AZUMAYA ALGEBRAS WITH ORTHOGONAL INVOLUTION ADMITTING AN IMPROPER ISOMETRY

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ABSTRACT. Let (A, σ) be an Azumaya algebra with orthogonal involution over a ring R with $2 \in R^{\times}$. We show that if (A, σ) admits an improper isometry, i.e., an element $a \in A$ with $\sigma(a)a = 1$ and $\operatorname{Nrd}_{A/R}(a) = -1$, then the Brauer class of A is trivial. An analogue of this statement also holds for Azumaya algebras with quadratic pair when $2 \notin R^{\times}$. We also show that at this level of generality, the hypotheses do not guarantee that A is a matrix algebra over R.

Let R be a (commutative) ring and let (A, σ, f) be an Azumaya algebra with a quadratic pair over R. This means that A is an Azumaya R-algebra of even degree, $\sigma : A \to A$ is an orthogonal involution and f is a semi-trace for (A, σ) , i.e., an R-linear map from $\operatorname{Sym}(A, \sigma) := \{a \in A : \sigma(a) = a\}$ to R satisfying $f(a + \sigma(a)) = \operatorname{Trd}_{A/R}(a)$ for all $a \in A$. See [4, §2.7, §4.4] or [9, §4] for all relevant definitions and also [11] for the case where R is a field.¹ The isometry group of (A, σ, f) , denoted $O(A, \sigma, f)$, is the subgroup of A^{\times} consisting of elements $a \in A$ satisfying $\sigma(a)a = 1$ and $f(axa^{-1}) = f(x)$ for all $x \in \operatorname{Sym}(A, \sigma)$. The functor mapping an R-ring S to the group $O(A_S, \sigma_S, f_S)$ (with $A_S = A \otimes_R S$, etc.) is represented by an affine R-group scheme $O(A, \sigma, f)$, and there is a unique R-group scheme homomorphism Δ from $O(A, \sigma, f)$ to the constant R-group scheme $(\mathbb{Z}/2\mathbb{Z})_R$ whose kernel is (fiberwise over Spec R) the identity connected component of $O(A, \sigma, f)$ [4, 4.4.0.43, 5.0.0.13, 2.7.0.32]. As usual, elements of $O(A, \sigma, f)$ are called isometries, and an isometry a is called proper if $\Delta(a) = 0 + 2\mathbb{Z}$ and improper if $\Delta(a) = 1 + 2\mathbb{Z}$.

When $2 \in \mathbb{R}^{\times}$, our setting becomes simpler: The map f must coincide with $\frac{1}{2} \operatorname{Trd}_{A/R}$, so we omit it from the notation and just write $O(A, \sigma)$ for the isometry group. The elements of $O(A, \sigma)$ are the $a \in A$ satisfying $\sigma(a)a = 1$, and if we identify $(\mathbb{Z}/2\mathbb{Z})_R$ with $\mu_{2,R}$ (the R-group scheme of square roots of 1), then the map Δ becomes the reduced norm $\operatorname{Nrd}_{A/R} : \mathbf{O}(A, \sigma) \to \mu_{2,R}$. An isometry $a \in O(A, \sigma)$ is therefore improper precisely when $\operatorname{Nrd}_{A/R}(a) = -1$.

Write Br R for the Brauer group of R and let [A] denote the Brauer class of A. When R is a field F of characteristic not 2, a result of Kneser [10, Lem. 2.6.1b] says that (A, σ, f) admits an improper isometry if and only if [A] = 0. A proof working in any characteristic was given later in [11, Cor. 13.43]. The statement was extended to the case where R is a semilocal ring with $2 \in R^{\times}$ in [6], where it was also shown that the "if" part of the statement is false for a general ring R. Here we complete the picture by proving that the "only if" part holds for any ring R, thus settling Question 3 in *op. cit*.

Theorem 1. Let (A, σ, f) be an Azumaya algebra with a quadratic pair over a ring R. If (A, σ, f) admits an improper isometry, then [A] = 0 in Br R.

