang[10], and we extend it to the case of Cartier divisors as follows.

Definition 1.6. Let X be a projective variety of dimension n defined over over arbitrary field k with characteristic zero and let $D_1, \ldots, D_q$ be q Cartier divisors of X. The distributive constant Δ of $\{D_1, \ldots, D_q\}$ in X is defined by

$$\Delta := \max_{\Gamma \subset \{1,\ldots,q\}} \frac{\sharp \Gamma}{n - dim((\cap_{j\in\Gamma} Supp D_j) \cap X(\overline{k}))}.$$

Here, we note that $dim\ \emptyset = -\infty$.

The second main theorem for holomorphic curves intersecting Cartier divisors with distributive constant in a complex projective variety is stated as follows.

Theorem 1.7. *Let X be a complex projective variety of dimension n. Let $D_1, \ldots, D_q$ be Cartier divisors of X with the distributive constant Δ in X. Let A be an ample Cartier divisor on X. Let $f : \mathbb{C} \to X$ be a holomorphic curve with Zariski-dense image. Then, for every $\epsilon > 0$,*

$$\sum_{j=1}^{q} \epsilon_{D_j}(A) m_f(r, D_j) \leq_{exc} (\Delta(n+1) + \epsilon)) T_{f,A}(r),$$

where $\leq_{exc}$ means the inequality holds for all $r \in \mathbb{R}^{\geq 0}$ outside a set of finite Lebesgue measure.

If $D_1, \ldots, D_q$ are in ℓ-subgeneral position with index κ, then by Remark 3.3 we have $\Delta \leq \frac{\ell - n + \kappa}{\kappa}$. Hence, Theorem 1.7 generalizes Theorem 1.1, Ji-Yan-Yu's work [5, Theorem 1.2] and He-Ru [3, Theorem 2.6].

Whether can Definition 1.6 be generalized to the closed subschemes in order to obtain a generalized second main theorem for closed subschemes with distributive constant? This is an open question.

The rest of this paper is structured as follows. In section 2, we briefly recall some basic notations and definitions about the closed subscheme and Nevanlinna theory. Some preliminary lemmas and the proof of Theorem 1.7 for Cartier divisors with distributive constant are given in section 3. Then Theorem 1.5 will be proved in section 4. According to the Vojta's dictionary ([16, 17]) of the analogy between Nevanlinna theory and Diophantine approximation, the Schmidt's subspace theorems will be given in section 5.

2. Preliminaries

2.1. **Seshadri constants.** Let X be a projective variety and Y be a closed subscheme of X, corresponding to a coherent sheaf of ideals $\mathcal{I}_Y$. Let $\mathcal{S} = \oplus_{d\geq 0}\mathcal{I}_Y^d$ be the sheaf of graded algebras, where $\mathcal{I}_Y^d$ is the d-th power of $\mathcal{I}_Y$, with the convention that $\mathcal{I}_Y^0 = \mathcal{O}_X$. Then $\tilde{X} := Proj\mathcal{S}$ is called the blowing-up of X with respect to $\mathcal{I}_Y$, or, the blowing-up of X along Y.

Let $\pi : \tilde{X} \to X$ be the blowing-up along Y. From Proposition II.7.13(a) in [2], the inverse image ideal sheaf $\tilde{\mathcal{I}}_Y = \pi^{-1}\mathcal{I}_Y \cdot \mathcal{O}_{\tilde{X}}$ is an invertible sheaf on $\tilde{X}$. Let E be an effective Cartier divisor in$\tilde{X}$ whose associated invertible sheaf is the dual of $\pi^{-1}\mathcal{I}_Y \cdot \mathcal{O}_{\tilde{X}}$.

Definition 2.1. Let Y be a closed subscheme of a projective variety X. Let $\pi : \tilde{X} \to X$ be the blowing-up of X along Y. Let A be a nef Cartier divisor on X. We define the Seshadri constant $\epsilon_Y(A)$ of Y with respect to A to be the real number

$$\epsilon_Y(A) = sup\{\gamma \in \mathbb{Q}_{\geq 0} | \pi^*A - \gamma E \ is \ \mathbb{Q} - nef\},$$

where E is an effective Cartier divisor on $\tilde{X}$ whose associated invertible sheaf is the dual of $\pi^{-1}\mathcal{I}_Y \cdot \mathcal{O}_{\tilde{X}}$.

2.2. **The Weil function and its properties.** In this section, we briefly recall the definition of the Weil function and its properties.

Definition 2.2. Let D be a Cartier divisor on a complex variety X. A local Weil function for D is a function $\lambda_D : (X \setminus suppD) \to \mathbb{R}$ such that for all $x \in X$ there is an open neighborhood U of x in X, a nonzero rational function f on X with $D\mid_U = (f)$, and a continuous function $\alpha : U \to \mathbb{R}$ such that

$$\lambda_D(x) = -\log |f(x)| + \alpha(x)$$

for all $x \in (U \setminus SuppD)$.

