Nonexistence of spontaneous symmetry breakdown of time-translation symmetry on general quantum systems: Any macroscopic order parameter moves not!

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Abstract

The Kubo-Martin-Schwinger (KMS) condition is a well-founded general definition of equilibrium states on quantum systems. The time invariance property of equilibrium states is one of its basic consequences. From the time invariance of any equilibrium state it follows that the spontaneous breakdown of time-translation symmetry is impossible. Moreover, triviality of the temporal long-rang-order is derived from the KMS condition. Therefore, the manifestation of genuine quantum time crystals is impossible as long as the standard notion of spontaneous symmetry breakdown is considered.

Key Words Genuine quantum time crystals; No-go theorem; KMS condition.
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1 Introduction

The manifestation of self-organized temporal periodic structures of quantum states was proposed by Wilczek [62]. The crystal structures in the time direction for *equilibrium* states are called *genuine* quantum time crystals [39] to distinguish them from non-equilibrium quantum time crystals which have been observed experimentally [20] [65] [46].

The existence of genuine quantum time crystals was questioned soon after the publication of [62]. The work [16, 17] by Bruno verified impossibility of spontaneously rotating time crystals for the original quantum-time-crystal model [62]. The work [60] by Watanabe-Oshikawa provided a general statement of absence of genuine quantum time crystals. In [60] and its extension [61] by Watanabe-Oshikawa-Koma (WOK), non-trivial behaviors of certain temporal correlation functions are identified with quantum time crystals. This criterion of quantum time crystals proposed by [60, 61] has been widely used in the research of quantum time crystals, see [35] [53].

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In this review, we provide other no-go statements of genuine quantum time crystals. We shall compare them with the above mentioned previous works [60, 61]. There are several reasons to do so. First, the time-invariance for equilibrium states (Proposition 3.1) is a basic fact of C^*-algebraic quantum statistical mechanics established already in the 1970s [13]. It immediately implies the impossibility of genuine quantum time crystals (Theorem 3.2). That's the end of the story of *genuine* quantum time crystals. Unfortunately, however, this simple consequence of [13] has been ignored in previous research works on quantum time crystals [33] [35] [53]. Second, the absence of temporal long-range order (Corollary 3.3) in the C^*-algebraic formulation and the no-go statement of [60, 61] are different in their formulations and mathematical derivations, although they look similar. Third, the C^*-algebraic no-go statements of genuine quantum time crystals are more thorough in terms of mathematical rigor and have much wider generality than [60, 61]. To establish clear argument, we shall specify exact meanings of the following key notions:

1. Quantum systems
2. Equilibrium
3. Quantum time evolutions
4. Spontaneous symmetry breakdown (SSB)

For the first in the above list, we formulate our quantum systems by quasi-local C^*-systems. For the second, we define equilibrium by the Kubo-Martin-Schwinger (KMS) condition. For the third, we formulate Heisenberg quantum time evolutions by C^*-dynamics. For the fourth, the notion of SSB is defined by the multiplicity of KMS states (or ground states) with respect to the group action of a given symmetry. Namely, we use the standard C^*-algebraic formalism [13] which will be given in Section 2.

In the general C^*-algebraic formulation as above, non-existence of time-translation symmetry breakdown is almost obvious. As a consequence, the notion of genuine quantum time crystals is negated with no effort. Even stronger, any non-trivial order in the time direction, such as periodic, quasi-periodic, droplet, and chaotic order is forbidden for equilibrium states. The precise statement is given in Theorem 3.2 in Section 3.

The no-go statement of genuine time crystals (Theorem 3.2) is actually not our finding; it is a direct consequence of some basic facts of C^*-algebraic quantum statistical mechanics [13]. As we will see in this review, the C^*-algebraic formulation has several advantages over the so called "the box-procedure method" or "Gibbs Ansatz" widely used in physics to investigate dynamical properties of equilibrium states (such as genuine quantum time crystals). In terms of mathematics, there are several general results of the KMS condition [13]. In terms of physics, the C^*-algebra gives a natural formulation of quantum time evolutions which are generically non-local. We refer to [27] by Kastler for some conceptual points of C^*-algebraic quantum statistical mechanics.

This review is written in a self-contained manner so that the readers can understand our statements without referring to the C^*-algebraic literature. We stress that our statements will not be guaranteed, unless their assumptions are completely satisfied. In particular, the original quantum crystal model [62] made by many-particles on an Aharonov-Bohm ring is beyond the scope of our C^*-algebraic formulation. We will not address *non-equilibrium* quantum time crystals, which do not conflict with fundamental laws of physics.

2 Mathematical formulation

In this section, we provide our mathematical formulation based on [13]. We make use of the C^*-algebraic definitions of equilibrium states, spontaneous symmetry breakdown, and long-range orders. In addition to the basic reference [13], we may refer to [54] [55].

2.1 Quantum systems

Let Γ denote an infinite space such as $\mathbb{R}^\mu$ and $\mathbb{Z}^\mu$ ($\mu \in \mathbb{N}$). Γ has a natural additive group structure: $\xi_x(y) := y + x \in \Gamma$ for $x, y \in \Gamma$. Let $\mathfrak{F}$ be a set of all subsets of Γ. If $\Lambda \in \mathfrak{F}$ has finite volume $|\Lambda| < \infty$, then we denote '$\Lambda \Subset \Gamma$'. Let $\mathfrak{F}_{\rm loc}$ denote the set of all finite subsets (or the set of sufficiently many finite subsets of certain shapes) of Γ. Let $\mathcal{A}$ denote a quasi-local C^*-system on Γ describing the infinite quantum system under consideration. The total system $\mathcal{A}$ includes a family of its subsystems $\{\mathcal{A}_\Lambda;\ \Lambda \in \mathfrak{F}\}$ indexed by $\mathfrak{F}$. The local algebra $\mathcal{A}_{\rm loc} := \bigcup_{\Lambda \in \mathfrak{F}_{\rm loc}} \mathcal{A}_\Lambda$ is a norm dense subalgebra of $\mathcal{A}$. For any two disjoint subsets Λ and Λ' the following commutation relations are satisfied:

$$[A,\ B] \equiv AB - BA = 0 \quad \forall A \in \mathcal{A}_\Lambda,\ \forall B \in \mathcal{A}_{\Lambda'}.$$

The above condition is called the local commutativity. Let $\{\tau_x \in \mathrm{Aut}(\mathcal{A}),\ x \in \Gamma\}$ denote the group of space-translation automorphisms on $\mathcal{A}$. The identity $\tau_x(\mathcal{A}_\Lambda) = \mathcal{A}_{\Lambda+x}$ holds for any $\Lambda \in \mathfrak{F}$ and $x \in \Gamma$.

A state on $\mathcal{A}$ is a normalized positive linear functional on $\mathcal{A}$. We denote the set of all states on $\mathcal{A}$ by $S(\mathcal{A})$. It is an affine space with the affine combination of states. For each $\omega \in S(\mathcal{A})$ the triplet $\big(\mathcal{H}_\omega,\ \pi_\omega,\ \Omega_\omega\big)$ denotes the Gelfand-Naimark-Segal (GNS) representation associated with ω. The GNS representation generates the von Neumann algebra $\mathfrak{M}_\omega := \pi_\omega(\mathcal{A})''$ on the GNS Hilbert space $\mathcal{H}_\omega$. The commutant of $\mathfrak{M}_\omega$ is given by $\mathfrak{M}_\omega' := \big\{X \in \mathfrak{B}(\mathcal{H}_\omega);\ [X,\ Y] = 0\ \forall Y \in \mathfrak{M}_\omega\big\}$, and the center of $\mathfrak{M}_\omega$ is given by $\mathfrak{Z}_\omega := \mathfrak{M}_\omega \cap \mathfrak{M}_\omega'$. The center $\mathfrak{Z}_\omega$ contains all macroscopic observables with respect to ω. Thereby the center $\mathfrak{Z}_\omega$ determines the macroscopic (thermodynamical) information about ω. In general, any order parameter that distinguishes different phases has its corresponding element in the center. For example, the energy density, the magnetization per unit volume for any space translation invariant state, and the staggered magnetization per unit cell for any space-periodic state belong to the center.

