

Stability of Erdős-Ko-Rado Theorems in Circle Geometries

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Abstract

Circle geometries are incidence structures that capture the geometry of circles on spheres, cones and hyperboloids in 3-dimensional space. In a previous paper, the author characterised the largest intersecting families in finite ovoidal circle geometries, except for Möbius planes of odd order. In this paper we show that also in these Möbius planes, if the order is greater than 3, the largest intersecting families are the sets of circles through a fixed point. We show the same result in the only known family of finite non-ovoidal circle geometries. Using the same techniques, we show a stability result on large intersecting families in all ovoidal circle geometries. More specifically, we prove that an intersecting family \mathcal{F} in one of the known finite circle geometries of order q , with $|\mathcal{F}| \geq \frac{1}{\sqrt{2}}q^2 + 2\sqrt{2}q + 8$, must consist of circles through a common point, or through a common nucleus in case of a Laguerre plane of even order.

Keywords. Erdős-Ko-Rado, Finite geometry, Möbius planes, Laguerre planes, Minkowski planes.

1 Introduction

In their seminal paper [EKR61], Erdős, Ko, and Rado proved the following theorem.

Theorem 1.1 ([EKR61]). *Choose integers k and n such that $0 < k \leq n/2$. Let \mathcal{F} be a family of subsets of size k of $\{1, \dots, n\}$ such that for each $F, G \in \mathcal{F}$, $F \cap G \neq \emptyset$. Then*

$$|\mathcal{F}| \leq \binom{n-1}{k-1}.$$

Moreover, if $k < n/2$, then equality holds if and only if \mathcal{F} is the family of all k -sets through a fixed element of $\{1, \dots, n\}$.

Since then, there has been a broad interest in the concept of *intersecting families*. In a general sense, this means the following. An *incidence structure* is a tuple $(\mathcal{P}, \mathcal{B})$, where we call the elements of \mathcal{P} *points*, and the elements of \mathcal{B} *blocks*, and where each block is a subset of \mathcal{P} . An intersecting family is a set $\mathcal{F} \subseteq \mathcal{B}$ such that any two elements of \mathcal{F} have non-empty intersection. A typical question in the vein of the Erdős-Ko-Rado theorem asks what the size and structure of the largest intersecting families in some incidence structure are.

Extremal combinatorics is concerned with characterising the largest (or smallest) combinatorial objects that satisfy some property. If all objects of maximum size have the same structure, one could wonder whether there exist large objects which have a significantly different structure. If such objects do not exist, we speak of a *stability result*.

In [Adr21], the author proved an Erdős-Ko-Rado type theorem in ovoidal circle geometries. More specifically, it was shown that if the circle geometry is of order $q > 2$, and not a Möbius plane of odd order, then the largest intersecting families are exactly the collections of circles through a fixed point, or through a fixed nucleus if the circle geometry is a Laguerre plane of even order. In this article, we extend this theorem to all known finite circle geometries, and to a stability result. The two main theorems are as follows.

Theorem 1.2. *Consider one of the known finite circle geometries of order $q > 2$. In case of a Möbius plane, assume that $q > 3$. The largest intersecting families consist of all circles through a fixed point, or through a fixed nucleus in case of a Laguerre plane of even order.*

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Theorem 1.3. *If \mathcal{F} is an intersecting family in one of the known finite circle geometries of order q , and $|\mathcal{F}| \geq \frac{1}{\sqrt{2}}q^2 + 2\sqrt{2}q + 8$, then all circles of \mathcal{F} go through a common point, or a common nucleus in case of a Laguerre plane of even order.*

As a corollary, we obtain stability results of intersecting families in $\text{PGL}(2, q)$ and quadratic polynomials over finite fields.

The structure of the paper is as follows. In §2, we give the necessary background on circle geometries and algebraic graph theory. In §3, we discuss association schemes related to circle geometries. In particular, we are interested in their eigenvalues, as they will be essential for the further proofs. In the next two sections, we prove that if \mathcal{F} is a large intersecting family in a known circle geometry, then all of the circles of \mathcal{F} go through a common point P . This is done in two steps. In §4, we find a candidate point P , that does not lie on “few” circles of \mathcal{F} . In §5, we prove that if \mathcal{F} is a large intersecting family in a circle geometry, then every point lies on either few or many circles of \mathcal{F} . In §6, we put these two results together to find a point that lies on many circles of \mathcal{F} , and prove that it in fact lies on all circles of \mathcal{F} . This yields the two theorems discussed above. These results also imply stability results of large intersecting families in $\text{PGL}(2, q)$ and in the set of polynomials over a finite field of degree at most two.

