Golod-Shafarevich-Vinberg type theorems and finiteness conditions for potential algebras

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Abstract

We obtain a lower estimate for the Hilbert series of Jacobi algebras and their completions by providing analogue of the Golog-Shafarevich-Vinberg theorem for potential case. We especially treat non-homogeneous situation. This estimate allows to answer number of questions arising in the work of Wemyss-Donovan-Brown on noncommutative singularities and deformation theory. In particular, we prove that the only case when a potential algebra or its completion could be finite dimensional or of linear growth, is the case of two variables and potential having terms of degree three.

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1 Introduction

The Golod-Shafarevich theorem provides a lower estimate for the Hilbert series of an algebra given by generators and relations in terms of degrees of generators and defining relations [3]. Vinberg generalised the theorem for the case when relations are not homogeneous [7].

We work here in the situation when relations are not arbitrary, but obtained as (non-commutative) derivatives of one polynomial, called the potential. Such algebras are called potential or Jacobi and they frequently appear in various contexts, for example, in physics. This additional constrain allows to improve the estimate, and as a consequence, obtain important results on possible dimensions, on conditions of finite-dimensionality, other finiteness conditions, on conditions of linearity of the growth of potential algebras, which are needed in the study of contraction algebras [2, 1]. Contraction algebras serve as a noncommutative invariants of curve contractions and appear to be, as shown by Van den Bergh, potential algebras. Thus, it is important to know conditions when potential algebras are finite dimensional or of linear growth. We provide an answer to these questions in the theorem below.

The methods we develop and apply in this section work for many varieties of twisted potential algebras as well. We restrict ourselves to potential algebras for the sake of clarity.

Denote by F_k the kth graded component of the potential F, and by $A^{(n)} = \mathbb{K}\langle X \rangle/(I+J^n)$, where $A = \mathbb{K}\langle X \rangle/I$ and J^n is an ideal generated by all monomials of degree strictly bigger than n.

We also denote here by $P_A = \sum \dim A^{(n)} t^n$ the generating function of dimensions of truncated algebras $A^{(n)}$, in stead of usual Poincare series of dimensions of components of the filtration.

We obtain a lower estimate for the Hilbert series of A by estimating this series P_A , which is obviously componentwise smaller than the Hilbert series of A:

$$H_A \geqslant P_A \geqslant (1-t)^{-1}(1-nt+nt^{k-1}-t^k)^{-1}$$
.

Note that the series P_A does coincide with the Hilbert series $H_{\bar{A}}$ of the completion \bar{A} of algebra A.

Here we give two equivalent definitions of the completion of an ideal and of an algebra (??).

Definition. Consider the decreasing sequence of ideals $I^{(n)} = I + J^n$, where J^n as above is an ideal generated by monomials of degree strictly bigger than n:

$$I^n \supset I^{(n+1)} \supset \dots$$

Call $\bar{I} = \cap I^{(n)}$ a completion of the ideal I, and corresponding algebra $\bar{A} = \mathbb{K}\langle X \rangle / \bar{I}$, the completion of an algebra A.

In spite of what the term of completion of an algebra should naively suggest, we have $A \supset \bar{A}$, however for an ideal the situation is not counterintuitive: $\bar{I} \supset I$.

It is easy to see that we get the same algebra, if we take quotient of the formal power series $\mathbb{K}\langle\langle X\rangle\rangle$ by the ideal generated by I in $\mathbb{K}\langle\langle X\rangle\rangle$.

Definition. $\bar{A} = \mathbb{K}\langle\langle X \rangle\rangle/id_{\mathbb{K}\langle\langle X \rangle\rangle}I$.

One could find these definitions for example in [2].

Theorem 1.1. Let $F \in \mathbb{K}^{\text{cyc}}\langle x_1, \ldots, x_n \rangle$ be such that $F_0 = \ldots = F_{k-1} = 0$, and let $n, k \in \mathbb{N}$ be such that $n \geq 2$, $k \geq 3$ and $(n, k) \neq (2, 3)$. Then \bar{A}_F and thereof A_F are infinite dimensional.

Furthermore, \bar{A}_F and A_F have at least cubic growth if (n,k) = (2,4) or (n,k) = (3,3) with cubic growth being possible in both cases, and they have exponential growth otherwise.