A state $\omega \in S(\mathcal{A})$ is called a factor state if its center is trivial $\mathfrak{Z}_\omega = \mathbb{C}\mathrm{I}$, where I is the identity operator on $\mathcal{H}_\omega$. The set of all factor states on $\mathcal{A}$ is denoted by $S_{\rm factor}(\mathcal{A})$. Any $\omega \in S_{\rm factor}(\mathcal{A})$ is known to satisfy the uniform cluster property with respect $\{\tau_x \in \mathrm{Aut}(\mathcal{A}),\ x \in \Gamma\}$. Hence factor states are identified with pure phases. Each $\omega \in S(\mathcal{A})$ has its factorial (central) decomposition:

$$\omega = \int d\mu(\omega_\lambda)\omega_\lambda, \quad \omega_\lambda \in S_{\mathrm{factor}}(\mathcal{A}), \tag{2.1}$$

where μ denotes the unique probability measure on $S_{\mathrm{factor}}(\mathcal{A})$ determined by ω. Note that $\omega \in S(\mathcal{A})$ is not necessarily translation invariant.

2.2 Quantum time evolution

Assume that our stationary Heisenberg-type quantum time evolution is given by a C^*-dynamics, i.e. a one-parameter group of automorphisms $\{\alpha_t \in \mathrm{Aut}(\mathcal{A}),\ t \in \mathbb{R}\}$ on the quasi-local C^*-algebra $\mathcal{A}$. We need continuity for $\{\alpha_t \in \mathrm{Aut}(\mathcal{A}),\ t \in \mathbb{R}\}$ with respect to $t \in \mathbb{R}$. We assume the strong continuity:

$$\lim_{t\to 0} \alpha_t(A) = A \text{ in norm for each fixed } A \in \mathcal{A}. \tag{2.2}$$

It is known that any short-range quantum spin lattice model generates a strongly continuous time evolution $\{\alpha_t \in \mathrm{Aut}(\mathcal{A}),\ t \in \mathbb{R}\}$, see the pioneer work [50] and [13]. For continuous quantum systems, we may assume σ-weakly continuity for quantum time evolutions in terms of the GNS representation of sufficiently many states (including all equilibrium states) on $\mathcal{A}$. In mathematics, these quantum systems are called W^*-dynamical systems, see [13] [24].

2.3 Equilibrium states

2.3.1 The KMS condition

We define equilibrium states on $\mathcal{A}$ by the Kubo-Martin-Schwinger (KMS) condition, which names after Kubo [40], Martin and Schwinger [45]. We capture its mathematical formulation due to Haag-Hugenholz-Winnink [32] in the following.

Let $\beta \in \mathbb{R}_+ \equiv \{a \in \mathbb{R};\ a \geq 0\}$ denote an inverse temperature. Let $D_\beta := \Big\{z \in \mathbb{C};\ 0 \leq \mathbf{Im}z \leq \beta\Big\}$ and $\overset{\circ}{D}_\beta := \Big\{z \in \mathbb{C};\ 0 < \mathbf{Im}z < \beta\Big\}$. A state φ on $\mathcal{A}$ is called a β-KMS state for the quantum time evolution $\{\alpha_t \in \mathrm{Aut}(\mathcal{A}),\ t \in \mathbb{R}\}$ if there exists a complex function $F_{A,B}(z)$ of $z \in D_\beta$ for every $A, B \in \mathcal{A}$ such that $F_{A,B}(z)$ is analytic in $\overset{\circ}{D}_\beta$, and the following relation holds;

$$F_{A,B}(t) = \varphi\big(A\alpha_t(B)\big), \quad F_{A,B}(t+i\beta) = \varphi\big(\alpha_t(B)A\big), \quad \forall t \in \mathbb{R}. \tag{2.3}$$

The set of β-KMS states for $\{\alpha_t \in \mathrm{Aut}(\mathcal{A}),\ t \in \mathbb{R}\}$ is denoted by $S_{\alpha_t,\beta}(\mathcal{A})$. Ground states are defined by the KMS condition at $\beta = \infty$. The set of ground states for $\{\alpha_t \in \mathrm{Aut}(\mathcal{A}),\ t \in \mathbb{R}\}$ is denoted by $S_{\alpha_t,\infty}(\mathcal{A})$. We denote the set of all equilibrium states for all positive and infinite β with respect to the same quantum time evolution $\{\alpha_t \in \mathrm{Aut}(\mathcal{A}),\ t \in \mathbb{R}\}$ by $S^{\mathrm{Equil}}_{\alpha_t}(\mathcal{A})$.

The KMS condition is a well-founded definition of equilibrium in quantum systems. Every KMS state satisfies the minimum-free-energy condition. Vice versa, any minimum-free-energy state given by the variational formula satisfies the KMS condition, see [6] [7] [51]. (For $\beta = \infty$, the minimum energy condition for ground states is given in [12].) The KMS condition implies the passivity formulated by Pusz-Woronowicz [49]. It is a kind of "the second law of thermodynamics": No work can be obtained from an isolated system in equilibrium by varying adiabatically external parameters of the quantum time evolution [42].

2.3.2 Time invariance and factorial decomposition of equilibrium states

To discuss temporal properties of equilibrium states, the KMS condition has several advantages over the "Gibbs Ansatz" that is based on local Gibbs ensembles on finite boxes, since the KMS condition is tightly related to the quantum time evolution by definition. As a notable example, the time invariance of *any* $\varphi \in S^{\mathrm{Equil}}_{\alpha_t}(\mathcal{A})$ follows from the KMS-condition (2.3), namely,

$$\varphi\left(\alpha_t(A)\right) = \varphi(A) \quad \text{for all } A \in \mathcal{A} \quad (\forall t \in \mathbb{R}) \tag{2.4}$$

holds regardless of whether $\varphi \in S^{\mathrm{Equil}}_{\alpha_t}(\mathcal{A})$ is a factor state (pure phase) or not (statistical mixture of multiple phases).

Next, let us address the structure of the set of equilibrium states. For any finite $\beta \in \mathbb{R}_+$, the affine space $S_{\alpha_t,\beta}(\mathcal{A})$ is a Choquet simplex, and the set of extremal points $S^{\mathrm{ext}}_{\alpha_t,\beta}(\mathcal{A})$ in $S_{\alpha_t,\beta}(\mathcal{A})$ coincides with $S_{\alpha_t,\beta}(\mathcal{A}) \cap S_{\mathrm{factor}}(\mathcal{A})$. Hence each $\varphi \in S_{\alpha_t,\beta}(\mathcal{A})$ is uniquely written as an affine sum of $S^{\mathrm{ext}}_{\alpha_t,\beta}(\mathcal{A})$:

$$\varphi = \int d\mu(\varphi_\lambda)\varphi_\lambda, \quad \varphi_\lambda \in S^{\mathrm{ext}}_{\alpha_t,\beta}(\mathcal{A}), \tag{2.5}$$

where μ is a unique probability measure determined by φ. Any factor state φ_λ appearing in the above factorial decomposition (2.5) is a KMS state, which is obviously time-invariant. The above general structure of KMS states follows from the fact that the center $\mathfrak{Z}_\varphi$ of any $\varphi \in S_{\alpha_t,\beta}(\mathcal{A})$ is pointwise fixed under the time evolution [3] [13].

For $\beta = \infty$, a similar statement holds as follows. For any $\varphi \in S_{\alpha_t,\infty}(\mathcal{A})$, consider its state-decomposition $\varphi = \int d\mu(\varphi_\lambda)\varphi_\lambda$, $\varphi_\lambda \in S(\mathcal{A})$. Then each φ_λ belongs to $S_{\alpha_t,\infty}(\mathcal{A})$ by the face property of $S_{\alpha_t,\infty}(\mathcal{A})$. Thus, it is obviously invariant under the time evolution. The above general structure of ground states follows from the fact that the commutant $\mathfrak{M}'_\varphi$ (and therefore $\mathfrak{Z}_\varphi$) of any $\varphi \in S_{\alpha_t,\infty}(\mathcal{A})$ is pointwise fixed under the time evolution [9] [13].

Remark 1. The KMS condition allows multiple equilibrium phases (non-factor KMS states). Thus its properties (such as the time-invariance) hold irrespective of the existence of SSB.

Remark 2. The KMS condition does not require specific dynamical assumptions such as ergodicity, mixing, thermalization and so on. The work [35] suggested special treatments depending on thermalization and non-thermalization dynamics for the proof of [60]. At least, it is sure that such divided treatments are unnecessary for our statements.

Remark 3. It looks that the Gibbs-Ansatz is a more familiar description of equilibrium states on quantum systems than the KMS condition. For quantum spin lattice systems, there is a Gibbssian formulation due to Araki-Ion [5]. The Araki-Ion Gibbsian condition is reduced to the DLR condition for classical interactions. Although the DLR condition is a natural mathematical formulation of Gibbs states on infinite classical systems [55], the Araki-Ion Gibbsian condition seems not allow such an intuitive interpretation; it is an involved formula given in terms of the Dyson's perturbation method.[1] We only mention some recent proposals [1] [34] of how to define "Gibbs states" in quantum spin lattice systems. Furthermore, the KMS condition may have wider equilibrium states than those determined by the Gibbs ansatz [56]. Thus, we use the KMS condition as the primary notion of quantum equilibrium states. We will come back to this point in Section 4.4.2.

