COMPOSITION OPERATORS ON HERZ-TYPE TRIEBEL-LIZORKIN SPACES WITH APPLICATION TO SEMILINEAR PARABOLIC EQUATIONS

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ABSTRACT. Let $G: \mathbb{R} \to \mathbb{R}$ be a continuous function. In the first part of this paper, we investigate sufficient conditions on G such that

$$\{G(f): f \in \dot{K}_{p,q}^{\alpha} F_{\beta}^{s}\} \subset \dot{K}_{p,q}^{\alpha} F_{\beta}^{s}$$

holds. Here $\dot{K}^{\alpha}_{p,q}F^s_{\beta}$ are Herz-type Triebel-Lizorkin spaces. These spaces unify and generalize many classical function spaces such as Lebesgue spaces of power weights, Sobolev and Triebel-Lizorkin spaces of power weights. In the second part of this paper we will study local and global Cauchy problems for the semilinear parabolic equations

$$\partial_t u - \Delta u = G(u)$$

with initial data in Herz-type Triebel-Lizorkin spaces. Our results cover the results obtained with initial data in some know function spaces such us fractional Sobolev spaces. Some limit cases are given.

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1. Introduction

Let $G: \mathbb{R} \to \mathbb{R}$ be a function. In this paper we consider the Cauchy problem for semilinear parabolic equations on \mathbb{R}^n of the following form:

$$\frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + G(u(t,x)), \quad (t,x) \in (0,\infty) \times \mathbb{R}^n$$
(1.1)

subject to the initial value condition

$$u(0,x) = u_0(x)$$
 on \mathbb{R}^n .

The most classical examples of such equations are the semilinear heat equations

$$\frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + u|u|^{\mu-1}, \quad (t,x) \in (0,\infty) \times \mathbb{R}^n, \mu > 1,$$
(1.2)

the Burgers viscous equations

$$\frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + \partial_x(|u|^{\mu}), \quad (t,x) \in (0,\infty) \times \mathbb{R}^n, \mu > 1$$

and the Navier-Stokes equation

$$\frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + \mathcal{P}\nabla(u \otimes u), \quad (t,x) \in (0,\infty) \times \mathbb{R}^n, \mu > 1,$$

where \mathcal{P} denotes the projector on the divergence free vector field. Let us recall briefly some results on most known function spaces. For Lebesgue space, Weissler in [62] and [63] studied (1.2) with singular data in certain Lebesgue spaces L^p . In [62] he proved the local existence of (1.2) with initially data in L^{p_c} with $p_c = \frac{n(\mu-1)}{2} > 1$ and the solution belongs

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to $C([0,T),L^p)$, and that T can be taken as infinity for sufficiently small data in L^{p_c} . Giga [30] proved that the solution belongs to $L^q([0,T),L^p)$ with $\frac{1}{q}=\frac{n}{2}(\frac{1}{p_c}-\frac{1}{p}), p,q>p_c$ and $q>\mu$.

Weissler [62] proved the local existence of (1.2) for initial values in L^p with $p > p_c$ and $p \ge \mu$. See [30] for further results.

In case of $1 there exist some non-negative initial data in <math>L^p$ for which there is no non-negative solution for any positive time T > 0, see e.g. [3] and [63].

Further results, for the well-posedness of the Cauchy problem of (1.2) can be found in [17], [55], [56] and [61].

In the framework of fractional Sobolev spaces, [45] established local well-posedness of problem (1.1) with some suitable assumptions on G and obtained existence of global small solutions in $H_p^{\frac{n}{p}-\frac{2}{\mu}}$. Miao and Zhang, [42] establish the local well-posedness and small global well-posedness in Besov spaces $B_{p,2}^s$. Also, they establish the local well-posedness and small global well-posedness of problem (1.1) in the critical space $B_{p,2}^{\frac{n}{p}}$.

In [27] the author study the equation (1.1) with

$$|G(x) - G(y)| \le |x - y|(|x|^{\mu - 1} + |y|^{\mu - 1}), \quad x, y \in \mathbb{R}, \mu > 1, G(0) = 0$$
 (1.3)

and initial data in Herz spaces $\dot{K}_{p,q}^{\alpha}$. Herz spaces play an important role in Harmonic Analysis. After they have been introduced in [29], the theory of these spaces had a remarkable development in part due to its usefulness in applications. For instance, they appear in the characterization of multipliers on Hardy spaces [4], in the summability of Fourier transforms [28] and in regularity theory for elliptic equations in divergence form [43]. They unify and generalize the classical Lebesgue spaces of power weights. More precisely, if $\alpha = 0$ and p = q, then $\dot{K}_{p,p}^0$ coincides with the Lebesgue spaces L^p and

$$\dot{K}_{p,p}^{\alpha} = L^p(\mathbb{R}^n, |\cdot|^{\alpha p}),$$
 (Lebesgue space equipped with power weight).

The aims of the present paper is to study the equation (1.1) in Herz-type Triebel-Lizorkin spaces $\dot{K}^{\alpha}_{p,q}F^s_{\beta}$. These spaces unify and generalize the classical Lebesgue spaces of power weights, fractional Sobolev spaces of power weights and Triebel-Lizorkin spaces of power weights. We will assume that G belongs to $G \in Lip\mu$, see Section 3 for the definition of the spaces $Lip\mu$.

We recall that the solution in the function space $K_{p,q}^{\alpha}F_{\beta}^{s}$ of the integral equation

$$u(t,x) = e^{t\Delta}u_0 + \int_0^t e^{(t-\tau)\Delta}G(u)(\tau,x)d\tau$$
(1.4)

is usually defined as the mild solution of the Cauchy problem (1.1). Under some assumption on p, q β, α and s we prove that for all initial data u_0 in $\dot{K}_{p,q}^{\alpha} F_{\beta}^s$ with $s > \bar{s} = \frac{n}{p} + \alpha - \frac{2}{\mu - 1}$, there exists a maximal solution u to (1.4) in $C([0, T_0), \dot{K}_{p,q}^{\alpha} F_{\beta}^s)$ with $T_0 \geqslant C \|u_0\|_{\dot{K}_{p,q}^{\alpha} F_{\beta}^s}^{-1}$. If $\theta < (s - \bar{s})(\mu - 1)$, then we prove that

$$u - e^{t\Delta}u_0 \in C([0, T_0), \dot{K}_{p,q}^{\alpha} F_{\beta}^{s+\theta}).$$
 (1.5)

Now if $\theta = (s - \bar{s})(\mu - 1), s > 1$ with $G \in Lips_0$ and

$$s_0 = \frac{\frac{n}{p} + \alpha}{\frac{n}{p} + \alpha - s + 1},$$

then we have (1.5), which was not treated in [45]. Our results cover the corresponding results of [45]. Moreover, we present the limit case

$$s = 1 + \frac{\mu - 1}{\mu} \left(\frac{n}{p} + \alpha \right)$$

and the case when $s > \frac{n}{p} + \alpha$. To study (1.1) we investigate sufficient conditions on G such that

$$\{G(f): f \in \dot{K}_{p,q}^{\alpha} F_{\beta}^{s}\} \subset \dot{K}_{p,q}^{\alpha} F_{\beta}^{s}.$$

In Sobolev space, [41] have presented the necessary and sufficient conditions on G such that

$$G(W_n^1(\mathbb{R}^n)) \subset W_n^1(\mathbb{R}^n),$$

except the case $p = n \ge 2$. A complete characterization of this problem in Sobolev spaces has been given by Bourdaud in [6] and [11]. The surprise result in Sobolev spaces is that under some assumptions there is no non-trivial function G which acts via left composition on such spaces. More precisely, in 1978 Dahlberg [18] proved that

$$G(f) \in W_p^m(\mathbb{R}^n), \quad f \in W_p^m(\mathbb{R}^n), \quad 1$$

implies G(t) = ct for some $c \in \mathbb{R}$. In the framework of Sobolev spaces with fractional order, $H^s(\mathbb{R}), 0 < s < 1, s \neq 2$, Igari in [33] gave the necessary and sufficient conditions on G such that $G(H^s(\mathbb{R})) \subset H^s(\mathbb{R})$. He observed the necessity of local Lipschitz continuity for the first time. See [35] for the Hardy-Sobolev space $F_2^{1,2}(\mathbb{R}^n)$.

The extension of the above results to Besov and Triebel-Lizorkin spaces is given by Bourduad in [7] and [8], Runst in [46], and Sickel in [51], [52] and [53]. Further results concerning the composition operators in Besov and Triebel-Lizorkin spaces are given [5], [9], [10], [12], [14] and [47]. Recently, Bourdaud and Moussai [13] proved the continuity of the composition operator in $W_p^m(\mathbb{R}^n) \cap \dot{W}_{mp}^1(\mathbb{R}^n)$ to itself, for every integer $m \geq 2$ and any $1 \leq p < \infty$ and in Sobolev spaces $W_p^m(\mathbb{R}^n)$, with $m \geq 2$ and $1 \leq p < \infty$. The author in [24] and [25] gave the necessary and sufficient conditions on G such that

$$G(W_p^m(\mathbb{R}^n,|\cdot|^{\alpha})) \subset W_p^m(\mathbb{R}^n,|\cdot|^{\alpha}),$$
 (Sobolev space of power weight),

with some suitable assumptions on m, p and α . The extension of Dahlberg result to Triebel-Lizorkin spaces of power weights $F_{p,q}^s(\mathbb{R}^n, |\cdot|^{\alpha})$ is given in [26].

1.1. Notation and conventions. Throughout this paper, we denote by \mathbb{R}^n the n-dimensional real Euclidean space, \mathbb{N} the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The letter \mathbb{Z} stands for the set of all integer numbers. The expression $f \lesssim g$ means that $f \leqslant c g$ for some independent constant c (and non-negative functions f and g), and $f \approx g$ means $f \lesssim g \lesssim f$. As usual for any $x \in \mathbb{R}$, $\lfloor x \rfloor$ stands for the largest integer smaller than or equal to x.

For $x \in \mathbb{R}^n$ and r > 0 we denote by B(x,r) the open ball in \mathbb{R}^n with center x and radius r. By supp f we denote the support of the function f, i.e., the closure of its non-zero set. If $E \subset \mathbb{R}^n$ is a measurable set, then |E| stands for the (Lebesgue) measure of E and χ_E denotes its characteristic function. For any u > 0, we set $C(u) = \{x \in \mathbb{R}^n : \frac{u}{2} < |x| \le u\}$. By c we denote generic positive constants, which may have different values at different occurrences.

Given a measurable set $E \subset \mathbb{R}^n$ and $0 , we denote by <math>L^p(E)$ the space of all functions $f: E \to \mathbb{C}$ equipped with the quasi-norm

$$||f||_{L^p(E)} = \left(\int_E |f(x)|^p dx\right)^{1/p} < \infty$$

with 0 and

$$||f||_{L^{\infty}(E)} = \operatorname{ess-sup}_{x \in E} |f(x)| < \infty.$$

If $E = \mathbb{R}^n$, then we put $L^p(\mathbb{R}^n) = L^p$ and $\|f\|_{L^p(\mathbb{R}^n)} = \|f\|_p$.

Let w denote a positive, locally integrable function and $0 . Then the weighted Lebesgue space <math>L^p(\mathbb{R}^n, w)$ contains all measurable functions f such that

$$||f||_{L^p(\mathbb{R}^n,w)} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx\right)^{1/p} < \infty.$$

If $1 \leqslant p \leqslant \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, then p' is called the conjugate exponent of p.

By $\mathcal{S}(\mathbb{R}^n)$ we denote the Schwartz space of all complex-valued, infinitely differentiable and rapidly decreasing functions on \mathbb{R}^n and by $\mathcal{S}'(\mathbb{R}^n)$ the dual space of all tempered distributions on \mathbb{R}^n . We define the Fourier transform of a function $f \in \mathcal{S}(\mathbb{R}^n)$ by

$$\mathcal{F}(f)(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) dx.$$

Its inverse is denoted by $\mathcal{F}^{-1}f$. Both \mathcal{F} and \mathcal{F}^{-1} are extended to the dual Schwartz space $\mathcal{S}'(\mathbb{R}^n)$ in the usual way.

For $v \in \mathbb{Z}$ and $m = (m_1, ..., m_n) \in \mathbb{Z}^n$, let $Q_{v,m}$ be the dyadic cube in \mathbb{R}^n , $Q_{v,m} = \{(x_1, ..., x_n) : m_i \leq 2^v x_i < m_i + 1, i = 1, 2, ..., n\}$. Also, we set $\chi_{j,m} = \chi_{Q_{j,m}}, j \in \mathbb{Z}, m \in \mathbb{Z}^n$.

Recall that $\eta_{R,m}(x) = R^n(1+R|x|)^{-m}$, for any $x \in \mathbb{R}^n$ and m, R > 0. Note that $\eta_{R,m} \in L^1(\mathbb{R}^n)$ when m > n and that $\|\eta_{R,m}\|_1 = c_m$ is independent of R, where this type of function was introduced in [19] and [31].

2. Function spaces

In this section we present the Fourier analytical definition of Herz-type Triebel-Lizorkin spaces and we present their basic properties such us Sobolev embeddings. We start by recalling the definition and some properties of Herz spaces. For convenience, we set

$$B_k = B(0, 2^k), \quad \bar{B}_k = \{x \in \mathbb{R}^n : |x| \le 2^k\}, \quad k \in \mathbb{Z}$$

and

$$R_k = B_k \setminus B_{k-1}, \quad \chi_k = \chi_{R_k}, \quad k \in \mathbb{Z}.$$

Definition 2.1. Let $0 < p, q \leqslant \infty$ and $\alpha \in \mathbb{R}$. The homogeneous Herz space $\dot{K}_{p,q}^{\alpha}$ is defined as the set of all $f \in L_{loc}^{p}(\mathbb{R}^{n} \setminus \{0\})$ such that

$$||f||_{\dot{K}_{p,q}^{\alpha}} = \left(\sum_{k \in \mathbb{Z}} 2^{k\alpha q} ||f\chi_k||_p^q\right)^{1/q} < \infty$$

(with the usual modifications when $q = \infty$).

Remark 2.2. Let $0 < p, q \leq \infty$ and $\alpha \in \mathbb{R}$.

(i) The space $\dot{K}_{p,p}^{\alpha}$ coincides with the Lebesgue space $L^p(\mathbb{R}^n,|\cdot|^{\alpha p})$. In addition

$$\dot{K}_{p,p}^0 = L^p.$$

(ii) Let $0 < q_1 \leqslant q_2 \leqslant \infty$. Then

$$\dot{K}^{\alpha}_{p,q_1} \hookrightarrow \dot{K}^{\alpha}_{p,q_2}.$$

(iii) The spaces $\dot{K}^{\alpha}_{p,q}$ are quasi-Banach spaces and if $\min(p,q)\geqslant 1$ then $\dot{K}^{\alpha}_{p,q}$ are Banach spaces.

Remark 2.3. A detailed discussion of the properties of Herz spaces my be found in [32] and [40], and references therein.

To present the definition of Herz-type Triebel-Lizorkin spaces, we first need the concept of a smooth dyadic resolution of unity. Let ψ be a function in $\mathcal{S}(\mathbb{R}^n)$ satisfying

$$0 \leqslant \psi \leqslant 1$$
 and $\psi(x) = \begin{cases} 1, & \text{if } |x| \leqslant 1, \\ 0, & \text{if } |x| \geqslant \frac{3}{2}. \end{cases}$

We put $\mathcal{F}\varphi_0 = \psi$, $\mathcal{F}\varphi_1 = \psi(\frac{\cdot}{2}) - \psi$ and $\mathcal{F}\varphi_j = \mathcal{F}\varphi_1(2^{1-j}\cdot)$ for $j = 2, 3, \ldots$ Then $\{\mathcal{F}\varphi_j\}_{j \in \mathbb{N}_0}$ is a smooth dyadic resolution of unity, $\sum_{j=0}^{\infty} \mathcal{F}\varphi_j(x) = 1$ for all $x \in \mathbb{R}^n$. Thus we obtain the Littlewood-Paley decomposition

$$f = \sum_{j=0}^{\infty} \varphi_j * f$$

of all $f \in \mathcal{S}'(\mathbb{R}^n)$ (convergence in $\mathcal{S}'(\mathbb{R}^n)$).

We are now in a position to state the definition of Herz-type Triebel-Lizorkin spaces.

Definition 2.4. Let $\alpha, s \in \mathbb{R}, 0 < p, q < \infty$ and $0 < \beta \leqslant \infty$. The Herz-type Triebel-Lizorkin space $\dot{K}_{p,q}^{\alpha} F_{\beta}^{s}$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^{n})$ such that

$$||f||_{\dot{K}_{p,q}^{\alpha}F_{\beta}^{s}} = \left\| \left(\sum_{j=0}^{\infty} 2^{js\beta} |\varphi_{j} * f|^{\beta} \right)^{1/\beta} \right||_{\dot{K}_{p,q}^{\alpha}} < \infty,$$

with the obvious modification if $\beta = \infty$.

Remark 2.5. Let $s \in \mathbb{R}, 0 < p, q < \infty, 0 < \beta \leqslant \infty$ and $\alpha > -\frac{n}{p}$. The spaces $\dot{K}_{p,q}^{\alpha} F_{\beta}^{s}$ are independent of the particular choice of the smooth dyadic resolution of unity $\{\mathcal{F}\varphi_{j}\}_{j\in\mathbb{N}_{0}}$ (in the sense of equivalent quasi-norms). In particular $\dot{K}_{p,q}^{\alpha} F_{\beta}^{s}$ are quasi-Banach spaces and if $p, q, \beta \geqslant 1$, then they are Banach spaces. Further results, concerning, for instance, lifting properties, Fourier multiplier and local means characterizations can be found in [20]-[21]-[22], [65] and [66].

Now we give the definition of the spaces $F_{p,\beta}^s$.

Definition 2.6. Let $s \in \mathbb{R}$, $0 and <math>0 < \beta \leq \infty$. The Triebel-Lizorkin space $F_{p,\beta}^s$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\left\|f\right\|_{F_{p,\beta}^{s}} = \left\|\left(\sum_{j=0}^{\infty} 2^{js\beta} \left|\varphi_{j} * f\right|^{\beta}\right)^{1/\beta}\right\|_{p} < \infty.$$

The theory of the spaces $F_{p,\beta}^s$ has been developed in detail in [48], [58] and [59] but has a longer history already including many contributors; we do not want to discuss this here. Clearly, for $s \in \mathbb{R}$, $0 and <math>0 < \beta \leq \infty$,

$$\dot{K}_{p,p}^0 F_{\beta}^s = F_{p,\beta}^s.$$

Let $w \in \mathcal{A}_{\infty}$, Muckenhoupt classes, $s \in \mathbb{R}$, $0 < \beta \leq \infty$ and $0 . We define weighted Triebel-Lizorkin space <math>F_{p,\beta}^s(\mathbb{R}^n, w)$ to be the set of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\left\| f \right\|_{F_{p,\beta}^s(\mathbb{R}^n,w)} = \left\| \left(\sum_{j=0}^{\infty} 2^{js\beta} \left| \varphi_j * f \right|^{\beta} \right)^{1/\beta} \right\|_{L^p(\mathbb{R}^n,w)}$$

is finite. In the limiting case $\beta = \infty$ the usual modification is required.