A continuous (fiber) metric $||\cdot||$ on the line sheaf $\mathcal{O}_X(D)$ determines a Weil function for D given by

$$\lambda_D(x) = -\log ||s(x)||$$

where s is the rational section of $\mathcal{O}_X(D)$ such that $D = (s)$. For example, the Weil function for the hyperplanes $H = \{a_0x_0 + \ldots + a_nx_n = 0\}$ is given by

$$\lambda_H(x) = \log \frac{\max_{0\le i\le n} |x_i| \max_{0\le i\le n} |a_i|}{|a_0x_0 + \ldots + a_nx_n|}.$$

The Weil functions with respect to divisors satisfy the following properties.

(a) Additivity: If λ_1 and λ_2 are Weil functions for Cartier divisors D_1 and D_2 on X, respectively, then $\lambda_1 + \lambda_2$ extends uniquely to a Weil function for $D_1 + D_2$.

(b) Functoriality: If λ is a Weil function for a Cartier divisor D on X, and if $\phi : X' \to X$ is a morphism such that $\phi(X') \not\subset suppD$, then $x \mapsto \lambda(\phi(x))$ is a Weil function for the Cartier divisor $\phi^* D$ on X'.

(c) Uniqueness: If both λ_1 and λ_2 are Weil functions for a Cartier divisor on X, then $\lambda_1 = \lambda_2 + O(1)$.

(d) Boundedness from below: If D is an effective divisor and λ is a Weil function for D, then λ is bounded from below.

For the closed subschemes case, let Y be a closed subscheme of a projective variety X. Then one can associate a Weil function $\lambda_Y : X \setminus SuppY \to \mathbb{R}$, well-defined up to $O(1)$. The following lemma indicates that a closed subscheme can be expressed by some Cartier divisors.

Lemma 2.3. [15, Lemma 2.2] *Let Y be a closed subscheme of a projective variety X. There exist effective Cartier divisors $D_1, \ldots, D_\ell$ such that*

$$Y = \cap_{i=1}^{\ell} D_i.$$

Definition 2.4. Let Y be a closed subscheme of a projective variety X. We define the Weil function for Y as

$$\lambda_Y = \min\{\lambda_{D_1}, \ldots, \lambda_{D_\ell}\} + O(1),$$

where $Y = \cap_{i=1}^{\ell} D_i$ (by Lemma 2.3, such D_i exist). Then there is a Weil function for the closed subscheme $\lambda_Y : X \setminus SuppY \to \mathbb{R}$, which does not depend on the choice of Cartier divisors.

We briefly recall the natural operations on subschemes which are similar to the case of divisors, more details can be found in [15, section 2]. Let Y, Z be closed subschemes of X,

(i) The sum of Y and Z, denoted by $Y + Z$, is the subscheme of X with ideal sheaf $\mathcal{I}_Y \mathcal{I}_Z$.

(ii) The intersection of Y and Z, denoted by $Y \cap Z$, is the subscheme of X with ideal sheaf $\mathcal{I}_Y + \mathcal{I}_Z$.

(iii) The union of Y and Z, denoted by $Y \cup Z$, is the subscheme of X with ideal sheaf $\mathcal{I}_Y \cap \mathcal{I}_Z$.

(iv) Let $\phi : X' \to X$ be a morphism of projective varieties, the inverse image of Y is the subscheme of X' with ideal sheaf $\phi^{-1}\mathcal{I}_Y \cdot \mathcal{O}_{X'}$, denoted by $\phi^* Y$.

In addition, some properties of the Weil functions for closed subschemes are written as follows.

If Y and Z are two closed subschemes of X, and $\phi : X' \to X$ is a morphism of projective varieties,

(i) $\lambda_{Y \cap Z} = \min\{\lambda_Y, \lambda_Z\}$.

(ii) $\lambda_{Y+Z} = \lambda_Y + \lambda_Z$.

(iii) If $Y \subset Z$, $\lambda_Y \leq \lambda_Z$.

(iv) $\lambda_Y(\phi(x)) = \lambda_{\phi^* Y}(x)$.

Lemma 2.5. [15] *Let Y be a closed subscheme of X, and let $\tilde{X}$ be the blowing-up of V along Y with exceptional divisor E. Then for $P \in \tilde{X} \setminus SuppE$,*

$$\lambda_Y(\pi(P)) = \lambda_E(P) + O_v(1).$$

2.3. **Characteristic function and proximity function.** Let X be a complex projective variety and $f : \mathbb{C} \to X$ be a holomorphic map. Let $L \to X$ be a positive line bundle. Denote by $|| \cdot ||$ a Hermitian metric in L and by ω its Chern form. We define the characteristic function of f with respect to L by

$$T_{f,L}(r) = \int_0^r \frac{dt}{t} \int_{|z|<t} f^* \omega.$$

Since any line bundle can be written as $L = L_1 \otimes L_2^{-1}$ with L_1, L_2 are both positive, we define $T_{f,L}(r) = T_{f,L_1}(r) - T_{f,L_2}(r)$.

