

ON GENERALIZED LIST \mathcal{G} -FREE COLOURINGS OF GRAPHSYASER ROWSHAN¹

ABSTRACT. For given graph H and graphical property P , the conditional chromatic number $\chi(H, P)$ of H , is the smallest number k , so that $V(H)$ can be decomposed into sets V_1, V_2, \dots, V_k , in which $H[V_i]$ satisfies the property P , for each $1 \leq i \leq k$. When property P be that each color class contains no copy of G , we write $\chi_G(H)$ instead of $\chi(G, P)$, which is called the G -free chromatic number. Due to this, we say H has a k - G -free coloring if there is a map $c : V(H) \rightarrow \{1, \dots, k\}$, so that each of the color classes of c be G -free. Assume that for each vertex v of a graph H is assigned a set $L(V)$ of colors, called a color list. Set $g(L) = \{g(v) : v \in V(H)\}$, that is the set of colors chosen for the vertices of H under g . An L -coloring g is called G -free, so that:

- $g(v) \in L(v)$, for any $v \in V(H)$.
- $H[V_i]$ is G -free for each $i = 1, 2, \dots, L$.

If there exists an L -coloring of H , then H is called L - G -free-colorable. A graph H is said to be k - G -free-choosable if there exists an L -coloring for any list-assignment L satisfying $|L(v)| \geq k$ for each $v \in V(H)$, and $H[V_i]$ be G -free for each $i = 1, 2, \dots, L$. Let graph H and a collection of graphs \mathcal{G} are given, the $\chi_G^L(H)$ of H is the last integer k , so that H is k - \mathcal{G} -free-choosable i.e. $H[V_i]$ is \mathcal{G} -free for each $i = 1, 2, \dots, k$ i.e. contains no copy of any member of \mathcal{G} . In this article, we determine some upper bounds for $\chi_G^L(H)$, in term of the $\Delta(H), |V(H)|$ and $\delta(G)$. In particular, we show that $\chi_G^L(H) = \chi_G(H)$ for some graph H and G , $\chi_G^L(H \oplus H') \leq \chi_G^L(H) + \chi_G^L(H')$ for each G, H , and H' . Also, we show that $\chi_G(H \oplus K_n) = \chi_G^L(H \oplus K_n)$, where \mathcal{G} is a collection of all d -regular graphs, and some n .

1. INTRODUCTION

All graphs G considered here are undirected, simple, and finite graphs. For given graph $H = (V(H), E(H))$, the maximum degree of H is denoted by $\Delta(H)$, and the minimum degree of H is denoted by $\delta(H)$. If v be a vertex of H , the degree and neighbors of v are denoted by $\deg_H(v)$ ($\deg(v)$) and $N_H(v)$, respectively. The join of two graphs G and H is denoted by $G \oplus H$ and obtained from G and H by joining each vertex of G to all vertices of H . We use $\chi(H)$ to denote the chromatic number of H . For given graph H , let each vertex v of H be assigned a set $L(V)$ of colors, called a color list. An L -coloring of H is a vertex-coloring c so that:

- For any $v \in V(H)$, $c(v) \in L(v)$.
- $c(v) \neq c(v')$ for each $vv' \in E(H)$.

If there exists an L -coloring of H , then H is called L -colorable. A graph H is said to be k -choosable if there exists L -coloring for any list-assignment L satisfying $|L(v)| \geq k$ for each $v \in V(H)$. The choice number $\chi_L(H)$ of H is the minimum integer k so that H is k -choosable. Note that $\chi(H) \leq \chi_L(H)$ for any graph H , however, equality does not necessarily hold. The following particular case has been proved by Ohba [9]:

Theorem A. [9] *If $|V(H)| \leq \chi(H) + \sqrt{2\chi(H)}$, then:*

$$\chi_L(H) = \chi(H).$$

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Theorem B. [9] *If $|V(H)| + |V(G)| \leq \chi(H) + \chi(G) + \sqrt{2(\chi(H) + \chi(G))}$, then:*

$$\chi_L(H \oplus G) = \chi(H) + \chi(G).$$

One can refer to [4, 5, 7, 11] and [14] and their references for further studies about L -coloring of graph.

1.1. Conditional Coloring. For given graph H and graphical property P , the conditional chromatic number $\chi(H, P)$ of H , is the smallest number k , so that $V(H)$ can be decomposed into sets V_1, V_2, \dots, V_k , in which $H[V_i]$ satisfies the property P , for each $1 \leq i \leq k$. This extension of graph coloring was stated by Harary in 1985 [2]. A particular state, when P is the property of being acyclic, $\chi(H, P)$ is said the vertex arboricity of H . The vertex arboricity of H is shown $\alpha(H)$ and is defined as the last number of subsets in a partition of the vertex set of H , so that any subset induces an acyclic subgraph. When P is the property that each color class contains no copy of G , we write $\chi_G(H)$ instead of $\chi(G, P)$, which is called the G -free chromatic number. Due to this, we say a graph H has a k - G -free coloring if there exists a map $g : V(H) \rightarrow \{1, \dots, k\}$, so that each of the color classes of g be G -free.