The spaces $F_{p,\beta}^s(\mathbb{R}^n,w)=F_{p,\beta}^s(w)$ are independent of the particular choice of the smooth dyadic resolution of unity $\{\mathcal{F}\varphi_j\}_{j\in\mathbb{N}_0}$ appearing in their definitions. They are quasi-Banach spaces (Banach spaces for $p,q\geqslant 1$). Moreover, for $w\equiv 1$ we obtain the usual (unweighted) Triebel-Lizorkin spaces. We refer, in particular, to the papers [15] and [34] for a comprehensive treatment of weighted function spaces. Let w_γ be a power weight, i.e., $w_\gamma(x)=|x|^\gamma$ with $\gamma>-n$. Then we have

$$F_{p,\beta}^s(w_\gamma) = \dot{K}_{p,p}^{\frac{\gamma}{p}} F_{\beta}^s,$$

in the sense of equivalent quasi-norms.

Definition 2.7. (i) Let $1 and <math>s \in \mathbb{R}$. Then the Herz-type Bessel potential space $\dot{k}_{p,s}^{\alpha,q}$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$||f||_{\dot{k}_{p,s}^{\alpha,q}} = ||(1+|\xi|^2)^{\frac{s}{2}} * f||_{\dot{K}_{p,q}^{\alpha}} < \infty.$$

(ii) Let $1 and <math>m \in \mathbb{N}$. The homogeneous Herz-type Sobolev space $\dot{W}_{p,m}^{\alpha,q}$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$||f||_{\dot{W}_{p,m}^{\alpha,q}} = \sum_{|\beta| \leqslant m} \left\| \frac{\partial^{\beta} f}{\partial x^{\beta}} \right\|_{\dot{K}_{p,q}^{\alpha}} < \infty,$$

where the derivatives must be understood in the sense of distribution.

In the following, we will present the connection between the Herz-type Triebel-Lizorkin spaces and the Herz-type Bessel potential spaces; see [39] and [64]. Let $1 < p, q < \infty$ and $-\frac{n}{p} < \alpha < n(1-\frac{1}{p})$. If $s \in \mathbb{R}$, then

$$\dot{K}_{p,q}^{\alpha} F_2^s = \dot{k}_{p,s}^{\alpha,q}$$

with equivalent norms. If $s = m \in \mathbb{N}$, then

$$\dot{K}_{p,q}^{\alpha} F_2^m = \dot{W}_{p,m}^{\alpha,q}$$

with equivalent norms. In particular

$$\dot{K}^{\alpha}_{p,p}F^m_2=W^p_m(\mathbb{R}^n,|\cdot|^{\alpha p})\quad \text{(Sobolev spaces of power weights)}$$

and

$$\dot{K}_{p,p}^0 F_2^m = W_m^p \quad \text{(Sobolev spaces)}, \quad \dot{K}_{p,q}^\alpha F_2^0 = \dot{K}_{p,q}^\alpha. \tag{2.8}$$

Let $0 < \theta < 1, 0 < p_0, p_1, q_0, q_1 < \infty, 0 < \beta_0, \beta_1 \leq \infty \text{ and } \alpha_0, \alpha_1, s_0, s_1 \in \mathbb{R}$. We set

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad \frac{1}{\beta} = \frac{1-\theta}{\beta_0} + \frac{\theta}{\beta_1}$$

and

$$\alpha = (1 - \theta)\alpha_0 + \theta\alpha_1, \quad s = (1 - \theta)s_0 + \theta s_1.$$

As an immediate consequence of Hölder's inequality we have the so-called interpolation inequalities:

$$||f||_{\dot{K}^{\alpha}_{p,q}F^{s}_{\beta}} \leq ||f||_{\dot{K}^{\alpha_{0}}_{p_{0},q_{0}},F^{s_{0}}_{\beta_{0}}}^{1-\theta} ||f||_{\dot{K}^{\alpha_{1}}_{p_{1},q_{1}}F^{s_{1}}_{\beta_{1}}}^{\theta}$$

$$(2.9)$$

holds for all $f \in \dot{K}^{\alpha_0}_{p_0,q_0} F^{s_0}_{\beta_0} \cap \dot{K}^{\alpha_1}_{p_1,q_1} F^{s_1}_{\beta_1}$. We collect some embeddings on these functions spaces as obtained in [21].

Theorem 2.10. Let $\alpha_1, \alpha_2, s_1, s_2 \in \mathbb{R}, 0 < s, p, q, r < \infty, 0 < \beta \leq \infty, \alpha_1 > -\frac{n}{s}$ and $\alpha_2 > -\frac{n}{q}$. We suppose that

$$s_1 - \frac{n}{s} - \alpha_1 = s_2 - \frac{n}{q} - \alpha_2.$$

Let $0 < q \leqslant s < \infty$ and $\alpha_2 \geqslant \alpha_1$. The embedding

$$\dot{K}_{q,r}^{\alpha_2} F_{\infty}^{s_2} \hookrightarrow \dot{K}_{s,p}^{\alpha_1} F_{\beta}^{s_1}$$

holds if $0 < r \le p < \infty$.

Let $0 < p, q < \infty$. For later use, we introduce the following abbreviations:

$$\sigma_p = n \max\left(\frac{1}{p} - 1, 0\right)$$
 and $\sigma_{p,q} = n \max\left(\frac{1}{p} - 1, \frac{1}{q} - 1, 0\right)$.

In the next we shall interpret L_{loc}^1 as the set of regular distributions, see [23].

Theorem 2.11. Let $0 < p, q < \infty, 0 < \beta \leqslant \infty, \alpha > -\frac{n}{p}$ and $s > \max(\sigma_p, \frac{n}{p} + \alpha - n)$. Then

$$\dot{K}_{p,q}^{\alpha}F_{\beta}^{s}\hookrightarrow L_{\mathrm{loc}}^{1}.$$

For any a > 0, $f \in \mathcal{S}'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we denote, Peetre maximal function,

$$\varphi_j^{*,a} f(x) = \sup_{y \in \mathbb{R}^n} \frac{|\varphi_j * f(y)|}{(1 + 2^j |x - y|)^a}, \quad j \in \mathbb{N}_0.$$

We now present a fundamental characterization of the above spaces, which plays an essential role in this paper, see [66, Theorem 1].

Theorem 2.12. Let $s \in \mathbb{R}, 0 < p, q < \infty, 0 < \beta \leqslant \infty$ and $\alpha > -\frac{n}{p}$. Let $a > \frac{n}{\min(p,\beta)}$. Then

$$||f||_{\dot{K}_{p,q}^{\alpha}F_{\beta}^{s}}^{\star} = \left\| \left(\sum_{j=0}^{\infty} 2^{js\beta} (\varphi_{j}^{*,a}f)^{\beta} \right)^{1/\beta} \right\|_{\dot{K}_{p,q}^{\alpha}},$$

is an equivalent quasi-norm in $\dot{K}^{\alpha}_{p,q}F^{s}_{\beta}$.

3. Composition operators

Let $G: \mathbb{R} \to \mathbb{R}$ be a continuous function. To solve (1.4), we study the action of the nonlinear function G on Herz-type Triebel-Lizorkin spaces. Let us recall some results obtained in [26], where they proved for Triebel-Lizorkin spaces of power weights, but the results can be easily expanded to Herz-type Triebel-Lizorkin spaces. Let $1 < p, q < \infty, 0 < \beta \leqslant \infty, 0 \leqslant \alpha < n - \frac{n}{p}$. Let T_G be a composition operator, or Nemytzkij operators, such that

$$T_G(\dot{\mathbb{K}}_{p,q}^{\alpha}\mathbb{F}_{\beta}^s) \subset \dot{\mathbb{K}}_{p,q}^{\alpha}\mathbb{F}_{\beta}^s,$$
 (3.1)

where $\dot{\mathbb{K}}_{p,q}^{\alpha}\mathbb{F}_{\beta}^{s}$ is the real-valued part of $\dot{K}_{p,q}^{\alpha}F_{\beta}^{s}$. If $s>\frac{n}{p}+\alpha$, then $G'\in L_{\mathrm{loc}}^{\infty}(\mathbb{R})$ is necessary. In the the case $0< s\leqslant \frac{n}{p}+\alpha$, we have $G'\in L^{\infty}(\mathbb{R})$ is necessary.

Now, let $1 < p, q < \infty, 0 < \beta \leq \infty, 0 \leq \alpha < n - \frac{n}{p}$ and $G \in C^2(\mathbb{R})$. Let T_G be a composition operator with (3.1) and

$$1 + \frac{1}{p} < s < \frac{n}{p} + \alpha.$$

Then G(t) = ct for some constant c.

In this section we investigate sufficient conditions on G such that (3.1) holds. First we need the following lemma, which is basically a consequence of Hardy's inequality in the sequence Lebesgue space ℓ_q .

Lemma 3.2. Let 0 < a < 1 and $0 < q \le \infty$. Let $\{\varepsilon_k\}_{k \in \mathbb{N}_0}$ be a sequences of positive real numbers and denote $\delta_k = \sum_{j=0}^k a^{k-j} \varepsilon_j$ and $\eta_k = \sum_{j=k}^\infty a^{j-k} \varepsilon_j$, $k \in \mathbb{N}_0$. Then there exists a constant c > 0 depending only on a and q such that

$$\left(\sum_{k=0}^{\infty} \delta_k^q\right)^{1/q} + \left(\sum_{k=0}^{\infty} \eta_k^q\right)^{1/q} \leqslant c \left(\sum_{k=0}^{\infty} \varepsilon_k^q\right)^{1/q}.$$

As usual, we put

$$\mathcal{M}(f)(x) = \sup_{Q} \frac{1}{|Q|} \int_{Q} |f(y)| \, dy, \quad f \in L^{1}_{\text{loc}},$$

where the supremum is taken over all cubes with sides parallel to the axis and $x \in Q$. Also, we set $\mathcal{M}_{\sigma}(f) = (\mathcal{M}(|f|^{\sigma}))^{\frac{1}{\sigma}}, 0 < \sigma < \infty$.

Various important results have been proved in the space $\dot{K}^{\alpha}_{p,q}$ under some assumptions on α, p and q. The conditions $-\frac{n}{p} < \alpha < n(1-\frac{1}{p}), 1 < p < \infty$ and $0 < q \leqslant \infty$ is crucial in the study of the boundedness of classical operators in $\dot{K}^{\alpha}_{p,q}$ spaces. This fact was first realized by Li and Yang [38] with the proof of the boundedness of the maximal function. Some of our results of this paper are based on the following result, see Tang and Yang [54].

Lemma 3.3. Let $1 < \beta < \infty, 1 < p < \infty$ and $0 < q \leqslant \infty$. If $\{f_j\}_{j \in \mathbb{N}_0}$ is a sequence of locally integrable functions on \mathbb{R}^n and $-\frac{n}{p} < \alpha < n(1-\frac{1}{p})$, then

$$\left\| \left(\sum_{j=0}^{\infty} (\mathcal{M}(f_j))^{\beta} \right)^{1/\beta} \right\|_{\dot{K}_{p,q}^{\alpha}} \leqslant c \left\| \left(\sum_{j=0}^{\infty} |f_j|^{\beta} \right)^{1/\beta} \right\|_{\dot{K}_{p,q}^{\alpha}}.$$

Let $\mu > 0$ and $f \in L_{\text{loc}}^{\max(1,\mu)}$. Define

$$I_k^{\mu}(f)(x) = \int_{\bar{B}_{-k}} |f(x+z) - f(x)|^{\mu} dz, \quad x \in \mathbb{R}^n, k \in \mathbb{Z}.$$

Lemma 3.4. Let $0 < p, q < \infty, 0 < \beta \leqslant \infty, \alpha > -\frac{n}{p}$ and

$$\max\left(\sigma_{p,\beta}, \frac{n}{p} + \alpha - n\right) < s < \mu.$$

Then there exists a constant c > 0 such that

$$\left\| \left(\sum_{k=-\infty}^{\infty} 2^{(n+s)k\beta} |I_k^{\mu}(f)|^{\beta} \right)^{1/\beta} \right\|_{\dot{K}_{p,q}^{\alpha}} \leqslant c \|f\|_{\dot{K}_{p\mu,q\mu}^{\frac{\alpha}{\mu}} F_{\beta\mu}^{\frac{s}{\mu}}}^{\mu}$$
(3.5)

holds for all $f \in L^{\max(1,\mu)}_{\mathrm{loc}}$ with

$$f = \sum_{j=0}^{\infty} \varphi_j * f,$$

in L^{μ}_{loc} , with the obvious modification if $\beta = \infty$.

Proof. We will do the proof in two steps.

Step 1. We set $\Delta_y f(x) = f(x+y) - f(x), x, y \in \mathbb{R}^n$. A change of variable yields

$$2^{(n+s)k}I_k^{\mu}(f)(x) = 2^{sk} \int_{\bar{B}_0} |\Delta_{z2^{-k}}f(x)|^{\mu} dz \lesssim J_{1,k}(f)(x) + J_{2,k}(f)(x)$$

for all $x \in \mathbb{R}^n$, where the implicit constant is independent of x and k,

$$J_{1,k}(f)(x) = 2^{sk} \int_{\bar{B}_0} \left| \sum_{j=0}^k \Delta_{z2^{-k}} (\varphi_j * f)(x) \right|^{\mu} dz$$

and

$$J_{2,k}(f)(x) = 2^{sk} \int_{\bar{B}_0} \Big| \sum_{j=k+1}^{\infty} \Delta_{z2^{-k}}(\varphi_j * f)(x) \Big|^{\mu} dz.$$

Estimate of $J_{1,k}$. Let $\Psi, \Psi_0 \in \mathcal{S}(\mathbb{R}^n)$ be two functions such that $\mathcal{F}\Psi = 1$ and $\mathcal{F}\Psi_0 = 1$ on $\operatorname{supp}\varphi_1$ and $\operatorname{supp}\psi$, respectively. Using the mean value theorem we obtain for any $x \in \mathbb{R}^n$, $j \in \mathbb{N}_0$ and $|z| \leq 1$

$$|\Delta_{z2^{-k}}(\varphi_{j} * f)(x)| = |\Delta_{z2^{-k}}(\Psi_{j} * \varphi_{j} * f)(x)|$$

$$\leq 2^{-k} \sup_{|x-y| \leq c} \sum_{2^{-k}} |D^{\beta}(\Psi_{j} * \varphi_{j} * f)(y)|,$$

with some positive constant c independent of x, j and k, and

$$\Psi_j(\cdot) = 2^{(j-1)n} \Psi(2^{j-1} \cdot)$$
 for $j = 1, 2,$

We see that if $|\beta| = 1$ and a > 0

$$\left| D^{\beta}(\Psi_{j} * \varphi_{j} * f)(y) \right|$$

$$= 2^{(j-1)n} \left| \int_{\mathbb{R}^{n}} D^{\beta} \left(\Psi \left(2^{j-1} \left(y - z \right) \right) \right) \varphi_{j} * f(z) dz \right|$$

$$\leq 2^{(j-1)(n+1)} \int_{\mathbb{R}^{n}} \left| \left(D^{\beta} \Psi \right) \left(2^{j-1} \left(y - z \right) \right) \right| \left| \varphi_{j} * f(z) \right| dz.$$

$$(3.6)$$

The right-hand side in (3.6) may be estimated as follows:

$$c \ 2^{j(n+1)} \varphi_j^{*,a} f(y) \int_{\mathbb{R}^n} \left| \left(D^{\beta} \Psi \right) \left(2^{j-1} \left(y - z \right) \right) \right| \left(1 + 2^j \left| y - z \right| \right)^a dz$$

 $\leqslant c \ 2^j \varphi_j^{*,a} f(y).$

Then we obtain for any $x \in \mathbb{R}^n$, $|z| \leq 1$ and any $j, k \in \mathbb{N}_0$

$$|\Delta_{z2^{-k}}(\varphi_j * f)(x)| \leq c 2^{j-k} \sup_{|x-y| \leq c 2^{-k}} \varphi_j^{*,a} f(y)$$

$$\leq c 2^{j-k} (1+2^{j-k})^a \sup_{|x-y| \leq c 2^{-k}} \frac{\varphi_j^{*,a} f(y)}{(1+2^j |x-y|)^a}$$

$$\leq c 2^{j-k} \varphi_j^{*,a} f(x),$$

if $0 \leq j \leq k, k \in \mathbb{N}_0$ and $x \in \mathbb{R}^n$. Therefore

$$J_{1,k}(f)(x) \lesssim 2^{sk} \left(\sum_{j=0}^{k} 2^{j-k} \varphi_j^{*,a} f(x)\right)^{\mu},$$

where the implicit constant is independent of x and k, and this yields that

$$\left\| \left(\sum_{k=0}^{\infty} |J_{1,k}(f)|^{\beta} \right)^{1/\beta} \right\|_{\dot{K}_{p,q}^{\alpha}}$$

can be estimated by

$$c \left\| \left(\sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} 2^{(j-k)(1-\frac{s}{\mu})} 2^{j\frac{s}{\mu}} \varphi_{j}^{*,a} f \right)^{\mu\beta} \right)^{1/\mu\beta} \right\|_{\dot{K}^{\frac{\alpha}{\mu}}_{p\mu,q\mu}}^{\mu}.$$

Using Lemma 3.2 the last expression is bounded by

$$c \left\| \left(\sum_{k=0}^{\infty} \left(2^{k\frac{s}{\mu}} \varphi_k^{*,a} f \right)^{\mu \beta} \right)^{1/\mu \beta} \right\|_{\dot{K}^{\frac{\alpha}{\mu}}_{p\mu,q\mu}}^{\mu} \lesssim \left\| f \right\|_{\dot{K}^{\frac{\alpha}{\mu}}_{p\mu,q\mu} F_{\beta\mu}^{\frac{s}{\mu}}}^{\mu},$$

where we have used Theorem 2.12.