The most important step is the idea behind §5. Consider the following situation. Let G be a regular graph with independence number $\alpha(G)$. Suppose that we know cocliques C_1, \dots, C_n of size $\alpha(G)$, such that each C_i and its complement form an equitable partition of the graph (which happens for example if $\alpha(G)$ meets Hoffman’s ratio bound.) Take such a coclique C_i , and remove all edges from G which have no endpoint in C_i . We are left with a bipartite graph, which must also be biregular. If the eigenvalues of this bipartite graph can be calculated, we can apply the expander mixing lemma. The idea is that if we take another large coclique C' in G , the bipartite expander mixing lemma implies that if $|C_i \cap C'|$ is large, C' is completely contained in C_i . If we can prove that every large coclique C' has a large intersection with at least one of the C_i ’s, we have proven that the C_i ’s are the only large cocliques.

2 Preliminaries

2.1 Circle geometries

In this paper, we will investigate a certain type of incidence structures, called *circle geometries*. The blocks of a circle geometry are often called *circles*. To explain what a circle geometry is, we first need to introduce the concept of a *parallel relation*. This is a partition of the points into so-called *parallel classes*. We call two points *parallel* if they are in the same parallel class of some parallel relation.

A circle geometry is an incidence structure $(\mathcal{P}, \mathcal{B})$ with at most two parallel relations, such that the following properties hold:

1. Given three pairwise non-parallel points, there is a unique circle containing these three points.
2. Given a circle c , a point $P \in c$, and a point $Q \notin c$, not parallel with P , there is a unique circle through P and Q , which is *tangent* to c , i.e. it intersects c only in P .
3. Any circle contains a unique point from each parallel class.
4. Two parallel classes from different parallel relations intersect in a unique point.
5. Each circle contains at least three points, there exists a circle and a point not on this circle.

A circle geometry with 0, 1, or 2 parallel relations is called respectively a *Möbius plane*, *Laguerre plane*, or a *Minkowski plane*.

In this paper, we only concern ourselves with finite circle geometries, which means that we assume the number of points is finite. In this case, there exists some integer $q \geq 2$, called the *order* of the circle geometry, such that the following properties hold, where ρ denotes the number of parallel relations.

- $|\mathcal{P}| = \frac{q^2+1-\rho}{q+1-\rho}(q+1)$,
- $b := |\mathcal{B}| = q^3 + (1-\rho)q$,
- every circle contains $q+1$ points,

- every point lies on $q^2 + (1 - \rho)q$ circles,
- through any pair of non-parallel points, there are $q + 1 - \rho$ circles,
- every parallel class contains $q + \rho - 1$ points.

From now on, we denote a circle geometry of order q with ρ parallel classes as a $\text{CM}(\rho, q)$.

Given a point P in a $\text{CM}(\rho, q)$, we define the *residue* at P as the incidence structure $(\mathcal{P}', \mathcal{B}')$, where \mathcal{P}' consists of the points not parallel with P , and \mathcal{B}' consist the circles through P with P removed, and the parallel classes not containing P . If $\rho = 2$, every parallel class not containing P , contains a unique point parallel with P . These points are also removed from the blocks in \mathcal{B}' . The residue at any point P is an affine plane of order q .

Let \mathbb{F}_q denote the finite field of size q , where q is understood to be a prime power, and let $\text{PG}(n, q)$ denote the projective space arising from the vector space \mathbb{F}_q^{n+1} . A *quadric* in $\text{PG}(n, q)$ is the set of points whose coordinates satisfy some homogeneous quadratic equation. To capture the behaviour of a quadric, Buekenhout [Bue69] defined a *quadratic set* in $\text{PG}(n, q)$ as a set of points \mathcal{Q} satisfying the following properties.

1. Any line that intersects \mathcal{Q} in more than two points, is completely contained in it.
2. For any point $P \in \mathcal{Q}$, the union of all lines through P which intersect \mathcal{Q} in 1 or $q + 1$ points, is a hyperplane or the entire space $\text{PG}(n, q)$. We call this the *tangent hyperplane* of P .
3. \mathcal{Q} is not the union of two subspaces.

A point P is called *degenerate* if its tangent hyperplane is the entire space. A quadratic set is called degenerate if it has a degenerate point.

There are three types of quadratic sets in $\text{PG}(3, q)$, $q > 2$.

1. An *ovoid* is a set of $q^2 + 1$ point, no three on a line. It was proven by Barlotti [Bar55], that the only ovoids in $\text{PG}(3, q)$, q odd, are the so-called elliptic quadrics $\mathcal{Q}^-(3, q)$.
2. An *oval* is a set of $q + 1$ points in $\text{PG}(2, q)$, no three on a line. An *oval cone* in $\text{PG}(3, q)$ is a set constructed by taking an oval \mathcal{O} in a plane, taking some point R not in this plane, and taking the union of all lines through R which intersect \mathcal{O} . By Segre's famous result [Seg55], the only ovals in $\text{PG}(2, q)$, q odd, are the quadrics $\mathcal{Q}(2, q)$, also called non-degenerate conics. Therefore, there is up to isomorphism only one oval cone in $\text{PG}(3, q)$ if q is odd.
3. By a result by Buekenhout [Bue69], the only remaining quadratic sets are the hyperbolic quadrics $\mathcal{Q}^+(3, q)$. They can be constructed as follows. Take three pairwise disjoint lines l_0, l_1, l_2 in $\text{PG}(3, q)$. There are exactly $q + 1$ lines m_0, \dots, m_q , which intersect all of the lines l_0, l_1, l_2 . These lines are also pairwise disjoint. l_0, l_1, l_2 can be extended in a unique way to a set of $q + 1$ pairwise disjoint lines l_0, \dots, l_q , such that all lines m_i and l_j intersect in a point. Then $\bigcup_{i=0}^q m_i = \bigcup_{i=0}^q l_i$ as point sets. Such a point set is a hyperbolic quadric.