We use $\alpha_L(H)$ to denote the list vertex arboricity of H , which is the least integer k , such that there exists an acyclic L -coloring for each list assignment L of H , in which $k \leq |L(v)|$. So, $\alpha(H) \leq \alpha_L(H)$ for any graph H . It has been proved that for each graph H , $\alpha(H) \leq \lceil \frac{\Delta(H)+1}{2} \rceil$ [1], and $\alpha_L(H) \leq \lceil \frac{\Delta(H)+1}{2} \rceil$ [12]. When H is not a complete graph or a cycle of odd order, then $\alpha(H) \leq \lceil \frac{\Delta}{2} \rceil$ [6]. Also, it has been shown that $\alpha(H) \leq 3, \alpha_L(H) \leq 3$ if H be a planar graph, [1, 3, 12]. As a generalized result of Theorem A and B, Lingyan Zhe, and Baoyindureng Wu have proven the following theorem [13].

Theorem C. [13] *If $|V(H)| \leq 2\alpha(H) + \sqrt{2\alpha(H)} - 1$, then:*

$$\alpha_L(H) = \alpha(H).$$

Assume that each vertex $v \in V(H)$ is assigned a set $L(v)$ of colors, told a color list. Set $c(L) = \{c(v) : v \in V(H)\}$. An L -coloring c is called G -free, so that:

- $c(v) \in L(v)$ for each $v \in V(H)$.
- $H[V_i]$ is G -free for each $i = 1, 2, \dots, L$.

If there exists an L -coloring of H , then H is said to be L - G -free-colorable. A graph H is said k - G -free-choosable if there exists an L -coloring for any list-assignment L satisfying $|L(v)| \geq k$ for each $v \in V(H)$, and $H[V_i]$ be G -free for each $i = 1, 2, \dots, L$. For given two graphs H and G , the $\chi_G^L(H)$ of H is the minimum integer k , if H be k - G -free-choosable.

In this article, we shall use Ohba's idea to show similar results in terms of G -free coloring, and list G -free coloring of graphs. In particular, in this article, we prove the subsequent results.

Theorem 1. *Suppose that H, H' , and G are three graphs, where $\delta(G) = \delta$, H is a k - G -free choosable, and H' is a k' - G -free choosable. Suppose that S and S' be the maximum subsets of $V(H)$ and $V(H')$, respectively, so that $H[S]$ is G -free and $H'[S']$ is G -free. In this case, if either $(|S'| - 1)(|V(H)| + |S'|) \leq |S'|\delta(k + 1)$ or $(|S| - 1)(|V(H')| + |S|) \leq |S|\delta(k' + 1)$, then $H \oplus H'$ is a $(k + k')$ - G -free-choosable, that is:*

$$\chi_G^L(H \oplus H') \leq \chi_G^L(H) + \chi_G^L(H') = k + k'.$$

Theorem 2. *If $|V(H)| \leq \delta\chi_G(H) + \sqrt{\delta\chi_G(H)} - (\delta - 1)$, then:*

$$\chi_G^L(H) = \chi_G(H).$$

Theorem 3. For each two connected graphs H and G , we have:

$$\chi_G^L(H) \leq \lceil \frac{\Delta(H)}{\delta(G)} \rceil + 1.$$

Theorem 4. Assume that \mathbb{R} be a collection of all d -regular graphs. For each arbitrary graph H , there exists a non-negative integer n' , so that $\chi_{\mathbb{R}}(H \oplus K_n) = \chi_{\mathbb{R}}^L(H \oplus K_n)$, where n is integer so that for which $n' \leq n$.

2. PROOF OF THE MAIN RESULTS

Before proving the main theorems, we need some basic results. Suppose that H and G are two graphs, and let L be a list-assignment color to $V(H)$. Assume that $S = v_1, v_2, \dots, v_m \subseteq V(H)$. Set $L(S) = \cup_{i=1}^m L(v_i)$. Now, we have the following lemma.

Lemma 5. Suppose that H and G be two graphs, where $\delta(G) = \delta$. Let H is not L - G -free colorable. Hence there is a subset of $V(H)$ say S , such that:

$$|S| > \delta|L(S)|.$$

Proof. By contradiction, suppose that $|S| \leq \delta|L(S)|$ for each subset S of $V(H)$. Now, define the bipartite graph B with bipartition (Y, Y') , where $Y = V(H)$, and Y' is the δ copies of $L(V(H))$. Also, for any member of Y say y , and each member of Y' say y' , $yy' \in E(B)$ if $y' \in L(y)$. It is clear to say that $N_B(W) = \delta|L(W)|$, for each $W \subseteq V(H)$. Now, let $S \subseteq V(H)$. By the assumption, $N_B(S) = \delta|L(S)| \geq |S|$, therefore, by Hall's theorem, there exists a matching M that saturates $V(H)$. Consider y and color y with the color matched by it in M . As Y' is a δ copy of $L(V(H))$. Any member y' of $L(V(H))$ appears in at most δ times as an end vertex of some edges of the M . Which means that, any color of $L(H)$ is assigned in at most δ vertices of $V(H)$. Hence, we achieve a L - G -free coloring of $V(H)$, which is impossible. Therefore the proof is complete. \blacksquare

To prove the following lemma, we use Ohba's notion.