Estimate of $J_{2,k}$. We can distinguish two cases as follows:

• Case 1. $\min(p,\beta) > 1$. Therefore $s > \max\left(0,\frac{n}{p} + \alpha - n\right)$. Assume that $\alpha \ge n(1 - \frac{1}{p})$. Let $1 - \frac{s\min(p,\beta)}{n} < \lambda < \min(\frac{np}{n+\alpha p},\beta)$ be a strict positive real number, which is possible because of

$$s > \frac{n}{p} + \alpha - n > \frac{np(\frac{n}{p} + \alpha - n)}{\min(p, \beta)(n + \alpha p)} = \frac{n}{\min(p, \beta)} \left(1 - \frac{np}{n + \alpha p}\right).$$

Let $\frac{n}{\mu \min(p,\beta)} < a < \frac{s}{\mu(1-\lambda)}$. Then

$$\frac{s}{\mu} > a(1 - \lambda). \tag{3.7}$$

If $-\frac{n}{p} < \alpha < n(1-\frac{1}{p})$, then we take $\lambda = 1$. From this we deduce that for all $x \in \mathbb{R}^n$, $2^{-sk}J_{2,k}(f)(x)$ can be estimated by

$$c \sum_{j=k+1}^{\infty} 2^{(j-k)\varepsilon} \int_{\bar{B}_0} \left| \Delta_{z2^{-k}} (\varphi_j * f)(x) \right|^{\mu} dz$$

$$\lesssim \sum_{j=k+1}^{\infty} 2^{(j-k)\varepsilon} \sup_{x \in \bar{B}_0} \left| \Delta_{z2^{-k}} (\varphi_j * f)(x) \right|^{\mu(1-\lambda)} \int_{\bar{B}_0} \left| \Delta_{z2^{-k}} (\varphi_j * f)(x) \right|^{\mu\lambda} dz$$

where $0 < \frac{2\varepsilon}{\mu} \leqslant \frac{s}{\mu} - a(1-\lambda)$ and the positive constant c is independent of k and x. Observe that

$$\int_{\bar{B}_{0}} \left| \Delta_{z2^{-k}}(\varphi_{j} * f)(x) \right|^{\mu \lambda} dz$$

$$\lesssim \left| \varphi_{j} * f(x) \right|^{\mu \lambda} + 2^{kn} \int_{|y-x| \leq 2^{-k}} \left| \varphi_{j} * f(y) \right|^{\mu \lambda} dy$$

$$\lesssim \left| \varphi_{j} * f(x) \right|^{\mu \lambda} + \mathcal{M}(\left| \varphi_{j} * f \right|^{\mu \lambda})(x).$$

This estimate combined with

$$|\Delta_{z2^{-k}}(\varphi_j * f)(x)| \leqslant c \ 2^{(j-k)a} \varphi_j^{*,a} f(x) \tag{3.8}$$

for any $x \in \mathbb{R}^n$, $|z| \leq 1$ and any $j \geq k+1$, yield

$$J_{2,k}(f) \lesssim J_{2,k,1}(f) + J_{2,k,2}(f),$$

where

$$J_{2,k,1}(f) = \sum_{j=k+1}^{\infty} 2^{(j-k)(\varepsilon + a\mu(1-\lambda) - s)} \left(2^{j\frac{s}{\mu}} \varphi_j^{*,a} f \right)^{\mu(1-\lambda)} \left| 2^{j\frac{s}{\mu}} \varphi_j * f \right|^{\mu\lambda}$$

and

$$J_{2,k,2}(f) = \sum_{j=k+1}^{\infty} 2^{(j-k)(\varepsilon + a\mu(1-\lambda) - s)} \left(2^{j\frac{s}{\mu}} \varphi_j^{*,a} f \right)^{\mu(1-\lambda)} \mathcal{M} \left(2^{j\frac{s}{\mu}} |\varphi_j * f| \right)^{\mu\lambda}.$$

By similarity we estimate only $J_{2,k,2}(f)$. Using Lemma 3.2 and Hölder's inequality we get

$$\left(\sum_{k=0}^{\infty} (J_{2,k,2}(f))^{\beta}\right)^{1/\beta}
\lesssim \left(\sum_{k=0}^{\infty} \left(2^{k\frac{s}{\mu}} \varphi_k^{*,a} f\right)^{\mu(1-\lambda)\beta} \left(\mathcal{M}\left(2^{k\frac{s}{\mu}} | \varphi_k * f|\right)^{\mu\lambda}\right)^{\beta}\right)^{1/\beta}
\lesssim \left(\sum_{k=0}^{\infty} \left(2^{k\frac{s}{\mu}} \varphi_k^{*,a} f\right)^{\mu\beta}\right)^{(1-\lambda)/\beta} \left(\sum_{k=0}^{\infty} \left(\mathcal{M}\left(2^{k\frac{s}{\mu}} | \varphi_k * f|\right)^{\mu\lambda}\right)^{\beta/\lambda}\right)^{\lambda/\beta}.$$

Again by Hölder's inequality

$$\left\| \left(\sum_{k=0}^{\infty} (J_{2,k,2}(f))^{\beta} \right)^{1/\beta} \right\|_{\dot{K}_{p,q}^{\alpha}},$$

can be estimated by

$$c \left\| \left(\sum_{k=0}^{\infty} \left(2^{k \frac{s}{\mu}} \varphi_k^{*,a} f \right)^{\mu \beta} \right)^{1/\beta} \right\|_{\dot{K}_{p,q}^{\alpha}}^{1-\lambda}$$

$$\times \left\| \left(\sum_{k=0}^{\infty} \left(\mathcal{M} \left(2^{k \frac{s}{\mu}} | \varphi_k * f | \right)^{\mu \lambda} \right)^{\beta/\lambda} \right)^{\lambda/\beta} \right\|_{\dot{K}_{\frac{p}{\lambda}, \frac{q}{\lambda}}^{\alpha\lambda}}$$

$$\lesssim \left\| f \right\|_{\dot{K}_{p\mu,q\mu}^{\alpha\mu} F_{\beta\mu}^{\frac{s}{\mu}}}^{(1-\lambda)\mu} \left\| \left(\sum_{k=0}^{\infty} \left(2^{k \frac{s}{\mu}} | \varphi_k * f | \right)^{\mu \beta} \right)^{1/\mu \beta} \right\|_{\dot{K}_{p\mu,q\mu}^{\alpha\mu}}^{\lambda\mu}$$

where we have used Theorem 2.12 and Lemma 3.3. Obviously we can estimate the last term by

$$c \|f\|^{\mu}_{\dot{K}^{\frac{\alpha}{\mu}}_{p\mu,q\mu}F^{\frac{s}{\mu}}_{\beta\mu}}.$$

• Case 2. $\min(p,\beta) \leq 1$. If $-\frac{n}{p} < \alpha < n(1-\frac{1}{p})$, then $s > \frac{n}{\min(p,\beta)} - n$. Taking $\max(0,1-\frac{s\min(p,\beta)}{n}) < \lambda < \min(1,p,\beta)$. The same arguments as in Case 1 yield the desired estimate. Now assume that $\alpha \geq n(1-\frac{1}{n})$. Therefore

$$s > \max\left(\frac{n}{\min(p,\beta)} - n, \frac{n}{p} + \alpha - n\right).$$

Taking $\max(0, 1 - \frac{s \min(p, \beta)}{n}) < \lambda < \min(p, \frac{np}{n + \alpha p}, \beta)$. The desired estimate can be done in the same manner as in Case 1.

Step 2. We will estimate

$$\left\| \left(\sum_{k=-\infty}^{-1} 2^{(n+s)k\beta} |I_k^{\mu}(f)|^{\beta} \right)^{1/\beta} \right\|_{\dot{K}_{p,q}^{\alpha}}.$$

We employ the same notations as in Step 1. Recall that

$$f = \sum_{j=0}^{\infty} \varphi_j * f.$$

Define

$$M_{k,2}(f)(x) = \int_{\bar{B}_0} \left| \sum_{j=0}^{\infty} \Delta_{z2^{-k}}(\varphi_j * f)(x) \right|^{\mu} dz.$$

As in the estimation of $J_{2,k}$, we obtain

$$M_{2,k}(f) \lesssim M_{2,k,1}(f) + M_{2,k,2}(f),$$

where

$$M_{2,k,1}(f) = 2^{-ka\mu(1-\lambda)} \sum_{j=0}^{\infty} 2^{j(\varepsilon + a\mu(1-\lambda) - s)} \left(2^{j\frac{s}{\mu}} \varphi_j^{*,a} f \right)^{\mu(1-\lambda)} \left| 2^{j\frac{s}{\mu}} \varphi_j * f \right|^{\mu\lambda}$$

and

$$M_{2,k,2}(f) = 2^{-ka\mu(1-\lambda)} \sum_{j=0}^{\infty} 2^{j(\varepsilon + a\mu(1-\lambda) - s)} \left(2^{j\frac{s}{\mu}} \varphi_j^{*,a} f \right)^{\mu(1-\lambda)} \mathcal{M} \left(2^{j\frac{s}{\mu}} |\varphi_j * f| \right)^{\mu\lambda},$$

with the help of (3.8). By similarity we estimate only $M_{2,k,2}$. Obviously

$$M_{2,k,2}(f) \lesssim 2^{-ka\mu(1-\lambda)} \sup_{j \in \mathbb{N}_0} \left(\left(2^{j\frac{s}{\mu}} \varphi_j^{*,a} f \right)^{\mu(1-\lambda)} \mathcal{M} \left(2^{j\frac{s}{\mu}} |\varphi_j * f| \right)^{\mu\lambda} \right)$$

and this yields that

$$\Big(\sum_{k=-\infty}^{-1} 2^{sk\beta} |M_{2,k,2}|^{\beta}\Big)^{1/\beta} \lesssim \sup_{j \in \mathbb{N}_0} \Big(\Big(2^{j\frac{s}{\mu}} \varphi_j^{*,a} f\Big)^{\mu(1-\lambda)} \mathcal{M} \Big(2^{j\frac{s}{\mu}} |\varphi_j * f|\Big)^{\mu\lambda} \Big).$$

By the same arguments as used in Step 1 we obtain the desired estimate. The proof is complete. $\hfill\Box$

Now we present the case of $s = \mu$, where the proof is very similar to Lemma 3.4.

Lemma 3.9. Let $0 < p, q < \infty, \alpha > -\frac{n}{p}$ and

$$\max\left(\sigma_p, \frac{n}{p} + \alpha - n\right) < \mu.$$

Then there exists a positive constant c such that

$$\left\| \sup_{k \in \mathbb{Z}} 2^{(n+\mu)k} |I_k^{\mu}(f)| \right\|_{\dot{K}_{p,q}^{\alpha}} \leqslant c \|f\|_{\dot{K}_{p\mu,q\mu}^{\alpha} F_1^1}^{\mu}$$

holds for all $f \in L^{\max(1,\mu)}_{\mathrm{loc}}$ with

$$f = \sum_{j=0}^{\infty} \varphi_j * f,$$

in $L_{\rm loc}^{\mu}$.

Using the fact that $||f||_{\dot{K}^{\frac{\alpha}{\mu}}_{p\mu,q\mu}F^{\frac{s}{\mu}}_{\beta\mu}} \leq ||f||_{\dot{K}^{\alpha}_{p,q}F^{s}_{\beta}} ||f||_{\infty}^{\mu-1}$, we we immediately arrive at the following results.

Lemma 3.10. Let $0 < p, q < \infty, 0 < \beta \leqslant \infty, \alpha > -\frac{n}{p}$ and

$$\max\left(1, \sigma_{p,\beta}, \frac{n}{p} + \alpha - n\right) < s < \mu.$$

Then there exists a positive constant c such that

$$\left\| \left(\sum_{k=-\infty}^{\infty} 2^{(n+s)k\beta} |I_k^{\mu}(f)|^{\beta} \right)^{1/\beta} \right\|_{\dot{K}_{p,q}^{\alpha}} \leqslant c \|f\|_{\dot{K}_{p,q}^{\alpha} F_{\beta}^{s}} \|f\|_{\infty}^{\mu-1}$$

holds for all $f \in \dot{K}_{p,q}^{\alpha} F_{\beta}^{s} \cap L^{\infty}$.

Remark 3.11. Corresponding statements to Lemmas 3.4, 3.9 and 3.10 were proved by Runst [46, Lemma 1], with $\alpha=0, p=q$ and the case of bounded functions, while with $\alpha=0, p=q$ has been given by Sickel in [50, Lemmas 1,2]. In our proof we have used the ideas of [50, Lemmas 1, 2].

The next two lemmas are used in the proof of our results, see e.g. [2].

Lemma 3.12. Let $s \in \mathbb{R}$, $A, B > 0, 0 < p, q < \infty, 0 < \beta \leq \infty$ and $\alpha > -\frac{n}{p}$. Let $\{f_l\}_{l \in \mathbb{N}_0}$ be a sequence of functions such that

$$\operatorname{supp} \mathcal{F} f_0 \subseteq \{ \xi \in \mathbb{R}^n : |\xi| \leqslant A \}$$

and

$$\operatorname{supp} \mathcal{F} f_l \subseteq \left\{ \xi \in \mathbb{R}^n : B2^{l+1} \leqslant |\xi| \leqslant A2^{l+1} \right\}.$$

There exists a constant c > 0 such that the following inequality

$$\left\| \sum_{l=0}^{\infty} f_l \right\|_{\dot{K}_{p,q}^{\alpha} F_{\beta}^s} \leqslant c \left\| \left(\sum_{l=0}^{\infty} 2^{ls\beta} |f_l|^{\beta} \right)^{1/\beta} \right\|_{\dot{K}_{p,q}^{\alpha}}$$

holds.

Lemma 3.13. Let $A, B > 0, 0 < p, q < \infty, 0 < \beta \le \infty$ and $\alpha > -\frac{n}{p}$. Let $s > \max(\sigma_p, \frac{n}{p} + \alpha - n)$. Let $\{f_l\}_{l \in \mathbb{N}_0}$ be a sequence of functions such that

$$\operatorname{supp} \mathcal{F} f_l \subseteq \left\{ \xi \in \mathbb{R}^n : |\xi| \leqslant A 2^{l+1} \right\}.$$

Then it holds that

$$\left\| \sum_{l=0}^{\infty} f_l \right\|_{\dot{K}_{p,q}^{\alpha} F_{\beta}^{s}} \leqslant c \left\| \left(\sum_{l=0}^{\infty} 2^{ls\beta} |f_l|^{\beta} \right)^{1/\beta} \right\|_{\dot{K}_{p,q}^{\alpha}}.$$

Let $G: \mathbb{R} \to \mathbb{R}$ be a continuous function. We shall deal with sufficient conditions on G to guarantee an embedding

$$T_G(\dot{K}_{p,q}^{\alpha}F_{\beta}^s) = G(\dot{K}_{p,q}^{\alpha}F_{\beta}^s) \subset \dot{K}_{p,q}^{\alpha}F_{\beta}^s.$$

First we begin with the case where G is polynomial.

Theorem 3.14. Let $0 < p, q < \infty, 0 < \beta \leqslant \infty, s \geqslant \frac{n}{p} - \frac{n}{q}, \alpha \geqslant 0$ and

$$\max\left(0, \frac{n}{p} + \alpha - \frac{n}{m}\right) < s < \frac{n}{p} + \alpha, \quad m = 2, 3, \dots$$
 (3.15)

We put

$$s_m = s - (m-1)\left(\frac{n}{p} + \alpha - s\right).$$

Then

$$||f^m||_{\dot{K}^{\alpha}_{r,q}F^{sm}_{\rho}} \lesssim ||f||^m_{\dot{K}^{\alpha}_{r,q}F^{s}_{\rho}} \tag{3.16}$$

holds for all $f \in \dot{K}^{\alpha}_{p,q} F^s_{\beta}$.

Proof. We will do the proof into three steps.

Step 1. Preparation. Let $\{\mathcal{F}\varphi_j\}_{j\in\mathbb{N}_0}$ be a partition of unity and $f\in\mathcal{S}'(\mathbb{R}^n)$. We define the convolution operators Δ_j by the following:

$$\Delta_j f = \varphi_j * f, \quad j \in \mathbb{N} \quad \text{and} \quad \Delta_0 f = \varphi_0 * f = \mathcal{F}^{-1} \psi * f.$$

We define the convolution operators Q_j , $j \in \mathbb{N}_0$ by the following:

$$Q_j f = \mathcal{F}^{-1} \psi_j * f, \quad j \in \mathbb{N}_0,$$

where $\mathcal{F}^{-1}\psi_j = 2^{jn}\mathcal{F}^{-1}\psi(2^j\cdot)$ and we see that

$$Q_j f = \sum_{k=0}^j \Delta_k f, \quad j \in \mathbb{N}_0.$$

For all $f_i \in \mathcal{S}'(\mathbb{R}^n)$, i = 1, 2, ..., m the product $\prod_{i=1}^m f_i$ is defined by

$$\prod_{i=1}^{m} f_i = \lim_{j \to \infty} \prod_{i=1}^{m} Q_j f_i,$$

if the limit on the right-hand side exists in $\mathcal{S}'(\mathbb{R}^n)$. The following decomposition of this product is given in [47, Chapter 4]. We have the following formal decomposition:

$$\prod_{i=1}^{m} f_i = \sum_{k_1, \dots, k_m = 0}^{\infty} \prod_{i=1}^{m} (\Delta_{k_i} f_i).$$

The fundamental idea is to split $\prod_{i=1}^{m} f_i$ into two parts, both of them being always defined. Let N be a natural number greater than $1 + \log_2 3 (m - 1)$. Then we have the following decomposition:

$$\prod_{i=1}^{m} f_{i} = \sum_{j=0}^{\infty} \left[Q_{j-N} f_{1} \cdot \ldots \cdot Q_{j-N} f_{m-1} \cdot \Delta_{j} f_{m} + \ldots + (\prod_{l \neq k} Q_{j-N} f_{l}) \Delta_{k} f_{j} + \ldots + \Delta_{j} f_{1} \cdot Q_{j-N} f_{2} \cdot \ldots \cdot Q_{j-N} f_{m} \right] + \sum_{j=0}^{\infty} \sum_{j=0}^{j} (\Delta_{k_{1}} f_{1}) \cdot \ldots \cdot (\Delta_{k_{m}} f_{m}),$$

where the \sum^j is taken over all $k\in\mathbb{Z}_+^n$ such that

$$\max_{\ell=1,\dots,m} k_1 = k_{k_{m_0}} = j$$
 and $\max_{\ell \neq m_0} |\ell - k_{\ell}| < N$.