Given a quadratic set \mathcal{Q} in $\text{PG}(3, q)$, call a plane π an *oval plane* if $\pi \cap \mathcal{Q}$ is an oval in π . If \mathcal{Q} is non-degenerate, every plane is either a tangent or an oval plane. If \mathcal{Q} has a degenerate point P , the oval planes are exactly the planes missing P . Consider the incidence structure $(\mathcal{P}, \mathcal{B})$, where \mathcal{P} consist of the non-degenerate points of \mathcal{Q} , and $\mathcal{B} = \{\pi \cap \mathcal{Q} \mid \pi \text{ an oval plane}\}$. This incidence structure is a circle geometry. More specifically, it is a Möbius, Laguerre, or Minkowski plane, if \mathcal{Q} is respectively an ovoid, oval cone, or hyperbolic quadric. A circle geometry that arises in this way from a quadratic set is called *ovoidal*.

Dembowski [Dem64] and Heise [Hei74] proved that a Möbius, respectively Minkowski plane of even order must be ovoidal.

Consider an oval cone \mathcal{Q} in $\text{PG}(3, q)$ with vertex R and base \mathcal{O} in some plane $\pi \not\ni R$. If q is even, then \mathcal{O} has a *nucleus* N . This is a point in π such that the tangent lines to \mathcal{O} in π are exactly the lines through N (in π). Let \mathcal{Q}_+ denote $\mathcal{Q} \cup \langle R, N \rangle$. If L denotes the Laguerre plane arising from \mathcal{Q} , then let L_+ denote the incidence structure $(\mathcal{P}_+, \mathcal{B}_+)$ with $\mathcal{P}_+ = \mathcal{Q}_+ \setminus \{R\}$, and $\mathcal{B}_+ = \{\pi \cap \mathcal{Q}_+ \mid \pi \text{ an oval plane}\}$. We call this incidence structure an *extended Laguerre plane*. Combinatorially, the most important difference between

a Laguerre plane and an extended Laguerre plane, is that in an extended Laguerre plane, two distinct circles intersect in either 0 or in 2 points, but never in exactly 1.

Two oval planes intersect in the extended Laguerre plane if and only if they intersect in the original Laguerre plane. Thus, switching to the extended Laguerre plane makes no (significant) difference for the study of intersecting families.

We now describe the only known non-ovoidal circle geometries. Let A be a set containing more than three elements, and let Π be a sharply 3-transitive set of permutations of A . This means that if (a_1, a_2, a_3) and (a_4, a_5, a_6) are ordered triples of three distinct elements of A , there is a unique $\pi \in \Pi$ with $\pi(a_i) = a_{i+3}$ for $i = 1, 2, 3$. Suppose that Π contains the identity element. For each $\pi \in \Pi$, define its *graph* as $\{(a, \pi(a)) \mid a \in A\}$. Consider the incidence structure $(\mathcal{P}, \mathcal{B})$ with $\mathcal{P} = A \times A$ and \mathcal{B} the set of graphs of $\pi \in \Pi$. This incidence structure is a Minkowski plane, and each finite Minkowski plane can be constructed in this way.

Construction 2.1. Let q be a prime power, and let φ be a field automorphism of \mathbb{F}_q . For each $M \in \text{PGL}(2, q)$, define the map $f_M \in \text{P}\Gamma\text{L}(2, q)$ as

$$f_M(x) = \begin{cases} Mx & \text{if } M \in \text{PSL}(2, q), \\ Mx^\varphi & \text{otherwise.} \end{cases}$$

Then the set $\Pi_\varphi = \{f_M \mid M \in \text{PGL}(2, q)\}$ is sharply 3-transitive on $\text{PG}(1, q)$. Denote the corresponding Minkowski plane as $\text{CM}(2, q, \varphi)$.

If $\varphi = \text{id}$, then the corresponding Minkowski plane is the ovoidal one. For q even, $\text{PSL}(2, q) = \text{PGL}(2, q)$, and the choice of φ does not matter (recall that all Minkowski planes of even order are ovoidal). If q is odd, $|\text{PGL}(2, q) : \text{PSL}(2, q)| = 2$. A non-trivial field automorphism φ gives rise to a non-ovoidal circle geometry.