Lemma 6. Suppose that H, H' , and G are three graphs, where $\delta(G) = \delta$, $|V(H')| = n$, and H' is G -free, i.e $G \not\subseteq H'$. If H be k - G -free choosable, and $(n-1)(|V(H)|+n) \leq n\delta(k+1)$, then $H \oplus H'$ is $(k+1)$ - G -free choosable, that is:

$$\chi_G^L(H \oplus H') \leq \chi_G^L(H) + 1 = k + 1.$$

Proof. The proof proceeds by induction on $|V(H')|$. Suppose that L is a $(k+1)$ -list assignment of $H \oplus H'$. For the induction basis, first assume that $n = 1$, and $V(H') = \{v'\}$. We color v' by one of the members of $L(v')$, say c' . For each member of $V(H)$ say v , assume that $L(v) \setminus \{c'\} = L'(v)$. As for any member of $V(H)$ say v , $|L'(v)| \geq |L(v)| - 1 = k$, and H is k - G -free choosable. Thus $H \oplus H'$ is $(k+1)$ - G -free choosable. So, assume that $n \geq 2$. Suppose that $V' = V(H') = \{v'_1, v'_2, \dots, v'_n\}$. Now, by considering $\bigcap_{v' \in V'} L(v')$, we have two cases as follow:

Case 1: $\bigcap_{v' \in V'} L(v') \neq \emptyset$.

Set $c' \in \bigcap_{v' \in V'} L(v')$ and assign c' to each member of $V(H')$. Therefore, as $G \not\subseteq H'$, one can check that $H \oplus H'[V_{c'}]$ is G -free. Now, for each member of $V(H)$ say v , assume that $L(v) \setminus \{c'\} = L'(v)$. As for each member of $V(H)$ say v , $|L'(v)| \geq |L(v)| - 1 = k$, and H is k - G -free choosable, so $H \oplus H'$ is a $(k+1)$ - G -free choosable.

Case 2: $\bigcap_{v' \in V'} L(v') = \emptyset$.

By contradiction, assume that $H \oplus H'$ has no L - G -free coloring. By Lemma 5, there must exist $S \subseteq V(H \oplus H')$ with $|S| > \delta|L(S)|$. Suppose that S be the maximal subset of $V(H \oplus H')$ so that $|S| > \delta|L(S)|$.

Suppose that $V' \subseteq S$. As $\bigcap_{v' \in V'} L(v') = \emptyset$, any color in $L(V')$ appears in the lists of at most $n - 1$ vertices of V' , therefore one can say that:

$$(1) \quad |L(S)| \geq |L(V')| \geq \frac{n}{n-1}(k+1).$$

On the other hand, by the assumption, as $(n-1)(|V(H)| + n) \leq n\delta(k+1)$, it can be check that:

$$(2) \quad |S| \leq |V(H \oplus H')| \leq \delta \frac{n}{n-1}(k+1).$$

Therefore, by Equations 1, 2, we have $|S| \leq \delta|L(S)|$, a contradiction. So, we may suppose that $V' \not\subseteq S$, that is $|V' \setminus S| \neq 0$.

Set $V'' = V(H \oplus H') \setminus S$. Assume that $L(v'') \setminus L(S) = L'(v'')$ for each $v'' \in V''$. By considering $H[S]$ and $H[V'']$, we have two claims as follow:

Claim 7. $H[S]$ has a L - G -free coloring.

Proof of the Claim7. As $(n-1)(|V(H)| + n) \leq n\delta(k+1)$, so $(n-2)(|V(H)| + n - 1) \leq (n-1)\delta(k+1)$, because this operation is an increasing operation. Therefore, by the induction hypothesis, $H \oplus H' \setminus \{v'\}$ is $(k+1)$ - G -free list colorable, for any $v' \in V(H')$. Since $H[S] \subseteq H \oplus H' \setminus \{v'\}$ it can be said $H[S]$ has an L - G -free coloring. ■

Claim 8. $H[V'']$ has an L' - G -free coloring.

Proof of the Claim8. By contradiction, let $H[V'']$ is not L' - G -free colorable. Therefore, by Lemma 5, there exists a set S' of V'' , so that $|S'| > \delta|L'(S')|$. In the other hand, $|L(S \cup S')| = |L(S)| + |L'(S')| < \frac{1}{\delta}|S + S'|$, therefore, $\delta|L(S \cup S')| < |S + S'|$ which contradicts to the maximality of S . Hence, $|S'| \leq \delta|L'(S')|$ for each $S' \subseteq V''$, which means that $H[V'']$ has an L' - G -free coloring. ■

Therefore, by Claim 7, and Claim 8, we can say that $H \oplus H'$ has a L - G -free colorable, a contradiction. Hence:

$$\chi_G^L(H \oplus H') \leq \chi_G^L(H) + 1.$$

Which means that the proof is complete. ■

In the following, by using Lemma 6, we prove the first main result, namely Theorem 1.