2.3.3 Symmetry and spontaneous symmetry breakdown

We shall make a detour in order to establish "the absence of spontaneous breakdown of time-translation symmetry" which is essentially equivalent to the time invariance of equilibrium states given in Section 2.3.2. We prepare some notations related to the spontaneous symmetry breakdown (SSB) in the C^*-algebraic language.

Let G be a group with its unit element e and let (G, θ) denote a faithful representation of G into $\mathrm{Aut}(\mathcal{A})$. Namely,

$$\begin{aligned}\theta_g \in \mathrm{Aut}(\mathcal{A}), \quad g \in G,\\ \theta_e = \mathrm{id} \in \mathrm{Aut}(\mathcal{A}),\\ \theta_g \neq \mathrm{id} \in \mathrm{Aut}(\mathcal{A}) \text{ for } \forall g \neq e \in G,\\ \theta_{g_1} \circ \theta_{g_2} = \theta_{g_1 g_2}, \quad g_1, g_2 \in G.\end{aligned}$$

The action of G upon $S(\mathcal{A})$ is given by ${\theta_g}^* \omega := \omega \circ \theta_g \in S(\mathcal{A})$ $(g \in G)$ for each $\omega \in S(\mathcal{A})$. If ${\theta_g}^* \omega = \omega \in S(\mathcal{A})$ for all $g \in G$, then ω is called a G-invariant state. The set of all G-invariant states is denoted by $S_{\mathrm{inv}}^{\theta, G}(\mathcal{A})$.

A state $\omega \in S(\mathcal{A})$ that is invariant under the time evolution $\{\alpha_t \in \mathrm{Aut}(\mathcal{A}),\ t \in \mathbb{R}\}$ is called a time-invariant or stationary state. The set of all time-invariant states $S_{\mathrm{inv}}^{\alpha, \mathbb{R}}(\mathcal{A})$ will be denoted simply by $S_{\alpha_t}^{\mathrm{stat.}}(\mathcal{A})$. As we have seen in (2.4)

$$S_{\alpha_t}^{\mathrm{Equil}}(\mathcal{A}) \subset S_{\alpha_t}^{\mathrm{stat.}}(\mathcal{A}). \tag{2.6}$$

A state $\omega \in S(\mathcal{A})$ that is invariant under the space-translation group $\{\tau_x,\ x \in \Gamma\}$ is called a translation-invariant state. Let Δ be a crystallographic subgroup of Γ, an infinite sub-lattice of Γ such that the quotient group Δ/Γ is finite. If $\omega \in S(\mathcal{A})$ is invariant under $\{\tau_x,\ x \in \Delta\}$, then it is called a spatially periodic state with respect to Δ. We denote the sets of all translation-invariant states and all spatially periodic states by $S_{\mathrm{inv}}^{\tau, \Gamma}(\mathcal{A})$ and $S_{\mathrm{inv}}^{\tau, \Delta}(\mathcal{A})$, respectively. Together, translation invariant and spatially periodic states are called homogeneous states. We denote the set of all homogeneous states on $\mathcal{A}$ by $S_{\mathrm{homo.}}^{\tau}(\mathcal{A})$.

We define spontaneous symmetry breakdown (SSB) as follows. If

$$\alpha_t \circ \theta_g = \theta_g \circ \alpha_t \in \mathrm{Aut}(\mathcal{A}) \quad \text{for all } t \in \mathbb{R},\ g \in G, \tag{2.7}$$

[1] Prof. Araki told that the Araki-Ion Gibbsian condition was an intermediate technical condition by which the KMS condition and the variational principle can be connected.

then (G, θ) is called a dynamical symmetry group. For such (G, θ), if there exists a state $\psi \in S_{\alpha_t,\beta}(\mathcal{A})$ breaking the symmetry, namely ${\theta_g}^*\psi \neq \psi$ for some $g \in G$, then the dynamical symmetry group (G, θ) is spontaneously broken.

Remark 4. The above definition of SSB is based on the set of equilibrium states $S_{\alpha_t,\beta}(\mathcal{A})$ on the C^*-system $\mathcal{A}$. There is another definition of SSB called the Bogoliubov's method [11]. The relationship between these different definitions of SSB for spin lattice systems has been elucidated in III. 10 of [55] and Theorem 6.2.42 of [13]. For boson systems, known facts are rather limited in comparison with the case of spin lattice systems, see [64].

3 Nonexistence of time-translation symmetry breakdown

3.1 General no-go statements

In the preceding section, we recalled the following basic fact of equilibrium states.

Proposition 3.1. *Any equilibrium state on C^*-algebras satisfying the KMS condition is invariant under the time evolution. Each macroscopic observable in the center of the von Neumann algebra generated by the GNS representation is fixed under the time evolution, irrespective of whether such a state is a factorial state (pure phase) or a non-factorial state (statistical mixture of different phases).*

Remark 5. The frozen property of equilibrium states stated in Proposition 3.1 may be expressed as: "Any macroscopic order parameter moves not!" This is the subtitle of this review. Here macroscopic order parameters are identified with elements in the center which is a classical (commutative) algebra. Those are usually given by the thermodynamic limit of densities. A similar statement was established in the work [18] for the particular model [62]. In [59], a non-equilibrium quantum time crystal model similar to [62] was developed by using a non-conventional definition of SSB. This is a meta-stable state which does not conflict with the above no-go statement.

The impossibility of spontaneous breakdown for time-translation symmetry immediately follows from Proposition 3.1 as follows.

Theorem 3.2. *Suppose that a quantum time evolution is given by C^*-dynamics, and equilibrium states are given by the KMS condition with respect to the quantum time evolution. Then there exists no spontaneous breakdown of time-translation symmetry, and therefore, no temporal order exists in equilibrium states. In particular, periodic, quasi-periodic, and chaotic orders in the time direction are forbidden in equilibrium states.*

In the above theorem, the one-parameter group of automorphisms $\{\alpha_t \in \mathrm{Aut}(\mathcal{A}),\ t \in \mathbb{R}\}$ plays two roles. First, it denotes a Heisenberg quantum time evolution determining the quantum model under consideration. Second, it provides a special dynamical symmetry $(G, \theta) = (\mathbb{R}, \alpha)$ of the model, as the requirement (2.7) is satisfied by the following obvious commutative relation:

$$\alpha_t \circ \alpha_s = \alpha_s \circ \alpha_t = \alpha_{t+s} \in \mathrm{Aut}(\mathcal{A}) \quad \text{for all } t \in \mathbb{R},\ s \in \mathbb{R}.$$

Remark 6. Although it is unnecessary, we shall take a look at the $\beta = 0$ case (the infinite temperature) to compare Theorem 3.2 with [61] that provides special treatment for this case in II. C "Proof for $\beta = 0$". The KMS condition for $\beta = 0$ yields the identity $\varphi(AB) = \varphi(BA)$ for all $A, B \in \mathcal{A}$, so φ is identical to the tracial state tr on $\mathcal{A}$. The tracial state is a factor state (if $\mathcal{A}$ is a simple algebra as usual in quantum statistical mechanics). Hence, it satisfies the uniform cluster property with respect the space-translations. As the tracial state is invariant under any automorphism, it is automatically invariant under $\{\alpha_t \in \mathrm{Aut}(\mathcal{A}),\ t \in \mathbb{R}\}$. Hence the possibility of quantum time crystal is obviously negated.

Remark 7. KMS states for correspond to canonical ensembles determined by the inverse temperature $\beta \in \mathbb{R}_+$. In a similar manner, one can consider the KMS condition corresponding to grand canonical ensembles specified by β and the chemical potential(s) μ, see [32] and Sec. 5.4.3 of [13]. For grand canonical KMS states as well, we can obtain statements similar to Proposition 3.1 Theorem 3.2 and Corollary 3.3. If U(1)-symmetry generated by the number operator breaks spontaneously such as the Bose-Einstein condensation, then the expectation value of the field operator which is responsible for the U(1)-symmetry breakdown will oscillate periodically in time as noted in [58] [61]. However, the time invariance property (2.4) holds for all gauge invariant observables, i.e. elements in $\mathcal{A}$ that are fixed by the U(1)-transformation.

3.2 Absence of temporal long-range order

Proposition 3.1 and Theorem 3.2 require no particular assumption on the spatial structure. In this subsection, we specialize in the homogeneous case, where the states and the quantum time evolution under consideration are both assumed to be homogeneous (translation invariant or spatially periodic).