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¹Following [4] and [9], we call an *R*-involution σ on an Azumaya algebra *A* orthogonal if there is a faithfully flat *R*-ring *R'* splitting *A* and such that $\sigma_{R'} : A_{R'} \to A_{R'}$ is adjoint to a regular symmetric bilinear form over *R'*. We caution that this is different from the definition used in [11] if $2 \notin R^{\times}$.

The proof is very different from the argument in [6]. It relies on the existence of generic Azumaya algebras with quadratic pair constructed implicitly in [7], and earlier in [2, §5] assuming $2 \in \mathbb{R}^{\times}$, in order to reduce the theorem to the known case where \mathbb{R} is a field. It does not generalize to Azumaya algebras with orthogonal involutions over schemes, or more generally, locally ringed topoi, and so it remains open whether Theorem 1 holds in this broader context (consult [4] and [8, Dfn. 5.1, Ex. 7.4] for the definitions at this level of generality).

We show in Example 8 below that the conclusion [A] = 0 in Theorem 1 cannot be improved to $A \cong M_n(R)$ for some $n \in \mathbb{N}$, even if R is connected.

Remark 2. Theorem 1 addresses only the case of even-rank Azumaya algebras. However, if (A, σ) is an Azumaya algebra with orthogonal involution of constant odd rank n, then 2[A] = 0 because $A \cong A^{\text{op}}$ and n[A] = 0 by a theorem of Saltman [13], so we must have [A] = 0.

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PROOF OF THEOREM 1

In what follows, an R-ring means a commutative R-algebra and an R-domain is an R-ring which is moreover an integral domain.

We first prove the following result, which may be of independent interest. We use the same conventions about torsors as those in $[7, \S 2]$.

Theorem 3. Let R_0 be a noetherian ring and let G be a linear R_0 -group scheme which is an extension of a finite locally free R_0 -group scheme by a reductive R_0 -group scheme. Then every G-torsor over an affine R_0 -scheme arises as the base-change of a G-torsor over a smooth R_0 -scheme with connected geometric fibers.

Proof. Let R be an R_0 -ring and let E be a G-torsor over R. We need to show that there is a smooth R_0 -ring R', a G-torsor E' over R' and a morphism f: Spec $R \to$ Spec R' such that $E \cong E' \times_{\text{Spec } R'}$ Spec R as G-torsors over R and Spec $R' \to$ Spec R_0 has connected geometric fibers. Write R as a direct limit of its finitely generated R_0 -algebras $\{R_i\}_{i\in I}$. Since G-torsors over R_i are classified by the first Čech cohomology group $H^1_{\text{fppf}}(R_i, G)$ and since $H^1_{\text{fppf}}(-, G)$ commutes with direct limits of rings [12] (see also [1, Cor. 5.9, Rem. 5.14a]), there is $i \in I$ and a G-torsor E_i over R_i such that E is the base change of E_i along $R_i \to R$. Replacing R with R_i and E with E_i , we may assume that R is finitely generated over R_0 , hence noetherian. In particular, R has finite Krull dimension. The existence of E'and R' now follows from [7, Thm. 8.1].

Corollary 4. Let (A, σ, f) be an Azumaya algebra with quadratic pair of constant degree 2n over a ring R. Then there is a smooth \mathbb{Z} -domain S, an Azumaya Salgebra with quadratic pair (B, τ, g) and a homomorphism $\varphi : R \to S$ such that $(B_R, \tau_R, g_R) \cong (A, \sigma, f).$

Proof. Let \mathbf{PGO}_{2n} denote the automorphism \mathbb{Z} -group scheme of the split Azumaya algebra with quadratic pair $(M_{2n}(\mathbb{Z}), \eta_{2n}, f_{2n})$ over \mathbb{Z} constructed in [4, p. 57] or [9, Ex. 4.5b]; it is an extension of the finite contant \mathbb{Z} -group scheme $(\mathbb{Z}/2\mathbb{Z})_{\mathbb{Z}}$ by a reductive \mathbb{Z} -group scheme [4, 4.4.0.37, 8.1.0.55]. By [4, 4.4.0.34], there is an equivalence of fibered categories over Spec \mathbb{Z} between the \mathbf{PGO}_{2n} -torsors and the degree-2n Azumaya algebras with quadratic pair. With this at hand, the corollary is just Theorem 3 in the special case $R_0 = \mathbb{Z}$ and $G = \mathbf{PGO}_{2n}$. The resulting R_0 -ring S is a domain by the following lemma.