Proof of the Theorem 1. Without loss of generality (W.l.g), suppose that:

$$(|S'| - 1)(|V(H)| + |S'|) \leq |S'|\delta(k+1).$$

Where S' is the maximum subset of $V(H')$ so that $G \not\subseteq H'[S']$. Therefore, as H' is k' - G -free choosable and $\chi_G(H') \leq \chi_G^L(H')$, so $\chi_G(H') \leq k'$. For $i = 1, 2, \dots, k'$, set $V'_i \subseteq V(H')$ where V'_1 is the maximum subset of $V(H')$ so that $G \not\subseteq H'[V'_1]$ and for each $i \geq 2$, V'_i be the maximum subset of $V(H') \setminus (\cup_{j=1}^{i-1} V'_j)$, and $(H' \setminus (\cup_{j=1}^{i-1} V'_j))[V'_i]$ be G -free. It can be said that $|V'_1| = |S'|$, and $|V'_i| \leq |V'_{i-1}|$ for each $i \geq 2$. So, as $(|S'| - 1)(|V(H)| + |S'|) \leq |S'|\delta(k+1)$, S' is the maximum subset of $V(H')$, $G \not\subseteq H'[S']$ and $|V_1| = |S'|$, then by Lemma 6 we have the following:

$$(3) \quad \chi_G^L(H \oplus H'[V'_1]) \leq \chi_G^L(H) + 1.$$

For each $i \geq 1$, set $H_i = H_{i-1} \oplus H'[V'_i]$, where $H_0 = H$. Therefore, it can be say that the following claim is true:

Claim 9. For each $i \geq 1$, $H \oplus H'[\cup_{j=1}^i V'_j] \subseteq H_i$.

Proof of Claim 9. It is clear that $V(H \oplus H'[\cup_{j=1}^{j=i} V'_j]) = V(H_i)$, also one can say that $E(H \oplus H'[\cup_{j=1}^{j=i} V'_j]) = E(H_i) \setminus E'$, where $E' = \{v_j v_{j'} : j \leq j', v_j \in V'_j, v_{j'} \in V'_{j'}, \text{ for each } j \leq j' \leq i\}$, which means that the claim is true. \blacksquare

Also, since $(|S'| - 1)(|V(H)| + |S'|) \leq |S'| \delta(k + 1)$, $|V'_1| = |S'|$, and $|V'_i| \leq |V'_{i-1}|$ for each $i \geq 2$, it can be said that the following claim is true:

Claim 10. For each $i \in \{1, 2, \dots, k'\}$, we have:

$$(|V'_i| - 1)(|V(H)| + |V'_i|) \leq |V'_i| \delta(k + 1).$$

Proof of Claim 10. For $i = 1$, since $|V'_1| = |S'|$, and $(|S'| - 1)(|V(H)| + |S'|) \leq |S'| \delta(k + 1)$, the proof is complete. So, suppose that $2 \leq j \leq k'$, also for each $2 \leq j \leq k'$ suppose that $|V'_1| - |V'_j| = t_j$. Therefore, we need to show that $(|V'_j| - 1)(|V(H)| + |V'_j|) \leq |V'_j| \delta(k + 1)$. As $|V'_1| - |V'_j| = t_j$, so:

$$\begin{aligned} & (|V'_j| - 1)(|V(H)| + |V'_j|) = (|V'_1| - 1 - t_j)(|V(H)| + |V'_1| - t_j) \\ & = (|V'_1| - 1)(|V(H)| + |V'_1|) - (t_j(|V'_1| - 1) - t_j(|V(H)| + |V'_1| - t_j)) \\ & \leq (|V'_1|) \delta(k + 1) - (t_j(|V'_1| - 1) - t_j(|V(H)| + |V'_1| - t_j)) \\ & = (|V'_j|) \delta(k + 1) + t_j \delta(k + 1) - (t_j(|V'_1| - 1) - t_j(|V(H)| + |V'_1| - t_j)) \\ & = (|V'_j|) \delta(k + 1) + t_j(\delta(k + 1) - |V(H)| - 2|V'_1| + 1 + t_j) \\ & = (|V'_j|) \delta(k + 1) + t_j(k\delta + \delta - |V(H)| - |V'_1| - |V'_j| + 1) \end{aligned}$$

So for each $j \in \{2, 3, \dots, k'\}$ we have the following equation:

$$(4) \quad (|V'_j| - 1)(|V(H)| + |V'_j|) \leq (|V'_j|) \delta(k + 1) + t_j(k\delta + \delta - |V(H)| - |V'_1| - |V'_j| + 1)$$