Of course, if k < 0 we put $\Delta_k f = 0$. Probably $\sum_{j=1}^{j}$ becomes more transparent by restricting to a typical part, which can be taken to be

$$\left(\prod_{i\in I_1}\Delta_j f_i\right)\prod_{i\in I_2}Q_j f_i,$$

where

$$I_1, I_2 \subset \{1, ..., m\}, \quad I_1 \cap I_2 = \emptyset, \quad I_1 \cup I_2 = \{1, ..., m\} = I, \quad |I_1| \geqslant 2.$$

We introduce the following notations

$$\Pi_{1,k}(f_1, f_2, ..., f_m) = \sum_{j=N}^{\infty} \left(\prod_{i \neq k} Q_{j-N} f_i \right) \Delta_j f_k$$

and

$$\Pi_2(f_1, f_2, ..., f_m) = \sum_{j=0}^{\infty} \sum_{i=1}^{j} \left(\prod_{i=1}^{m} \Delta_{k_i} f_i \right).$$

The advantage of the above decomposition is based on

supp
$$\mathcal{F}\Big(\Big(\prod_{i\neq k}Q_{j-N}f_i\Big)\Delta_jf_k\Big)\subset \{\xi\in\mathbb{R}^n: 2^{j-1}\leqslant |\xi|\leqslant 2^{j+1}\},\quad j\geqslant N$$

and

supp
$$\mathcal{F}\left(\sum_{i=1}^{j}\left(\prod_{i=1}^{m}\Delta_{k_{i}}f_{i}\right)\right)\subset\left\{\xi\in\mathbb{R}^{n}:\left|\xi\right|\leqslant2^{j+N-2}\right\},\quad j\in\mathbb{N}_{0}.$$

Step 2. We will prove (3.16). Observe that we need only to estimate

$$\Pi_1(f, f, ..., f) = \sum_{j=N}^{\infty} (Q_{j-N}f)^{m-1} \Delta_j f$$

and

$$\Pi_2(f, f, ..., f) = \sum_{j=0}^{\infty} (\Delta_j f)^{|I_1|} (Q_j f)^{|I_2|}.$$

Define

$$\frac{1}{v} = \frac{1}{p} + (m-1)\left(\frac{1}{p} - \frac{s}{n}\right).$$

Therefore we have the following Sobolev embeddings

$$\dot{K}_{v,q}^{\alpha m} F_{\beta}^s \hookrightarrow \dot{K}_{p,q}^{\alpha} F_{\beta}^{s_m}.$$

Lemma 3.12 gives

$$\|\Pi_1(f,f,...,f)\|_{\dot{K}_{v,q}^{\alpha m}F_{\beta}^s}$$

can be estimated by

$$c \left\| \left(\sum_{j=N}^{\infty} |2^{js} (Q_{j-N} f)^{m-1} \Delta_j f|^{\beta} \right)^{\frac{1}{\beta}} \right\|_{\dot{K}_{v,q}^{\alpha m}}$$

$$\lesssim \left\| \left(\sup_{j \geqslant N} |Q_{j-N} f| \right)^{m-1} \left(\sum_{j=N}^{\infty} |2^{js} \Delta_j f|^{\beta} \right)^{\frac{1}{\beta}} \right\|_{\dot{K}_{v,q}^{\alpha m}}.$$

By Hölder's inequality we estimate the last term by

$$c \|\sup_{j\geqslant N} |Q_{j-N}f|\|_{\dot{K}^{\alpha}_{b,\infty}}^{m-1} \|f\|_{\dot{K}^{\alpha}_{p,q}F^{s}_{\beta}},$$

with $\frac{1}{b} = \frac{1}{p} - \frac{s}{n}$. Recall that

$$\begin{aligned} \|\sup_{j\geqslant N} |Q_{j-N}f|\|_{\dot{K}^{\alpha}_{b,\infty}} &\lesssim \|\sup_{j\geqslant N} |Q_{j-N}f|\|_{\dot{K}^{\alpha}_{b,b}} \\ &\lesssim \|f\|_{F^{0}_{b,2}(\mathbb{R}^{n},|\cdot|^{\alpha b})} \\ &\lesssim \|f\|_{\dot{K}^{\alpha}_{b,b}F^{0}_{2}}, \end{aligned} \tag{3.17}$$

see [15, Theorem 1.4], because of $-\frac{n}{b} < \alpha < n(1-\frac{1}{b})$. Since, $s \ge \frac{n}{p} - \frac{n}{q}$, thanks to the embedding

$$\dot{K}_{p,q}^{\alpha} F_{\beta}^{s} \hookrightarrow \dot{K}_{b,b}^{\alpha} F_{2}^{0}, \tag{3.18}$$

see Theorem 2.10, we obtain

$$\|\Pi_1(f, f, ..., f)\|_{\dot{K}_{\eta, \sigma}^{\alpha m} F_s^s} \lesssim \|f\|_{\dot{K}_{\eta, \sigma}^{\alpha} F_s^s}^m.$$

Now we estimate $\Pi_2(f, f, ..., f)$. Define

$$\frac{1}{u} = \frac{|I_1|}{p} + \frac{|I_2|}{b}, \quad \sigma - \frac{n}{u} - \alpha m = s_m - \frac{n}{p} - \alpha.$$

Observe that $\sigma = |I_1|s$. Hence

$$\dot{K}_{u,\frac{q}{|I_1|}}^{\alpha m} F_{\frac{\beta}{|I_1|}}^{|I_1|s} \hookrightarrow \dot{K}_{p,q}^{\alpha} F_{\beta}^{s_m}.$$

From (3.15) it follows that $\sigma > \max \left(0, \frac{n}{u} + \alpha m - n\right)$. Lemma 3.13 gives

$$\begin{split} & \left\| \Pi_{2}(f,f,...,f) \right\|_{\dot{K}_{u,\frac{q}{|I_{1}|}}^{\alpha m} F_{\frac{\beta}{|I_{1}|}}^{|I_{1}|s}} \\ & \lesssim \left\| \left(\sum_{j=0}^{\infty} |2^{j|I_{1}|s} (Q_{j}f)^{|I_{2}|} (\Delta_{j}f)^{|I_{1}|} |^{\frac{\beta}{|I_{1}|}} \right)^{\frac{|I_{1}|}{\beta}} \right\|_{\dot{K}_{u,\frac{q}{|I_{1}|}}^{\alpha m}} \\ & \lesssim \left\| (\sup_{j\geqslant 0} |Q_{j}f|)^{|I_{2}|} \left(\sum_{j=0}^{\infty} |2^{js}\Delta_{j}f|^{\beta} \right)^{\frac{|I_{1}|}{\beta}} \right\|_{\dot{K}_{u,\frac{q}{|I_{1}|}}^{\alpha m}}. \end{split}$$

Again, by Hölder's inequality we estimate the last term by

$$c \|\sup_{j\geqslant N} |Q_j f|\|_{\dot{K}^{\alpha}_{b,\infty}}^{|I_2|} \| \Big(\sum_{j=0}^{\infty} |2^{js} \Delta_j f|^{\beta} \Big)^{\frac{1}{\beta}} \|_{\dot{K}^{\alpha}_{p,q}}^{|I_1|} \lesssim \|f\|_{\dot{K}^{\alpha}_{p,q}F^s_{\beta}}^{m},$$

where we have used (3.17) and (3.18).

Theorem 3.19. Let $0 < p, q < \infty, 0 < \beta \leq \infty, \alpha \geq 0$ and

$$s > \max\left(0, \frac{n}{p} + \alpha - n\right), \quad m = 2, 3, \dots$$

Then

$$\left\|f^m\right\|_{\dot{K}^\alpha_{p,q}F^s_\beta}\lesssim \left\|f\right\|_{\dot{K}^\alpha_{p,q}F^s_\beta}\left\|f\right\|_\infty^{m-1}$$

holds for all $f \in \dot{K}^{\alpha}_{p,q} F^s_{\beta} \cap L^{\infty}$.

Proof. First, we estimate $\Pi_1(f, f, ..., f)$. Recall that

$$\sup_{j \in \mathbb{N}_0} |Q_j f| \lesssim \|f\|_{\infty} \quad \text{and} \quad \sup_{j \in \mathbb{N}_0} |\Delta_j f| \lesssim \|f\|_{\infty}. \tag{3.20}$$

Lemma 3.12 gives

$$\|\Pi_1(f,f,...,f)\|_{\dot{K}^{\alpha}_{p,q}F^s_{\beta}}$$

can be estimated by

$$c \left\| \left(\sum_{j=N}^{\infty} |2^{js} (Q_{j-N} f)^{m-1} \Delta_j f|^{\beta} \right)^{\frac{1}{\beta}} \right\|_{\dot{K}^{\alpha}_{p,q}}$$

$$\lesssim \left\| \left(\sup_{j \geqslant N} |Q_{j-N} f| \right)^{m-1} \left(\sum_{j=N}^{\infty} |2^{js} \Delta_j f|^{\beta} \right)^{\frac{1}{\beta}} \right\|_{\dot{K}^{\alpha}_{p,q}}$$

$$\lesssim \left\| f \right\|_{\infty}^{m-1} \left\| f \right\|_{\dot{K}^{\alpha}_{p,q} F^s_{\beta}},$$

where we used (3.20). Lemma 3.13 gives

$$\|\Pi_{2}(f, f, ..., f)\|_{\dot{K}_{p,q}^{\alpha} F_{\beta}^{s}} \lesssim \|\left(\sum_{j=0}^{\infty} |2^{js} (Q_{j}f)^{|I_{2}|} (\Delta_{j}f)^{|I_{1}|}|^{\beta}\right)^{\frac{1}{\beta}} \|_{\dot{K}_{p,q}^{\alpha}}$$

$$\lesssim \|f\|_{\infty}^{m-1} \|\left(\sum_{j=0}^{\infty} |2^{js} \Delta_{j}f|^{\beta}\right)^{\frac{1}{\beta}} \|_{\dot{K}_{p,q}^{\alpha}}$$

$$\lesssim \|f\|_{\infty}^{m-1} \|f\|_{\dot{K}_{p,q}^{\alpha} F_{\beta}^{s}},$$

with the help of (3.20).

Remark 3.21. Theorem 3.14 in the case m=2, p=q and $\alpha=0$ is contained in [68] and also in [49]. For m>2, p=q and $\alpha=0$ see [50, Remark 17] and [47, p. 291]. We refer the reader to the monograph [47] and the paper [36] for further details, historical remarks and more references on multiplication in Besov and Triebel-Lizorkin spaces.

Definition 3.22. Let $\mu > 0$. Let $L \in \mathbb{N}_0$, and let $0 < \nu \leqslant 1$ such that $\mu = L + \nu$. The spaces $Lip\mu$ is the collection of all $f \in C^{L,loc}(\mathbb{R})$ such that

$$f^{(l)}(0) = 0, \quad l = 0, 1, 2, ..., L$$

and

$$\sup_{t_0, t_1 \in \mathbb{R}} \frac{|f^{(L)}(t_0) - f^{(L)}(t_1)|}{|t_0 - t_1|^{\nu}} < \infty.$$

Then we put

$$||f||_{Lip\mu} = \sum_{j=0}^{L-1} \sup_{t \in \mathbb{R}} \frac{|f^{(j)}(t)|}{|t|^{\mu-j}} + \sup_{t_0, t_1 \in \mathbb{R}} \frac{|f^{(L)}(t_0) - f^{(L)}(t_1)|}{|t_0 - t_1|^{\nu}}.$$

Remark 3.23. $\|\cdot\|_{Lip\mu}$ defines not a norm, but for simplicity we will use this notation, see [47, p. 295]. A typical example of a function belongs to $Lip\mu$ is $f(t) = |t|^{\mu}, \mu > 1$. Recall that $Lip\mu$ is not monotone with respect to μ .

We follow the same notations as in [47, Chapter 5].

Definition 3.24. (i) For $f \in \mathcal{S}'(\mathbb{R}^n)$ we define a distribution \bar{f} by

$$\bar{f}(\varphi) = \overline{f(\bar{\varphi})}, \quad \varphi \in \mathcal{S}(\mathbb{R}^n).$$

(ii) The space of real-valued distributions $\mathbb{S}'(\mathbb{R}^n)$ is defined to be

$$\mathbb{S}'(\mathbb{R}^n) = \{ f \in \mathcal{S}'(\mathbb{R}^n) : \bar{f} = f \}.$$

(iii) Let A be a complex-valued, quasi-normed distribution space such that $A \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$. Then we define the real-valued part \mathbb{A} of A to be the restriction of A to $\mathbb{S}'(\mathbb{R}^n)$ equipped with the same quasi-norm as A.

Now we are in position to state the first result of this section.

Theorem 3.25. Let $0 < p, q < \infty, 0 < \beta \leqslant \infty, \mu > 1, \alpha \geqslant 0, s \geqslant \frac{n}{p} - \frac{n}{q}$ and

$$0 < s < \frac{n}{p} + \alpha.$$

We put

$$s_{\mu} = s - (\mu - 1)(\frac{n}{p} + \alpha - s).$$

Let $G \in Lip\mu$ and

$$\max\left(0, \frac{n}{p} + \alpha - n\right) < s_{\mu} < \mu. \tag{3.26}$$

Then

$$\|G(f)\|_{\dot{K}^{\alpha}_{p,q}F^{s_{\mu}}_{\beta}} \lesssim \|G\|_{Lip\mu} \|f\|^{\mu}_{\dot{K}^{\alpha}_{p,q}F^{s}_{\infty}}$$

holds for any $f \in \dot{\mathbb{K}}_{p,q}^{\alpha} \mathbb{F}_{\infty}^{s}$.

Proof. We will do the proof in three steps.

Step 1. Preparation. Consider the partition of the unity $\{\mathcal{F}\varphi_j\}_{j\in\mathbb{N}_0}$. Let $f\in \dot{K}_{p,q}^{\alpha}F_{\infty}^s$. We set

$$\frac{1}{b} = \frac{\frac{n}{p} + \alpha - s}{n + \alpha p}$$
 and $\alpha_1 = \frac{\alpha p}{b}$.

Then

$$\max(1, p) < b < \infty \quad \text{and} \quad -\frac{n}{b} < \alpha_1 < \min\left(\alpha, n - \frac{n}{b}\right).$$
 (3.27)

Hence

$$\dot{K}_{p,q}^{\alpha} F_{\infty}^{s} \hookrightarrow \dot{K}_{b,r}^{\alpha_1}, \quad \max(1,q,\mu) < r.$$
 (3.28)

Since $G \in Lip\mu$, $\frac{b}{\mu} > 1$ and $\alpha_1 \mu < n - \frac{n\mu}{b}$, we have

$$G(f) \in \dot{K}^{\alpha_1 \mu}_{\frac{b}{\mu}, \frac{r}{\mu}} \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$$

and so we can interpret G as a mapping of a subspace of $\mathcal{S}'(\mathbb{R}^n)$ into $\mathcal{S}'(\mathbb{R}^n)$. In addition

$$f = \sum_{j=0}^{\infty} \varphi_j * f$$
, in $\dot{K}_{\mu,r}^{\alpha_1 - \frac{n}{\mu} + \frac{n}{b}}$. (3.29)

Indeed, let

$$\varrho_k = \sum_{j=0}^k \varphi_j * f, \quad k \in \mathbb{N}_0.$$

Obviously $\{\varrho_k\}$ converges to f in $\mathcal{S}'(\mathbb{R}^n)$ and by the embedding (3.28) we derive that $\{\varrho_k\}\subset \dot{K}_{b,r}^{\alpha_1}$. Furthermore, $\{\varrho_k\}$ is a Cauchy sequences in $\dot{K}_{b,r}^{\alpha_1}$ and hence it converges to $g\in \dot{K}_{b,r}^{\alpha_1}$. Let us prove that f=g a.e. Let $\varphi\in\mathcal{D}(\mathbb{R}^n)$. We write

$$\langle f - g, \varphi \rangle = \langle f - \varrho_N, \varphi \rangle + \langle g - \varrho_N, \varphi \rangle, \quad N \in \mathbb{N}_0.$$

Here $\langle \cdot, \cdot \rangle$ denotes the duality bracket between $\mathcal{D}'(\mathbb{R}^n)$ and $\mathcal{D}(\mathbb{R}^n)$. Clearly, the first term tends to zero as $N \to \infty$, while by Hölder's inequality there exists a constant C > 0 independent of N such that

$$|\langle g - \varrho_N, \varphi \rangle| \leqslant C ||g - \varrho_N||_{\dot{K}_{b,n}^{\alpha_1}},$$

which tends to zero as $N \to \infty$. Therefore f = g almost everywhere. Consequently, $f = \sum_{j=0}^{\infty} \varphi_j * f$ in $\dot{K}_{b,r}^{\alpha_1}$. Finally, (3.29), follows by the embedding $\dot{K}_{b,r}^{\alpha_1} \hookrightarrow \dot{K}_{\mu,r}^{\alpha_1 - \frac{n}{\mu} + \frac{n}{b}}$. We have also

$$f = \sum_{j=0}^{\infty} \varphi_j * f$$
, in L_{loc}^{μ} ,

because of $\dot{K}_{\mu,r}^{\alpha_1-\frac{n}{\mu}+\frac{n}{b}}\hookrightarrow L_{\text{loc}}^{\mu}$. Indeed, let $B(0,2^M)\subset\mathbb{R}^n, M\in\mathbb{Z}$. Hölder's inequality and the fact that $\alpha_1-\frac{n}{\mu}+\frac{n}{b}=\frac{n}{p}+\alpha-\frac{n}{\mu}-s<0$, see (3.26), give

$$||f||_{L^{\mu}(B(0,2^{M}))}^{\mu} = \sum_{i=-\infty}^{M} ||f\chi_{R_{i}}||_{\mu}^{\mu}$$

$$= \sum_{i=-\infty}^{M} 2^{-i(\alpha_{1} - \frac{n}{\mu} + \frac{n}{b})\mu} 2^{i(\alpha_{1} - \frac{n}{\mu} + \frac{n}{b})\mu} ||f\chi_{R_{i}}||_{\mu}^{\mu}$$

$$\leqslant C(M) \Big(\sum_{i=-\infty}^{M} 2^{i(\alpha_{1} - \frac{n}{\mu} + \frac{n}{b})r} ||f\chi_{i}||_{\mu}^{r} \Big)^{\frac{\mu}{r}}$$

$$\lesssim ||f||_{\dot{K}_{\mu,r}^{\alpha_{1} - \frac{n}{\mu} + \frac{n}{b}}}^{\mu}.$$

We put $\mu = L + \nu$, where $0 < \nu \le 1$. The function G has the Taylor expansion

$$G(t) = \sum_{l=0}^{L-1} \frac{G^{(l)}(z)}{l!} (t-z)^l + R(t,z), \quad t, z \in \mathbb{R},$$

where

$$R(t,z) = \frac{1}{L!} \int_{z}^{t} (t-y)^{L-1} G^{(L)}(y) dy.$$

Since $f \in \dot{K}^{\alpha}_{p,q}F^s_{\infty}$ and $s > \max(0, \frac{n}{p} + \alpha - n)$ there exists a set A of Lebesgue-measure zero such that $|f(x)| < \infty$ for all $x \in \mathbb{R}^n \setminus A$. We can we suppose that $|f(x)| < \infty$ for all $x \in \mathbb{R}^n$. Therefore

$$G(f(y)) = \sum_{l=0}^{L-1} \frac{1}{l!} \sum_{j=0}^{l} (-1)^{l-j} C_j^l f^j(y) (\psi_k * f(x))^{l-j} G^{(l)}(\psi_k * f(x))$$

$$+ R_k(f(y), \psi_k * f(x)),$$

where, $x, y \in \mathbb{R}^n$,

$$\psi_k * f = \sum_{i=0}^k \varphi_i * f, \quad k \in \mathbb{N}_0$$

and

$$R_k(f(y), \psi_k * f(x)) = \frac{1}{L!} \int_{\psi_k * f(x)}^{f(y)} (f(y) - h)^{L-1} G^{(L)}(h) dh.$$

We put $K_{j,l} = (-1)^{l-j} C_j^l \frac{1}{l!}$, with $0 \leq l \leq L-1, 0 \leq j \leq l$. Consequently

$$\varphi_k * G(f)(x) = \int_{\mathbb{R}^n} \varphi_k(x - y) G(f(y)) dy = \sum_{l=0}^{L-1} \sum_{j=0}^l H_{k,1,j,l}(x) + H_{k,2}(x),$$

where

$$H_{k,1,j,l}(x) = K_{j,l}(\psi_k * f(x))^{l-j} G^{(l)}(\psi_k * f(x)) \int_{\mathbb{R}^n} \varphi_k(x - y) f^j(y) dy$$
$$= K_{j,l}(\psi_k * f(x))^{l-j} G^{(l)}(\psi_k * f(x)) \varphi_k * f^j(x)$$

with $0 \leqslant l \leqslant L - 1, 0 \leqslant j \leqslant l$ and

$$H_{k,2}(x) = \frac{1}{L!} \int_{\mathbb{R}^n} \varphi_k(x - y) \int_{\psi_{k*f}(x)}^{f(y)} (f(y) - h)^{L-1} G^{(L)}(h) dh dy.$$

We will estimate each term separately.