For $n, k, m \in \mathbb{N}$ such that $n \ge 2$ and $m \ge k \ge 3$, denote

$$\mathcal{P}_{n,k}^{(m)} = \{ F \in \mathbb{K}^{\text{cyc}} \langle x_1, \dots, x_n \rangle : F_j = 0 \text{ for } j < k \text{ and for } j > m \}.$$

Clearly, $\mathcal{P}_{n,k}^{(m)}$ is a vector space and $\mathcal{P}_{n,k}^{(k)} = \mathcal{P}_{n,k}$. Recall that for $j \in \mathbb{Z}_+$ and $F \in \mathcal{P}_{n,k}^{(m)}$, $A_F^{(j)}$ is the quotient of A_F by the ideal generated by the monomials of degree j+1.

Then the main lemma about zero divisors state.

Lemma 1.2. Let $n, k \in \mathbb{N}$, $n \ge 2$, $m \ge k \ge 3$ and $(n, k) \ne (2, 3)$, then for generic $F \in \mathcal{P}_{n,k}^{(m)}$, $x_1 a \ne 0$ in \bar{A}_F for every non-zero $a \in \bar{A}_F$.

Using this lemma and analogue of Golod-Shafarevich-Vinberg theorem for potential algebras we give a lower estimate, which implies the statements of the main theorem above.

Lemma 1.3. Let $n, k \in \mathbb{N}$, $n \ge 2$, $m \ge k \ge 3$ and $(n, k) \ne (2, 3)$, $F \in \mathcal{P}_{n, k}^{(m)}$ and $A = A_F$. Then $P_A \ge (1 - t)^{-1}(1 - nt + nt^{k-1} - t^k)^{-1}$.

Proof given here is a little more detailed version of the argument appeared in [4].

2 Preliminary homogeneous results

We will need two examples. We say that potential algebra is *exact* if associated potential complex is exact [5].

Example 2.1. Let n and k be integers such that $k \ge n \ge 2$, $k \ge 3$ and $(n, k) \ne (2, 3)$. Consider the potential $F \in \mathcal{P}_{n,k}$ given by

$$F = \sum_{\sigma \in S_{n-1}} x_n^{k-n+1} x_{\sigma(1)} \dots x_{\sigma(n-1)}^{\circlearrowleft},$$

where the sum is taken over all bijections from the set $\{1, ..., n-1\}$ to itself. Then the potential algebra $\bar{A}_F = A_F$ is exact. Furthermore, $x_1u \neq 0$ for every non-zero $u \in A$.

Example 2.2. Let n and k be integers such that $n > k \geqslant 3$. Order the generators by $x_n > x_{n-1} > \ldots > x_1$ and consider the left-to-right degree-lexicographical ordering on the monomials. Consider the set M of degree k-2 monomials in x_1, \ldots, x_{n-1} in which each letter x_j features at most once. Let m_1, \ldots, m_{n-1} be the top n-1 monomials in M enumerated in such a way that $m_{n-k+1} = x_{n-1} \ldots x_{n-k+2}$ (the biggest one). Now define the potential $F \in \mathcal{P}_{n,k}$ by

$$F = x_n x_{n-1} \dots x_{n-k+1}^{\circlearrowleft} + \sum_{\substack{1 \le j \le n-1 \\ j \ne n-k+1}} x_j x_n m_j^{\circlearrowleft}.$$

Then the potential algebra $A = A_F$ is exact. Furthermore, $x_1u \neq 0$ for every non-zero $u \in A$.

Lemma 2.3. Let \mathbb{K} be uncountable field, $n, k \in \mathbb{N}$, $n \geq 2$, $k \geq 3$ and $(n, k) \neq (2, 3)$. Then for a generic $F \in \mathcal{P}_{n,k}$, $x_1 a \neq 0$ in A_F for every non-zero $a \in A_F$.

Proof. Let F_0 be the potential provided by the appropriate (depending on whether $k \ge n$ or k < n) Example 2.1 or Example 2.2. Then $x_1 a \ne 0$ in A_{F_0} for every non-zero $a \in A_{F_0}$ and $H_{A_{F_0}} = (1 - nt + nt^{k-1} - t^k)^{-1}$. As was noticed by Ufnarovskij, the generic Hilbert series is minimal, hence $H_{A_F} = (1 - nt + nt^{k-1} - t^k)^{-1}$ for generic $F \in \mathcal{P}_{n,k}$. Applying Lemma 3.9 from [5] to the map $a \mapsto x_1 a$ from A_F to A_F , we now see that $\dim x_1(A_F)_j \ge \dim x_1(A_{F_0})_j$ for all j for generic $F \in \mathcal{P}_{n,k}$. Since $\dim x_1(A_{F_0})_j = \dim (A_{F_0})_j = \dim (A_F)_j$ for generic F, the map $a \mapsto x_1 a$ from A_F to itself is injective for generic F.