Now by Equation 4, it is sufficient to show that for each $j \in \{2, 3, \dots, k'\}$ we have the followings:

$$t_j(k\delta + \delta - |V(H)| - |V'_1| - |V'_j| + 1) \leq 0.$$

It is easy to say that $|V(H)| \geq (k - 1)\delta + 1$, also one can say that $|V'_1| + |V'_j| \geq 2\delta$, which means that:

$$t_j(k\delta + \delta - |V(H)| - |V'_1| - |V'_j| + 1) \leq 0.$$

Hence, for each $i \in \{1, 2, \dots, k'\}$:

$$(5) \quad (|V'_i| - 1)(|V(H)| + |V'_i|) \leq |V'_i| \delta(k + 1).$$

Therefore, Equation 5 shows that the proof of the claim is complete. \blacksquare

Hence, by Claim 10 and by Lemma 6, for each $i \in \{1, 2, \dots, k'\}$ we can say that:

$$(6) \quad \chi_G^L(H \oplus H'[V'_i]) \leq \chi_G^L(H) + 1 \leq k + 1.$$

So, regarding Equation 6, and $\chi_G(H'[\cup_{j=1}^{j=i} V'_j]) = i$, by assigning i new separate and unique colors to each V'_i , it can be concluded that $\chi_G^L(H \oplus H'[\cup_{j=1}^{j=i} V'_j]) \leq k + i$, for each $i \in \{1, 2, \dots, k'\}$. Also, since $H \oplus H' \cong H \oplus H'[\cup_{i=1}^{i=k'} V'_i]$, so:

$$(7) \quad \chi_G^L(H \oplus H') = \chi_G^L(H \oplus H'[\cup_{i=1}^{i=k'} V'_i]) \leq k + k' = \chi_G^L(H) + \chi_G^L(H').$$

Equation 7 means that the proof is complete. \blacksquare

Before proving Theorem 2, we need two lemmas, which we state and prove in the following. In the following lemma, we determine an upper bond for $\chi_G^L(H)$, for each graph H and G .

Lemma 11. *Suppose that H and G be two graphs, where $\delta(G) = \delta$, $|V(H)| = n$, then:*

$$\chi_G^L(H) \leq \lceil \frac{n}{\delta} \rceil$$

Proof. The proof proceeds by induction on $|V(H)|$. It is clear that $\chi_G^L(H) \leq \lceil \frac{n}{\delta} \rceil$ when $|V(H)| \in \{1, 2, \dots, \delta\}$. Hence, assume that $|V(H)| \geq \delta + 1$ and suppose that L is a $\lceil \frac{n}{\delta} \rceil$ -list apportion of H . If there exist $v_1, v_2, \dots, v_{\delta-1}$, and v_δ in H so that $L(v_1) \cap L(v_2) \cap \dots \cap L(v_\delta) \neq \emptyset$, then we catch a color $c \in L(v_1) \cap L(v_2) \cap \dots \cap L(v_\delta)$ and allocate c to $v_1, v_2, \dots, v_{\delta-1}$, and v_δ and set $L(v) \setminus \{c\} = L(v)$ for any $v \in V(H) \setminus \{v_1, v_2, \dots, v_\delta\}$. Regarding $|L(v)| \geq |L(v)| - 1 \geq \lceil \frac{n-\delta}{\delta} \rceil$, and by the induction hypothesis, $H \setminus \{v_1, v_2, \dots, v_\delta\}$ has an L - G -free coloring. Therefore, H is L - G -free colorable. Otherwise, if for any δ vertices $v_1, v_2, \dots, v_{\delta-1}$, and v_δ of H , $L(v_1) \cap L(v_2) \cap \dots \cap L(v_\delta) = \emptyset$, then it is easy to say that H is L - G -free colorable, by considering $\lceil \frac{n}{\delta} \rceil$ class of size at most δ . ■

Suppose that H and G are two graphs, where $\delta(G) = \delta$, $|V(H)| = n$, and we may suppose that $g : V(H) \rightarrow \{1, 2, \dots, \chi_G(H)\}$ is a $\chi_G(H)$ -coloring of H , so that for each $i \in [\chi_G(H)]$, $G \not\subseteq H[V_i]$ and $V_i = \{v \in V(H) : g(v) = i\}$. For each i , suppose that $|V_i| = n_i$, and w.l.g assume that $n_1 \geq n_2 \geq \dots \geq n_{\chi_G(H)}$. It is easy to say that $n_i \geq \delta$ for each $i \in [\chi_G(H) - 1]$. Now by considering n_i , we have the following lemma.

Lemma 12. *If $(n_1 - 1)|V(H)| \leq n_1 \delta \chi_G(H)$. Then:*

$$\chi_G^L(H) = \chi_G(H).$$

Proof. Since $\chi_G(H) \leq \chi_G^L(H)$, it do to prove that H is $\chi_G(H)$ - G -free choosable(list colorable). The proof proceeds by induction on $\chi_G(H)$. If H be G -free, then $\chi_G^L(H) = \chi_G(H) = 1$, so suppose that H has at least one copy of G as a subgraph, that is $2 \leq \chi_G(H)$. Since $n_1 \geq \delta$, then by lemma assumption one can say that:

$$(8) \quad |V(H)| \leq \frac{n_1 \delta}{n_1 - 1} \chi_G(H).$$

If $n_1 = \delta$, then as n_1 is maximal, so $\chi_G(H) = \lceil \frac{|V(H)|}{\delta} \rceil$, hence by Lemma 11, $\chi_G^L(H) = \chi_G(H)$. Now, suppose that $n_1 \geq \delta + 1$. Hence:

$$\begin{aligned} |V(H) \setminus V_1| &= |V(H)| - n_1 \\ &\leq \frac{n_1 \delta}{n_1 - 1} \chi_G(H) - n_1 \\ &= \frac{n_1}{n_1 - 1} (\delta \chi_G(H) - n_1 + 1) \\ &\leq \frac{n_1}{n_1 - 1} (\delta \chi_G(H) - \delta) \\ &\leq \frac{n_2 \delta}{n_2 - 1} (\chi_G(H) - 1) \end{aligned}$$

So, this inequality and induction hypothesis implies that $H \setminus V_1$ is $(\chi_G(H) - 1)$ - G -free choosable, because V_2 is the maximal color class of $H \setminus V_1$. Now, regarding Equation 8 and the fact that $H \setminus V_1$ is $(\chi_G(H) - 1)$ - G -free list colorable, Lemma 6 implies that H is $\chi_G(H)$ - G -free choosable, which means that the proof is complete. ■

In the following, we prove Theorem 2 and Theorem 3 by using Lemma 12.

Theorem 13. (Theorem2) Suppose that H and G are two graphs, where $\delta(G) = \delta$. If we have, $|V(H)| \leq \delta\chi_G(H) + \sqrt{\delta\chi_G(H)} - (\delta - 1)$, then:

$$\chi_G^L(H) = \chi_G(H).$$

Proof. Assume that $g : V(H) \rightarrow \{1, 2, \dots, \chi_G(H)\}$ is the $\chi_G(H)$ -coloring of H , so that for each $i \in [\chi_G(H)]$, $G \not\subseteq H[V_i]$ and $V_i = \{v \in V(H) : g(v) = i\}$. For each i , suppose that $|V_i| = n_i$, and w.l.g assume that $n_1 \geq n_2 \geq \dots \geq n_{\chi_G(H)}$. It is easy to say that $n_i \geq \delta$ for each $i \in [\chi_G(H) - 1]$, and $n_{\chi_G(H)} \geq 1$. It can be checked that:

$$(9) \quad n_1 \leq |V(H)| - (\delta\chi_G(H) - (2\delta - 1)).$$

Otherwise one can check that $\sum_{i=1}^{i=\chi_G(H)} n_i \geq |V(H)| + 1$, a contradiction. Therefore, by assumption lemma and by Equation 9 we have the next equation:

$$(10) \quad n_1 \leq \delta\chi_G(H) + \sqrt{\delta\chi_G(H)} - (\delta - 1) - (\delta\chi_G(H) - (2\delta - 1)) = \sqrt{\delta\chi_G(H)} + \delta.$$

Hence, by Equation 10 we have the following:

$$\begin{aligned} \frac{n_1}{n_1 - 1} \delta\chi_G(H) &\geq \frac{\sqrt{\delta\chi_G(H)} + \delta}{\sqrt{\delta\chi_G(H)} + (\delta - 1)} \times \delta\chi_G(H) \\ &= \delta\chi_G(H) + \frac{\delta\chi_G(H)}{\sqrt{\delta\chi_G(H)} + (\delta - 1)} \\ &> \delta\chi_G(H) + \frac{\delta\chi_G(H) - (\delta - 1)}{\sqrt{\delta\chi_G(H)} + (\delta - 1)} \\ &= \delta\chi_G(H) + \sqrt{\delta\chi_G(H)} - (\delta - 1) \geq |V(H)| \end{aligned}$$

So, by Lemma 12, $\chi_G^L(H) = \chi_G(H)$, which means that the proof is complete. \blacksquare

Now by Theorem 13, one can say that the following results are valid. For this, first we need the following definition.

Definition 14. \mathcal{G} -free graph: Suppose that \mathcal{G} is a collection of some graphs, a given graph H is called \mathcal{G} -free, if H contains no copy of G as a subgraph for each $G \in \mathcal{G}$. When $\mathcal{G} = \{G\}$, then H is G -free if $G \not\subseteq H$. In this attention, we say a graph H has a k - \mathcal{G} -free coloring if there exists a map $g : V(H) \rightarrow \{1, \dots, k\}$ so that each of the color classes of g is \mathcal{G} -free i.e. contain no copy of each member of \mathcal{G} as a subgraph. For given graphs H and \mathcal{G} , the $\chi_{\mathcal{G}}(H)$ of H is the minimum integer k so that H is k - \mathcal{G} -free coloring.