The characteristic function satisfies the following properties:

(i) Functoriality: If $\phi : X \to X'$ is a morphism and if L is a line bundle on X', then

$$T_{f,\phi^* L} = T_{\phi \circ f, L}(r) + O(1).$$

(ii) Additivity: If L_1 and L_2 are line bundle on X, then

$$T_{f,L_1 \otimes L_2(r)} = T_{f,L_1}(r) + T_{f,L_2}(r) + O(1).$$

(iii) Positivity: If L is positive and $f : \mathbb{C} \to X$ is non-constant, then

$$T_{f,L}(r) \to +\infty \; as \; r \to +\infty.$$

For an effective divisor D on X, we define a proximity function of r, for any holomorphic map $f : \mathbb{C} \to X$ with $f(\mathbb{C}) \not\subset SuppD$,

$$m_f(r, D) := \int_0^{2\pi} \lambda_D(f(re^{i\theta}))\frac{d\theta}{2\pi}.$$

Analogously, for a closed subscheme Y on X, we define a proximity function of r, for a holomorphic curve $f : \mathbb{C} \to X$ with $f(\mathbb{C}) \not\subset SuppY$,

$$m_f(r, Y) := \int_0^{2\pi} \lambda_Y(f(re^{i\theta}))\frac{d\theta}{2\pi}.$$

3. Some lemmas for divisors and the proof of Theorem 1.7

The following is a reformulation of [10, Lemma 3.1] by taking the logarithm of both sides of the inequality.

Lemma 3.1. [10, Lemma 3.1] *Let $t_0, t_1, \ldots, t_n$ be $n+1$ integers such that $1 = t_0 < t_1 < \ldots < t_n$, and let $\Delta = \max_{1\le s\le n} \frac{t_s - t_0}{s}$. Then for arbitrary real numbers $a_0, a_1, \ldots, a_{n-1}$ with $a_0 \ge a_1 \ge \ldots \ge a_{n-1} \ge 1$, we have*

$$\sum_{i=0}^{n-1}(t_{i+1} - t_i)\log a_i \le \Delta \sum_{i=0}^{n-1} \log a_i.$$

Next we need to introduce the definition of $(t_1, t_2, \ldots, t_n)$-subgeneral position for Cartier divisors, which was originally given by Quang[10] for hypersurfaces.

Definition 3.2. Let X be a projective variety of dimension n defined over a number field k and let $D_1, \ldots, D_q$ be q Cartier divisors of X. We say that $D_1, \ldots, D_q$ are in $(t_1, t_2, \ldots, t_n)$-subgeneral position with respect to X if for every $1 \le s \le n$ and $t_s + 1$ Cartier divisors $D_{j_0}, \ldots, D_{j_{t_s}}$, we have

$$\dim \cap_{i=0}^{t_s} SuppD_{j_i} \cap X \le n - s - 1,$$

where $t_0, t_1, \ldots, t_n$ are integers with $0 = t_0 < t_1 < \ldots < t_n$.

Remark 3.3. (i) If $D_1, \ldots, D_q$ are in $(t_1, \ldots, t_n)$-subgeneral position with respect to X, then their distributive constant in X satisfies

$$\Delta = \max_{1\le k\le n} \frac{t_k}{n-(n-k)} = \max_{1\le k\le n} \frac{t_k}{k}.$$

(ii) If $D_1, \ldots, D_q$ are in ℓ-subgeneral position with index κ with respect to X, then one has

$$\dim\left(\cap_{j=1}^{k} D_{i_j}\right) \le n - \kappa - (k - (\ell - n + \kappa - 1)) = \ell - k - 1.$$

Hence, $D_1, \ldots, D_q$ are in $(1, 2, \ldots, \kappa-1, \ell-n+\kappa, \ell-n+\kappa+1, \ldots, \ell-1, \ell)$-subgeneral position with respect to X and thus

$$\begin{aligned}\Delta &\leq \max\left\{\frac{1}{n-(n-1)}, \frac{2}{n-(n-2)}, \ldots, \frac{\kappa-1}{n-(n-\kappa+1)}, \ldots, \right.\\ &\qquad \left. \frac{(\ell-n)+\kappa}{n-(n-\kappa)}, \frac{(\ell-n)+\kappa+1}{n-(n-\kappa+1)}, \ldots, \frac{\ell}{n}\right\}\\ &= \max\left\{1, \frac{\ell-n}{\kappa}+1\right\}\\ &= \frac{\ell-n+\kappa}{\kappa}.\end{aligned}$$

The following lemma is just a special case of [10, Lemma 3.2].

Lemma 3.4. [10, Lemma 3.2] *Let k be a number field. Let $X \subset \mathbb{P}_k^M$ be a projective variety of dimension n. Let $H_0, \ldots, H_l$ be hyperplanes in X which are in $\{t_1, t_2, \ldots, t_n\}$-subgeneral position on X where $t_0, t_1, \ldots, t_n$ are integers with $0 = t_0 < t_1 < \ldots < t_n = l$. Let $L_1, \ldots, L_q$ be the normalized linear forms defining $H_1, \ldots, H_q$, respectively. Then there exist linear forms $L'_1, \ldots, L'_{n+1}$ of $M+1$ variables such that,*

(i) $L'_0 = L_0$;

(ii) For every $s \in \{1, \ldots, n\}$, $L'_s \in span_k(L_0, \ldots, L_{t_s})$;

(iii) Let $H'_j,\ j = 0, \ldots, n$, be the hyperplanes defined by $L'_j,\ j = 0, \ldots, n$. Then they are in general position on X.

We need also the following lemma [14, Theorem 1.22], which plays an important role in the arguments regarding dimension.