3 Main Lemma about zero divisors

Lemma 3.1. Let $n, k \in \mathbb{N}$, $n \ge 2$, $m \ge k \ge 3$ and $(n, k) \ne (2, 3)$, then for generic $F \in \mathcal{P}_{n,k}^{(m)}$, $x_1 a \ne 0$ in \bar{A}_F for every non-zero $a \in \bar{A}_F$.

Proof. Assume the contrary. Then there exist $j \in \mathbb{Z}_+$ and $a \in \mathbb{K}\langle x_1, \ldots, x_n \rangle$ such that $a \neq 0$ in $A_F^{(j)}$ and $x_1 a = 0$ in $A_F^{(j+1)}$. The latter means that

$$x_1 a = \sum_{j \in N} u_j r_{s(j)} v_j \pmod{J^{(j+1)}},$$

where $r_j = \delta_{x_j} F$, N is a finite set, s is a map from N to $\{1, \ldots n\}$, u_j, v_j are non-zero homogeneous elements of $\mathbb{K}\langle x_1, \ldots, x_n \rangle$ such that the degree of each $u_j v_j$ does not exceed j - k + 2, and the equality $f = g \pmod{J}$ means $f - g \in J$. Let m be the lowest degree of $u_j v_j$ and $N' = \{j \in N : \deg u_j v_j = m\}$. Then the smallest degree part of the above display reads

$$x_1 a_{m+k-2} = \sum_{j \in N'} u_j \rho_{s(j)} v_j \text{ in } \mathbb{K}\langle x_1, \dots, x_n \rangle,$$

where $\rho_j = \delta_{x_j} F_k$. Note that automatically $a_q = 0$ for q < m + k - 2.

Consider two cases, depending on whether the left hand side of the lower degree term is zero or not.

Case I. If $x_1 a_{m+k-2} = 0$, then we have an equality

$$0 = \sum u_j \rho_{s(j)} v_j,$$

which means that it is a syzygy of ρ_i th. We will use the following fact.

Definition. We say that $F \in \mathbb{K}^{\text{cyc}}\langle x_1, \ldots, x_n \rangle$ is *S-trivial* if the module of syzygies of A_F presented by generators x_1, \ldots, x_n and relations r_1, \ldots, r_n with $r_j = \delta_{x_j} F$ is generated by trivial syzygies and the syzygy $\sum [x_j, r_j]$.

Proposition 3.2. Any homogeneous potential in general position is S-trivial.

Proof. First, we observe that any 'extra' syzygy will 'drop' the dimension of the corresponding component of the ideal of relations thus increasing the dimension of the component of the algebra compared to the minimal Hilbert series. This follows from the fact that the module of syzygies of relations is generated by polynomials obtained from the resolutions of ambiguities of the Gröbner basis [6].

Thus, if Hilbert series is minimal, all syzygies must be generated by one syzygy. Now we remind the fact observed, for example, by Ufnarovskij [8], that generic series is minimal. \Box

Due to the above proposition we have

$$\sum_{j \in N} u_j \rho_{s(j)} v_j = \sum_k \alpha_k \sum_i [\rho_i, x_i] \beta_k.$$

We can rewrite the initial equality

$$x_1 a = \sum_{j \in N} u_j r_{s(j)} v_j \pmod{J^{(j+1)}},$$

adding the appropriate combination of syzygies (which is zero) to the right hand side:

$$x_1 a = \sum u_j r_{s(j)} v_j - \sum \alpha_k \sum_i [r_i, x_i] \beta_k \ (mod J^{(j+1)}).$$

Now the lowest term of the right hand side become bigger. Indeed, the lowest terms cancel, since the lowest degree term of $\sum_{j\in N} u_j r_{s(j)} v_j$ coincides with the lowest degree term of $\sum_{j\in N} u_j \rho_{s(j)} v_j$, and the lowest degree term of $\sum \alpha_k [r_i, x_i] \beta_k$ coincides with the lowest degree term of $\sum \alpha_k [\rho_i, x_i] \beta_k$.