Lemma 5. Let $f : X \to Y$ be a smooth morphism of schemes. If Y is irreducible (resp. integral) and f has connected fibers, then X is also irreducible (resp. integral).

Proof. The morphism f is smooth, hence open [15, Tag 056G]. For $y \in Y$, let X_y denote the scheme-theoretic fiber of f over y, i.e., $X \times_Y \operatorname{Spec} \kappa(y)$, where $\kappa(y)$ is the residue field of y; we use similar notation for open subschemes of X. Our assumptions on f imply that X_y is a smooth connected $\kappa(y)$ -scheme, hence integral. Note also that X_y is subspace of X when both are viewed as topological spaces.

We first prove that X is irreducible. Let U, U' be two nonempty open subsets of X. Then f(U), f(U') are nonempty open subsets of the irreducible scheme Y, hence $f(U) \cap f(U') \neq \emptyset$. Let $y \in f(U) \cap f(U')$. Then U_y and U'_y are nonempty open subschemes of X_y . Since X_y is irreducible, $U_y \cap U'_y \neq \emptyset$, so $U \cap U \neq \emptyset$. This proves that X is irreducible.

Suppose now that Y is integral and let y be its generic point. Choose an open affine covering $\{V_i\}_i$ of Y and, for each $i \in I$, an open affine covering $\{U_{ij}\}_j$ of $f^{-1}(V_i)$. Then each $U_{ij} \to V_i$ is smooth, and so corresponds to a smooth ring map $B_i \to A_{ij}$, where B_i is a domain with fraction field $\kappa(y)$. Since $B_i \to A_{ij}$ is flat, $A_{ij} \to \kappa(y) \otimes_{B_i} A_{ij}$ is injective. Observe that $\operatorname{Spec}(\kappa(y) \otimes_{B_i} A_{ij})$ is an open subscheme of X_y , which is integral, so $\kappa(y) \otimes_{B_i} A_{ij}$ is a domain. This means that each A_{ij} is a domain, and in particular reduced. We have therefore shown that X is covered by reduced open affine subschemes, so X is reduced. Since X is also irreducible, it is integral.

Remark 6. When $2 \in \mathbb{R}^{\times}$, we can give a shorter, more direct proof of Corollary 4 using the generic Azumaya algebras with involution of [2] as follows: We may forget about the semi-trace f. Regard \mathbb{R} as an algebra over $\Omega = \mathbb{Z}[\frac{1}{2}]$. Let $m \in \mathbb{N}$ denote the *Formanek number* of A as defined in [2, §5.1], let (B, τ) denote the Azumaya algebra with orthogonal involution (A(m, 2n), t) constructed in *op. cit.* using the coefficient ring Ω , and let S = Cent(B). By [2, Thm. 17] (see also [14, Cor. 2.9b]), S is a domain and there is a ring homomorphism $\phi : S \to \mathbb{R}$ such that $(A, \sigma) \cong (B_R, \tau_R)$. Furthermore, by [2, Prop. 20], S is a smooth Ω -algebra, hence smooth over \mathbb{Z} .

Lemma 7. Let R, (A, σ, f) be as in Corollary 4 and let a be an improper isometry of (A, σ, f) . Then there is an smooth \mathbb{Z} -domain S, an Azumaya S-algebra with quadratic pair (B, τ, g) admitting an improper isometry b, and a morphism φ : $R \to S$ such that $(B_R, \tau_R, g_R) \cong (A, \sigma, f)$ and the isomorphism maps $b \otimes 1$ to a.

Proof. We apply Corollary 4 to (A, σ, f) to get a smooth \mathbb{Z} -domain S and an Azumaya S-algebra with quadratic pair (B, τ, g) such that $(A, \sigma, f) \cong (B_R, \tau_R, g_R)$ (it will not be our final S).