Lemma 3.5. [14, 3] *Let X be a projective variety over an algebraic closed field k, and A be an ample Cartier divisor on X. Let $F \subset X$ be a proper irreducible subvariety. Then either $F \subset A$ or* $\dim(F \cap A) \leq \dim F - 1$.

For the proof of Theorem 1.5, we need to prove the second main theorem for Cartier divisors with distributive constant. The basic idea is to consider the distributive constant for Cartier divisors by making use of methods of Heier-Levin[4] and He-Ru[3].

Proof of Theorem 1.7. Fix a real number $\epsilon > 0$, choose a rational number $\delta > 0$ such that

$$\delta\Delta + \delta\Delta(n+1+\delta) < \epsilon,$$

and for a sufficiently small positive rational number δ' depending on δ, $\delta A - \delta' D_i$ is $\mathbb{Q}$-ample for all $i = 1, \ldots, q$. By the definition of the Seshadri constant, there exists a rational number $\epsilon_i > 0$ such that

$$\epsilon_{D_i}(A) - \delta' \leq \epsilon_i \leq \epsilon_{D_i}(A)$$

and that $A - \epsilon_i D_i$ is $\mathbb{Q}$-nef for all $i = 1, \dots, q$. Then we have

$$(1+\delta)A - (\epsilon_i + \delta')D_i = (A - \epsilon_i D_i) + (\delta A - \delta' D_i)$$

is a $\mathbb{Q}$-ample divisor for all i. Let N be a large enough natural number such that $N(1+\delta)A$ and $N[(1+\delta)A - (\epsilon_i + \delta')D_i]$ become very ample integral divisors for all i.

We claim that if $\{D_1, \dots, D_q\}$ is in $(t_1, t_2, \dots, t_n)$-subgeneral position with respect to X, then we can construct divisors $div(s_i)$ on X, $s_i \in H^0(X, N(1+\delta)A - N(\epsilon_i + \delta')D_i)$, $i = 1, \dots, q$, such that,

(i) $div(s_i) \sim N(1+\delta)A, i = 1, \dots, q$.

(ii) The divisors $div(s_1), \dots, div(s_q)$ are in $\{t_1, t_2, \dots, t_n\}$-subgeneral position on X.

We define $div(s_1), \dots, div(s_q)$ by induction as follows. Assume that, for some $j \in \{1, \dots, q\}$, $div(s_1)$, $\dots$, $div(s_{j-1})$ with desired property have been defined and $div(s_1), \dots, div(s_{j-1}), D_j, \dots, D_q$ are in $\{t_1, t_2, \dots, t_n\}$-subgeneral position on X (for $j = 1$, this reduces to the hypothesis that $D_1, \dots, D_q$ are in $\{t_1, t_2, \dots, t_n\}$-subgeneral position).

By the definition of $\{t_1, t_2, \dots, t_n\}$-subgeneral position,

(a) For all $1 \le s \le n, I' \subset \{1, \dots, j-1\}, J' \subset \{j+1, \dots, q\}$, and $\sharp I' + \sharp J' = t_s$, we have

$$\dim(\cap_{i\in I'} div(s_i) \cap (\cap_{k\in J'} D_k) \cap X) \le n - s,$$

and

$$\dim(D_j \cap (\cap_{i\in I'} div(s_i)) \cap (\cap_{k\in J'} D_k) \cap X) \le n - s - 1.$$

We can find a non-zero section $s_j \in H^0(X, N(1+\delta)A - N(\epsilon_j + \delta')D_j)$ such that s_j does not vanish entirely on any irreducible components of $\cap_{i\in I'} div(s_i) \cap (\cap_{k\in J'} D_k)$, for all $1 \le s \le n, I' \subset \{1, \dots, j-1\}, J \subset \{j+1, \dots, q\}$, and $\sharp I' + \sharp J' = t_s$. Then by Lemma 3.5

$$\begin{aligned} & \dim\{div(s_j) \cap (\cap_{i\in I'} div(s_i)) \cap (\cap_{k\in J'} D_k) \cap X\} \\ \le\ & \max\left(\dim D_j \cap (\cap_{i\in I'} div(s_i)) \cap (\cap_{k\in J'} D_k) \cap X, n - s - 1\right) \\ =\ & n - s - 1. \end{aligned}$$

(b) For all $1 \le s \le n, I \subset \{1, \dots, j-1\}, J \subset \{j+1, \dots, q\}$, and $\sharp I + \sharp J = t_s + 1$,

$$\dim(\cap_{i\in I} div(s_i) \cap (\cap_{k\in J} D_k) \cap X) \le n - s - 1.$$

From this, we get that $div(s_1), \dots, div(s_{j-1}), div(s_j), D_{j+1}, \dots, D_q$ are in $\{t_1, \dots, t_n\}$-subgeneral position on X. Thus, the divisors $div(s_1), \dots, div(s_q)$ satisfy the above required properties (i) and (ii). Hence, the claim is proved.

Denote $F_i = div(s_i)$ for $i = 1, \ldots, q$. It's easy to see that the distributive constant of $\{F_1, \ldots, F_q\}$ is equal to the distributive constant of $\{D_1, \ldots, D_q\}$. Now we're going to prove the theorem.