Now for newly rewritten equality we again write down the lowest terms of the left and right hand side. If for the left hand side it is again zero: $x_1 a_{m'+k-2} = 0$, we continue the process of lifting up the lowest degree on the right hand side, as described in Case I.

Case II. Otherwise we are in Case II, when the left hand side $x_1 a_{m'+k-2} \neq 0$. The lowest term is from the left hand side now. We will find a presentation, where the lower degree of the left hand side is bigger.

Due to the fact that the statement of our lemma 3.1 holds true in the case of homogeneous potential, which is proved in Lemma 2.3, we can deduce that

$$a_{m'+k-2} = \sum_{p \in M} f_p \rho_{t(p)} g_p,$$

where M is a finite set, t is a map from M to $\{1, \ldots n\}$, f_p, g_p are non-zero homogeneous elements of $\mathbb{K}\langle x_1, \ldots, x_n \rangle$, such that the degree of each f_pg_p is m-1. Now we replace a by

$$a' = a - \sum_{p \in M} f_p r_{t(p)} g_p.$$

Note that a = a' in \bar{A}_F and therefore a = a' in $A_F^{(j)}$ and $x_1 a = x_1 a'$ in $A_F^{(j+1)}$. So a' satisfies the same properties as a with the only essential difference being that $a'_{m'+k-2} = 0$.

Now we can repeat the process chipping off the homogeneous degree-components of left and right hand sides of equality from bottom up one by one until at the final step we arrive to a contradiction with $a \neq 0$ in $A_F^{(j)}$.

4 Proof of the main estimate using Golod-Shafarevich-Vinberg type argument

Lemma 4.1. Let $n, k \in \mathbb{N}$, $n \ge 2$, $m \ge k \ge 3$ and $(n, k) \ne (2, 3)$, $F \in \mathcal{P}_{n, k}^{(m)}$ and $A = \bar{A}_F$. Then $P_A \ge (1 - t)^{-1}(1 - nt + nt^{k-1} - t^k)^{-1}$.

Proof. First, observe that exchanging the ground field \mathbb{K} for a field extension does not affect the series P_A . Thus we can without loss of generality assume that \mathbb{K} is uncountable. For $j \in \mathbb{Z}_+$, let b_j be Taylor coefficients of the rational function $Q(t) = (1-t)^{-1}(1-nt+nt^{k-1}-t^k)^{-1}$ (that is, $Q(t) = \sum b_j t^j$) and $a_j = \min\{\dim A_G^{(j)} : G \in \mathcal{P}_{n,k}^{(m)}\}$. The proof will be complete if we show that $a_j = b_j$ for all $j \in \mathbb{Z}_+$. Denote $P = \sum a_j t^j$. First, note that Examples 2.1 and 2.2, provide $G \in \mathcal{P}_{n,k} \subseteq \mathcal{P}_{n,k}^{(m)}$ for which $H_G = (1-nt+nt^{k-1}-t^k)^{-1}$. It immediately follows that $P_G = Q$. By definition of P (minimality of a_j), we then have $P \leqslant Q$, that is, $a_j \leqslant b_j$ for all $j \in \mathbb{Z}_+$.

For a generic $G \in \mathcal{P}_{n,k}^{(m)}$ $P_{A_G} = P$ according to nonhomogeneous generalisation of Ufnarovskij's observation. Moreover, by lemma 3.1, for a generic $G \in \mathcal{P}_{n,k}^{(m)}$, we have that

 $x_1 a \neq 0$ in \bar{A}_G for every non-zero $a \in \bar{A}_G$. In particular, we can pick a single $G \in \mathcal{P}_{n,k}^{(m)}$ such that for $B = \bar{A}_G$, $P_B = P$ and $x_1 a \neq 0$ in \bar{A}_G for every non-zero $a \in \bar{A}_G$.

According to Lemma 3.1, we then have that for each $j \in \mathbb{Z}_+$, $x_1b \neq 0$ in $B^{(j+1)}$ for every $b \in \mathbb{K}\langle x_1, \ldots, x_n \rangle$ such that $b \neq 0$ in $B^{(j)}$. This property allows us to pick inductively (starting with $M_0 = \{1\}$) sets M_j of monomials of degree j such that $M_{j+1} \supseteq x_1M_j$ and $N_j = M_0 \cup \ldots \cup M_j$ is a linear basis in $B^{(j)}$ for each $j \in \mathbb{Z}_+$. For every j, let B_j^+ be the linear span of $N_j \setminus x_1N_{j-1}$ in $\mathbb{K}\langle x_1, \ldots, x_n \rangle$. Clearly $P_B = \sum (\dim B_j^+)t^j$ and therefore $a_j = \dim B_j^+$ for all $j \in \mathbb{Z}_+$.

