ON CENTRALIZERS IN AZUMAYA DOMAINS

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ABSTRACT. We prove a positive characteristic analogue of the classical result that the centralizer of a nonconstant differential operator in one variable is commutative. This leads to a new, short proof of that classical characteristic zero result, by reduction modulo p.

1. INTRODUCTION

The theory of commuting differential operators in one variable goes back at least a century [4]. Of special interest is the following fundamental result:

Theorem. *Let $P = a_n \frac{d^n}{dx} + \cdots + a_1 \frac{d}{dx} + a_0, a_i \in \mathbb{C}[x]$, be a differential operator of positive degree. Then the algebra of differential operators that commute with P is commutative.*

The modern version of this result is attributed to Flanders in [1] and has been given at least two rather different proofs [1, 2]. In this note, we are interested in an analogous statement in positive characteristic. Namely for the first Weyl algebra $A_1(k)$ over a field k of positive characteristic. We prove a generalization of the following:

Theorem. *Let k be a field of positive characteristic and let $P \in A_1(k)$ be noncentral. Then the centralizer of P is a commutative algebra.*

We also prove that the fraction field of the centralizer is isomorphic to the fraction field of $Z[P]$, where Z is the center of $A_1(k)$. Note that this is simpler than the analogous assertion for a complex differential operator Q, for which the fraction field of the centralizer is in general only a finite extension of the field of rational functions in Q, see [1, Corollary 1]. Unfortunately, neither the elementary methods from [1] (because of operators of degree divisible by the characteristic), nor those from [2] (because all operators are algebraic over the center, which is of dimension 2) do adapt to the positive characteristic. Nevertheless, we present here a simple argument based on a dimension count. We note finally that, in the spirit of [6] and [3], this provides a new and very short proof of the classical characteristic zero theorem above, by reduction modulo primes.

2. MAIN RESULT

Here is the main result of this note.

Lemma 2.0.1. *Let A be a domain that is a finitely generated module of rank p^2 over its center Z, for p a prime number. Then the centralizer B_a of every noncentral element a is a commutative ring. Moreover, the natural embedding $Z[a] \subseteq B_a$ induces an isomorphism of fields of fractions.*

Proof. Let $a \in A$ be an element which does not belong to the center Z, and let $B = B_a$ be the centralizer of a in A. We let K be the fraction field of Z, L be the fraction field of the center R of B, and L' be the fraction field of $Z[a]$. Note that $L' \subseteq L$.

Consider $A_K := K \otimes_Z A$, $B_L := L \otimes_R B$, and $B_{L'} := L' \otimes_{Z[a]} B$. Since these are all localizations of domains, they are domains. Note that A_K is of finite dimension p^2 over its central subfield K. Hence it is a division ring, as multiplication by a nonzero element is injective and hence invertible, by the finite dimension. Thus there are natural inclusions of the localizations $B_{L'} \subseteq B_L \subseteq A_K$, and B_L and $B_{L'}$ are also division rings, since they are finite dimensional over their central subfield K. Moreover the dimension $\dim_K(B_L)$ (resp. $\dim_K(B_{L'})$) divides $\dim_K(A_K) = p^2$. Thus $\dim_K(B_L)$ (resp. $\dim_K(B_{L'})$) is $1, p$ or p^2. But the inclusions $K \subsetneq B_L \subsetneq A_K$ (resp. $K \subsetneq B_{L'} \subsetneq A_K$) are proper, since a is a noncentral element. Hence $\dim_K(B_L) = p = \dim_K(B_{L'})$. Thus $B_{L'} = B_L$.

Finally, we have that $\dim_L(B_L)$ (resp. $\dim_{L'}(B_{L'})$) divides $\dim_K(B_L) = p$. Thus $\dim_L(B_L)$ (resp. $\dim_{L'}(B_{L'})$) is either 1 or p. But the inclusion $K \subsetneq L$ (resp. $K \subsetneq L'$) is proper as a is not central, hence $\dim_L(B_L) < \dim_K(B_L)$ (resp. $\dim_{L'}(B_{L'}) < \dim_K(B_{L'})$). We conclude that $\dim_L(B_L) = 1 = \dim_{L'}(B_{L'})$, thus $B \subseteq L$ is commutative and the fraction field of B is $B_L = B_{L'} = L'$. $\square$

Remark 2.0.2. *If A is as in the lemma and is also an algebra over a field k, then $k[a]$ is included in B_a. However B_a is not necessarily finite over $k[a]$. For example, in the first Weyl algebra over a field k of positive characteristic p with coordinate x, we have $B_x = k[x, (\frac{d}{dx})^p]$. This algebra is not a finitely generated module over $k[x]$.*

Remark 2.0.3. *In the case of a domain of higher rank p^n over its center, the centralizer of a noncentral element is not necessarily commutative. For example, the centralizer of x in the second Weyl algebra $A_2(k)$, with coordinates x and y, is generated over the positive characteristic p field k by x, ∂_x^p, y and ∂_y, and is thus not commutative. Nevertheless, we believe that the rank of a centralizer over the center is a useful invariant here too, and hope to consider it in a future work.*

3. A COROLLARY

The lemma applies in particular to the first Weyl algebra over a field of positive characteristic. This leads to a proof by reduction modulo p of the following classical result [1, 2].

Theorem 3.0.1. *Let k be a field of characteristic zero. Then the centralizer of every nonconstant polynomial differential operator in one variable over k is a commutative ring.*

Proof. For an arbitrary commutative ring k', we denote by $A(k')$ the ring of polynomial differential operators in one variable over k', i.e. the first Weyl algebra over k'. Moreover, for all $a \in A(k')$, we let the total degree $\text{tot}(a)$ of a be the degree of the total symbol of a as a k'-polynomial in 2 variables.

Let $a \in A(k)$ be nonconstant and let $P, Q \in A(k)$ be operators commuting with a. We want to show that the commutator $C := [P, Q]$ vanishes. We let S be the ring generated by the coefficients of a, P, and Q, lying in k. We have $a, P, Q, C \in A(S) \subseteq A(k)$.

Let n be the total degree of $a = \Sigma_{i+j \leqslant n} a_{ij} x^i \frac{d^j}{dx}$, and let $u = N \times \Pi_{i+j=n} a_{ij}$, where N is the factorial of n and the product Π is taken over nonzero coefficients only. We note that the ring $S[\frac{1}{u}]$ is Jacobson [5, Cor.10.4.6]. Hence the closed points of $\text{Spec}(S[\frac{1}{u}])$ form a dense subset, and for each closed point s, the residue field $k(s)$ is a finite field. Let us denote by a_s, P_s, Q_s, and C_s the images of a, P, Q, and C in $A(k(s))$, respectively. Then by the choice of u, for all closed point s of $\text{Spec}(S[\frac{1}{u}])$, the element a_s is of positive total degree which is prime to the characteristic of $k(s)$. Hence a_s is not central, P_s and Q_s commute with a_s, and $C_s = [P_s, Q_s]$.

Since the ring $A(k(s))$ is a domain of rank p_s^2 over its center [7, Thm. 2 and §4], for p_s the characteristic of $k(s)$, we can apply the lemma. Hence we conclude that $C_s = 0$ for all closed points s of $\mathrm{Spec}(S[\frac{1}{u}])$. This holds generically since the closed points of $\mathrm{Spec}(S[\frac{1}{u}])$ are dense. Thus $C = 0$. □

References

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