For a survey on circle geometries, see e.g. Hartmann [Har04] or Delandtsheer [Del95, §5]. It is worth mentioning that circle geometries are also called *Benz planes* and Möbius planes are also called *inversive planes*.

2.2 Graph theory

All graphs considered in this paper are simple, undirected and without loops. Let $G = (V, E)$ be a graph. If two vertices x and y are adjacent, we denote this as $x \sim y$. The *adjacency matrix* of G is the matrix $A(G)$ whose rows and columns are labelled by the vertices of G , such that

$$A(G)_{x,y} = \begin{cases} 1 & \text{if } x \sim y, \\ 0 & \text{otherwise.} \end{cases}$$

We denote the eigenvalues of $A(G)$ as $\lambda_1(G) \geq \lambda_2(G) \geq \dots \geq \lambda_n(G)$. If G is clear from context, we just write $\lambda_1, \dots, \lambda_n$. These eigenvalues can convey a lot of information about the graph, as illustrated by the following lemmata. Let $e(S, T)$ denote the number of edges with an endpoint in S and an endpoint in T .

Lemma 2.2 ([Hae95, Theorem 5.1] Bipartite expander mixing lemma). *Let G be a bipartite graph, with bipartition L and R . Take sets $S \subseteq L$ and $T \subseteq R$. Suppose that every vertex in L has degree d_L and every vertex in R has degree d_R . Then*

$$\left| e(S, T) - \frac{d_L}{|R|} |S| |T| \right| \leq \lambda_2 \sqrt{|S| |T| \left(1 - \frac{|S|}{|L|}\right) \left(1 - \frac{|T|}{|R|}\right)}.$$

This lemma has an interesting counterpart in non-bipartite graphs, of which we state a specific case. For a set $S \subseteq V$, let $e(S)$ denote the number of edges with both endpoints in S .

Lemma 2.3 ([Hae95, Theorem 3.5] Non-bipartite expander mixing lemma). *Let G be a d -regular graph with n vertices. Then*

$$\frac{d}{n} |S|^2 + \lambda_n |S| \left(1 - \frac{|S|}{n}\right) \leq 2e(S) \leq \frac{d}{n} |S|^2 + \lambda_2 |S| \left(1 - \frac{|S|}{n}\right).$$

2.3 Association schemes

Definition 2.4. Let $\mathcal{A} = \{A_0, \dots, A_d\}$ be a set of non-zero $n \times n$ -matrices over \mathbb{C} with only 0 and 1 as entries. We call these matrices a *d-class association scheme* if the following properties are satisfied.

1. $A_0 = I_n$,
2. $A_0 + \dots + A_d = J_n$, the $n \times n$ all-one matrix,
3. the set $\{A_0, \dots, A_d\}$ is closed under transposition,
4. there exist numbers p_{ij}^k , called *intersection numbers*, such that for every $i, j = 0, \dots, d$, $A_i A_j = \sum_{k=0}^d p_{ij}^k A_k$, and $p_{ij}^k = p_{ji}^k$.

If the rows and columns of the matrices A_0, \dots, A_d of an association scheme are indexed by some set X , then there is a natural correspondence between the matrix A_i and the relation

$$R_i = \{(x, y) \in X \times X \mid A_{xy} = 1\}.$$

We also say that the relations R_0, \dots, R_d are an association scheme, since they clearly convey the same information.

The algebra $\mathbb{C}[\mathcal{A}]$ spanned by A_0, \dots, A_d over \mathbb{C} is called the *Bose-Mesner algebra* of the association scheme. It readily follows from the definition of an association scheme that this algebra is $(d + 1)$ -dimensional.

Theorem 2.5. *The Bose-Mesner algebra of a d-class association scheme \mathcal{A} has a basis E_0, E_1, \dots, E_d of idempotents such that $E_i E_j = \delta_{ij} E_i$, $E_0 = \frac{1}{n} J_n$, and $\sum_{i=0}^d E_i = I_n$.*

Let V_i denote the column space of E_i . The previous theorem implies that the matrices A_0, \dots, A_d are simultaneously diagonalisable, and each eigenspace of each A_i is a direct sum of some V_j 's.

The transition matrix from the A_i basis to the E_j basis is called the *matrix of eigenvalues* (or sometimes the character table) and usually denoted by P .

Given a group G of order n , let $C_0 = \{1\}, C_1, \dots, C_d$ denote its conjugacy classes. The relations $R_i = \{(x, y) \in G \times G \mid yx^{-1} \in C_i\}$ constitute an association scheme, sometimes called the *conjugacy class scheme*. Let A_0, \dots, A_d denote the corresponding 01-matrices.

The basis of idempotents and their corresponding eigenvalues can be found as follows. Let $\psi_0 = 1, \psi_1, \dots, \psi_d$ denote the irreducible characters of G over \mathbb{C} . Define the matrix E_i , whose rows and columns are indexed by G , as $(E_i)_{xy} = \frac{\psi_i(1)}{n} \psi_i(yx^{-1})$. Then E_0, \dots, E_d are the basis of idempotents, and $A_i E_j = |C_i| \frac{\overline{\psi_j(c)}}{\psi_j(1)} E_j$ for any $c \in C_i$. Furthermore, $\dim V_j = \psi_j(1)^2$.