It is sufficient for us to consider the case where $\Delta < \frac{q}{n+1}$. Note that $\Delta \geq 1$, and hence $q > n+1$. If there exits $i \in \{1, \ldots, q\}$ such that $\cap_{j=1, j\neq i}^{q} F_j^* \cap X \neq \emptyset$, then

$$\Delta \geq \frac{q-1}{n} \geq \frac{q}{n+1}.$$

This is a contradiction. Therefore, $\cap_{j=1,j\neq i}^{q} F_j^* \cap X = \emptyset$ for all $i \in \{1, \ldots, q\}$.

We denote by $\mathcal{I}$ the set of all permutations of the set $\{1, \ldots, q\}$. Denote by n_0 the cardinality of $\mathcal{I}$, $n_0 = q!$ and we write $\mathcal{I} = \{I_1, \ldots, I_{n_0}\}$, where $I_i = \{I_i(0), \ldots, I_i(q-1)\} \in \mathbb{N}^q$ and $I_1 < I_2 < \ldots < I_{n_0}$ in the lexicographic order.

For each $I_i \in \mathcal{I}$, since $\cap_{j=1}^{q-1} F_{I_i(j)} \cap X = \emptyset$, there exists $n+1$ integers $t_{i,0}, t_{i,1}, \ldots, t_{i,n}$ with $0 = t_{i,0} < \ldots < t_{i,n} = l_i$, where $l_i \leq q-2$ such that $\cap_{j=0}^{l_i} F_{I_i(j)} \cap X = \emptyset$ and

$$\dim(\cap_{j=0}^{s} F_{I_i(j)}) \cap X = n - u \ \forall\ t_{i,u-1} \leq s < t_{i,u},\ 1 \leq u \leq n.$$

Then, $\Delta > \frac{t_{i,u} - t_{i,0}}{u}$ for all $1 \leq u \leq n$.

It means that $\{F_1, \ldots, F_q\}$ is in $(t_{i,1}, t_{i,2}, \ldots, t_{i,n})$-subgeneral position with respect to X. Denote by $\phi : X \to \mathbb{P}^{\tilde{N}}(k)$ the canonical embedding associated to the very ample divisor $N(1+\delta)A$ and let $H_0, \ldots, H_{q-1}$ be the hyperplanes in $\mathbb{P}^{\tilde{N}}(k)$ with $F_j = \phi^* H_{j-1}$ for $j = 1, \ldots, q$. We denote $L_0, \ldots, L_{q-1}$ to be the linear forms defining $H_0, \ldots, H_{q-1}$ respectively. By Lemma 3.4, there exist hyperplanes $\hat{H}_0, \ldots, \hat{H}_n$ with defining linear forms $\hat{L}_0, \ldots, \hat{L}_n$, such that $\hat{L}_0 = L_0$, and for every $s \in \{1, \ldots, n\}$, $\hat{L}_s \in span_k(L_0, \ldots, L_{t_s})$ and $\phi^* \hat{H}_0, \ldots, \phi^* \hat{H}_n$ are located in general position on X. Applying Theorem 1.1 to $\phi^* \hat{H}_0, \ldots, \phi^* \hat{H}_n$, we conclude that there exists a Zariski-closed set Z such that for all $x \in X(k) \backslash Z$,

$$\int_0^{2\pi} \sum_{i=0}^{n} \lambda_{\phi^* \hat{H}_i}(x) \frac{d\theta}{2\pi} \leq [(n+1) + \delta] T_{f, N(1+\delta)A}(r).$$

Consider a point $P = \phi(x) \in \mathbb{P}^{\tilde{N}}(k)$, fix a element $I_i \in \mathcal{I}$, we arrange such that

$$||L_{I_i(0)}(P)|| \leq ||L_{I_i(1)}(P)|| \leq \ldots \leq ||L_{I_i(q-1)}||,$$

which implies

$$\sum_{j=I_i(0)}^{I_i(q-1)} \lambda_{H_j(P)} \leq \sum_{j=t_{i.0}}^{t_{i,n}} \lambda_{H_j(P)} + O(1). \tag{3.1}$$

(The proof of (3.1) is similar to that of Lemma 20.7 in [17] which is standard and is omitted here.) Thus using the construction of $\hat{L}_0, \ldots, \hat{L}_n$, we have

$$||\hat{L}_u(P)|| \leq B \max_{0 \leq j \leq t_{i,u}} ||L_j(P)|| = B||L_{t_{i,u}}(P)||$$

for all $u = 0, \ldots, n$ and some constant $B > 0$. Thus, by the definition of Weil function,