3.2.1 Densities as macroscopic observables

Each local observable gives a macroscopic observable on the GNS Hilbert space for any homogeneous (translation invariant or spatially periodic) state by its Cesàro sum under space translations. Let us give this statement a precise formula below following [30] [13].

We prepare some notations regarding infinite-volume limits. Let Γ denote our infinite space, and Δ denote a crystallographic subgroup of Γ as before. Suppose that a net $\{\Lambda;\ \Lambda \Subset \Gamma\}$ of finite subsets of Γ eventually includes any $\mathrm{I} \Subset \Gamma$. Then, we denote $\{\Lambda \uparrow \Gamma;\ \Lambda \Subset \Gamma\}$ or simply $\{\Lambda \uparrow \Gamma\}$. Similarly, if a net $\{\Lambda;\ \Lambda \Subset \Delta\}$ eventually includes any $\mathrm{I} \Subset \Delta$, then we denote $\{\Lambda \uparrow \Delta;\ \Lambda \Subset \Delta\}$ or simply $\{\Lambda \uparrow \Delta\}$.

Take any $A \in \mathcal{A}$. For any finite subset $\Lambda \Subset \Gamma$ (or $\Lambda \Subset \Delta$) we take the following averaged sum:

$$m_\Lambda(A) := \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \tau_x(A) \in \mathcal{A}. \tag{3.1}$$

For $\omega \in S(\mathcal{A})$ let $m^\omega_\Lambda(A) := \pi_\omega\left(m_\Lambda(A)\right) \in \mathfrak{M}_\omega$. For any $\omega \in S^{\tau,\Gamma}_{\mathrm{inv}}(\mathcal{A})$ and any $\{\Lambda \uparrow \Gamma;\ \Lambda \Subset \Gamma\}$, the net of uniformly bounded operators $\left\{m^\omega_\Lambda(A) \in \mathfrak{M}_\omega;\ \Lambda \uparrow \Gamma;\ \Lambda \Subset \Gamma\right\}$ has at least one and more accumulation point(s) in the center $\mathfrak{Z}_\omega$. Precisely, there exists a subnet that converges to some element of the center in the weak-operator topology. We denote any such accumulation point by $\widehat{A}^{\omega}_{\infty}$. Heuristically, we write

$$\widehat{A}^{\omega}_{\infty} \equiv \lim_{\bar{\Lambda}\uparrow\Gamma;\ \bar{\Lambda}\Subset\Gamma} m^{\omega}_{\bar{\Lambda}}(A) \in \mathfrak{Z}_{\omega}, \tag{3.2}$$

where $\{\bar{\Lambda} \uparrow \Delta;\ \bar{\Lambda} \Subset \Gamma\}$ is some subnet of $\{\Lambda \uparrow \Gamma;\ \Lambda \Subset \Gamma\}$. Any such $\widehat{A}^{\omega}_{\infty}$ belongs to the center $\mathfrak{Z}_{\omega}$ and is called a macroscopic observable (thermodynamic density, or observable at infinity [41]). When ω is a non-factor state, there can be multiple accumulation points. Similarly, we consider $\omega \in S^{\tau,\Delta}_{\mathrm{inv}}(\mathcal{A})$. Then every accumulation point of $\left\{m^{\omega}_{\Lambda}(A) \in \mathfrak{M}_{\omega};\ \Lambda \uparrow \Delta;\ \Lambda \Subset \Delta\right\}$ in the weak-operator topology belongs to $\mathfrak{Z}_{\omega}$. If $\omega \in S^{\tau}_{\mathrm{homo.}}(\mathcal{A})$ is a factor state, these macroscopic observables are sharply given as the scalars $\omega(A)\mathrm{I}$ with no dispersion for $A \in \mathcal{A}$.

3.2.2 Long-range order in the C^*-algebraic formulation

We give a general formulation of long-range order (LRO) in the C^*-algebraic formulation. Take $\omega \in S^{\tau}_{\mathrm{homo.}}(\mathcal{A})$ and $A, B \in \mathcal{A}$. Denote their densities by $\widehat{A}^{\omega}_{\infty}, \widehat{B}^{\omega}_{\infty} \in \mathfrak{Z}_{\omega}$ as in (3.2). Consider the following two-point correlation function with respect to ω

$$f^{\omega}_{A,B} \equiv \left(\Omega_{\omega},\ \widehat{A}^{\omega}_{\infty}\widehat{B}^{\omega}_{\infty}\Omega_{\omega}\right), \tag{3.3}$$

where the GNS representation of ω is used. If $f^{\omega}_{A,B}$ is non-zero, then ω is said to exhibit LRO for $A, B \in \mathcal{A}$. If $f^{\omega}_{A,A}$ is non-zero for some $A \in \mathcal{A}$, then ω is said to exhibit LRO and $A \in \mathcal{A}$ is called a local order parameter.

Now we consider the group action (G, θ). For $A \in \mathcal{A}$ and $g \in G$, we define

$$\widehat{A}^{\omega}_{\infty}(g) := \widehat{\theta_g(A)}^{\omega}_{\infty} \in \mathfrak{Z}_{\omega}, \tag{3.4}$$

by substituting $\theta_g(A)$ for A in (3.2). We consider the following two-point correlation function with respect to $\omega \in S^{\tau}_{\mathrm{homo.}}(\mathcal{A})$:

$$f^{\omega}_{A,B}(g) \equiv \left(\Omega_{\omega},\ \widehat{A}^{\omega}_{\infty}(g)\widehat{B}^{\omega}_{\infty}\Omega_{\omega}\right),\ g \in G. \tag{3.5}$$

If $f^{\omega}_{A,B}(g)$ is a non-constant function of $g \in G$, then ω is said to exhibit G-dependent LRO for $A, B \in \mathcal{A}$. If $f^{\omega}_{A,A}(g)$ is a non-constant function of $g \in G$ for some $A \in \mathcal{A}$, then ω is said to exhibit G-dependent LRO, and $A \in \mathcal{A}$ is called a local order parameter with respect to (G, θ)-symmetry.

Next, we consider the quantum time evolution. Let $\varphi \in S^{\mathrm{Equil}}_{\alpha_t}(\mathcal{A}) \cap S^{\tau}_{\mathrm{homo.}}(\mathcal{A})$, i.e. an arbitrary homogeneous equilibrium state for $\{\alpha_t \in \mathrm{Aut}(\mathcal{A}),\ t \in \mathbb{R}\}$. Substituting φ for $\omega \in S^{\tau}_{\mathrm{homo.}}(\mathcal{A})$, $(\mathbb{R}, \alpha)$ for (G, θ), and $t \in \mathbb{R}$ for $g \in G$ of $\widehat{A}^{\omega}_{\infty}(g)$ defined above, we obtain a macroscopic observable

$$\widehat{A}^{\varphi}_{\infty}(t) \in \mathfrak{Z}_{\varphi}.$$

Let $A, B \in \mathcal{A}$. Consider the following two-point temporal correlation function with respect to φ:

$$f^{\varphi}_{A,B}(t) \equiv \left(\Omega_{\varphi},\ \widehat{A}^{\varphi}_{\infty}(t)\widehat{B}^{\varphi}_{\infty}\Omega_{\varphi}\right),\ t \in \mathbb{R}. \tag{3.6}$$

By the pointwise invariance of the center $\mathfrak{Z}_\varphi$ under the time evolution $\{\alpha_t \in \mathrm{Aut}(\mathcal{A}),\ t \in \mathbb{R}\}$ as stated in Proposition 3.1,

$$\widehat{A}^{\varphi}_{\infty}(t) = \widehat{A}^{\varphi}_{\infty}, \quad \forall t \in \mathbb{R}. \tag{3.7}$$

It implies the fixed temporal correlation function:

$$f^{\varphi}_{A,B}(t) = f^{\varphi}_{A,B}(0) \quad \text{for all } t \in \mathbb{R}. \tag{3.8}$$

Thus, the absence of non-trivial temporal LRO is proved.

Corollary 3.3. *Assume the same assumption of Theorem 3.2. Assume further that the time evolution is homogeneous in space. Then, there exists no non-trivial temporal LRO for any homogeneous equilibrium state.*

Remark 8. We have defined $\widehat{A}^{\omega}_{\infty} \in \mathfrak{Z}_\omega$ for all $A \in \mathcal{A}$ in (3.2) which are not necessarily strictly local. This is essential for $f^{\varphi}_{A,B}(t)$ in (3.6) to be well defined, since generically the time development $\alpha_t(A)$ of $A \in \mathcal{A}_{\mathrm{loc}}$ does not stay in $\mathcal{A}_{\mathrm{loc}}$.