More on association schemes can be found e.g. in [GM16, §3] or [BCN89, §2]. For more on the conjugacy class scheme, see [GM16, §11].

3 Association schemes from ovoidal circle geometries

Given a CM(ρ, q), define the following relations on the circles. We say that circles c_1 and c_2 are in relation R_0, R_1, R_2, R_3 if $|c_1 \cap c_2|$ equals respectively $q + 1$ (i.e. $c_1 = c_2$), 1, 2, or 0. In some circle geometries, these relations constitute an association scheme. We list these cases, and give the corresponding matrix of eigenvalues, see [Adr21].

Möbius planes of even order $q > 2$.

$$P = \begin{pmatrix} 1 & q^2 - 1 & \frac{q^2}{2}(q + 1) & \frac{q}{2}(q - 1)(q - 2) \\ 1 & q - 1 & -q & 0 \\ 1 & -2 & q \frac{q-1}{2} & -(q + 1) \frac{q-2}{2} \\ 1 & -(q + 1) & 0 & q \end{pmatrix}$$

Ovoidal Laguerre planes of odd order.

$$P = \begin{pmatrix} 1 & q^2 - 1 & q\frac{q^2-1}{2} & q\frac{(q-1)^2}{2} \\ 1 & -1 & q\frac{q-1}{2} & -q\frac{q-1}{2} \\ 1 & q-1 & -q & 0 \\ 1 & -(q+1) & 0 & q \end{pmatrix}$$

Minkowski planes of even order $q > 2$.

$$P = \begin{pmatrix} 1 & q^2 - 1 & q(q+1)\frac{q-2}{2} & (q-1)\frac{q^2}{2} \\ 1 & q-1 & -q & 0 \\ 1 & -(q+1) & 0 & q \\ 1 & 0 & \frac{q^2-q-2}{2} & -(q-1)\frac{q}{2} \end{pmatrix}$$

Using the description of an ovoidal Minkowski plane as the graphs of $\text{PGL}(2, q)$, this association scheme and the matrix of eigenvalues can be found in [Ban91, page 172].

Extended Laguerre plane of even order $q > 2$.

In this case, we need to drop the relation R_1 , since it is empty.

$$P = \begin{pmatrix} 1 & \binom{q+2}{2}(q-1) & (q-1)^2\frac{q}{2} \\ 1 & -\frac{q+2}{2} & \frac{q}{2} \\ 1 & (q+1)\frac{q-2}{2} & -(q-1)\frac{q}{2} \end{pmatrix}$$

Ovoidal Möbius and Minkowski planes of odd order.

In ovoidal Möbius and Minkowski planes of odd order, the above relations do not constitute an association scheme. Some of the relations need to be spliced.

Let q be an odd prime power. Let \mathcal{Q}^ε denote the elliptic quadric $\mathcal{Q}^-(3, q)$ if $\varepsilon = -1$, and the hyperbolic quadric $\mathcal{Q}^+(3, q)$ if $\varepsilon = +1$. There exists some symmetric bilinear form $b(x, y)$ on \mathbb{F}_q^4 , and an associated quadratic form $\kappa(x) = b(x, x)$, such that \mathcal{Q}^ε consists of the projective points whose coordinate vectors satisfy $\kappa(x) = 0$. Define the polarity \perp by mapping a point P with coordinate vector x , to the plane P^\perp consisting of the points with coordinate vectors y satisfying $b(x, y) = 0$. Write $P \perp R$ if $P \in R^\perp$ (or equivalently $R \in P^\perp$). Note that the tangent and oval planes are the planes P^\perp with $P \in \mathcal{Q}^\varepsilon$ respectively $P \notin \mathcal{Q}^\varepsilon$.

Lemma 3.1. *Let l be a line in $\text{PG}(3, q)$, not contained in \mathcal{Q}^ε . Then*

$$|l^\perp \cap \mathcal{Q}^\varepsilon| = \begin{cases} 2 - |l \cap \mathcal{Q}^\varepsilon| & \text{if } \varepsilon = -1, \\ |l \cap \mathcal{Q}^\varepsilon| & \text{if } \varepsilon = +1. \end{cases}$$

Proof. Let i denote $|l \cap \mathcal{Q}^\varepsilon|$ and let x denote the number of tangent planes through l . Note that if $Q \in \mathcal{Q}^\varepsilon$, then $Q \in l$ if and only if $l^\perp \subset Q^\perp$, thus $x = |l^\perp \cap \mathcal{Q}^\varepsilon|$. If $\varepsilon = -1$, then every tangent and oval plane through l contains $1 - i$ respectively $q + 1 - i$ points of $\mathcal{Q}^\varepsilon \setminus l$, and $|\mathcal{Q}^\varepsilon| = q^2 + 1$. Hence,

$$i + x(1 - i) + (q + 1 - x)(q + 1 - i) = q^2 + 1 \quad \implies \quad x = 2 - i.$$