Let $\mathbf{O}^+(B,\tau,g)$ denote the S-group scheme of proper isomerties of (B,τ,g) ; it is semisimple [4, 8.1.0.55] and therefore smooth and has connected geometric fibers over S. We further let $\mathbf{O}^-(B,\tau,g)$ be the pullback of $\Delta : \mathbf{O}^+(B,\tau,g) \rightarrow$ $(\mathbb{Z}/2\mathbb{Z})_S$ along the $(1+2\mathbb{Z})$ -section $u: \operatorname{Spec} S \rightarrow (\mathbb{Z}/2\mathbb{Z})_S$. Then $\mathbf{O}^-(B,\tau,g)$ the affine S-group scheme representing the functor mapping an S-ring T to the set of improper isometries of (B_T,τ_T,g_T) , denoted $O^-(B_T,\tau_T,g_T)$. The product in the group $O(B_T,\tau_T,g_T)$ restricts to an action of $O^+(B_T,\tau_T,g_T)$ on $O^-(B_T,\tau_T,g_T)$, which is free provided $O^-(B_T,\tau_T,g_T) \neq \emptyset$. By [4, 4.4.0.46, 4.4.0.37], there is a faithfully flat étale S-algebra T with $O^-(B_T,\tau_T,g_T) \neq \emptyset$, so $\mathbf{O}^-(B,\tau,g)$ is a $\mathbf{O}^+(B,\tau,g)$ -torsor over S. Since $\mathbf{O}^+(B,\tau,S) \rightarrow \operatorname{Spec} S$. Our choice of S and Lemma 5 now imply that $\mathbf{O}^-(B,\tau,g)$ is an integral scheme that is smooth over Spec Z. Let S' denote the coordinate ring of the affine scheme $\mathbf{O}^-(B, \tau, g)$; it is a smooth \mathbb{Z} -domain by what we have shown. Put $B' = B_{S'}, \tau' = \tau_{S'}, g' = g_{S'}$. The identity map $S' \to S'$ corresponds to a improper isometry $b' \in O^-(B', \tau', g')$, which may be thought of as the universal improper isometry of (B, τ, g) . Since $(A, \sigma, f) \cong (B_R, \tau_R, g_R)$, the improper isometry $a \in O^-(A, \sigma, f) = \mathbf{O}^-(B, \tau, g)(R)$ corresponds to an S-ring homomorphism $\alpha : S' \to R$. Viewing R as an S'-algebra via α , we get $(B', \tau', g') \otimes_{S'} R \cong (B, \tau, g) \otimes_S R \cong (A, \sigma, f)$. Moreover, the equality $\alpha \circ \operatorname{id}_{S'} = \alpha$ translates into the fact the image of $b \in O^-(B, \tau', g')$ under the map $O^-(B, \tau', g') \to O^-(A, \sigma, f)$ is a. Thus, the data $\alpha : S' \to R$, (B', τ', g') , b' are what we were looking for.

We can now deduce Theorem 1 from its known special case where R is a field.

Proof of Theorem 1. Recall that deg A denotes the function from Spec R to N mapping \mathfrak{p} to $\sqrt{\dim_{\kappa(\mathfrak{p})} A_{\kappa(\mathfrak{p})}}$, where $\kappa(\mathfrak{p})$ is the fraction field of R/\mathfrak{p} . If deg A is not constant, then we can write R as a product of rings $R = \prod_{i=1}^{t} R_i$ such that deg A_{R_i} is constant for all *i*. It is enough to prove that $[A_{R_i}] = 0$ in Br R_i for each *i*, so we may work with each factor separately. We thus restrict to the case where deg A is constant.

Let a be an improper isometry of (A, σ, f) and let $\varphi : S \to R$, (B, τ, g) and b be as in Lemma 7. Since $\varphi_* : \operatorname{Br} S \to \operatorname{Br} R$ maps [B] to [A], it is enough to prove that [B] = 0.

By construction, S is a regular domain. Let K be the fraction field of S. By the Auslander–Goldmann–Grothendieck Theorem [3, Thm. 7.2] (here we need S to be regular), the map Br $S \to$ Br K is injective, so it enough to show that $[B_K] = 0$ in Br K. But this follows from [10, Lem. 2.6.1b] and [11, Cor. 13.43], because (B_K, τ_K, g_K) is a central simple K-algebra with quadratic pair admitting an improper isometry.