Remark 9. Corollary 3.3 has the following obvious generalizations. Let $\varphi \in S^{\mathrm{Equil}}_{\alpha_t}(\mathcal{A}) \cap S^{\tau}_{\mathrm{homo.}}(\mathcal{A})$ and $A, B \in \mathcal{A}$ as in Corollary 3.3. Consider

$$\begin{aligned} &\left(\Omega_\varphi,\ \widehat{A}^{\varphi}_{\infty}(t)\pi_\varphi(B)\Omega_\varphi\right),\ t \in \mathbb{R}, \\ &\left(\Omega_\varphi,\ \pi_\varphi\left(\alpha_t(A)\right)\widehat{B}^{\varphi}_{\infty}\Omega_\varphi\right),\ t \in \mathbb{R}. \end{aligned}$$

Then from (3.7) and the time-invariance of φ it follows that these temporal two-point correlation functions are constant with respect to $t \in \mathbb{R}$. On the other hand, generically, the two-point function $(\pi_\varphi\left(\alpha_t(A)\right)\pi_\varphi(B)\Omega_\varphi)$ is not constant in $t \in \mathbb{R}$ as noted in [60].

4 Comparison

In this section, we compare our no-go statements of genuine quantum time crystals given in Section 3 with the previous works by Watanabe-Oshikawa-Koma which will be summarized in the first subsection.

4.1 Summary of the result of Watanabe-Oshikawa-Koma

We recall the formulation and the main result of [60, 61] adding some minor modifications for comparison purposes. We always explicitly write the Λ-dependence of subsystems as $\mathcal{A}_\Lambda$, as we consider such specification is crucial. Furthermore, any subsystem $\mathcal{A}_\Lambda$ is embedded into the total system $\mathcal{A}$. We shall take the cubic lattice $\mathbb{Z}^\mu$ as in [61]. Define the metric on $\Gamma \equiv \mathbb{Z}^\mu$ by $\|x-y\| := \max_{i\in\{1,2,\cdots,\mu\}} |x_i - y_i|$ for $x = (x_i),\ y = (x_i) \in \Gamma$. Let $\mathrm{diam}(\Lambda) \equiv \{\sup \|x-y\|;\ x, y \in \Lambda\}$ for $\Lambda \subset \Gamma$.

Suppose that the Hamiltonian on the total system is formally given by

$$\hat{H} := \sum_{x\in\Gamma} \hat{h}_x, \tag{4.1}$$

where each local Hamiltonian $\hat{h}_x \in \mathcal{A}_{\rm loc}$ is a finite-range self-adjoint operator with its support $\mathrm{supp}(\hat{h}_x)$ centered at site $x \in \Gamma$. We assume that the range and the norm of $\{\hat{h}_x;\ x \in \Gamma\}$ are uniformly bounded over $x \in \Gamma$:

$$\mathrm{diam}(\mathrm{supp}(\hat{h}_x)) \le R_h, \quad \|\hat{h}_x\| \le N_h, \tag{4.2}$$

where R_h and N_h are some positive constants.

For each $\Lambda \Subset \Gamma$, let

$$\hat{H}_\Lambda := \sum_{x\in\Lambda} \hat{h}_x \in \mathcal{A}_{\Lambda\cup\partial_{\rm ext}\Lambda}, \tag{4.3}$$

where $\partial_{\rm ext}\Lambda$ denotes some outer surface region surrounding but not intersecting Λ. By (4.2) the ration $\frac{|\partial_{\rm ext}\Lambda|}{|\Lambda|}$ tends to 0 as $\Lambda \uparrow \Gamma$. One may choose the free-boundary local Hamiltonian $\hat{H}^{\rm free}_\Lambda := \sum_{x\in\Lambda\setminus\partial_{\rm inside}} \hat{h}_x \in \mathcal{A}_\Lambda$, where $\partial_{\rm inside}$ is the smallest subset within Λ such that the above sum is in $\mathcal{A}_\Lambda$. In the following, we use the above local Hamiltonian (4.3) as in [61].

For $\Lambda \Subset \Gamma$, we define the local Heisenberg time evolution by

$$\alpha_{\Lambda,t}(A) := e^{it\hat{H}_\Lambda} A e^{-it\hat{H}_\Lambda} \ \text{ for } \ A \in \mathcal{A}, \quad t \in \mathbb{R}. \tag{4.4}$$

For $\Lambda \Subset \Gamma$, we define the local Gibbs state $\rho^{\beta,\rm Gibbs}_\Lambda \in S(\mathcal{A})$ at inverse temperature β by the same local Hamiltonian $\hat{H}_\Lambda$ as

$$\rho^{\beta,\rm Gibbs}_\Lambda(A) := \frac{1}{\mathrm{tr}(e^{-\beta\hat{H}_\Lambda})}\mathrm{tr}(e^{-\beta\hat{H}_\Lambda}A) \quad \text{for} \ \ A \in \mathcal{A}, \tag{4.5}$$

where tr denotes the tracial state on $\mathcal{A}$. Note that the local Gibbs state $\rho^{\beta,\rm Gibbs}_\Lambda$ defined on the total system $\mathcal{A}$ is the unique β-KMS state for $\{\alpha_{\Lambda,t} \in \mathrm{Aut}(\mathcal{A}),\ t \in \mathbb{R}\}$. Similarly, let $\rho^{\infty,\rm Ground}_\Lambda \in S(\mathcal{A})$ denote a ground state for the local Hamiltonian $\hat{H}_\Lambda$, equivalently, a ground state for $\{\alpha_{\Lambda,t} \in \mathrm{Aut}(\mathcal{A}),\ t \in \mathbb{R}\}$ as defined in Section 2.3.1. Note that such ground state surely exists $\rho^{\infty,\rm Ground}_\Lambda \in S(\mathcal{A})$ if the local Hamiltonian $\hat{H}_\Lambda$ is a bounded element; this is always the case for quantum spin lattice systems. It is often the case that there non-unique ground states.

Take a set of local operators $\{A_x;\ x \in \Gamma\}$, where each A_x is a local operator with its support $\mathrm{supp}(A_x)$ centered at $x \in \Gamma$. Assume that the range and the norm for $\{A_x;\ x \in \Gamma\}$ are uniformly bounded over $x \in \Gamma$: there are positive constants r and a such that for all $x \in \Gamma$

$$\mathrm{diam}(\mathrm{supp}(A_x)) \le r, \quad \|A_x\| \le a. \tag{4.6}$$

Then for $\Lambda \Subset \Gamma$, we define

$$\widehat{A}_\Lambda \equiv m_\Lambda(\{A_x;\ x \in \Gamma\}) := \frac{1}{|\Lambda|}\sum_{x\in\Lambda} A_x \in \mathcal{A}_{\rm loc}. \tag{4.7}$$

The notation $\widehat{A}_\Lambda$ above corresponds to $\widehat{A}$ in [61]. If A_0 is a local operator with its support centered at the origin and $A_x = \tau_x(A_0)$ for all $x \in \Gamma$, then such $\{A_x;\ x \in \Gamma\}$ is said to be covariant in space-translations, and $\widehat{A}_\Lambda$ is equal to $m_\Lambda(A)$ of (3.1) with $A := A_0 \in \mathcal{A}$.

Consider $\{A_x;\ x \in \Gamma\}$ and $\{B_x;\ x \in \Gamma\}$, both satisfying (4.6), and then take their $\widehat{A}_\Lambda$ and $\widehat{B}_\Lambda$ as in (4.7) for each $\Lambda \Subset \Gamma$. For each $\Lambda \Subset \Gamma$, define the following temporal two-point correlation function for the local Gibbs state (4.5) under the local Heisenberg time evolution (4.4):

$$f^{\beta,\mathrm{WOK}}_{\Lambda;\,\widehat{A}_\Lambda,\widehat{B}_\Lambda}(t) \equiv \rho^{\beta,\mathrm{Gibbs}}_{\Lambda}\left(\alpha_{\Lambda,\,t}\big(\widehat{A}_\Lambda\big)\widehat{B}_\Lambda\right), \quad t \in \mathbb{R}. \tag{4.8}$$

Similarly, for the case of $\beta = \infty$, let

$$f^{\infty,\mathrm{WOK}}_{\Lambda;\,\widehat{A}_\Lambda,\widehat{B}_\Lambda}(t) \equiv \rho^{\infty,\mathrm{Ground}}_{\Lambda}\left(\alpha_{\Lambda,\,t}\big(\widehat{A}_\Lambda\big)\widehat{B}_\Lambda\right), \quad t \in \mathbb{R}. \tag{4.9}$$

The above functions will be called WOK temporal correlation functions. The main statement of [60, 61] is as follows. For each fixed $t \in \mathbb{R}$,

$$\lim_{\Lambda\uparrow\Gamma}\left| f^{\beta,\mathrm{WOK}}_{\Lambda;\,\widehat{A}_\Lambda,\widehat{B}_\Lambda}(t) - f^{\beta,\mathrm{WOK}}_{\Lambda;\,\widehat{A}_\Lambda,\widehat{B}_\Lambda}(0)\right| = 0 \quad \text{for any} \quad \beta \in \mathbb{R}_+, \tag{4.10}$$

and

$$\lim_{\Lambda\uparrow\Gamma}\left| f^{\infty,\mathrm{WOK}}_{\Lambda;\,\widehat{A}_\Lambda,\widehat{B}_\Lambda}(t) - f^{\infty,\mathrm{WOK}}_{\Lambda;\,\widehat{A}_\Lambda,\widehat{B}_\Lambda}(0)\right| = 0 \tag{4.11}$$

are satisfied. By the triviality of the (Griffiths-type) LRO with respect to $t \in \mathbb{R}$ as in (4.10) (4.11), Watanabe-Oshikawa-Koma concluded "absence of quantum time crystals for equilibrium states".