Step 2. Estimate of $H_{k,1,j,l}$. First assume that $0 < j \le L - 1$. Recall that

$$s_i = s - (i-1)(\frac{n}{p} + \alpha - s), \quad i > 1$$

and $s_i \leq s_v, i \geq v > 1$. Define

$$p_1 = \frac{n + \alpha p}{(\mu - j)(\frac{n}{p} + \alpha - s)}$$

and

$$p_2 = \frac{n + \alpha p}{\bar{s} - s_j + \frac{n}{p} + \alpha},\tag{3.30}$$

where

$$s_{\mu} < \bar{s} < \min(\mu, s_L),$$

with $0 < j \le l, 0 \le l \le L - 1$. Since $s_j - \frac{n}{p} - \alpha = -j(\frac{n}{p} + \alpha - s) < 0$, (3.30) is well defined. We put $\frac{1}{\bar{p}} = \frac{1}{p_1} + \frac{1}{p_2}$. Hence

$$\bar{s} - \frac{n + \alpha p}{\bar{p}} = s_{\mu} - \frac{n}{p} - \alpha, \quad \bar{p}$$

In addition

$$s - \frac{n}{p} - \alpha = \frac{-n}{(\mu - j)p_1} - \frac{\alpha p}{(\mu - j)p_1}$$
 and $s_j - \frac{n}{p} - \alpha = \bar{s} - \frac{n}{p_2} - \frac{\alpha p}{p_2}$.

These choices guarantee the Sobolev embeddings

$$\dot{K}_{p,q}^{\alpha}F_{p,\beta}^{s} \hookrightarrow \dot{K}_{(\mu-j)p_{1},\infty}^{\frac{\alpha p}{(\mu-j)p_{1}}}F_{1}^{0}, \quad \dot{K}_{\bar{p},q}^{\frac{\alpha p}{\bar{p}}}F_{\infty}^{\bar{s}} \hookrightarrow \dot{K}_{p,q}^{\alpha}F_{\beta}^{s_{\mu}} \tag{3.31}$$

and

$$\dot{K}^{\alpha}_{p,q} F^{s}_{p,\beta} \hookrightarrow \dot{K}^{\alpha}_{p,q} F^{s_j}_r \hookrightarrow \dot{K}^{\frac{\alpha p}{p_2}}_{p_2,q} F^{\bar{s}}_r, \quad 0 < r \leqslant \infty, \tag{3.32}$$

see Theorem 2.10. We will prove that

$$\left\| \sup_{k \in \mathbb{N}_0} 2^{k\bar{s}} |H_{k,1,j,l} + H_{k,2}| \right\|_{\dot{K}^{\frac{\alpha_p}{\bar{p}}}, \frac{s}{\bar{p}}} \lesssim \|f\|_{\dot{K}^{\alpha}_{p,q} F^s_{p,\beta}}^{\mu}.$$

By Hölder's inequality and the fact that

$$|G^{(l)}(t)| \le ||G||_{Linu}|t|^{\mu-l}, \quad t \in \mathbb{R}, l = 0, ..., L - 1$$
 (3.33)

we obtain that

$$2^{k\bar{s}} \|H_{k,1,j,l}\|_{\dot{K}^{\frac{\alpha p}{\bar{p}}}_{\bar{p},q}} \lesssim \||\psi_{k} * f|^{l-j} G^{(l)}(\psi_{k} * f)\|_{\dot{K}^{\frac{\alpha p}{\bar{p}}}_{p_{1},\infty}} 2^{k\bar{s}} \|\varphi_{k} * f^{j}\|_{\dot{K}^{\frac{\alpha p}{\bar{p}}}_{p_{2},q}}$$

$$\lesssim \|G\|_{Lip\mu} \|\psi_{k} * f\|_{\dot{K}^{(\mu-j)p_{1}}_{(\mu-j)p_{1},\infty}}^{\mu-j} 2^{k\bar{s}} \|\varphi_{k} * f^{j}\|_{\dot{K}^{\frac{\alpha p}{\bar{p}}}_{p_{2},q}}$$

$$\lesssim \|G\|_{Lip\mu} \|f\|_{\dot{K}^{(\mu-j)p_{1}}_{(\mu-j)p_{1},\infty}}^{\mu-j} \|f^{j}\|_{\dot{K}^{\frac{\alpha p}{\bar{p}}}_{p_{2},q}}^{\bar{s}}$$

$$\lesssim \|G\|_{Lip\mu} \|f\|_{\dot{K}^{\alpha}_{p,q}F^{s}_{p,\beta}}^{\mu-j} \|f^{j}\|_{\dot{K}^{\alpha}_{p,q}F^{s}_{r}}^{s_{j}}$$

$$\lesssim \|G\|_{Lip\mu} \|f\|_{\dot{K}^{\alpha}_{p,q}F^{s}_{p,\beta}}^{\mu}$$

for any $k \in \mathbb{N}_0$, where we have used the embeddings (3.31) and (3.32), and Theorem 3.14 with the fact that

$$s > \max\left(0, \frac{n}{p} + \alpha - \frac{n}{\mu}\right),$$

see (3.26). Now we estimate $H_{k,1,0,l}$. Let us recall some properties of our system $\{\mathcal{F}\varphi_k\}_{k\in\mathbb{N}_0}$. It holds

$$\int_{\mathbb{R}^n} \varphi_k(y) dy = 0 \quad \text{and} \quad \int_{\mathbb{R}^n} \varphi_0(y) dy = c \neq 0 \quad k \in \mathbb{N}.$$

Therefore we need only to estimate $H_{0,1,0,l}$, $0 \le l \le L-1$. We have, again by (3.33),

$$||H_{0,1,0,l}||_{\dot{K}^{\frac{\alpha p}{\bar{p}}}_{\bar{p},q}} \lesssim ||G||_{Lip\mu} |||\varphi_0 * f|^{\mu}||_{\dot{K}^{\frac{\alpha p}{\bar{p}}}_{\bar{p},q}} \lesssim ||G||_{Lip\mu} ||f||^{\mu}_{\dot{K}^{\frac{\alpha p}{\bar{p}\mu}}_{\bar{p}\mu},q}^{0}_{\infty}.$$

Thanks to the embeddings

$$\dot{K}_{p,q}^{\alpha}F_{\beta}^{s} \hookrightarrow \dot{K}_{\bar{p}\mu,q}^{\frac{\alpha p}{\bar{p}\mu}}F_{\infty}^{\frac{\bar{s}}{\mu}} \hookrightarrow \dot{K}_{\bar{p}\mu,q}^{\frac{\alpha p}{\bar{p}\mu}}F_{\infty}^{0}, \tag{3.34}$$

because of

$$\bar{s} - \frac{n + \alpha p}{\bar{p}} = \mu(s - \frac{n}{p} - \alpha)$$
 and $\bar{p}\mu > p$,

we obtain

$$\big\|H_{0,1,0,l}\big\|_{\dot{K}^{\frac{\alpha p}{\overline{p}}}_{\overline{p},q}} \lesssim \big\|G\big\|_{Lip\mu} \big\|f\big\|^{\mu}_{\dot{K}^{\alpha}_{p,q}F^{s}_{\beta}}.$$

Step 3. Estimate of $H_{k,2}$. We have

$$\int_{\psi_{k}*f(x)}^{f(y)} (f(y) - h)^{L-1} G^{(L)}(h) dh$$

$$= G^{(L)}(\psi_{k} * f(x)) \frac{(f(y) - \psi_{k} * f(x))^{L}}{L}$$

$$+ \int_{\psi_{k}*f(x)}^{f(y)} (f(y) - h)^{L-1} (G^{(L)}(h) - G^{(L)}(\psi_{k} * f(x))) dh$$

$$= H_{k,2,1}(x, y) + H_{k,2,2}(x, y).$$

The estimation of $H_{k,2,1}$ can be obtained by the same arguments given in Step 2. We estimate $H_{k,2,2}$. Using the fact that

$$|G^{(L)}(t_0) - G^{(L)}(t_1)| \le ||G||_{Lip\mu} |t_0 - t_1|^{\nu}, \quad t_0, t_1 \in \mathbb{R},$$

we obtain

$$|H_{k,2,2}(x,y)| \lesssim ||G||_{Lip\mu} |\psi_k * f(x) - f(y)|^{\mu}, \quad x, y \in \mathbb{R}^n.$$

Obviously

$$|\psi_k * f(x) - f(y)| \le |\psi_k * f(x) - f(x)| + |f(x) - f(y)|,$$

which yields that

$$\int_{\mathbb{R}^n} |\varphi_k(x-y)| |H_{k,2,2}(x,y)| dy \leqslant S_{k,1}(f)(x) + S_{k,2}(f)(x),$$

where

$$S_{k,1}(f)(x) = \|G\|_{Lip\mu} \int_{\mathbb{R}^n} |\varphi_k(x-y)| |\psi_k * f(x) - f(x)|^{\mu} dy$$

$$\lesssim \|G\|_{Lip\mu} |\psi_k * f(x) - f(x)|^{\mu}$$

and

$$S_{k,2}(f)(x) = ||G||_{Lip\mu} \int_{\mathbb{R}^n} |\varphi_k(x-y)||f(x) - f(y)|^{\mu} dy.$$

First we estimate $S_{k,1}(f)$. Observe that

$$f - \psi_k * f = \sum_{i=k+1}^{\infty} \varphi_i * f, \quad k \in \mathbb{N}_0.$$

Therefore

$$\left\| \sup_{k \in \mathbb{N}_{0}} \left(2^{k\bar{s}} S_{k,1}(f) \right) \right\|_{\dot{K}^{\frac{\alpha p}{\bar{p}}}_{\bar{p},q}} \lesssim \left\| G \right\|_{Lip\mu} \left\| \sup_{k \in \mathbb{N}_{0}} 2^{k\frac{\bar{s}}{\bar{\mu}}} \left(\sum_{i=k+1}^{\infty} |\varphi_{i} * f| \right) \right\|_{\dot{K}^{\frac{\alpha p}{\bar{p}\mu}}_{\bar{p}\mu,q}}^{\mu}$$

$$\lesssim \left\| G \right\|_{Lip\mu} \left\| \sup_{k \in \mathbb{N}_{0}} 2^{k\frac{\bar{s}}{\bar{\mu}}} |\varphi_{k} * f| \right\|_{\dot{K}^{\frac{\alpha p}{\bar{p}\mu}}_{\bar{p}\mu,q}}^{\mu}$$

$$\lesssim \left\| G \right\|_{Lip\mu} \left\| f \right\|_{\dot{K}^{\frac{\alpha p}{\bar{p}\mu}}_{\bar{p}\mu},F^{\frac{\bar{s}}{\mu}}}^{\mu},$$

where we used Lemma 3.2. We conclude our desired estimate by the embeddings (3.34). Now we estimate $S_{k,2}(f)$. Since $\psi, \varphi \in \mathcal{S}(\mathbb{R}^n)$, this yields

$$|\varphi_k(z)| \lesssim \eta_{2^k,M}(z), \quad z \in \mathbb{R}^n,$$

where M is an arbitrary positive real number and the implicit constant is independent of z and $k \in \mathbb{N}_0$. By means of this inequality we find

$$\int_{\mathbb{R}^{n}} |\varphi_{k}(-z)| |f(x) - f(x+z)|^{\mu} dz$$

$$\lesssim \int_{\bar{B}_{k}} |\varphi_{k}(-z)| |f(x) - f(x+z)|^{\mu} dz$$

$$+ \sum_{l=0}^{\infty} \int_{\bar{B}_{k-l-1} \setminus B_{k-l}} |\varphi_{k}(-z)| |f(x) - f(x+z)|^{\mu} dz$$

$$\lesssim 2^{kn} \sum_{l=0}^{\infty} 2^{-lM} \int_{\bar{B}_{k-l-1}} |f(x) - f(x+z)|^{\mu} dz$$

$$\lesssim 2^{kn} \sum_{l=0}^{\infty} 2^{-lM} I_{k-l}^{\mu}(f)(x),$$

where the implicit constant is independent of x and k. Let $d = \min(1, \bar{p})$. Taking M large enough such that $M - n - \bar{s} - 1 > 0$ and using Lemma 3.4, we obtain

$$\begin{aligned} & \left\| \sup_{k \in \mathbb{N}_{0}} 2^{k\bar{s}} |S_{k,2}(f)| \right\|_{\dot{K}^{\frac{\alpha p}{\bar{p}}}_{\bar{p},q}}^{d} \\ & \lesssim & \left\| G \right\|_{Lip\mu}^{d} \sum_{l=0}^{\infty} 2^{-lMd} \left\| \sup_{k \in \mathbb{N}_{0}} \left(2^{k(n+\bar{s})} I_{k-l}^{\mu}(f) \right) \right\|_{\dot{K}^{\frac{\alpha p}{\bar{p}}}_{\bar{p},q}}^{d} \\ & \lesssim & \left\| G \right\|_{Lip\mu}^{d} \sum_{l=0}^{\infty} 2^{-l(M-n-\bar{s})d} \left\| \sup_{i \geqslant -l} \left(2^{i(n+\bar{s})} I_{i}^{\mu}(f) \right) \right\|_{\dot{K}^{\frac{\alpha p}{\bar{p}}}_{\bar{p},q}}^{d} \\ & \lesssim & \left\| G \right\|_{Lip\mu}^{d} \left\| f \right\|_{\dot{K}^{\frac{\alpha p}{\bar{p}}_{1,q}}_{\bar{p},q},E^{\frac{\bar{s}}{\mu}}_{\infty}}^{d}. \end{aligned}$$

Our desired estimate follows by the embedding (3.34). The proof is complete.

From Theorem 3.25 and the fact that $G(t) = |f|^{\mu} \in Lip\mu, \mu > 1$, we immediately arrive at the following result.

Corollary 3.35. Under the hypotheses of Theorem 3.25, we have

$$||f|^{\mu}||_{\dot{K}^{\alpha}_{p,q}F^{s\mu}_{\beta}} \leqslant c||G||_{Lip\mu}||f||^{\mu}_{\dot{K}^{\alpha}_{p,q}F^{s}_{\infty}}$$

holds for any $f \in \dot{\mathbb{K}}_{p,q}^{\alpha} \mathbb{F}_{\infty}^{s}$.

Remark 3.36. The valued s_{μ} in Theorem 3.25 is optimal. Indeed, we put

$$f_{\kappa}(x) = \theta(x)|x|^{\kappa},$$

where $\kappa > 0$ and θ is a smooth cut-off function with $\operatorname{supp} \theta \subset \{x : |x| \leq \vartheta\}, \ \vartheta > 0$ sufficiently small. As in [24] we can prove that $f_{\kappa} \in \dot{K}_{p,q}^{\alpha} F_{\beta}^{s}$ if and only if $s < \frac{n}{p} + \alpha + \kappa$. Let $G(x) = |x|^{\mu}, \mu > 1, x \in \mathbb{R}$. Then

$$G(f_{\kappa}) \notin \dot{K}_{p,q}^{\alpha} F_{\beta}^{d}$$

if $d \geqslant \frac{n}{p} + \alpha + \kappa \mu > s_{\mu}$.

Theorem 3.37. Let $0 < p, q < \infty, 0 < \beta \leq \infty, \mu > 1, \alpha \geq 0$ and

$$\max\left(0, \frac{n}{p} + \alpha - n\right) < s < \mu.$$

Let $G \in Lip\mu$. Then

$$||G(f)||_{\dot{K}_{p,q}^{\alpha}F_{\beta}^{s}} \le c||G||_{Lip\mu}||f||_{\dot{K}_{p,q}^{\alpha}F_{\beta}^{s}}||f||_{\infty}^{\mu-1}$$

holds for any $f \in \dot{\mathbb{K}}_{p,q}^{\alpha} \mathbb{F}_{\beta}^{s} \cap \mathbb{L}^{\infty}$.

Proof. We employ the notation of Theorem 3.25. We will prove that

$$\left\| \sup_{k \in \mathbb{N}_0} 2^{ks} |H_{k,1,j,l} + H_{k,2}| \right\|_{\dot{K}_{p,q}^{\alpha}} \lesssim \|G\|_{Lip\mu} \|f\|_{\dot{K}_{p,q}^{\alpha} F_{\beta}^{s}} \|f\|_{\infty}^{\mu-1}.$$

Thanks to (3.33) and Theorem 3.19 it follows

$$\begin{split} 2^{ks} & \| H_{k,1,j,l} \|_{\dot{K}^{\alpha}_{p,q}} & \lesssim & \| |\psi_k * f|^{l-j} G^{(l)}(|\psi_k * f|) \|_{\infty} 2^{ks} \| \varphi_k * f^j \|_{\dot{K}^{\alpha}_{p,q}} \\ & \lesssim & \| G \|_{Lip\mu} \| \psi_k * f \|_{\infty}^{\mu-j} 2^{ks} \| \varphi_k * f^j \|_{\dot{K}^{\alpha}_{p,q}} \\ & \lesssim & \| G \|_{Lip\mu} \| f \|_{\infty}^{\mu-j} \| f^j \|_{\dot{K}^{\alpha}_{p,q}F^s_{\infty}} \\ & \lesssim & \| G \|_{Lip\mu} \| f \|_{\infty}^{\mu-1} \| f \|_{\dot{K}^{\alpha}_{p,q}F^s_{\infty}}, \end{split}$$

where we used $\|\psi_k * f\|_{\infty} \lesssim \|f\|_{\infty}$, by Young's inequality. Now

$$\begin{aligned} \|H_{0,1,0,l}\|_{\dot{K}^{\alpha}_{p,q}} &\lesssim \|G\|_{Lip\mu} \||\varphi_{0} * f|^{\mu}\|_{\dot{K}^{\alpha}_{p,q}} \\ &\lesssim \|G\|_{Lip\mu} \|f\|_{\infty}^{\mu-1} \|\varphi_{0} * f\|_{\dot{K}^{\alpha}_{p,q}} \\ &\lesssim \|G\|_{Lip\mu} \|f\|_{\infty}^{\mu-1} \|f\|_{\dot{K}^{\alpha}_{\infty},F^{s}_{\infty}}. \end{aligned}$$

Observe that

$$S_{k,1}(f)(x) \lesssim \|G\|_{Lip\mu} |\psi_k * f(x) - f(x)|^{\mu} \lesssim \|G\|_{Lip\mu} \|f\|_{\infty}^{\mu-1} |\psi_k * f(x) - f(x)|.$$

Then

$$\left\| \sup_{k \in \mathbb{N}_{0}} \left(2^{ks} S_{k,1}(f) \right) \right\|_{\dot{K}_{p,q}^{\alpha}} \lesssim \|G\|_{Lip\mu} \|f\|_{\infty}^{\mu-1} \|\sup_{k \in \mathbb{N}_{0}} 2^{ks} \left(\sum_{i=k+1}^{\infty} |\varphi_{i} * f| \right) \|_{\dot{K}_{p,q}^{\alpha}}$$

$$\lesssim \|G\|_{Lip\mu} \|f\|_{\infty}^{\mu-1} \|\sup_{k \in \mathbb{N}_{0}} 2^{ks} |\varphi_{k} * f| \|_{\dot{K}_{p,q}^{\alpha}}$$

$$\lesssim \|G\|_{Lip\mu} \|f\|_{\infty}^{\mu-1} \|f\|_{\dot{K}_{n,q}^{\alpha}F_{\infty}^{s}},$$

by Lemma 3.2. Using Lemma 3.4, we obtain

$$\begin{aligned} & \left\| \sup_{k \in \mathbb{N}_{0}} 2^{ks} |S_{k,2}(f)| \right\|_{\dot{K}_{p,q}^{\alpha}}^{d} \\ & \lesssim & \left\| G \right\|_{Lip\mu}^{d} \sum_{l=0}^{\infty} 2^{-lMd} \left\| \sup_{k \in \mathbb{N}_{0}} \left(2^{k(n+s)} I_{k-l}^{\mu}(f) \right) \right\|_{\dot{K}_{p,q}^{\alpha}}^{d} \\ & \lesssim & \left\| G \right\|_{Lip\mu}^{d} \sum_{l=0}^{\infty} 2^{-l(M-n-\bar{s})d} \left\| \sup_{i \geqslant -l} \left(2^{i(n+s)} I_{i}^{\mu}(f) \right) \right\|_{\dot{K}_{p,q}^{\alpha}}^{d} \\ & \lesssim & \left\| G \right\|_{Lip\mu}^{d} \left\| f \right\|_{\dot{K}_{pl,ql}^{\alpha}}^{d\mu} F_{\infty}^{\frac{\bar{s}}{\mu}}. \end{aligned}$$

The desired estimate follows by the fact that

$$||f||_{\dot{K}_{p,q}^{\frac{\alpha}{\mu}}F_{\infty}^{\frac{s}{\mu}}}^{\mu} \lesssim ||f||_{\infty}^{\mu-1} ||f||_{\dot{K}_{p,q}^{\alpha}F_{\beta}^{s}}.$$

The proof is completed.