Corollary 15. Suppose that \mathcal{G} be a collection of all d -regular graphs, where for each $G \in \mathcal{G}$ we have:

$$|V(H)| \leq d\chi_G(H) + \sqrt{d\chi_G(H)} - (d - 1).$$

Then:

$$\chi_{\mathcal{G}}^L(H) = \chi_{\mathcal{G}}(H).$$

Proof. Consider $G \in \mathcal{G}$, so that for which $\chi_{G'}(H) \leq \chi_G(H)$ for each $G' \in \mathcal{G}$. Therefore, one can say that $\chi_{\mathcal{G}}(H) \leq \chi_G(H)$, also by assumption and by Theorem 13 we have $\chi_G(H) = \chi_G^L(H)$, that is $\chi_{\mathcal{G}}(H) \leq \chi_G^L(H)$. So as $\chi_G^L(H) \leq \chi_{\mathcal{G}}^L(H)$ we have $\chi_{\mathcal{G}}(H) \leq \chi_{\mathcal{G}}^L(H)$. Now we have to show that $\chi_{\mathcal{G}}^L(H) \leq \chi_{\mathcal{G}}(H)$. Consider $G'' \in \mathcal{G}$, so that for which $\chi_{G'}^L(H) \leq \chi_{G''}^L(H)$. Therefore, by assumption and by Theorem 13, we have $\chi_{G''}(H) = \chi_{G''}^L(H)$, and as $\chi_{G''}(H) \leq \chi_{\mathcal{G}}(H)$, it can be said that $\chi_{G''}^L(H) \leq \chi_{\mathcal{G}}(H)$, which means that the proof is complete. \blacksquare

In corollary 15, if we take $d = 2$ hence $\mathcal{G} = \{C_n, n \geq 3\}$ and for any arbitrary graph for H , then it is easy to say that we get Theorem C. Also, for $d = 1$, we have $\mathcal{G} = \{K_2\}$. Hence, as $|V(H)| \leq \chi(H) + \sqrt{\chi(H)} \leq \chi(H) + \sqrt{2\chi(H)}$, therefore by corollary 15, if we take $d = 1$, then we get Theorem A.

Theorem 16. (Theorem 3) *For each two connected graphs H and G , we have:*

$$\chi_G^L(H) \leq \lceil \frac{\Delta(H)}{\delta(G)} \rceil + 1.$$

Proof. Suppose that H and G are two connected graphs that satisfy the theorem conditions. Assume that L is a k -list assignment of H , where $k = \lceil \frac{\Delta(H)}{\delta(G)} \rceil + 1$. Suppose that H is not G -free. Otherwise, it is easy to say that $\chi_G^L(H) = 1 \leq \lceil \frac{\Delta(H)}{\delta(G)} \rceil + 1$. Hence assume that $G \subseteq H$, that is $2 \leq \chi_G(H)$. Suppose that v_1, v_2, \dots, v_n be an ordering of $V(H)$ so that for each i , vertex v_i , have at most $\Delta(H)$ low-indexed neighbors in H . Now we color the vertices of H from their lists by this order. Assume that v_1, v_2, \dots, v_i has been colored and every subset of v_1, v_2, \dots, v_i which colored with the same color be G -free. Since v_{i+1} has at most $\Delta(H) + 1 - \delta(G)$ neighbors in v_1, v_2, \dots, v_i , there are at most $\lceil \frac{\Delta(H)}{\delta(G)} \rceil$ of this G containing at least δ neighbors of v_{i+1} . Therefore, it can be said that for v_{i+1} , there exists at least one color available in $L(v_{i+1})$ which appears at most once among the low-indexed neighbor of v_{i+1} . We assign such color to v_{i+1} . Therefore, we gain a G -free coloring of $H[v_1, v_2, \dots, v_{i+1}]$. By continuing this method until $i + 1 = n$, we obtain a G -free L -coloring of H . ■

2.1. Proof of Theorem 4. Before proving Theorem 4, we need to some results. Suppose that \mathbb{R} is a collection of all d -regular graphs and let H be an arbitrary graph. Hence, we have the following lemma:

Lemma 17. *Suppose that n be a positive integer, and H be an arbitrary graph. So:*

$$\chi_{\mathbb{R}}(H \oplus K_n) \geq \frac{\chi_{\mathbb{R}}(H) + n}{d}.$$

Proof. Suppose that $t = \chi_{\mathbb{R}}(H \oplus K_n)$ and let g be a t - \mathcal{G} -free coloring of $H \oplus K_n$. For each $i \in \{1, 2, \dots, \chi_{\mathbb{R}}(H)\}$, let:

$$V_i = \{v \in V(H \oplus K_n) : g(v) = i\}.$$

Since $H[V_i]$ is a \mathbb{R} -free, and \mathbb{R} is a collection of all d -regular graphs, so $K_{d+1} \not\subseteq H[V_i]$ for each i . Therefore, each V_i contains at most d vertices of $V(K_n)$. Now, if $t_j = |\{i : 1 \leq i \leq t \text{ and } |V_i \cap V(K_n)| = j\}|$, and $k_j \neq 0$, then $j \leq d$. Therefore, $t = t_0 + t_1 + \dots + t_d$, and it is easy to say that $n = \sum_{m=1}^{m=d} mt_m$, also one can say that $\chi_{\mathbb{R}}(H) \leq \sum_{m=0}^{m=d-1} t_m$. Hence,

$$\begin{aligned} dt &= d \left(\sum_{m=0}^{m=d} t_m \right) = dt_0 + \sum_{m=1}^{m=d-1} (d-m)t_m + \sum_{m=1}^{m=d} mt_m \\ &\geq (d-1)t_0 + \sum_{m=0}^{m=d-1} t_m + \sum_{m=1}^{m=d} mt_m \\ &\geq \chi_{\mathbb{R}}(H) + n \end{aligned}$$