Remark 10. Equation (4.10) does not assert the identity $\lim_{\Lambda\uparrow\Gamma} f^{\beta,\mathrm{WOK}}_{\Lambda;\,\widehat{A}_\Lambda,\widehat{B}_\Lambda}(t) = \lim_{\Lambda\uparrow\Gamma} f^{\beta,\mathrm{WOK}}_{\Lambda;\,\widehat{A}_\Lambda,\widehat{B}_\Lambda}(0)$. The existence of this limit is not known.

4.2 On different formulations of LRO

There are variant formulations of LRO (long-range order). The well-known definition based on the box procedure-method is due to Griffiths [29]. Precisely, it is formulated by a net of local Gibbs states (under some boundary condition) together with averaged local observables. Another definition of LRO is defined in terms of states on a quasi-local C^*-algebra and macroscopic observables (which belong to the von Neumann algebra not in the given C^*-algebra), we refer to [54]. Let us call the former the Griffiths-type LRO, and the latter the C^*-algebraic LRO. The Griffiths-type LRO has produced remarkable results on several statistical-physics models [29] [25], whereas the C^*-algebraic LRO is a mathematical formula which is useful for general discussion but not for practical analysis of concrete models. Corollary 3.3 is based on the C^*-algebraic LRO, whereas the work [60, 61] relies on the Griffiths-type LRO as we have seen in Section 4.1. Before going into the in-depth discussion, let us recall general information on these two different LROs in terms of SSB:

- The existence of non-trivial C^*-algebraic LRO is *equivalent* to the existence of multiple phases. Here, the states are assumed to be homogeneous states on the quasi-local C^*-algebra $\mathcal{A}$, but not necessarily equilibrium states. See §5.2 of [54] for this equivalence relation.

- Non-trivial Griffiths-type LRO appeared in a (classical or quantum) spin lattice model implies a corresponding spontaneous symmetry breakdown. For the precise statement, see Theorem 1.3 of [25], [37], [38], and §5.5 of [54], Sec.III.10 of [55]. On the other hand, the converse implication is not known in general (even for classical lattice models). From the state of the art of mathematical rigorous statistical mechanics, there are few non-trivial cases for which the converse implication is justified.

Watanabe-Oshikawa-Koma defined a Griffiths-type LRO with the time parameter by $\lim_{\Lambda\uparrow\Gamma} f^{\beta,\mathrm{WOK}}_{\Lambda;\widehat{A}_\Lambda,\widehat{B}_\Lambda}(t)$ $(t\in\mathbb{R})$. And they postulated that the periodicity of this quantity in time identifies with emergent quantum time crystal. However, in view of the general status of Griffiths-type LRO mentioned above, the absence of time translation symmetry breakdown for equilibrium states cannot be concluded solely by the triviality of the Griffiths-type LRO (4.10) (4.11). As far as we understand, the essential idea of [60, 61] is owing to the method of finding SSB in quantum spin lattice models given in [38]. However, it is not certain whether non-detection of SSB by this specific method yields a *complete* proof of the absence of SSB. On the other hand, the KMS condition completely excludes temporal SSB.

4.3 On limit procedure

Our C^*-algebraic formulation and the works [60, 61] are very different in the treatment of the infinite-volume-limit. We now introduce some notaions concerning the infinite-volume limit and recall some related facts.

Let $\varrho^{\beta,\lim\mathrm{Gibbs}}$ denote an arbitrary accumulation point of the net of local Gibbs states $\{\rho^{\beta,\mathrm{Gibbs}}_\Lambda;\ \Lambda\Subset\Gamma\}$ (4.5). Heuristically, we write

$$\varrho^{\beta,\lim\mathrm{Gibbs}}(A)=\lim_{\Lambda\uparrow\Gamma}\rho^{\beta,\mathrm{Gibbs}}_\Lambda(A),\quad A\in\mathcal{A}. \tag{4.12}$$

Any accumulation point $\varrho^{\beta,\lim\mathrm{Gibbs}}$ is called a limiting Gibbs state. Such $\varrho^{\beta,\lim\mathrm{Gibbs}}$ is not necessarily unique. Let $S^{\lim}_{\mathrm{Gibbs},\beta}(\mathcal{A})$ denote the set of all such $\varrho^{\beta,\lim\mathrm{Gibbs}}$. The quantum time evolution $\{\alpha_t\in\mathrm{Aut}(\mathcal{A}),\ t\in\mathbb{R}\}$ is called approximately inner if

$$\alpha_t(A)=\lim_{\Lambda\uparrow\Gamma}\alpha_{\Lambda,t}(A)\quad\text{for each } A\in\mathcal{A} \text{ and }\ t\in\mathbb{R}, \tag{4.13}$$

where $\alpha_{\Lambda,t}$ denites the local Heisenberg time evolution (4.4), and the convergence is with respect to the norm (or σ-weak topology introduced by the GNS representation of a chosen state). The existence of at least one and more limiting Gibbs states $\varrho^{\beta,\lim\mathrm{Gibbs}}$ as in (4.12) and the existence of a unique strongly continuous approximately inner $\{\alpha_t\in\mathrm{Aut}(\mathcal{A}),\ t\in\mathbb{R}\}$ as in (4.13) have been verified for any short-range quantum spin lattice model, see [50] [48], and Theorem 6.2.4 of [13]. Also, it has been known that under the same assumption every limiting Gibbs state $\varrho^{\beta,\lim\mathrm{Gibbs}}$ satisfies the KMS condition with respect to $\{\alpha_{\Lambda,t}\in\mathrm{Aut}(\mathcal{A}),\ t\in\mathbb{R}\}$ for β. Thus, the following inclusion holds:

$$S^{\lim}_{\mathrm{Gibbs},\beta}(\mathcal{A})\subset S_{\alpha_t,\beta}(\mathcal{A}) \tag{4.14}$$

4.3.1 How to define LRO in the infinite volume limit?

The WOK temporal correlation function $f^{\beta,\mathrm{WOK}}_{\Lambda;\,\widehat{A}_\Lambda,\widehat{B}_\Lambda}(t)$ defined in (4.8) has the same Λ-dependence on the local Gibbs state $\rho_\Lambda^{\beta,\mathrm{Gibbs}}$, the cut-off time-translation symmetry $\alpha_{\Lambda,\,t}$, and the local order parameters $\widehat{A}_\Lambda$ and $\widehat{B}_\Lambda$. These three Λs are taken to infinity at the same time. As noted in Remark 10, however, the existence of $\lim_{\Lambda\uparrow\Gamma} f^{\beta,\mathrm{WOK}}_{\Lambda;\,\widehat{A}_\Lambda,\widehat{B}_\Lambda}(t)$ has not been verified. On the other hand, if the three limits are taken in the specified order as follows, then a C^*-algebraic LRO (whose existence is surely verified for short-range quantum spin-lattice models) will appear:

$$\lim_{\Lambda_3\uparrow\Gamma}\lim_{\Lambda_2\uparrow\Gamma}\lim_{\Lambda_1\uparrow\Gamma}\rho_{\Lambda_1}^{\beta,\mathrm{Gibbs}}\Big(\alpha_{\Lambda_2,\,t}\big(\widehat{A}_{\Lambda_3}\big)\widehat{B}_{\Lambda_3}\Big) = f^{\varrho^{\beta,\lim\mathrm{Gibbs}}}_{A_0,B_0}(t), \tag{4.15}$$

where the right-hand side is given by the formula $f^{\varphi}_{A,B}(t)$ (3.6) in Section 3.2.2 with the approximately inner time evolution $\{\alpha_t \in \mathrm{Aut}(\mathcal{A}),\ t \in \mathbb{R}\}$ for the chosen C^*-dynamics, and $A = A_0$, $B = B_0$, $\varphi = \varrho^{\beta,\lim\mathrm{Gibbs}}$.