$$\lambda_{\hat{H}_u}(P) \geq \lambda_{H_{t_{i,u}}}(P) + O(1).$$

Therefore, by Lemma 3.1 and (3.1) we have

$$\begin{aligned}
(3.2) \qquad & \sum_{i=0}^{q-1} \lambda_{H_j}(P) \\
= \ & \sum_{j=t_{i,0}}^{I_i(q-1)} \lambda_{H_j}(P) \\
\leq \ & \sum_{j=t_{i,0}}^{t_{i,1}} \lambda_{H_j}(P) + \sum_{j=t_{i,1}+1}^{t_{i,2}} \lambda_{H_j}(P) + \cdots + \sum_{j=t_{i,n-1}+1}^{t_{i,n}} \lambda_{H_j}(P) + O(1) \\
\leq \ & (t_{i,1} - t_{i,0})\lambda_{H_{t_{i,0}}}(P) + (t_{i,2} - t_{i,1})\lambda_{H_{t_{i,1}}}(P) + \cdots \\
& + (t_{i,n} - t_{i,n-1})\lambda_{H_{t_{i,n-1}}}(P) + \lambda_{H_{t_{i,n}}}(P) + O(1) \\
\leq \ & \Delta \sum_{j=0}^{n-1} \lambda_{H_{t_{i,j}}}(P) + \lambda_{H_{t_{i,n}}}(P) + O(1) \\
\leq \ & \Delta \sum_{j=0}^{n} \lambda_{H_{t_{i,j}}}(P) + O(1) \\
\leq \ & \Delta \sum_{u=0}^{n} \lambda_{\hat{H}_u}(P) + O(1).
\end{aligned}$$

By the functoriality of Weil function, we have $\lambda_{H_j}(P) = \lambda_{H_j}(\phi(x)) = \lambda_{\phi^* H_j}(x) = \lambda_{F_{j+1}}(x)$, thus,

$$\begin{aligned}
\int_0^{2\pi} \sum_{i=1}^{q} \lambda_{F_i}(x) \frac{d\theta}{2\pi} &= \int_0^{2\pi} \sum_{j=0}^{q-1} \lambda_{H_j}(P) \frac{d\theta}{2\pi} \\
&\leq \Delta \int_0^{2\pi} \sum_{i=0}^{n} \lambda_{\hat{H}_i}(P) \frac{d\theta}{2\pi} + O(1) \\
&\leq \Delta[(n+1) + \delta] T_{f, N(1+\delta)A}(r) + O(1).
\end{aligned}$$

From the construction of divisors $F_i = div(s_i)$ for $i = 1, \ldots, q$, we know that $F_i - N(\epsilon_i + \delta')D_i$ is effective for each $i = 1, \ldots, q$. By the definition of the Seshadri constant and the Boundedness from below of a Weil function,

$$\lambda_{F_i}(x) \geq N(\epsilon_i + \delta')\lambda_{D_i} + O(1) \geq N\epsilon_{D_i}(A)\lambda_{D_i}(x) + O(1).$$

Then,

$$\begin{aligned} N\int_0^{2\pi}\sum_{i=1}^{q}\epsilon_{D_i}(A)\lambda_{D_i}(x)\frac{d\theta}{2\pi} &\leq \int_0^{2\pi}\sum_{i=1}^{q}\lambda_{F_i}(x)\frac{d\theta}{2\pi} \\ &\leq \Delta[(n+1)+\delta]T_{f,N(1+\delta)A}(r)+O(1). \end{aligned}$$

Note that $T_{f,N(1+\delta)A}(r) = N(1+\delta)T_{f,A}(r)$, hence we get the inequality as follow,

$$\int_0^{2\pi}\sum_{i=1}^{q}\epsilon_{D_i}(A)\lambda_{D_i}(x)\frac{d\theta}{2\pi} \leq \Delta[(n+1)+\delta](1+\delta)T_{f,A}(r).$$

It implies that

$$\sum_{i=1}^{q}\epsilon_{D_i}(A)m_f(r,D_i) \leq \Delta[(n+1)+\delta](1+\delta)T_{f,A}(r).$$

Recall the choice of δ, we get for every $\epsilon > 0$,

$$\sum_{j=1}^{q}\epsilon_{D_j}(A)m_f(r,D_j) \leq_{exc} (\Delta(n+1)+\epsilon))T_{f,A}(r).$$

□

4. Proof of theorem 1.5

Before the proof of Theorem 1.5, we need a key lemma.

Lemma 4.1. [6, Lemma 5.4.24] *Let X be a projective variety, $\mathcal{I}$ be a coherent ideal sheaf. Let $\pi : \tilde{X} \to X$ be the blowing-up of $\mathcal{I}$ with exceptional divisor E. Then there exists an integer $p_0 = p_0(\mathcal{I})$ with the property that if $p \geq p_0$, then $\pi_*\mathcal{O}_{\tilde{X}}(-pE) = \mathcal{I}^p$, and moreover, for any divisor D on X,*

$$H^i(X, \mathcal{I}^p(D)) = H^i(\tilde{X}, \mathcal{O}_{\tilde{X}}(\pi^*D - pE))$$

for all $i \geq 0$.

We now prove the second main theorem for closed subschemes in ℓ-subgeneral position with index κ.

Proof of Theorem 1.5. The proof basically follows Heier-Levin[4] and He-Ru[3]. Denote by $\mathcal{I}_i$ the ideal sheaf of Y_i, $\pi_i : \tilde{X} \to X$ the blowing-up of X along Y_i, and E_i the exceptional divisor on $\tilde{X}_i$. Then, fix a real number $\epsilon > 0$, choose a rational number $\delta > 0$ such that

$$\frac{\delta(\ell-n+\kappa)}{\kappa} + \frac{\delta(\ell-n+\kappa)}{\kappa}(n+1+\delta) < \epsilon,$$

and for a sufficiently small positive rational number δ' depending on δ, $\delta A - \delta' E_i$ is $\mathbb{Q}$-ample for all $i = 1, \dots, q$. By the definition of the Seshadri constant, there exists a rational number $\epsilon_i > 0$ such that

$$\epsilon_{E_i}(A) - \delta' \le \epsilon_i \le \epsilon_{E_i}(A),$$

and that $A - \epsilon_i E_i$ is $\mathbb{Q}$-nef for all $i = 1, \dots, q$. Then we have

$$(1+\delta)A - (\epsilon_i + \delta')E_i = (A - \epsilon_i E_i) + (\delta A - \delta' E_i)$$

is a $\mathbb{Q}$-ample divisor for all i. Let N be a large enough natural number such that $N(1+\delta)A$ and $N[(1+\delta)A - (\epsilon_i + \delta')E_i]$ become very ample integral divisors for all i.