If $\varepsilon = 1$, then every tangent plane through l contains $2q + 1 - i$ points of $\mathcal{Q}^\varepsilon \setminus l$, and $|\mathcal{Q}^\varepsilon| = (q + 1)q(q - 1)$. Hence,

$$i + x(2q + 1 - i) + (q + 1 - x)(q + 1 - i) = (q + 1)q(q - 1) \quad \implies \quad x = i.$$

□

Take a point $P \notin \mathcal{Q}^\varepsilon$. Let x be a coordinate vector for P . Then the other coordinate vectors for P are of the form αx , $\alpha \in \mathbb{F}_q^*$. For every such α , $\kappa(\alpha x) = \alpha^2 \kappa(x)$ is a square if and only if $\kappa(x)$ is a square. Let S_q and \overline{S}_q denote respectively the non-zero squares and the non-squares of \mathbb{F}_q . Then we write $\kappa(P) = S_q$ if $\kappa(x) \in S_q$, and $\kappa(P) = \overline{S}_q$ if $\kappa(x) \in \overline{S}_q$. For a scalar α and subsets A and B of \mathbb{F}_q , let αAB denote the set $\{\alpha ab \mid a \in A, b \in B\}$. Given a conic C in $\text{PG}(2, q)$, we call a point $P \notin C$ *external* or *internal* when P lies on 2 respectively 0 tangent lines to C .

Lemma 3.2. *Take two points P and R in $\text{PG}(3, q)$, not in \mathcal{Q}^ε , with $P \perp R$. Then P is external to $R^\perp \cap \mathcal{Q}^\varepsilon$ if and only if*

$$-\kappa(P)\kappa(R) = \begin{cases} \overline{S}_q & \text{if } \varepsilon = -1, \\ S_q & \text{if } \varepsilon = +1. \end{cases}$$

Proof. P is external to $R^\perp \cap \mathcal{Q}^\varepsilon$ if and only if there are two points Q_1 and Q_2 in \mathcal{Q}^ε with $Q_1, Q_2 \in R^\perp$ and PQ_1, PQ_2 tangent lines. Note that since $P \notin \mathcal{Q}^\varepsilon$, PQ_i is a tangent line if and only if $P \in Q_i^\perp$ if and only if $Q_i \in P^\perp$. Therefore, P is external to $R^\perp \cap \mathcal{Q}^\varepsilon$ if and only if $(PR)^\perp$ is a 2-secant to \mathcal{Q}^ε .

Let x and y be coordinate vectors for P and R respectively. First suppose that $\varepsilon = -1$. Then $(PR)^\perp$ is a 2-secant if and only if PR is a 0-secant. Equivalently,

$$\kappa(x + \alpha y) = \kappa(x) + 2b(x, y)\alpha + \kappa(y)\alpha^2 = 0$$

has no solutions. Since $P \perp R$, $b(x, y) = 0$, and the above is equivalent to $-\kappa(x)/\kappa(y)$ being a non-square. Thus, P is external to $R^\perp \cap \mathcal{Q}^\varepsilon$ if and only if $-\kappa(P)/\kappa(R) = -\kappa(P)\kappa(R) = \overline{S}_q$.

The case $\varepsilon = +1$ works analogously, but then $(PR)^\perp$ is a 2-secant if and only if PR is a 2-secant. \square

Hence, given two oval planes P^\perp and R^\perp there are two options. Either $\kappa(P) = \kappa(R)$ and each point of $(PR)^\perp$ is external in P^\perp if and only if it is external in R^\perp . Or $\kappa(P) \neq \kappa(R)$, and each point of $(PR)^\perp$ is external in P^\perp if and only if it is internal in R^\perp . This has us hoping that if we define relations on the circles of our circle geometry depending on the size of the intersection of two circles and whether they correspond to planes P^\perp and R^\perp with $\kappa(P) = \kappa(R)$ or $\kappa(P) \neq \kappa(R)$, these relations constitute an association scheme. We note that if P^\perp and R^\perp intersect in a unique point of \mathcal{Q}^ε , then all other points of $(PR)^\perp$ are external in P^\perp and R^\perp , which implies that $\kappa(P) = \kappa(R)$. Thus, this would be a 5-class association scheme.

Unfortunately, it is not straightforward to find all intersection numbers of this association scheme. However, for $\varepsilon = +1$, we can use the alternative representation of the ovoidal Minkowski plane as graphs of $\text{PGL}(2, q)$. We can then construct the association scheme described above as a so-called subscheme of the conjugacy class scheme of $\text{PGL}(2, q)$, which means that we need to join together different relations of the conjugacy class scheme.