Now we present some limit case.

Theorem 3.38. Let $0 < p, q < \infty, \alpha \geqslant 0, \mu \geqslant \frac{\frac{n}{p} + \alpha}{\frac{n}{q} + \alpha + 1}$ and

$$\max\left(1, \frac{n}{p} + \alpha - n\right) < \mu < \frac{n}{p} + \alpha. \tag{3.39}$$

Let $G \in Lip\mu$ and

$$s = 1 + \frac{\mu - 1}{\mu} \left(\frac{n}{p} + \alpha \right). \tag{3.40}$$

Then

$$||G(f)||_{\dot{K}^{\alpha}_{p,q}F^{\mu}_{\infty}} \leqslant c||G||_{Lip\mu} ||f||^{\mu}_{\dot{K}^{\alpha}_{p,q}F^{s}_{\infty}}$$

holds for any $f \in \dot{\mathbb{K}}_{p,q}^{\alpha} \mathbb{F}_{\infty}^{s}$.

Proof. We employ the notation of the proof of Theorem 3.25. From (3.39) and (3.40), we obtain $\mu < s < \frac{n}{p} + \alpha$. With the help of (3.39) we get (3.27), $\frac{b}{\mu} > 1$ and $\alpha_1 \mu < n - \frac{n\mu}{b}$. Consequently the embedding (3.28) holds. We have $s_{\mu} = \mu$ and we will take $\bar{s} = s_{\mu}$ and $\bar{p} = p$. The proof is very similar as in Theorem 3.25, but here we use Lemma 3.9 instead of Lemma 3.4.

From Theorem 3.38 and the fact that $G(t) = |f|^{\mu} \in Lip\mu, \mu > 1$, we get the following result:

Corollary 3.41. Under the hypotheses of Theorem 3.38, we have

$$||f|^{\mu}||_{\dot{K}_{p,q}^{\alpha}F_{\infty}^{\mu}} \le c||G||_{Lip\mu}||f||_{\dot{K}_{p,q}^{\alpha}F_{\infty}^{s}}^{\mu}$$

holds for any $f \in \dot{\mathbb{K}}_{p,q}^{\alpha} \mathbb{F}_{\infty}^{s}$

Remark 3.42. Corresponding statements to Theorems 3.25, 3.37 and 3.38 were proved in [47] and [49, Theorem 6] with $\alpha = 0$, see also [46].

4. Semilinear parabolic equations in Herz-Triebel-Lizorkin spaces

4.1. Heat kernel estimates. Let $t > 0, x \in \mathbb{R}^n$ and $f \in \mathcal{S}'(\mathbb{R}^n)$. We put

$$e^{t\Delta}f(x) = \mathcal{F}^{-1}(\exp(-t|\xi|^2)\mathcal{F}f)(x).$$

Recall that

$$g(x) = \mathcal{F}^{-1}(\exp(-t|\xi|^2))(x) = (4\pi t)^{-\frac{n}{2}}\exp(-4t^{-1}|x|^2), \quad x \in \mathbb{R}^n.$$

We will give some key estimates of heat kernel $e^{t\Delta}$ needed in the proofs of the main statements. First, we estimate the heat kernel $e^{t\Delta}$ in Herz-type Triebel-Lizorkin spaces. We follows the arguments of [1] and [60]. We need the so called molecular and wavelet characterizations of Herz-type Triebel-Lizorkin spaces.

Definition 4.1. Let $K, L \in \mathbb{N}_0$ and M > 0. A K-times continuous differentiable function μ is called a [K, L, M]-molecule concentrated in $Q_{j,m}$ if for some $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$

$$|D^{\gamma}\mu(x)| \le 2^{|\gamma|j} (1 + 2^j |x - 2^{-j}m|)^{-M}, \quad 0 \le |\gamma| \le K$$

and

$$\int_{\mathbb{R}^n} x^{\gamma} \mu(x) dx = 0 \quad if \quad 0 \leqslant |\gamma| < L, j \in \mathbb{N}.$$

Notice that for L=0 or j=0 there are no moment conditions on μ . If μ is a molecule concentrated in $Q_{j,m}$, then it is denoted $\mu_{j,m}$.

We introduce the sequence spaces associated with the function spaces $\dot{K}_{p,q}^{\alpha}F_{\beta}^{s}$. Let $\alpha, s \in \mathbb{R}, 0 < p, q < \infty$ and $0 < \beta \leqslant \infty$. We set

$$\dot{K}^{\alpha}_{p,q}f^{s}_{\beta} = \{\lambda = \{\lambda_{j,m}\}_{j \in \mathbb{N}_{0}, m \in \mathbb{Z}^{n}} \subset \mathbb{C} : \|\lambda\|_{\dot{K}^{\alpha}_{p,q}f^{s}_{\beta}} < \infty\},\,$$

where

$$\|\lambda\|_{\dot{K}^{\alpha}_{p,q}f^{s}_{\beta}} = \left\| \left(\sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^{n}} 2^{js\beta} |\lambda_{j,m}|^{\beta} \chi_{j,m} \right)^{1/\beta} \right\|_{\dot{K}^{\alpha}_{p,q}}.$$

Now we come to the molecule decomposition theorem for $\dot{K}_{p,q}^{\alpha}F_{\beta}^{s}$ spaces. For the proof, see [21] and [67].

Theorem 4.2. Let $s \in \mathbb{R}, 0 < p, q < \infty, 0 < \beta \leqslant \infty$ and $\alpha > -\frac{n}{p}$. Furthermore, let $K, L \in \mathbb{N}_0$ and let M > 0 with

$$L > \sigma_{p,\beta} - s$$
, $K > s$ and M large enough.

If $a_{j,m}$ are [K, L, M]-molecules concentrated in $Q_{j,m}$ and

$$\lambda = \{\lambda_{j,m}\}_{j \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in \dot{K}_{p,q}^{\alpha} f_{\beta}^s,$$

then the sum

$$f = \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{j,m} a_{j,m} \tag{4.3}$$

converges in $\mathcal{S}'(\mathbb{R}^n)$ and

$$||f||_{\dot{K}_{p,q}^{\alpha}F_{\beta}^{s}} \lesssim ||\lambda||_{\dot{K}_{p,q}^{\alpha}f_{\beta}^{s}}.$$

Let $J \in \mathbb{N}$ and $\psi_F, \psi_M \in C^J(\mathbb{R})$ be real-valued compactly supported Daubechies wavelets with

$$\mathcal{F}\psi_F(0) = (2\pi)^{-\frac{1}{2}}, \quad \int_{\mathbb{R}} x^l \psi_M(x) dx = 0, \quad l \in \{0, ..., J-1\}$$

and

$$\|\psi_F\|_2 = \|\psi_M\|_2 = 1.$$

We have that

$$\{\psi_F(x-m), 2^{\frac{j}{2}}\psi_M(2^jx-m)\}_{j\in\mathbb{N}_0, m\in\mathbb{Z}^n}$$

is an orthonormal basis in $L^2(\mathbb{R})$. This orthonormal basis can be generalized to the \mathbb{R}^n by the usual multiresolution procedure. Let

$$G = \{G_1, ..., G_n\} \in G^0 = \{F, M\}^n$$

which means that G_r is either F or M. Let

$$G = \{G_1, ..., G_n\} \in G^j = \{F, M\}^{n^*}, \quad j \in \mathbb{N},$$

where indicates that at least one of the components of G must be an M. Let

$$\Psi_{G,m}^{j}(x) = 2^{j\frac{n}{2}} \prod_{r=1}^{n} \psi_{G_r}(2^{j}x_r - m_r), \quad G \in G^j, m \in \mathbb{Z}^n, x \in \mathbb{R}^n, j \in \mathbb{N}_0.$$

Then

$$\Psi = \{ \Psi_{G,m}^j : \quad j \in \mathbb{N}_0, G \in G^j, m \in \mathbb{Z}^n \}$$

is an orthonormal basis in $L^2(\mathbb{R}^n)$.

Let $\alpha, s \in \mathbb{R}, 0 < p, q < \infty$ and $0 < \beta \leq \infty$. We set

$$\dot{K}^{\alpha}_{p,q}\tilde{f}^{s}_{\beta}=\{\lambda=\{\lambda^{G}_{j,m}\}_{j\in\mathbb{N}_{0},G\in G^{j},m\in\mathbb{Z}^{n}}\subset\mathbb{C}:\left\Vert \lambda\right\Vert _{\dot{K}^{\alpha}_{p,q}\tilde{f}^{s}_{\beta}}<\infty\},$$

where

$$\|\lambda\|_{\dot{K}^{\alpha}_{p,q}\tilde{f}^{s}_{\beta}} = \left\| \left(\sum_{j=0}^{\infty} \sum_{G \in G^{j}} \sum_{m \in \mathbb{Z}^{n}} 2^{js\beta} |\lambda_{j,m}^{G}|^{\beta} \chi_{j,m} \right)^{1/\beta} \right\|_{\dot{K}^{\alpha}_{p,q}}.$$

Theorem 4.4. Let $\alpha, s \in \mathbb{R}, 0 < p, q < \infty, 0 < \beta \leqslant \infty$ and $\alpha > -\frac{n}{p}$. Let $\{\Psi_{G,m}^j\}$ be the wavelet system with $J > \max(\sigma_{p,\beta} - s, s)$. Let $f \in \mathcal{S}'(\mathbb{R}^n)$. Then $f \in \dot{K}_{p,q}^{\alpha} F_{\beta}^s$ if and only if

$$f = \sum_{j=0}^{\infty} \sum_{G \in G^j} \sum_{m \in \mathbb{Z}^n} \lambda_{j,m}^G 2^{-j\frac{n}{2}} \Psi_{G,m}^j, \quad \lambda \in \dot{K}_{p,q}^{\alpha} \tilde{f}_{\beta}^s$$

$$(4.5)$$

with unconditional convergence in $S'(\mathbb{R}^n)$ and in any space $\dot{K}_{p,q}^{\alpha}F_{\beta}^{\sigma}$ with $\sigma < s$. The representation (4.5) is unique. We have

$$\lambda_{j,m}^G = \lambda_{j,m}^G(f) = 2^{j\frac{n}{2}} \langle f, \Psi_{G,m}^j \rangle$$

and

$$I: f \longmapsto \{\lambda_{i,m}^G(f)\}$$

is an isomorphic map from $\dot{K}^{\alpha}_{p,q}F^s_{\beta}$ into $\dot{K}^{\alpha}_{p,q}\tilde{f}^s_{\beta}$. In particular, it holds

$$||f||_{\dot{K}_{p,q}^{\alpha}F_{\beta}^{s}} \approx ||\lambda||_{\dot{K}_{p,q}^{\alpha}\tilde{f}_{\beta}^{s}}.$$

For the proof, see again, [21] and [67]. To estimate the heat kernel $e^{t\Delta}$ in Herz-type Triebel-Lizorkin spaces, we need the following lemma.

Lemma 4.6. Let $s > 0, \theta \geqslant 0, 0 < t < T, 0 < p, q < \infty, 0 < \beta \leqslant \infty$ and $\alpha > -\frac{n}{n}$. We set

$$b_{G,m}^{j}(x,t) = 2^{-j\frac{n}{2}}e^{t\Delta}\Psi_{G,m}^{j}(x).$$

Then there exists C > 0 such that the functions

$$b_{G,m}^{j}(x,t)_{\theta} = C2^{j\theta} t^{\frac{\theta}{2}} b_{G,m}^{j}(x,t), \quad j \in \mathbb{N}_{0}, G \in G^{*}, m \in \mathbb{Z}^{n}$$
 (4.7)

[K, L, M]-molecules for any fixed t with $2^j t^{\frac{1}{2}} \geqslant 1$, provided that $L \leqslant J, K \leqslant J, L+n-1 < 1$ $M < J + n - \theta$ and $\theta \leqslant J - L + 1$. Assume that

$$J > \theta + \max(s, \sigma_{p,\beta}).$$

Then, the numbers K, L, M can be chosen such that for some C > 0 and any t with $2^{j}t^{\frac{1}{2}} \geqslant 1$, such that (4.7) are molecules for $\dot{K}_{p,q}^{\alpha}F_{\beta}^{s+\theta}$.

Proof. We use the arguments of [1, Proposition 3.1] and we need only to prove the second part of the Lemma. Let $L = [\sigma_{p,\beta}] + 1$, which yields that $L > \sigma_{p,\beta} - s - \theta$. Since $J > \sigma_{p,\beta}$ it follows that $J \geqslant L$. Hence

$$\int_{\mathbb{R}^n} x^{\nu} b_{G,m}^j(x,t)_{\theta} dx = 0, \qquad 0 \leqslant |\nu| < L, j \in \mathbb{N}.$$

Let M large enough be such that $\sigma_{p,\beta} + n < M < J + n - \theta$. Then M > L + n - 1 and $\theta < J - \sigma_{p,\beta} < J - L + 1$. Regarding the derivatives of $b_{G,m}^j(x,t)_\theta$ we claim $s + \theta < K \leqslant 1$

We present one of the main tools used in this section.

Lemma 4.8. Let $s > 0, \theta \geqslant 0, 0 < t < T, 1 < p, q < \infty, 1 < \beta \leqslant \infty \ and -\frac{n}{p} < \alpha < n - \frac{n}{p}$. Then there exists a positive constant C(T) > 0 independent of t such that

$$\|e^{t\Delta}f\|_{\dot{K}^{\alpha}_{p,q}F^{s+\theta}_{\beta}} \leqslant C(T)t^{-\frac{\theta}{2}}\|f\|_{\dot{K}^{\alpha}_{p,q}F^{s}_{\beta}}$$

for any $f \in \dot{K}_{n,a}^{\alpha} F_{\beta}^{s}$.

Proof. Let $k \in \mathbb{N}$ be such that $2^{-2k} < \frac{t}{T} \leqslant 2^{-2(k-1)}$. From Theorem 4.4 we have f = $f_{1,k} + f_{2,k}$, with

$$f_{1,k} = \sum_{j=0}^{k-1} \sum_{G \in G^j} \sum_{m \in \mathbb{Z}^n} \lambda_{j,m}^G 2^{-j\frac{n}{2}} \Psi_{G,m}^j$$

and

$$f_{2,k} = \sum_{j=k}^{\infty} \sum_{G \in G^j} \sum_{m \in \mathbb{Z}^n} \lambda_{j,m}^G 2^{-j\frac{n}{2}} \Psi_{G,m}^j,$$

where $\lambda \in \dot{K}_{p,q}^{\alpha} \tilde{f}_{\beta}^{s}$.