We claim that if $\{Y_1, \dots, Y_q\}$ is in ℓ-subgeneral position with index κ with respect to X, then we can construct divisors F_i on X, $i = 1, \dots, q$, such that,

(i) $F_i \sim N(1+\delta)A, i = 1, \dots, q$.

(ii) $\pi^* F_i \ge N(\epsilon_i + \delta')E_i, i = 1, \dots, q$.

(iii) The divisors $F_1, \dots, F_q$ are in ℓ-subgeneral position with index κ on X.

Like the special divisor case in the proof of preparation theorem, we can construct by induction. Assume that, for some $j \in \{1, \dots, q\}$, $div(s_1)$, $\dots$, $div(s_{j-1})$ with desired property have been defined and F_1, $\dots$, F_{j-1}, Y_j, $\dots$, Y_q ($F_i = div(s_i)$) are in ℓ-subgeneral position with index κ on X (for $j = 1$, this reduces to the hypothesis that $Y_1, \dots, Y_q$ are in ℓ-subgeneral position with index κ). To find F_j, we let $\tilde{F}_i^{(j)} = \pi_j^* F_i, i = 1, \dots, j-1$ and $\tilde{Y}_i^j = \pi_j^* Y_i$ for $i = j+1, \dots, q$. Since in particular, $F_1, \dots, F_{j-1}, Y_{j+1}, \dots, Y_q$ are in ℓ-subgeneral position with index κ on X, and by noticing that π_j^{-1} is an isomorphism outside of Y_j, we know that $\tilde{F}_1, \dots, \tilde{F}_{j-1}, \tilde{Y}_{j+1}, \dots, \tilde{Y}_q$ are in ℓ-subgeneral position with index κ on $\tilde{X}_j$ outside of E_j. Thus it reduces to the construction in the divisor case, and by the argument in the divisor case, there are sections

$$\tilde{s}_j \in H^0(\tilde{X}_j, \mathcal{O}_{\tilde{X}_j}(N(1+\delta)\pi_j^* A - N(\epsilon_i + \delta')E_j)),$$

such that $\tilde{F}_1, \dots, \tilde{F}_{j-1}, div(\tilde{s}_j), \tilde{Y}_{j+1}, \dots, \tilde{Y}_q$ are in ℓ-subgeneral position with index κ on $\tilde{X}_j$ outside of E_j, where we regard $H^0(\tilde{X}_j, \mathcal{O}_{\tilde{X}_j}(N((1+\delta)\pi_j^* A - (\epsilon_j + \delta')E_j)))$ as a subspace of $H^0(\tilde{X}_j, \mathcal{O}_{\tilde{X}_j}(N(1+\delta)\pi_j^* A))$.

To guarantee that F_j has the required properties, by Lemma 4.1, we have, for a big enough N,

$$H^0(X, \mathcal{O}_X(N(1+\delta)A) \otimes \mathcal{I}_j^{N(\epsilon_i+\delta')}) = H^0(\tilde{X}_j, \mathcal{O}_{\tilde{X}_j}(N((1+\delta)\pi_j^* A - (\epsilon_j + \delta')E_j))).$$

Therefore there is an effective divisor $F_j \sim N(1+\delta)A$ on X such that $div(\tilde{s}_j) = \pi_j^* F_j$. Since $\tilde{s}_j \in H^0(\tilde{X}_j, \mathcal{O}_{\tilde{X}_j}(N(1+\delta)\pi_j^*A - N(\epsilon_i + \delta')E_j))$, we have $\pi^* F_j \geq N(\epsilon_i + \delta')E_j$ on $\tilde{X}_j$. $\tilde{F}_1, \ldots, \tilde{F}_{j-1}, div(\tilde{s}_j), \tilde{Y}_{j+1}, \ldots, \tilde{Y}_q$ are in ℓ-subgeneral position with index κ on $\tilde{X}_j$ outside of E_j, and π_i is an isomorphism above the complement of Y_j, thus $F_1, \ldots, F_{j-1}, F_j, Y_{j+1}, \ldots, Y_q$ are in ℓ-subgeneral position with index κ on X_j outside of Y_j. Since Y_j is in ℓ-subgeneral position with index κ with $F_1, \ldots, F_{j-1}, Y_{j+1}, \ldots, Y_q$, it implies that $F_1, \ldots, F_{j-1}, F_j, Y_{j+1}, \ldots, Y_q$ are in ℓ-subgeneral position with index κ on X. Thus, we obtain divisors $F_1, \ldots, F_q$ with the required properties. Hence, the claim is proved.