Let also $\pi^{(j)}$ be the natural projection of $\mathbb{K}\langle x_1,\ldots,x_n\rangle$ onto the linear span of monomials of length $\leq j$ along $J^{(j)}$ (in fact, $\pi^{(j)}:\mathbb{K}[X]\to\mathbb{K}[X]^{(j)}$). As usual, let V be the linear span of $x_1,\ldots,x_n,\ r_j=\delta_{x_j}G,\ R$ be the linear span of r_1,\ldots,r_n and I be the ideal generated by r_1,\ldots,r_n (=the ideal of relations of B). For the sake of brevity denote $\Phi=\mathbb{K}\langle x_1,\ldots,x_n\rangle$.

Now we argue in a way similar to the Golod-Shafarevich-Vinberg theorem ??, but incorporating at some point the syzygy $\sum [x_j, r_j] = 0$ which holds for any potential algebra.

Obviously, $I = VI + R\Phi$. Then $\pi^{(j+1)}(I) = V\pi^{(j)}(I) + \pi^{(j+1)}(R\Phi)$ for every $j \in \mathbb{Z}_+$. Using the definition of B_i^+ and the fact that each r_i starts at degree $\geq k-1$, we obtain

$$\pi^{(j+1)}(I) = V\pi^{(j)}(I) + \pi^{(j+1)}(RB_{j+2-k}^+).$$

Since $\sum [x_i, r_i] = 0$ in Φ , we can get rid of r_1x_1 :

$$V\pi^{(j)}(I) + \pi^{(j+1)}(RB_{j+2-k}^+) = V\pi^{(j)}(I) + \pi^{(j+1)}(R'B_{j+2-k}^+ + r_1B_{j+2-k}^{++})$$

where R' is the linear span of r_2, \ldots, r_n . Thus

$$\pi^{(j+1)}(I) = V\pi^{(j)}(I) + \pi^{(j+1)}(R'B_{j+2-k}^+ + r_1B_{j+2-k}^{++}).$$

Hence

$$\dim \pi^{(j+1)}(I) \leqslant \dim V \pi^{(j)}(I) + \dim R' B_{j+2-k}^+ + \dim r_1 B_{j+2-k}^{++}$$
$$= n \dim \pi^{(j)}(I) + (n-1) \dim B_{j+2-k}^+ + \dim B_{j+2-k}^{++}.$$

Plugging the equalities dim $B_j^+ = a_j$, dim $B_j^{++} = a_j - a_{j-1}$ (since multiplication by x_1 is injective, we assume also $a_s = 0$ for s < 0), and dim $\pi^{(j)}(I) = 1 + n + \ldots + n^j - a_j$ into the inequality in the above display, we get

$$1 + \ldots + n^{j+1} - a_{j+1} \le n + \ldots + n^{j+1} - na_j + na_{j+2-k} - a_{j+1-k}$$

Hence $a_{j+1} \geqslant na_j - na_{j+2-k} + a_{j+1-k} - 1$ for $j \in \mathbb{Z}_+$. On the other hand, it is easy to see that the Taylor coefficients b_j of Q satisfy $b_{j+1} = nb_j - nb_{j+2-k} + b_{j+1-k} - 1$ for $j \geqslant k-1$. It is also elementary to verify that $a_j = b_j$ for $0 \leqslant j \leqslant k-1$. Now for $c_j = b_j - a_j$, we have $c_j = 0$ for $0 \leqslant j \leqslant k-1$, $c_j \geqslant 0$ for $j \geqslant k$ and $c_{j+1} \leqslant nc_j - nc_{j+2-k} + c_{j+1-k}$ for $j \geqslant k-1$. The only sequence satisfying these conditions is easily seen to be the zero sequence. Hence $a_j = b_j$ for all $j \in \mathbb{Z}_+$, which completes the proof.

Now Theorem 1.1 is a direct consequence of Lemma 4.1. Indeed, every potential F on n variables starting in degree $\geq k$ belongs to $\mathcal{P}_{n,k}^{(m)}$ for m large enough and Lemma 4.1 provides at least cubic growth of \bar{A}_F in the case (n,k)=(3,3) or (n,k)=(2,4) and exponential growth otherwise.

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