Estimate of $f_{1,k}$. We claim that

$$|e^{t\Delta}(\varphi_j * f_{1,k})(x)| \lesssim \mathcal{M}(\varphi_j * f_{1,k})(x), \quad x \in \mathbb{R}^n, j \in \mathbb{N}_0,$$
 (4.9)

where the implicit constant is independent of x, k, j and t. Using the estimate (4.9) and Lemma 3.3 we obtain

$$\begin{aligned} \|e^{t\Delta}f_{1,k}\|_{\dot{K}_{p,q}^{\alpha}F_{\beta}^{s+\theta}} &= \left\| \left(\sum_{j=0}^{\infty} 2^{j(s+\theta)\beta} |\varphi_{j} * e^{t\Delta}f_{1,k}|^{\beta} \right)^{1/\beta} \right\|_{\dot{K}_{p,q}^{\alpha}} \\ &= \left\| \left(\sum_{j=0}^{\infty} 2^{j(s+\theta)\beta} |e^{t\Delta}(\varphi_{j} * f_{1,k})|^{\beta} \right)^{1/\beta} \right\|_{\dot{K}_{p,q}^{\alpha}} \\ &\lesssim \left\| \left(\sum_{j=0}^{\infty} 2^{j(s+\theta)\beta} |\mathcal{M}(\varphi_{j} * f_{1,k})|^{\beta} \right)^{1/\beta} \right\|_{\dot{K}_{p,q}^{\alpha}} \\ &\lesssim \left\| \left(\sum_{j=0}^{\infty} 2^{j(s+\theta)\beta} |\varphi_{j} * f_{1,k}|^{\beta} \right)^{1/\beta} \right\|_{\dot{K}_{p,q}^{\alpha}}. \tag{4.10} \end{aligned}$$

In view of the definition of the spaces $\dot{K}_{p,q}^{\alpha}F_{\beta}^{s+\theta}$, (4.10) is just $||f_{1,k}||_{\dot{K}_{p,q}^{\alpha}F_{\beta}^{s+\theta}}$. Thanks to Theorem 4.4 we get

$$||f_{1,k}||_{\dot{K}^{\alpha}_{p,q}F^{s+\theta}_{\beta}} \lesssim ||\left(\sum_{j=0}^{k-1}\sum_{G\in G^{j}}\sum_{m\in\mathbb{Z}^{n}}2^{j(s+\theta)\beta}|\lambda_{j,m}^{G}|^{\beta}\chi_{j,m}\right)^{1/\beta}||_{\dot{K}^{\alpha}_{p,q}}$$

$$\lesssim 2^{k\theta}||\left(\sum_{j=0}^{k-1}\sum_{G\in G^{j}}\sum_{m\in\mathbb{Z}^{n}}2^{js\beta}|\lambda_{j,m}^{G}|^{\beta}\chi_{j,m}\right)^{1/\beta}||_{\dot{K}^{\alpha}_{p,q}}$$

$$= ct^{-\frac{\theta}{2}}||\lambda||_{\dot{K}^{\alpha}_{p,q}F^{s}_{\beta}}.$$

$$\lesssim t^{-\frac{\theta}{2}}||f||_{\dot{K}^{\alpha}_{p,q}F^{s}_{\beta}}.$$

$$(4.11)$$

Substituting (4.11) into (4.10), this gives the desired estimate. Now we prove our claim. Since $g \in \mathcal{S}(\mathbb{R}^n)$, we have

$$|e^{t\Delta}(\varphi_j * f_{1,k})(x)| \lesssim \eta_{t^{-\frac{1}{2}},m} * |\varphi_j * f_{1,k}|(x), \quad m > n,$$

which can be estimated by

$$c\eta_{t^{-\frac{1}{2}},m}\chi_{B(x,2t^{\frac{1}{2}})} * |\varphi_{j} * f_{1,k}|(x) + \eta_{t^{-\frac{1}{2}},m}\chi_{\mathbb{R}^{n}\setminus B(x,2t^{\frac{1}{2}})} * |\varphi_{j} * f_{1,k}|(x).$$

$$(4.12)$$

Obviously, the first term of (4.12) is bounded by $c\mathcal{M}(\varphi_i * f_{1,k})(x)$. We have

$$\eta_{t^{-\frac{1}{2},m}} \chi_{\mathbb{R}^{n} \setminus B(x,2t^{\frac{1}{2}})} * |\varphi_{j} * f_{1,k}|(x)$$

$$= \sum_{i=1}^{\infty} \eta_{t^{-\frac{1}{2},m}} \chi_{B(x,2^{i+1}t^{\frac{1}{2}}) \setminus B(x,2^{i}t^{\frac{1}{2}})} * |\varphi_{j} * f_{1,k}|(x)$$

$$\leq \sum_{i=1}^{\infty} 2^{-im} \eta_{t^{-\frac{1}{2},m}} \chi_{B(x,2^{i+1}t^{\frac{1}{2}})} * |\varphi_{j} * f_{1,k}|(x)$$

$$\leq \mathcal{M}(\varphi_{j} * f_{1,k})(x) \sum_{i=1}^{\infty} 2^{i(n-m)}$$

$$\leq \mathcal{M}(\varphi_{j} * f_{1,k})(x).$$

Estimate of $f_{2,k}$. If $j \geqslant k$, then $2^j(\frac{t}{T})^{\frac{1}{2}} > 2^{j-k} \geqslant 1$, which yields that

$$e^{t\Delta} f_{2,k} = \sum_{j=k}^{\infty} \sum_{G \in G^{j}} \sum_{m \in \mathbb{Z}^{n}} 2^{-j\theta} (\frac{t}{T})^{-\frac{\theta}{2}} \lambda_{j,m}^{G} 2^{-j\frac{n}{2}} 2^{j\theta} (\frac{t}{T})^{\frac{\theta}{2}} e^{t\Delta} \Psi_{G,m}^{j}$$
$$= \sum_{j=k}^{\infty} \sum_{G \in G^{j}} \sum_{m \in \mathbb{Z}^{n}} \mu_{j,m}^{G} b_{G,m}^{j}(x,t)_{\theta},$$

where

$$C\mu_{j,m}^G = 2^{-j\theta} (\frac{t}{T})^{-\frac{\theta}{2}} \lambda_{j,m}^G$$
 and $b_{G,m}^j (x,t)_{\theta} = 2^{-j\frac{n}{2}} 2^{j\theta} (\frac{t}{T})^{\frac{\theta}{2}} \Psi_{G,m}^j$,

and C as in (4.7). Let

$$\mu^* = \{2^{-j\theta} (\frac{t}{T})^{-\frac{\theta}{2}} \lambda_{j,m}^G, j \in \mathbb{N}_0, G \in G^*, m \in \mathbb{Z}^n\}.$$

Again, from Theorem 4.4 we obtain

$$\begin{aligned} \|e^{t\Delta}f_{2,k}\|_{\dot{K}^{\alpha}_{p,q}F^{s+\theta}_{\beta}} &\lesssim \|\mu^*\|_{\dot{K}^{\alpha}_{p,q}\tilde{f}^{s+\theta}_{\beta}} \\ &= ct^{-\frac{\theta}{2}} \|\Big(\sum_{j=k}^{\infty} \sum_{G \in G^j} \sum_{m \in \mathbb{Z}^n} 2^{js\beta} |\lambda_{j,m}^G|^{\beta} \chi_{j,m}\Big)^{1/\beta} \|_{\dot{K}^{\alpha}_{p,q}} \\ &= ct^{-\frac{\theta}{2}} \|\lambda\|_{\dot{K}^{\alpha}_{p,q}\tilde{f}^{s}_{\beta}} \\ &\lesssim t^{-\frac{\theta}{2}} \|f\|_{\dot{K}^{\alpha}_{p,q}F^{s}_{\beta}} \end{aligned}$$

and this completes the proof.

The following lemmas was proved in [27].

Lemma 4.13. Let $\alpha_1, \alpha_2 \in \mathbb{R}, 0 < t < \infty$ and $1 < p, \kappa, q, r < \infty$. We suppose that $1 < q \leqslant p < \infty$ and $-\frac{n}{p} < \alpha_1 \leqslant \alpha_2 < n - \frac{n}{q}$. Then there exists a positive constant C > 0 independent of t such that

$$\|e^{t\Delta}f\|_{\dot{K}_{p,r}^{\alpha_1}} \le Ct^{-\frac{1}{2}(\frac{n}{q}-\frac{n}{p}+\alpha_2-\alpha_1)}\|f\|_{\dot{K}_{q,\delta}^{\alpha_2}}$$

for any $f \in \dot{K}_{q,\delta}^{\alpha_2}$, where

$$\delta = \begin{cases} r, & if \quad \alpha_2 = \alpha_1, \\ \kappa, & if \quad \alpha_2 > \alpha_1. \end{cases}$$

4.2. **The results and their proofs.** We look for mild solutions of (1.1) i.e. for solutions of integral equation

$$u(t,x) = e^{t\Delta}u_0(x) + \int_0^t e^{(t-\tau)\Delta}G(u)(\tau,x)d\tau.$$
 (4.14)

We set

$$F(u)(t,x) = \int_0^t e^{(t-\tau)\Delta} G(u)(\tau,x) d\tau.$$

We study Cauchy problem for semilinear parabolic equations (1.1) with initially data in Herz-type Triebel-Lizorkin spaces and will assume that G belongs to $G \in Lip\mu$. We set

$$\bar{s} = \frac{n}{p} + \alpha - \frac{2}{\mu - 1}$$
 and $\vartheta = \frac{s - \bar{s}}{2}$.

We now state the existence of mild solutions of (4.14).

Theorem 4.15. Let $1 < p, q < \infty, 1 < \beta \leqslant \infty, \mu > 1, 0 \leqslant \alpha < n - \frac{n}{p}, s \geqslant \frac{n}{p} - \frac{n}{q}$ and

$$0 < s < \frac{n}{p} + \alpha.$$

Let $G \in Lip\mu$ and

$$0 < s_{\mu} < \mu$$
.

- (i) For all initial data u_0 in $\dot{K}_{p,q}^{\alpha}F_{\beta}^s$ with $s>\bar{s}$, there exists a maximal solution u to (4.14) in $C([0,T_0), \dot{K}_{p,q}^{\alpha}F_{\beta}^s)$ with $T_0 \geqslant C \|u_0\|_{\dot{K}_{p,q}^{\alpha}F_{\beta}^s}^{-1}$. (ii) Let $\theta < 2\vartheta(\mu-1)$ or $\theta = 2\vartheta(\mu-1), s > 1$ and $G \in Lips_0$ with

$$s_0 = \frac{\frac{n}{p} + \alpha}{\frac{n}{p} + \alpha - s + 1}.$$

We have

$$u - e^{t\Delta}u_0 \in C([0, T_0), \dot{K}_{p,q}^{\alpha} F_{\beta}^{s+\theta}).$$

Proof. We will do the proof into two steps. Our arguments are based on [45].

Step 1. We prove part (i) of the theorem.

Substep 1.1. In this step we prove the existence of a solution to (4.14). Recall that

$$F(u)(t,x) = \int_0^t e^{(t-\tau)\Delta} G(u)(\tau,x) d\tau$$
 and $\frac{1}{\tilde{p}} = \frac{1}{p} + \frac{\alpha - s}{n}$.

For simplicity, we consider the spaces

$$Y = C([0,T), \dot{K}_{p,q}^{\alpha} F_{\beta}^{s})$$
 and $X = C([0,T), \dot{K}_{\tilde{p},q}^{0}).$

Further, we consider the sequence of functions

$$u^{0} = e^{t\Delta}u_{0}$$
 and $u^{j+1} = u^{0} + F(u^{j}), \quad j \in \mathbb{N}.$ (4.16)

From Lemma 4.13 and Sobolev embedding $\dot{K}^{\alpha}_{p,q}F^s_{\beta}\hookrightarrow\dot{K}^0_{\tilde{p},q}$, see Theorem 2.10, we deduce that

$$||u^0||_{\dot{K}_{\bar{p},q}^0} \lesssim ||u_0||_{\dot{K}_{\bar{p},q}^0} \lesssim ||u_0||_{\dot{K}_{p,q}^{\alpha}F_{\beta}^s}.$$
 (4.17)

Let $u, v \in X$. Since, $\frac{\tilde{p}}{\mu} > 1$, again, by Lemma 4.13 we obtain

$$\begin{aligned} & \|F(u)(t,\cdot) - F(v)(t,\cdot)\|_{\dot{K}^{0}_{\tilde{p},q}} \\ & \leq \int_{0}^{t} \|e^{(t-\tau)\Delta}(G(u)(\tau,\cdot) - G(v)(\tau,\cdot))\|_{\dot{K}^{0}_{\tilde{p},q}} d\tau \\ & \leq \int_{0}^{t} \|e^{(t-\tau)\Delta}(G(u)(\tau,\cdot) - G(v)(\tau,\cdot))\|_{\dot{K}^{0}_{\tilde{p},\frac{q}{\mu}}} d\tau \\ & \leq C \int_{0}^{t} (t-\tau)^{-\frac{n(\mu-1)}{2\tilde{p}}} \|G(u)(\tau,\cdot) - G(v)(\tau,\cdot)\|_{\dot{K}^{0}_{\frac{\tilde{p}}{\mu},\frac{q}{\mu}}} d\tau, \end{aligned}$$

$$(4.18)$$

where the second estimate follows by the embedding $\dot{K}^0_{\tilde{p},\frac{q}{u}} \hookrightarrow \dot{K}^0_{\tilde{p},q}$ and the positive constant C is independent of t. Observe that

$$|G(u)(\tau,\cdot) - G(v)(\tau,\cdot)| \le |u - v|(|u|^{\mu-1} + |v|^{\mu-1})$$

and

$$\frac{\mu}{\tilde{p}} = \frac{1}{\tilde{p}} + \frac{\mu - 1}{\tilde{p}}, \quad \frac{\mu}{q} = \frac{1}{q} + \frac{\mu - 1}{q}.$$

Therefore, by Hölder's inequality

$$\|G(u)(\tau,\cdot) - G(v)(\tau,\cdot)\|_{\dot{K}^{0}_{\frac{\bar{\rho}}{\mu},\frac{q}{\mu}}}$$

$$\leq \|u(\tau,\cdot) - v(\tau,\cdot)\|_{\dot{K}^{0}_{\bar{p},q}} \Big(\|u(\tau,\cdot)\|_{\dot{K}^{0}_{\bar{p},q}}^{\mu-1} + \|v(\tau,\cdot)\|_{\dot{K}^{0}_{\bar{p},q}}^{\mu-1} \Big).$$

$$(4.19)$$

Substituting (4.19) into (4.18) and then using

$$\frac{n(\mu - 1)}{2\tilde{p}} = \frac{(\mu - 1)}{2}(\frac{n}{p} + \alpha - s) = 1 - \frac{(\mu - 1)(s - \bar{s})}{2},$$

this gives

$$||F(u) - F(v)||_X \leqslant CT^{\frac{(\mu-1)(s-\bar{s})}{2}} ||u - v||_X (||u||_X^{\mu-1} + ||v||_X^{\mu-1}). \tag{4.20}$$

In view of (4.16), (4.20) and (4.17), we obtain

$$\begin{aligned} \|u^{j+1}\|_{X} &\lesssim \|u^{0}\|_{X} + \|F(u^{j})\|_{X} \\ &\leqslant \|u_{0}\|_{\dot{K}_{n,a}^{\alpha}F_{s}^{s}} + CT^{\frac{(\mu-1)(s-\bar{s})}{2}} \|u^{j}\|_{X}^{\mu} \end{aligned}$$

and

$$||u^{j+1} - u^j||_X \leqslant CT^{\frac{(\mu-1)(s-\bar{s})}{2}} ||u^j - u^{j-1}||_X (||u^j||_X^{\mu-1} + ||u^{j-1}||_X^{\mu-1}).$$

Let

$$F = (\frac{1}{C})^{\frac{2}{(\mu-1)(s-\bar{s})}} \left(\mu^{\frac{-1}{\mu-1}} - \mu^{\frac{-\mu}{\mu-1}}\right)^{\frac{2}{s-\bar{s}}}.$$

As in [27] and [37], the fixed point argument shows that if

$$T < F 2^{\frac{-2}{(\mu-1)(s-\bar{s})}} \left(1 - \frac{1}{\mu}\right)^{\mu-1} \left\| u_0 \right\|_{\dot{K}^{\alpha}_{p,q} F^{s}_{\beta}}^{\frac{-2}{s-\bar{s}}}, \tag{4.21}$$

then the sequence $\{u^j\}_j$ converges strongly in X to a limit u which is a solution of the integral equation (4.14).

Substep 1.2. In this step we prove that the solution of the integral equation (4.14) belongs to Y. We employ the notation of Substep 1.1. We claim that

$$||u^{j+1}||_{Y} \leqslant ||u_{0}||_{\dot{K}_{n_{\alpha}F_{\beta}}^{\alpha}F_{\beta}} + CT^{\frac{(\mu-1)(s-\bar{s})}{2}} ||u^{j}||_{Y}^{\mu}. \tag{4.22}$$

From (4.21) and (4.22), the sequence $\{u^j\}_j$ is bounded. Then we can extract a subsequence $\{u^{j_i}\}_i$ converges weakly to $\tilde{u} \in Y$. From Step 1, $\{u^{j_i}\}_i$ converges weakly to u, so $u = \tilde{u} \in Y$. Now we prove the claim. Let $u \in Y$. By Lemma 4.8 and Theorem 3.25 we obtain

$$\begin{split} \big\| F(u)(t,\cdot) \big\|_{\dot{K}^{\alpha}_{p,q}F^{s}_{\beta}} & \leqslant \int_{0}^{t} \big\| e^{(t-\tau)\Delta}(G(u)(\tau,\cdot)) \big\|_{\dot{K}^{\alpha}_{p,q}F^{s}_{\beta}} d\tau \\ & \leqslant C \int_{0}^{t} (t-\tau)^{-\frac{s-s\mu}{2}} \big\| G(u)(\tau,\cdot) \big\|_{\dot{K}^{\alpha}_{p,q}F^{s\mu}_{\beta}} d\tau \\ & \leqslant C \int_{0}^{t} (t-\tau)^{-\frac{s-s\mu}{2}} \big\| u \big\|_{\dot{K}^{\alpha}_{p,q}F^{s}_{\beta}}^{\mu} d\tau \\ & \leqslant C T^{1-\frac{s-s\mu}{2}} \big\| u \big\|_{Y}^{\mu}. \end{split}$$

This leads to (4.22), with the help of the fact that

$$||u^0||_{\dot{K}^{\alpha}_{p,q}F^s_{\beta}} \leqslant C||u_0||_{\dot{K}^{\alpha}_{p,q}F^s_{\beta}}$$

by, Lemma 4.8 and

$$1 - \frac{s - s\mu}{2} = \frac{(\mu - 1)(s - \bar{s})}{2} > 0.$$

From (4.21), we easily obtain $T_0 \geqslant C \|u_0\|_{\dot{K}_{n,q}^{\alpha} F_{\beta}^s}^{\frac{-2}{s-\bar{s}}}$.

Substep 1.3. We will prove the uniqueness of the solution of (4.14). Let $u, v \in Y$ be two solutions for the same initial data u_0 . Using the fact that u and u solve (4.14), we obtain

$$\|u-v\|_X = \|F(u)-F(v)\|_X \leqslant 2CT^{\frac{(\mu-1)(s-\bar{s})}{2}}A^{\mu-1}\|u-v\|_X$$

where

$$A = \sup_{t \in [0,T]} (\|u(t \cdot)\|_{\dot{K}^0_{\tilde{p},q}}^{\mu-1}, \|v(t \cdot)\|_{\dot{K}^0_{\tilde{p},q}}^{\mu-1}), \quad T < \max(T_0(u), T_0(v)).$$

Taking T small enough such that

$$2CT^{\frac{(\mu-1)(s-\bar{s})}{2}}A^{\mu-1} < \frac{1}{2}$$

we obtain u = v on [0,T]. We iterate this to prove that $T_0(u) = T_0(v)$ and u = v on $[0, T_0(u))$, which ensures the uniqueness of the solution of (4.14).

Step 2. We prove part (ii) of the theorem. We split our considerations into the cases $\theta < 2\vartheta(\mu - 1)$ and $\theta = 2\vartheta(\mu - 1)$.

• Case 1. $\theta < 2\vartheta(\mu - 1)$. Let $u \in Y$ be a solution of (4.14) with initial data u_0 . Observe that $2\vartheta(\mu-1)=(\mu-1)(s-\bar{s})=2-s+s_{\mu}$. Thanks to Lemma 4.8 and Theorem 3.25 it follows

$$\begin{aligned} \|u - e^{t\Delta} u_0\|_{\dot{K}^{\alpha}_{p,q} F^{s+\theta}_{\beta}} & \leq \int_0^t \|e^{(t-\tau)\Delta} (G(u)(\tau, \cdot))\|_{\dot{K}^{\alpha}_{p,q} F^{s+\theta}_{\beta}} d\tau \\ & \leq C \int_0^t (t-\tau)^{-\frac{\theta}{2} - \frac{s-s\mu}{2}} \|G(u)(\tau, \cdot)\|_{\dot{K}^{\alpha}_{p,q} F^{s\mu}_{\beta}} d\tau \\ & \leq C T_0^{1 - \frac{\theta}{2} - \frac{s-s\mu}{2}} \|u\|_{V}^{\mu}, \end{aligned}$$

since $1 - \frac{\theta}{2} - \frac{s - s_{\mu}}{2} = -\frac{\theta}{2} + \frac{(\mu - 1)(s - \bar{s})}{2} > 0$. • Case 2. $\theta = 2\vartheta(\mu - 1)$. Observe that $s_{\mu} < \mu$, this gives

$$\mu > \frac{\frac{n}{p} + \alpha}{\frac{n}{p} + \alpha - s + 1} = s_0 > 1$$
 and $s = 1 + \frac{s_0 - 1}{s_0} \left(\frac{n}{p} + \alpha\right)$.