Since $\Delta \leq \frac{\ell - n + \kappa}{\kappa}$, as a special case of the result in the proof of Theorem 1.7 in the above section, we can also get

$$\int_0^{2\pi} \sum_{i=1}^{q} \lambda_{F_i}(x) \frac{d\theta}{2\pi} \leq \left[\frac{\ell - n + \kappa}{\kappa}(n+1+\delta) \right] T_{f,N(1+\delta)A}(r) + O(1) \tag{4.1}$$

on $X(k) \setminus Z$ where Z is a proper Zariski-closed subset of X. By functoriality, additivity, and the fact that Weil functions of divisors are bounded from below, for all $P \in \tilde{X}_i \setminus SuppE_i$,

$$\begin{aligned} \lambda_{F_i}(\pi_i(P)) &= \lambda_{\pi_i^* F_i}(P) + O(1) \\ &\geq N(\epsilon_i + \delta')\lambda_{E_i}(\pi_i(P)) + O(1) \\ &= N(\epsilon_i + \delta')\lambda_{Y_i}(\pi_i(P)) + O(1) \\ &\geq N(\epsilon_{Y_i})\lambda_{Y_i}(\pi_i(P)) + O(1). \end{aligned}$$

Together with (4.1),

$$\int_0^{2\pi} \sum_{i=1}^{q} N\epsilon_{Y_i}\lambda_{Y_i}(x) \frac{d\theta}{2\pi} \leq \left[\frac{\ell - n + \kappa}{\kappa}(n+1+\delta) \right] T_{f,N(1+\delta)A}(r) + O(1).$$

Then, by the choice of ϵ, we have

$$\sum_{j=1}^{q} \epsilon_{Y_j}(A) m_f(r, Y_j) \leq_{exc} \left(\frac{(\ell - n + \kappa)(n+1)}{\kappa} + \epsilon \right) T_{f,A}(r).$$

□

5. Schmidt's subspace theorem

In this section, we give the counterpart in Diophantine approximation of our main results. The standard notations in Schmidt's subspace Theorem can be seen in [4],[3],[16],[17]).

Let k be a number field. Denote by M_k the set of places of k and by k_v the completion of k for each $v \in M_k$. Norms $|| \cdot ||_v$ on k are normalized so that

$$||x||_v = |\sigma(x)|^{[k_v:\mathbb{R}]} \quad or \quad ||p||_v = p^{-[k_v:\mathbb{Q}_p]}$$

if $v \in M_k$ is an Archimedean place corresponding to an embedding $\sigma : k \to \mathbb{C}$ or a non-Archimedean place lying above the rational prime p, respectively.

An M_k-constant is a collection $(c_v)_{v\in M_k}$ of real constants such that $c_v = 0$ for all but finitely many v. Hights are logarithmic and relative to the number field used as a base field which is always denoted by k. For $\mathbf{x} = (x_0, \ldots, x_n) \in k^{n+1}$, define

$$||\mathbf{x}||_v := \max\{||x_0||_v, \ldots, ||x_n||_v\}, \quad v \in M_k.$$

The absolute logarithmic height of a point $\mathbf{x} = [x_0 : \cdots : x_n] \in \mathbb{P}^n(k)$ is defined by

$$h(\mathbf{x}) := \sum_{v\in M_k} \log ||\mathbf{x}||_v.$$

For each $v \in M_k$, we can associate the local Weil functions $\lambda_{Y,v}$ which have similar properties as the Weil function introduced in Section 2.

Similar discussions as in Nevanlinna theory, one can easily obtain the counterparts of Theorem 1.7, Theorem 1.5 for Schmidt's subspace theorems in Diophantine approximation. Thus we omit the details.

Theorem 5.1. *Let X be a projective variety of dimension n defined over a number field k. Let S be a finite set of places of k. For each $v \in S$, let $D_{1,v}, \ldots, D_{q,v}$ be Cartier divisors of X, defined over k, and with the distributive constant Δ. Let A be an ample Cartier divisor on X. Then, for $\epsilon > 0$, there exists a Zariski-closed set $Z \subset X$ such that for all points $x \in X(k) \setminus Z$,*

$$\sum_{v\in S} \sum_{j=1}^{q} \epsilon_{D_{j,v}}(A)\lambda_{D_{j,v},v} < (\Delta(n+1) + \epsilon))h_A(x).$$

Theorem 5.1 generalizes Theorem 2.6 in [3].

Theorem 5.2. *Let X be a projective variety of dimension n defined over a number field k. Let S be a finite set of places of k. For each $v \in S$, let $Y_{1,v}, \ldots, Y_{q,v}$ be closed subschemes of X, defined over k, and in ℓ-subgeneral position with index κ in X, and $\ell \geq n$. Let A be an ample Cartier divisor on X. Then, for $\epsilon > 0$, there exists a Zariski-closed set $Z \subset X$ such that for all points $x \in X(k) \setminus Z$,*

$$\sum_{v\in S} \sum_{j=1}^{q} \epsilon_{Y_{j,v}}(A)\lambda_{Y_{j,v},v} < \left(\frac{(\ell - n + \kappa)(n+1)}{\kappa} + \epsilon\right) h_A(x).$$

Theorem 5.2 gives a generalization of Theorem 1.3 in [4] and the Main Theorem (Arithmetic Part) in [3].

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