Therefore, $t = \chi_{\mathbb{R}}(H \oplus K_n) \geq \frac{\chi_{\mathbb{R}}(H) + n}{d}$, which means that proof is complete. ■

Theorem 18. *Assume that \mathbb{R} is a collection of all d -regular graphs where $d \geq 1$. For each arbitrary graph H , there exists a non-negative integer n' , so that $\chi_{\mathbb{R}}(H \oplus K_n) = \chi_{\mathbb{R}}^L(H \oplus K_n)$, for any integer n with $n \geq n'$.*

Proof. Suppose that $m = |V(H)|$, and $g : V(H) \rightarrow \{1, 2, \dots, \chi_{\mathbb{R}}(H \oplus K_n)\}$ be an optimal \mathbb{R} -free coloring of $H \oplus K_n$, with a color class of sizes $n_1 \geq n_2 \geq \dots \geq n_{\chi_{\mathbb{R}}(H \oplus K_n)}$. One can check that $d \leq n_1 \leq m + d$ for sufficiently large n . By Lemma 17, $\chi_{\mathbb{R}}(H \oplus K_n) \geq \frac{\chi_{\mathbb{R}}(H) + n}{d}$. Observe that $m + n \leq \frac{d(m+d)}{m+d-1} \frac{dn_1}{n_1+1-d} \leq \frac{dn_1}{n_1+1-d} \chi_{\mathbb{R}}(H \oplus K_n)$ for sufficiently large n . Therefore,

$$(n_1 + 1 - d)(m + n) \leq dn_1 \chi_{\mathbb{R}}(H \oplus K_n)$$

Hence, by Lemma 12, $\chi_{\mathbb{R}}(H \oplus K_n) = \chi_{\mathbb{R}}^L(H \oplus K_n)$, which means that the proof is complete. \blacksquare

The following theorem has been proved by Ohba [9]:

Theorem 19. [9] *Let H be a graph. Hence there is a positive integer n' so that $\chi(H \oplus K_n) = \chi_L(H \oplus K_n)$, for each integer n with $n' \leq n$.*

As a generalized result of Theorem 19, Lingyan Zhe, and Baoyindureng Wu have proven the following theorem [13].

Theorem 20. [13] *Let H be a graph. Hence there is a positive integer n' so that $\alpha(H \oplus K_n) = \alpha_L(H \oplus K_n)$, for each integer n with $n' \leq n$.*

In Theorem 18, if we take $d = 1$ hence $\mathbb{R} = \{K_2\}$, then we get Theorem 19. Also, by setting $d = 2$, we have $\mathbb{R} = \{C_n, n \geq 3\}$ and for any arbitrary graph for H , then it is easy to say that we get Theorem 20.

2.2. Some research problems related to the contents of this paper. In this section, we propose some research problems related to the contents of this paper. The following conjecture has been proposed by Ohba [9]:

Conjecture 1. [9] *If $|V(H)| \leq 2\chi(H) + 1$, then:*

$$\chi_L(H) = \chi(H).$$

Jonathan A. Noel et.al in [8] have proven the conjecture1. The first problem concerns Theorem 2 and Conjecture 1, as we address below:

Problem 20.1. *For each two connected graphs H and G , find a constant M , so that if $|V(H)| \leq M\chi_G(H)$, Then:*

$$\chi_G^L(H) = \chi_G(H).$$

Y.Rowshan and A.Taherkhani in [10] have proven the following theorem.

Theorem D. *Suppose that H and G be two connected graphs while H satisfies the followingf items:*

- *If G is regular, then $H \not\cong G$.*
- *If $G \cong K_{\delta(G)+1}$, then H is not $K_{k\delta(G)+1}$.*
- *If $G \cong K_2$, then H is neither a complete graph nor an odd cycle.*

Then:

$$\chi_G(H) \leq \lceil \frac{\Delta(H)}{\delta(G)} \rceil.$$

And one of those color classes is a maximum induced G -free subgraph in H .

The second problem concerns Theorem 3 and D, as we address below:

Problem 20.2. *Suppose that H and G be two connected graphs while H satisfies the following items:*

- *If G is regular, then $H \not\cong G$.*
- *If $G \cong K_{\delta(G)+1}$, then H is not $K_{k\delta(G)+1}$.*
- *If $G \cong K_2$, then H is neither a complete graph nor an odd cycle.*

Then:

$$\chi_G^L(H) \leq \lceil \frac{\Delta(H)}{\delta(G)} \rceil.$$

And one of those color classes is a maximum induced G -free subgraph in H .

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