In addition

$$0 < s_{\mu} < s_0 < \frac{n}{p} + \alpha$$
 and $s_0 \geqslant \frac{\frac{n}{p} + \alpha}{\frac{n}{q} + \alpha + 1}$.

Assume that $s + \theta = 2 + s_{\mu} < s_0$. Let $2 + s_{\mu} < s_1 < s_0$ and $0 < \gamma < 1$ be such that $s + \theta = \gamma s_{\mu} + (1 - \gamma) s_1$. From interpolation inequality (2.9), Lemma 4.8, Theorems 3.25 and 3.38 we get

$$||u - e^{t\Delta}u_0||_{\dot{K}_{p,q}^{\alpha}F_{\beta}^{s+\theta}}$$

can be estimated by

$$\int_{0}^{t} \|e^{(t-\tau)\Delta}(G(u)(\tau,\cdot))\|_{\dot{K}_{p,q}^{\alpha}F_{\beta}^{s+\theta}} d\tau
\leq C \int_{0}^{t} \|e^{(t-\tau)\Delta}G(u)(\tau,\cdot)\|_{\dot{K}_{p,q}^{\alpha}F_{\beta}^{s\mu}}^{\gamma} \|e^{(t-\tau)\Delta}G(u)(\tau,\cdot)\|_{\dot{K}_{p,q}^{\alpha}F_{\beta}^{s_{1}}}^{s_{1}} d\tau
\leq C \int_{0}^{t} \|G(u)(\tau,\cdot)\|_{\dot{K}_{p,q}^{\alpha}F_{\gamma\beta}^{s\mu}}^{\gamma} \|G(u)(\tau,\cdot)\|_{\dot{K}_{p,q}^{\alpha}F_{\infty}^{s_{0}}}^{1-\gamma} d\tau
\leq C T_{0} \|u\|_{Y}^{\gamma\mu} \|u\|_{Y}^{(1-\gamma)s_{0}}.$$

Now assume that $\theta + s = 2 + s_{\mu} \geqslant s_0$. Let $\kappa > 0$ be such that $2 + s_{\mu} - s_0 < \kappa < 2$. Let $0 < \varrho < 1$ be such that $s + \theta = \varrho s_{\mu} + (1 - \varrho)(s_0 + \kappa)$. Again, from interpolation inequality we obtain

$$||u - e^{t\Delta}u_0||_{\dot{K}_{p,q}^{\alpha}F_{\beta}^{s+\theta}}$$

is bounded by

$$\int_{0}^{t} \left\| e^{(t-\tau)\Delta}(G(u)(\tau,\cdot)) \right\|_{\dot{K}_{p,q}^{\alpha}F_{\beta}^{s+\theta}} d\tau$$

$$\leqslant C \int_{0}^{t} \left\| e^{(t-\tau)\Delta}G(u)(\tau,\cdot) \right\|_{\dot{K}_{p,q}^{\alpha}F_{\beta\rho}^{s\mu}}^{s\mu} \left\| e^{(t-\tau)\Delta}G(u)(\tau,\cdot) \right\|_{\dot{K}_{p,q}^{s}F_{\infty}^{s0+\kappa}}^{1-\varrho} d\tau.$$

Applying Hölder's inequality, Lemma 4.8, Theorems 3.25 and 3.38, we estimate the last expression by

$$C\left(\int_{0}^{t} \|e^{(t-\tau)\Delta}G(u)(\tau,\cdot)\|_{\dot{K}^{\alpha}_{p,q}F^{s_{\mu}}_{\beta\varrho}}d\tau\right)^{\varrho} \\ \times \left(\int_{0}^{t} \|e^{(t-\tau)\Delta}G(u)(\tau,\cdot)\|_{\dot{K}^{\alpha}_{p,q}F^{s_{0}+\kappa}_{\infty}}d\tau\right)^{1-\varrho} \\ \leqslant C\left(\int_{0}^{t} \|G(u)(\tau,\cdot)\|_{\dot{K}^{\alpha}_{p,q}F^{s_{\mu}}_{\beta\varrho}}d\tau\right)^{\varrho} \\ \times \left(\int_{0}^{t} (t-\tau)^{-\frac{\kappa}{2}} \|G(u)(\tau,\cdot)\|_{\dot{K}^{\alpha}_{p,q}F^{s_{0}}_{\infty}}d\tau\right)^{1-\varrho} \\ \leqslant CT_{0}^{1+(\varrho-1)\frac{\kappa}{2}} \|u\|_{Y}^{\varrho\mu} \|u\|_{Y}^{(1-\varrho)s_{0}}.$$

The proof is completed.

Using a combination of the arguments used in the proof of Theorem 4.15 with the help of Theorem 3.38 we get the following result:

Theorem 4.23. Let
$$0 < p, q < \infty, 0 \leqslant \alpha < n - \frac{n}{p}, \mu \geqslant \frac{\frac{n}{p} + \alpha}{\frac{n}{q} + \alpha + 1}$$
 and
$$1 < \mu < \frac{n}{p} + \alpha.$$

Let $G \in Lip\mu$ and

$$s = 1 + \frac{\mu - 1}{\mu} \left(\frac{n}{p} + \alpha \right).$$

- (i) For all initial data u_0 in $\dot{K}_{p,q}^{\alpha} F_{\beta}^s$ with $s > \bar{s}$, there exists a maximal solution u to (4.14) in $C([0,T_0),\dot{K}_{p,q}^{\alpha}F_{\beta}^s)$ with $T_0 \geqslant C \|u_0\|_{\dot{K}_{\alpha,p}^{\alpha}F_{\beta}^s}^{s}$.
- (ii) Let $\theta \leq 2\vartheta(\mu 1)$. We have

$$u - e^{t\Delta}u_0 \in C([0, T_0), \dot{K}_{p,q}^{\alpha} F_{\beta}^{s+\theta}).$$

Let $s > \frac{n}{p} + \alpha$. Using Theorem 3.37, the embedding $\dot{K}_{p,q}^{\alpha} F_{\beta}^{s} \hookrightarrow L^{\infty}$, we immediately arrive at the following result. We omit the proof since is essentially similar to the proof of Theorem 4.15.

Theorem 4.24. Let $1 < p, q < \infty, 1 < \beta < \infty, \mu > 1$ and $0 \leqslant \alpha < n - \frac{n}{p}$. Let $G \in Lip\mu$ and

$$\frac{n}{p} + \alpha < s < \mu.$$

- (i) For all initial data u_0 in $\dot{K}^{\alpha}_{p,q}F^s_{\beta}$ with $s > \bar{s}$, there exists a maximal solution u to (4.14) in $C([0,T_0),\dot{K}^{\alpha}_{p,q}F^s_{\beta})$ with $T_0 \geqslant C \|u_0\|_{\dot{K}^{\alpha}_{p,q}F^s_{\beta}}^{-1}$.
- (ii) Let $\theta < 2$. We have

$$u - e^{t\Delta}u_0 \in C([0, T_0), \dot{K}_{p,q}^{\alpha} F_{\beta}^{s+\theta}).$$

Remark 4.25. Corresponding statements to Theorem 4.15 were proved by Ribaud [45], with $\theta < 2\vartheta(\mu - 1)$, $\alpha = 0$, p = q and $\beta = 2$, under the assumption

$$\frac{n}{p} - \frac{n}{\mu p} < s < \min\left(\frac{\left(1 + \frac{n}{p}\right)(\mu - 1)}{\mu}, \frac{n}{p}\right). \tag{4.26}$$

Here we are requiring

$$\max\left(0, \frac{n}{p} - \frac{n}{\mu}\right) < s < \min\left(1 + \frac{\mu - 1}{\mu} \frac{n}{p}, \frac{n}{p}\right),$$

which improve (4.26).

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References

- [1] F. Baaske and H.-J. Schmeißer, On a generalized nonlinear heat equation in Besov and Triebel-Lizorkin spaces. Math. Nachr 290 (14-15) (2017), 2111–2131.
- [2] B. H. Brahim and D. Drihem, Multiplication on Besov and Triebel-Lizorkin spaces of power weights. Submitted.
- [3] H. Brezis and T. Cazenave, A nonlinear heat equation with singular initial data. Journal D'Analyse Mathematique 68 (1996), no 1, 277–304.
- [4] A. Baernstein II and E. T. Sawyer, Embedding and multiplier theorems for $H^p(\mathbb{R}^n)$. Mem. Amer. Math. Soc 53, no. 318, 1985.
- [5] G. Bourdaud and D. Kateb, Fonctions qui opèrent sur certains espaces de Besov. Ann. Inst. Fourier 40 (1990), 153–162.
- [6] G. Bourdaud, Le calcul fonctionnel dans les espaces de Sobolev. Invent. Math 104 (1991), 435–446.
- [7] G. Bourdaud, Fonctions qui opérent sur les espaces de Besov et de Triebel. Annales de L'I.H.P. (Analyse non linéaire) 10 (1993), 413–422.
- [8] G. Bourdaud, *The functional calculus in Sobolev spaces*. In: Function spaces, differential operators and nonlinear analysis. Teubner-Texte Math. 133, Teubner, Stuttgart, Leipzig, 1993, 127–142.

- [9] G. Bourdaud, M. Lanza de Cristoforis and W. Sickel, Superposition operators and functions of bounded p-variation. Revista Mat. Iberoamer 22 (2006), 455–487.
- [10] G. Bourdaud and M. Lanza de Cristoforis, Regularity of the symbolic calculus in Besov algebras. Studia Math 184 (3) (2008), 271–298.
- [11] G. Bourdaud, Superposition in homogeneous and vector valued Sobolev spaces. Trans. Amer. Math. Soc **362** (2010), 6105–6130.
- [12] G. Bourdaud, M. Moussai and W. Sickel, Composition operators in Lizorkin-Triebel spaces. J. Funct. Anal 259 (2010), 1098–1128.
- [13] G. Bourdaud and M. Moussai, Continuity of composition operators in Sobolev spaces, Ann. I.H. Poincaré-AN **36** (2019), 2053-2063.
- [14] H. Brezis and P. Mironescu, Gagliardo-Nirenberg, composition and products in fractional Sobolev spaces. J. Evol. Equ 1 (4) (2001), 387–404.
- [15] H. Q. Bui, Weighted Besov and Triebel spaces: interpolation by the real method. Hiroshima Math J 12 (1982), 581–605.
- [16] H. Q. Bui, Characterizations of weighted Besov and Triebel-Lizorkin spaces via temperatures. J. Funct. Anal 55 (1984), 39–62.
- [17] T. Cazenave, F. Dickstein, I. Naumkin and F. B. Weissler, Perturbations of self-similar solutions. Dynamics of PDE 16 (2019), 151–183.
- [18] B. J. Dahlberg, A note on Sobolev spaces. Proc. Symp. Pure Math 35 (1) (1979), 183–85.
- [19] L. Diening, P. Hästö and S. Roudenko, Function spaces of variable smoothness and integrability. J. Funct. Anal 256 (2009), 1731–1768.
- [20] D. Drihem, Embeddings properties on Herz-type Besov and Triebel-Lizorkin spaces. Math. Ineq and Appl 16 (2) (2013), 439–460.
- [21] D. Drihem, Sobolev embeddings for Herz-type Triebel-Lizorkin spaces. Function Spaces and Inequalities. P. Jain, H.-J.Schmeisser (ed.). Springer Proceedings in Mathematics and Statistics. Springer, 2017.
- [22] D. Drihem, Complex interpolation of Herz-type Triebel-Lizorkin spaces. Math. Nachr 291 (13) (2018), 2008–2023.
- [23] D. Drihem, Caffarelli-Kohn-Nirenberg inequalities on Besov and Triebel-Lizorkin-type spaces. arXiv:1808.08227.
- [24] D. Drihem, Nemytzkij operators on Sobolev spaces with power weights. I. Submitted.
- [25] D. Drihem, Nemytzkij operators on Sobolev spaces with power weights. II. Preprint.
- [26] D. Drihem, Composition operators on Besov and Triebel-Lizorkin spaces with power weights: necessary conditions. arXiv:2108.00718.
- [27] D. Drihem, Semilinear parabolic equations in Herz spaces. Submitted.
- [28] H. G. Feichtinger and F. Weisz, Herz spaces and summability of Fourier transforms. Math. Nachr 281 (3) (2008), 309–324.
- [29] C. Herz, Lipschitz spaces and Bernstein's theorem on absolutely convergent Fourier transforms. J. Math. Mech 18 (1968), 283–324.
- [30] Y. Giga, Solutions for semilinear parabolic equations in L^p and the regularity of weak solutions of the Navier Stokes system. J Diff Equations **61** (1986), 186–222.
- [31] L. I. Hedberg and Y.V. Netrusov, An axiomatic approach to function spaces, spectral synthesis, and Luzin approximation. Mem. Amer. Math. Soc 188 (2007), no. 882, vi+97 pp.
- [32] E. Hernandez and D. Yang, Interpolation of Herz-type spaces and applications. Math. Nachr 42 (1998), 564–581.
- [33] S. Igari, Sur les fonctions qui opèrent sur l'espace \hat{A}^2 . Ann. Inst. Fourier (Grenoble) 15 (1965), 525–536.
- [34] M. Izuki and Y. Sawano, Atomic decomposition for weighted Besov and Triebel-Lizorkin spaces. Math. Nachr 285 (2012), 103–126.
- [35] S. Janson, Harmonic analysis and partial differential equations. Proceedings, El Escorial 1987. (Lect. Notes Math., vol. 1384, pp. 193–301) Berlin Heidelberg New York: Springer 1989
- [36] J. Johnsen, Pointwise multiplication of Besov and Triebel-Lizorkin spaces. Math. Nachr 175 (1995), 85–133.

- [37] H. Kozono and M. Yamazaki, Semilinear heat equations and the Navier-Stokes equation with distributions in new function spaces as initial data. Comm. in Partial Differential Equations 19 (1994), 959–1014.
- [38] X. Li and D. Yang, Boundedness of some sublinear operators on Herz spaces. Illinois J. Math 40 (1996), 484–501.
- [39] S. Lu and D. Yang, Herz-type Sobolev and Bessel potential spaces and their applications. Sci. in China (Ser. A) 40 (1997), 113–129.
- [40] S. Lu, D. Yang and G. Hu, Herz type spaces and their applications. Beijing: Science Press, 2008
- [41] M. Marcus and V. J. Mizel, Complete characterization of functions which act, via superposition, on Sobolev spaces. Trans. Am. Math. Soc 251 (1979), 187–218.
- [42] C. Miao and B. Zhang, The Cauchy problem for semilinear parabolic equations in Besov spaces. Houston J. Math **30** (2004), 829–878.
- [43] M. A. Ragusa, Homogeneous Herz spaces and regularity results. Nonlinear Anal 71 (2009), e1909–e1914.
- [44] F. Ribaud, Semilinear parabolic equation with distributions as initial data. Discrete Cont. Dynam. Sys 3 (1997), 305–316.
- [45] F. Ribaud, Cauchy problem for semilinear parabolic equation with data in $H_p^s(\mathbb{R}^n)$ spaces. Rev. Mat. Iberoramericana 14 (1998), 1–45.
- [46] T. Runst, Mapping properties of non-linear operators in spaces of Triebel-Lizorkin and Besov type. Anal. Math 12 (1986), 313–346.
- [47] T. Runst and W. Sickel, Sobolev spaces of fractional order, Nemytskij operators and nonlinear partial differential equations. de Gruyter, Berlin 1996.
- [48] Y. Sawano, Theory of Besov spaces. Developments in Math. 56, Springer, Singapore, 2018.
- [49] W. Sickel, On pointwise multipliers in Besov-Triebel-Lizorkin spaces. Seminar Analysis 1986 (ed. by B.-W. Schulze and H. Triebel), Teubner-Texte Math., 96, Teubner, Leipzig 1987.
- [50] W. Sickel, On boundedness of superposition operators in spaces of Triebel-Lizorkin type. Czechoslovak. Math. J **39** (114) (1989), 323–347.
- [51] W. Sickel, Necessary conditions on composition operators acting on Sobolev spaces of fractional order. The critical case 1 < s < n/p. Forum Math 9 (1997), 267–302.
- [52] W. Sickel, Necessary conditions on composition operators acting between Sobolev spaces of fractional order. The critical case 1 < s < n/p. II. Forum Math 10 (1998), 199–231.
- [53] W. Sickel, Necessary conditions on composition operators acting between Besov spaces. The case 1 < s < n/p. III. Forum Math 10 (1998), 303–327.
- [54] L. Tang and D. Yang, Boundedness of vector-valued operators on weighted Herz spaces. Approx. Th. & its Appl 16 (2000), 58–70.
- [55] S. Tayachi and F. B. Weissler, The nonlinear heat equation involving highly singular initial values and new blowup and life span results, Journal of Elliptic and Parabolic Equations 4 (2018), 141–176.
- [56] E. Terraneo, Non-uniqueness for a critical non-linear heat equation. Comm. Partial Differential Equations 27 (2002), 185–218.
- [57] Y. Tsutsui, The Navier-Stokes equations and weak Herz spaces. Adv. Differential Equations 16 (2011), 1049–1085.
- [58] H. Triebel, Theory of function spaces. Birkhäuser Verlag, Basel, 1983.
- [59] H. Triebel, Theory of function spaces II. Birkhäuser Verlag, Basel, 1992.
- [60] H. Triebel, *Hybrid function spaces, Heat and Navier-Stokes Equations*. European Math. Soc. Publishing House, Zürich, 2014.
- [61] H. Umakoshi. A semilinear heat equation with initial data in negative Sobolev spaces. Discrete & Continuous Dynamical Systems S 14 (2) 2021, 745–767.
- [62] F. B. Weissler, Semilinear evolution equations in Banach spaces. J. Funcf. Anal 32 (1979), 277–296.
- [63] F. B. Weissler, Local existence and nonexistence for semilinear parabolic equations in L^p. Indiana Univ. Math. J 29 (1980), 79–102.
- [64] J. Xu and D. Yang, Applications of Herz-type Triebel-Lizorkin spaces. Acta. Math. Sci (Ser. B) 23 (2003), 328–338.
- [65] J. Xu and D. Yang, Herz-type Triebel-Lizorkin spaces, I. Acta Math. Sci (English Ed.) 21 (3) (2005), 643–654.

- [66] J. Xu, Equivalent norms of Herz type Besov and Triebel-Lizorkin spaces. J Funct Spaces Appl 3 (2005), 17–31.
- [67] J. Xu, Decompositions of non-homogeneous Herz-type Besov and Triebel-Lizorkin spaces. Sci. China. Math 47 (2) (2014), 315–331.
- [68] M. Yamazaki, A quasi-homogeneous version of paradifferential operators, I. Boundedness on spaces of Besov type. J. Fac. Sci. Univ. Tokyo, Sect. IA Math 33 (1986), 131–174.

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