Example 8. We now give an example of a connected ring R with $2 \in R^{\times}$ and an Azumaya R-algebra with an orthogonal involution (A, σ) having an improper isometry, but such that $A \ncong M_n(R)$ for all $n \in \mathbb{N}$.

Let R be a Dedekind domain with $2 \in R^{\times}$ and $\operatorname{Pic} R \cong \mathbb{Z}/2\mathbb{Z}$, and let L be a rank-1 projective R-module representing the nontrivial element of $\operatorname{Pic} R$; such rings R exist, e.g., use [5]. In fact, any integral domain R admitting a rank-1 projective module L such that 2[L] = 0 in $\operatorname{Pic} R$ and $[L] \notin 2\operatorname{Pic} R$ will work. The former condition implies that there exists an R-module isomorphism $\phi : L \otimes_R L \to R$.

Put $M = R \oplus L$ (we write elements of M as column vectors), $A = \operatorname{End}_R(M)$, and define a symmetric R-bilinear form $b: M \times M \to R$ by $b(\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} z \\ w \end{bmatrix}) = xz + \phi(y \otimes w)$. If we were to choose L = R, then b would be a diagonal bilinear form $(\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} z \\ w \end{bmatrix}) = xz + \alpha yw$, where $\alpha \in R^{\times}$ depends on ϕ . Since L becomes isomorphic to R over some Zariski covering of R, this means that the R-linear map $\hat{b}: M \to M^* =$ $\operatorname{Hom}_R(M, R)$ given by $(\hat{b}x)y = b(x, y)$ is an isomorphism Zariski-locally on R, hence an isomorphism; otherwise said, b is *regular*. Thus, b is adjoint to an orthogonal involution $\sigma: A \to A$; for $a \in A$, the element $\sigma(a)$ is the unique R-endomorphism of M satisfying $b(ax, y) = b(x, \sigma(a)y)$ for all $x, y \in M$.

Let $a = \operatorname{id}_R \oplus (-\operatorname{id}_L) \in A$. Then $\sigma(a) = a$, hence $a\sigma(a) = a^2 = 1_A$ and $a \in O(A, \sigma)$. Writing F for the fraction field of R, we have

$$\operatorname{Nrd}_{A/R}(a) = \operatorname{Nrd}_{\operatorname{M}_2(F)/F}(\operatorname{id}_F \oplus (-\operatorname{id}_F)) = -1$$

so a is an improper isometry of (A, σ) .

We now show that $A \ncong M_n(R)$ as *R*-algebras for any $n \in \mathbb{N}$. For the sake of contradiction, suppose that such an isomorphism exists; rank considerations then force n = 2. Consider R^2 as a left *A*-module via the isomorphism $\operatorname{End}_R(R^2) \cong$

 $M_2(R) \cong A$ and put $P = \operatorname{Hom}_A(R^2, M)$. We claim that P is a projective R-module of rank 1. That P is finitely generated and projective follows readily from the fact that that M and R^2 are projective left A-modules. This also implies that $P \otimes_R k \cong$ $\operatorname{Hom}_{A \otimes k}(k^2, M \otimes_R k)$ for every R-field k. Since A is an Azumaya R-algebra of degree 2 with trivial Brauer class, $A \otimes_R k \cong M_2(k)$, and under this isomorphism $M \otimes_R k \cong$ k^2 as left $M_2(k)$ -modules. Thus, $\dim_k P \otimes_R k = \dim_k \operatorname{End}_{M_2(k)}(k^2) = 1$, proving that P is of rank 1. Consider the morphism $\varphi : P \otimes_R R^2 = \operatorname{Hom}_A(R^2, M) \otimes_R R^2 \to$ M given by $\varphi(p \otimes x) = p(x)$ $(p \in P, x \in R^2)$. It is an isomorphism because the source and target are finitely generated projective R-modules, and because $\varphi \otimes_R \operatorname{id}_k$ is an isomorphism for any R-field k by our earlier observations. Thus, $P^2 \cong P \otimes_R R^2 \cong M = R \oplus L$ as R-modules. Taking the second exterior power of both sides, we find that $P \otimes_R P \cong L$, or rather, 2[P] = [L] in Pic R. This contradicts our choice of L, so an isomorphism $A \to M_n(R)$ cannot exist.

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