The characters of $\text{PGL}(2, q)$ can e.g. be found in [MS11, §3]. We also give the description here for q odd. The conjugacy classes of $\text{PGL}(2, q)$ are as follows:

1. The identity I_2 .
2. The matrices with one linearly independent eigenvector. All of these matrices are conjugate to $U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.
3. The matrices with two linearly independent eigenvectors. Each such matrix is conjugate to a matrix of the form $D_x = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}$, $x \neq 0, 1$. Note that D_x and D_y are conjugate if and only if $x = y$ or $x = y^{-1}$. Thus, this gives us $\frac{q-1}{2}$ conjugacy classes.
4. The matrices without eigenvectors. Each such matrix is conjugate to a matrix of the form $V_r = \begin{pmatrix} 0 & 1 \\ -r^{q+1} & r + r^q \end{pmatrix}$ for some $r \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$. The matrices V_r and V_s are conjugate if and only if $r\mathbb{F}_q^*$ and $s\mathbb{F}_q^*$ are equal or each others inverses in $\mathbb{F}_{q^2}^*/\mathbb{F}_q^*$. This gives us $\frac{q+1}{2}$ conjugacy classes.

Next we describe the irreducible characters of $\text{PGL}(2, q)$. Define the map

$$\delta_q : \mathbb{F}_q^* \rightarrow \{-1, 1\} : x \mapsto \begin{cases} 1 & \text{if } x \in S_q, \\ -1 & \text{otherwise.} \end{cases}$$

For every group morphism $\gamma : \mathbb{F}_q^* \rightarrow \mathbb{C}^*$ of order greater than 2, there is an irreducible character η_γ , and $\eta_\gamma = \eta_{\gamma'}$ if and only if $\gamma = \gamma'$ or $\gamma(x) = \gamma'(x^{-1})$.

For every group morphism $\beta : \mathbb{F}_{q^2}^*/\mathbb{F}_q^* \rightarrow \mathbb{C}^*$ of order greater than two, there is an irreducible character ν_β , and $\nu_\beta = \nu_{\beta'}$ if and only if $\beta = \beta'$ or $\beta(x) = \beta'(x^{-1})$.

The characters can be read in the following table. For each conjugacy class, we give a representative, which might have a parameter, list the number of parameters that give a representative of a different

| | Representative | I_2 | U | $D_x, x \neq -1$ | D_{-1} | $V_r, r \mathbb{F}_q^* \neq i \mathbb{F}_q^*$ | V_i |
|----------------|-----------------|-------|-----------|------------------------------|-------------------|---|--------------------|
| | No. | 1 | 1 | $\frac{q-3}{2}$ | 1 | $\frac{q-1}{2}$ | 1 |
| | Size | 1 | $q^2 - 1$ | $q(q+1)$ | $q \frac{q+1}{2}$ | $q(q-1)$ | $q \frac{q-1}{2}$ |
| Character | No. | | | | | | |
| λ_1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| λ_{-1} | 1 | 1 | 1 | $\delta_q(x)$ | $\delta_q(-1)$ | $\delta_{q^2}(r)$ | $\delta_{q^2}(i)$ |
| ψ_1 | 1 | q | 0 | 1 | 1 | -1 | -1 |
| ψ_{-1} | 1 | q | 0 | $\delta_q(x)$ | $\delta_q(-1)$ | $-\delta_{q^2}(r)$ | $-\delta_{q^2}(i)$ |
| η_β | $\frac{q-1}{2}$ | $q-1$ | -1 | 0 | 0 | $-(\beta(r) + \beta(r^{-1}))$ | $-2\beta(i)$ |
| ν_γ | $\frac{q-3}{2}$ | $q+1$ | 1 | $\gamma(x) + \gamma(x^{-1})$ | $2\gamma(-1)$ | 0 | 0 |

Table 1: Character table of $\text{PGL}(2, q)$, q odd. By slight abuse of notation, $\beta(r)$ denotes $\beta(r \mathbb{F}_q^*)$.

conjugacy class, and give the size of the conjugacy classes. Let i denote an element of the unique coset in $\mathbb{F}_{q^2}^*/\mathbb{F}_q^*$ of order 2.

We remark that $\delta_q(x)$ and $\delta_{q^2}(r)$ equal 1 if and only if D_x respectively V_r is an element of $\text{PSL}(2, q)$.

To obtain the desired subscheme, we need to know which element of $\text{PGL}(2, q)$ corresponds to which oval plane of $\mathcal{Q}^+(3, q)$ in the two isomorphic representations of the ovoidal Minkowski plane. This correspondence is very straightforward. We can choose coordinates of $\text{PG}(3, q)$ such that the quadratic form of $\mathcal{Q}^+(3, q)$ is $\kappa(X_1, X_2, X_3, X_4) = X_1X_4 - X_2X_3$. Then the oval planes are the planes of the form P^\perp with $P = (x_1, x_2, x_3, x_4)$ and $x_1x_4 - x_2x_3 \neq 0$. Then P^\perp corresponds to the element $M_P = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$ of $\text{PGL}(2, q)$. It follows that $\kappa(P) = \kappa(R)$ if and only if $M_R M_P^{-1} \in \text{PSL}(2, q)$. Furthermore, $|P^\perp \cap R^\perp \cap \mathcal{Q}^+(3, q)|$ equals the number of points Q of $\text{PG}(1, q)$ with $M_P Q = M_R Q$, which equals the number of linearly independent eigenvectors of $M_R M_P^{-1}$.