4.3.2 How to formulate quantum time evolutions?

We now discuss a crucial problem of how to formulate quantum time evolutions. It appears directly relevant to physics; it is not merely a matter of mathematical rigor. In [60, 61], the local Gibbs state and the local time-translation symmetry are given by the *same* local Hamiltonian $\hat{H}_\Lambda$. This assumption is essential for the proof of (4.10) (4.11). However, to establish non-existence of something completely, one should take (infinitely many) possibilities into account. Different choices of local Hamiltonians for a local Gibbs state and a local time-translation on the same Λ may be possible; there is no reason to exclude them. Furthermore, the same-local-Hamiltonian prescription used in [60, 61] needs justification, because the true quantum time evolution instantly evolves local observables to nonlocal ones, whereas the cut-off time evolution on Λ used in [60, 61] unnaturally confines local observables of $\mathcal{A}_\Lambda$ in its slightly larger subsystem $\mathcal{A}_{\Lambda\cup\partial_{\mathrm{ext}}\Lambda}$ eternally not allowing them to escape from the given region. In the following, we estimate the difference between the cut-off and infinite-volume time evolutions for short-range quantum spin lattice models.

Proposition 4.1. *Let $\mathcal{A}$ denote a quantum spin system on the lattice $\mathbb{Z}^\mu$. Suppose that the time evolution $\{\alpha_t \in \mathrm{Aut}(\mathcal{A}),\ t \in \mathbb{R}\}$ is translation invariant, strongly continuous, and approximately inner. Let $\{\alpha_{\Lambda,\,t} \in \mathrm{Aut}(\mathcal{A}),\ t \in \mathbb{R}\}$ denote the local Heisenberg time evolution for $\Lambda \Subset \Gamma$ as given in (4.4). Let $\{A_x = \tau_x(A_0);\ x \in \Gamma\}$, where A_0 is a local operator with its support centered at the origin of $\mathbb{Z}^\mu$. Let $\varepsilon > 0$ and $t_0 > 0$. Then for sufficiently large $\Lambda \Subset \Gamma$ the following estimate holds for all $t \in [-t_0,\ t_0]$:*

$$\Big\|\alpha_t\big(\widehat{A}_\Lambda\big) - \alpha_{\Lambda,\,t}\big(\widehat{A}_\Lambda\big)\Big\| \le \varepsilon. \tag{4.16}$$

Proof. For each $n \in \mathbb{N}$ define

$$\Lambda_0(n) := \Big\{(x_1, x_2, \cdots, x_\mu) \in \mathbb{Z}^\mu;\ 0 \le |x_i| \le \frac{n}{2}\Big\} \Subset \mathbb{Z}^\mu. \tag{4.17}$$

It is a box region centered at the origin $0 \in \mathbb{Z}^\mu$ and $\mathrm{diam}(\Lambda_0(n)) = n$ or $n-1$. Let $\Lambda_x(n) := \Lambda_0(n) + x$, i.e. the translation of $\Lambda_0(n)$ by $x \in \mathbb{Z}^\mu$. From the assumption (4.13) we have

$$\alpha_t(A) = \lim_{n\to\infty} \alpha_{\Lambda_0(n),\,t}(A) \quad \text{for each } A \in \mathcal{A} \text{ and } \ t \in \mathbb{R}. \tag{4.18}$$

By (4.7) for any $\Lambda \Subset \Gamma$

$$\alpha_{\Lambda,\,t}(\widehat{A}_\Lambda) = \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \alpha_{\Lambda,\,t}(A_x). \tag{4.19}$$

Since the time evolution under consideration is space translation invariant, for any fixed $\varepsilon > 0$ and $t_0 > 0$, there exists a constant $m(>r) \in \mathbb{N}$ (that is independent of $x \in \Gamma$) such that the following estimate holds

$$\left\| \alpha_t(A_x) - \alpha_{\mathrm{I}_x,\,t}(A_x) \right\| < \varepsilon/2 \quad \text{for any } t \in [-t_0,\ t_0] \text{ and } x \in \Gamma, \tag{4.20}$$

where I_x is any finite subset that includes the box region $\Lambda_x(m)$ centered at x:

$$\mathrm{I}_x \supset \Lambda_x(m). \tag{4.21}$$

We now take a sufficiently large $\Lambda \Subset \Gamma$ such that $\Lambda \ni 0$. We divide Λ into the following two complement regions:

$$\breve{\Lambda}_\varepsilon := \{x \in \Lambda;\ \Lambda_x(m) \subset \Lambda\}, \quad \partial_{\mathrm{ext}}\breve{\Lambda}_\varepsilon := \Lambda \setminus \breve{\Lambda}_\varepsilon, \tag{4.22}$$

where the subscript indicates ε-dependence, but t_0 dependence is omitted as there is no fear of confusion. Hence by the obvious inclusion $\Lambda \supset \bigcup_{x\in\breve{\Lambda}_\varepsilon} \Lambda_x(m)$, from (4.20) (4.21) it follows that

$$\left\| \alpha_t(A_x) - \alpha_{\Lambda,\,t}(A_x) \right\| < \varepsilon/2 \quad \text{for any } t \in [-t_0,\ t_0] \text{ and } x \in \breve{\Lambda}_\varepsilon. \tag{4.23}$$

Let $a := \|A_0\|$. By using (4.19) (4.23) we obtain

$$\begin{aligned}
&\left\| \alpha_t(\widehat{A}_\Lambda) - \alpha_{\Lambda,\,t}(\widehat{A}_\Lambda) \right\| = \frac{1}{|\Lambda|} \left\| \sum_{x\in\Lambda} \left(\alpha_t(A_x) - \alpha_{\Lambda,\,t}(A_x) \right) \right\| \\
&\le \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \left\| \alpha_t(A_x) - \alpha_{\Lambda,\,t}(A_x) \right\| \\
&= \frac{1}{|\Lambda|} \sum_{x\in\breve{\Lambda}_\varepsilon} \left\| \alpha_t(A_x) - \alpha_{\Lambda,\,t}(A_x) \right\| \\
&\quad + \frac{1}{|\Lambda|} \sum_{x\in\partial_{\mathrm{ext}}\breve{\Lambda}_\varepsilon} \left\| \alpha_t(A_x) - \alpha_{\Lambda,\,t}(A_x) \right\| \\
&\le \frac{1}{|\Lambda|} \sum_{x\in\breve{\Lambda}_\varepsilon} \left\| \alpha_t(A_x) - \alpha_{\Lambda,\,t}(A_x) \right\| + \frac{1}{|\Lambda|} \sum_{x\in\partial_{\mathrm{ext}}\breve{\Lambda}_\varepsilon} \|2A_x\| \\
&\le \frac{1}{|\Lambda|} \cdot |\breve{\Lambda}_\varepsilon| \cdot \frac{\varepsilon}{2} + \frac{1}{|\Lambda|} |\partial_{\mathrm{ext}}\breve{\Lambda}_\varepsilon| \cdot 2a \\
&\le \frac{\varepsilon}{2} + 2a \frac{|\partial_{\mathrm{ext}}\breve{\Lambda}_\varepsilon|}{|\Lambda|}
\end{aligned} \tag{4.24}$$

By (4.22) for each fixed $\varepsilon > 0$ we have

$$\lim_{\Lambda \rightsquigarrow \infty} \frac{|\partial_{\mathrm{ext}} \breve{\Lambda}_{\varepsilon}|}{|\Lambda|} = 0, \tag{4.25}$$

where the above infinite-volume limit $\Lambda \rightsquigarrow \infty$ is the so called van Hove limit, see Sec. 6.2.4 of [13]. By (4.24) and (4.25) for sufficiently large $\Lambda \Subset \Gamma$ we obtain the estimate (4.16). □

Lemma 4.1 shows that the difference between these two different time evolutions of any local order parameter disappears in the infinite-volume limit. The following result follows.

Proposition 4.2. *Under the same assumption as Lemma 4.1, the following identity holds:*

$$\lim_{\Lambda_2 \uparrow \Gamma} \lim_{\Lambda_1 \uparrow \Gamma} \rho_{\Lambda_1}^{\beta,\mathrm{Gibbs}} \left(\alpha_{\Lambda_2,t}\big(\widehat{A}_{\Lambda_2}\big) \widehat{B}_{\Lambda_2} \right) = f_{A_0,B_0}^{\varrho^{\beta,\lim \mathrm{Gibbs}}}(t), \tag{4.26}$$

where the right-hand side is given by the formula $f_{A,B}^{\varphi}(t)$ (3.6) *in Section 3.2.2 with the approximately inner time evolution* $\{\alpha_t \in \mathrm{Aut}(\mathcal{A}),\ t \in \mathbb{R}\}$ *as our* C^**-dynamics,* $A = A_0$*,* $B = B_0$*, and* $\varphi = \varrho^{\beta,\lim \mathrm{Gibbs}}$*.*

Remark 11. Proposition 4.2 does not imply that

$$\lim_{\Lambda \uparrow \Gamma} f_{\Lambda;\widehat{A}_\Lambda,\widehat{B}_\Lambda}^{\beta,\mathrm{WOK}}(t) = f_{A_0,B_0}^{\varrho^{\beta,\lim \mathrm{Gibbs}}}(t), \quad t \in \mathbb{R}. \tag{4.27}$$

This equation seems to be difficult to prove or disprove. We thus consider that Corollary 3.3 based on the C^*-algebraic LRO is independent of the no-go statement of [60, 61] based on the Griffiths-type LRO.

4.4 On generality of assumptions

We have seen that the formulations of ours and [60, 61] are different. Those may describe different physics situations, and thereby, the meaning of "non-existence of periodic temporal correlation functions" by us and that of [60, 61] are not same, cf. Remark 11. Nevertheless, we shall compare our work with [60, 61] in terms of the generality of assumptions in order to find the precise validity of these similar but different no-go statements.

4.4.1 Peridoic and aperiodic crystals

Theorem 3.2 thoroughly excludes any type of quantum time crystals such as periodic space-time crystals as in [43] and also aperiodic time crystals as in [10] [28] for equilibrium states. One may imagine temporal orders similar to interfaces (domain walls) [23] or turbulent crystals (or chaotic crystals) [52]. Those are negated by Theorem 3.2 as well. On the other hand, such inhomogeneous quantum time crystals can not be precluded by [60, 61], as it essentially requires the spatial homogeneity in its proof.

4.4.2 Equilibrium states under consideration

The inclusion of (4.14) becomes identity for some general translation invariant classical spin lattice models as shown in Theorem 6.63 [26]. However, for quantum spin lattice models, it is not known whether the inclusion of (4.14) is strict or not. In the method of [38] on which Watanabe-Oshikawa-Koma's argument relies, the sequence of symmetric local Gibbs states is taken. However, according to [21], there is an equilibrium state that can not be obtained by limits of symmetric local Gibbs states. For the boson system, the inclusion of (4.14) is strict, as there are KMS states which are not limiting Gibbs states [56]. Thus, the set of translation invariant equilibrium states considered in this review is much larger than that considered in [60, 61].

4.4.3 Is the Lieb-Robinson bound argument really necessary?

In [60, 61] the Lieb-Robinson bound estimate [44] for local Hamiltonians is used in the derivation of (4.10). However, is the Lieb-Robinson bound essential for the absence of genuine quantum time crystals? It has been known [13] that the Lieb-Robinson bound estimate yields approximately inner C^*-dynamics, which is our important assumption. On the other hand, the converse implication is not true in general. There are C^*-dynamics not satisfying the Lieb-Robinson bound; examples are given by quasi-free automorphisms on the fermion lattice system [4]. Proposition 3.1, Theorem 3.2 and Corollary 3.3 can be applied to such long-range C^*-dynamics, whereas the no-go statement of [60, 61] cannot. Let us mention other long-range models [15] for which our results are valid.

4.4.4 Limitations of C^*-algebraic approach

So far we have emphasized wide generality of our results. We now mention restrictions of our results based on the existence of C^*-dynamics. We note that concrete examples of C^*-dynamics are rather exceptional such as short-range quantum spin lattice models and non-interacting quantum field models [13]. In general, construction of C^*-dynamics (or W^*-dynamics) for quantum-field models is formidable. Some long-range quantum spin lattice models do not have their C^*-dynamics. For example, a strong coupling BCS model does not have its infinite-volume time evolution as C^*-dynamics; it only exists in a state-dependent manner [57] [14]. On the other hand, the formulation of WOK temporal correlation functions is more flexible. Kozin-Kyriienko [39] showed that some infinite-range Hamiltonian generates a nontrivial periodic WOK temporal correlation function and they claimed that it is an example of genuine quantum crystals. Obviously, Corollary 3.3 cannot be applied to Kozin-Kyriienko's model. (The feasibility of the long-range Hamiltonian and the highly entangled ground states of Kozin-Kyriienko's model has been critically argued in [36].)

4.5 Is theory of relativity relevant?

Why are quantum time crystals impossible, whereas spatial crystals in equilibrium states are common? In the article [63] to general audience, Wilczek [62] addresses the theory of relativity as his motivation to pursue the above question. In [60]the theory of relativity is also addressed. Although we cannot specify the intension of these authors, we highlight the following seemingly relevant facts of the theory of relativity and quantum equilibrium states.

1. Thermal equilibrium states in a fixed Lorentz-frame can be characterized by the KMS condition [19]. The KMS states violate the Lorentz-symmetry [47], while they always preserve the time-translation symmetry as shown in Proposition 3.1.

2. The spectrum condition of local quantum physics [31] postulates that the spectrum of Hamiltonian and momentum operators on the Hilbert space of a vacuum state is included in the forward-light cone in Lorentz space-time. (Here the vacuum state is not necessarily Lorentz-invariant.) The spectrum condition forbids crystalline structure in *space* [2]. See also Theorem 4.6 [9] and Theorem 3.2.4 [31]. Thus relativistic vacuum states allow crystalline structure *neither* in the space direction nor in the time direction.

It looks that the theory of relativity will give certain restrictions upon possible crystal structure on space-time. From a purely scientific perspective, we cannot find a meaningful link between the notion of genuine quantum crystals and the theory of relativity.

4.6 On non-equilibrium quantum time crystals

In the final part of [61], quantum time crystals by non-Gibbsian states are discussed. In the C^*-algebraic language as well, a general formula of certain non-equilibrium quantum time crystals can be given putting aside their concrete realization. It is a naive generalization of the notion of SSB to stationary states by using the identification of factor states and pure phases as follows. Suppose that $\tilde{\psi}$ is an invariant state under the time-translation symmetry $\{\alpha_t \in \mathrm{Aut}(\mathcal{A}),\ t \in \mathbb{R}\}$ but it is not an equilibrium state. Suppose that $\tilde{\psi}$ has the following specific factorial decomposition: For some $p > 0$

$$\tilde{\psi} = \int_0^p dt\, \alpha_t^* \psi, \quad \psi \in S_{\mathrm{factor}}(\mathcal{A}), \tag{4.28}$$

where ψ is a factor state breaking the time-translation symmetry $\{\alpha_t \in \mathrm{Aut}(\mathcal{A}),\ t \in \mathbb{R}\}$ but invariant under its discrete subgroup $\{\alpha_t \in \mathrm{Aut}(\mathcal{A}),\ t \in p\mathbb{Z}\}$. If ψ is a homogeneous state, then by the equivalence of the existence of non-trivial C^*-algebraic LRO and that of multiple phases [54], the function $f^{\tilde{\psi}}_{A,A}(t)$ defined by the formula (3.6) oscillates periodically in time for some local order parameter $A \in \mathcal{A}$.

5 Discussion

Based on the KMS condition, we gave no-go statements of genuine quantum time crystals in the C^*-algebraic formulation. The above no-go statements based on C^*-dynamics have wider generality than [60, 61] in several points, although they were essentially obtained in 1970s. Our viewpoint upon the notion of genuine quantum time crystals and the no-go statement used in the physics literature is contrasting to [33] [35] [53].

Now let us come back to the following fundamental problem addressed in the discussion of [60]: Why are quantum time crystals impossible for equilibrium states, whereas spatial crystals in equilibrium states are common? In Section 4.5, we emphasized that the theory of relativity is irrelevant to the above question. We consider that the impossibility of genuine quantum time crystals is due to rigid stability of quantum equilibrium states. The KMS condition is known to imply and to be implied by several characterizations of equilibrium, see [13]. Thus, Theorem 3.2 given in terms of the KMS condition can be rephrased as follows:

• The variational principle for equilibrium states forbids the existence of genuine quantum time crystals.

• The passivity by Pusz-Woronowicz forbids the existence of genuine quantum time crystals.

This review has narrowed the possibility of genuine quantum time crystals. But as we have noted, there are many quantum models that can not be formulated in the C^*-algebraic formulation. Although we consider that our no-go statements have wider genarality beyond the C^*-algebraic formulation, these should not be applied to such models unless there is a rigorous proof.

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