We describe the relations on the circles of the Minkowski plane in the following table. For each point $P \notin \mathcal{Q}^+(3, q)$ of $\text{PG}(3, q)$, let $c_P = P^\perp \cap \mathcal{Q}^+(3, q)$ denote its corresponding circle. We describe the condition on P and R for (c_P, c_R) to be in a certain relation. We also describe to which element of $\text{PGL}(2, q)$ $M_R^{-1} M_P$ must then be conjugate.

| Relation | Geometric condition | Conjugate element to $M_R M_P^{-1}$ |
|----------|--|---|
| R_0 | $P = R$ | I_2 |
| R_1 | $ c_P \cap c_R = 1$ | U |
| R_2 | $ c_P \cap c_R = 2, \kappa(P) = \kappa(R)$ | $D_x, x \in S_q \setminus \{1\}$ |
| R_3 | $ c_P \cap c_R = 2, \kappa(P) \neq \kappa(R)$ | $D_x, x \in \overline{S_q}$ |
| R_4 | $ c_P \cap c_R = 0, \kappa(P) = \kappa(R)$ | $V_r, r \in S_{q^2} \setminus \mathbb{F}_q$ |
| R_5 | $ c_P \cap c_R = 0, \kappa(P) \neq \kappa(R)$ | $V_r, r \in \overline{S_{q^2}}$ |

Table 2: Relations on the ovoidal Minkowski plane of odd order q

Let A_i denote the matrix corresponding to relation R_i . The matrices A_i lie in the Bose-Mesner algebra of the conjugacy class scheme of $\text{PGL}(2, q)$. For each irreducible character ψ of $\text{PGL}(2, q)$, let E_ψ denote its corresponding idempotent. We will prove that for each A_i , its eigenvalue corresponding to E_{η_β} (resp. E_{ν_γ}) is independent of β (resp. γ). This implies that the A_i matrices lie in the span of $E_{\lambda_1}, E_{\lambda_{-1}}, E_{\psi_1}, E_{\psi_{-1}}, E_\eta = \sum_\beta E_{\eta_\beta}$, and $E_\nu = \sum_\gamma E_{\nu_\gamma}$. Since these matrices are also idempotents, this means that the A_i matrices span a 6-dimensional algebra. It is easy to check that each A_i is symmetric. Then the fact that they span a 6-dimensional algebra is sufficient to prove that they constitute an association scheme. Let $D_{S_q}, D_{\overline{S_q}}, V_{S_{q^2}}$, and $V_{\overline{S_{q^2}}}$ denote the elements of $\text{PGL}(2, q)$ conjugate to respectively D_x with $x \in S_q \setminus \{1\}$, D_x with $x \in \overline{S_q}$, V_r with $r \in S_{q^2} \setminus \mathbb{F}_q$, and V_r with $r \in \overline{S_{q^2}}$. Then we need to prove that for each $C \in \{D_{S_q}, D_{\overline{S_q}}, V_{S_{q^2}}, V_{\overline{S_{q^2}}}\}$, the value $\sum_{M \in C} \eta_\beta(M)$ does not depend on the choice of β and $\sum_{M \in C} \nu_\gamma(M)$ does not depend on γ . This only leaves a few non-trivial cases to check. We use the following lemma, which is a variation on a well-known property.

Lemma 3.3. *Let G be a finite group, $H \leq G$, and $\varphi : G \rightarrow \mathbb{C}^*$ be a morphism of order greater than $|G : H|$. Then for each coset S of H , $\sum_{x \in S} \varphi(x) = 0$.*

Proof. There is some element $h \in H$ with $\varphi(h) \neq 1$. Otherwise, $H \leq \text{Ker}(\varphi)$, and the order of $\text{Im}(\varphi)$ is at most $|G : H|$, contradicting that the order of φ is greater than $|G : H|$. Take a right coset $S = Hg$ of

H. Then

$$\sum_{x \in S} \varphi(x) = \sum_{x \in H} \varphi(xg) = \sum_{x \in H} \varphi(hxg) = \varphi(h) \sum_{x \in H} \varphi(xg) = \varphi(h) \sum_{x \in S} \varphi(x).$$

Since $\varphi(h) \neq 1$, this implies that $\sum_{x \in S} \varphi(x) = 0$. Analogous for left cosets of H . \square

Now take a morphism $\gamma : \mathbb{F}_q^* \rightarrow \mathbb{C}^*$ of order greater than 2.

$$\begin{aligned} \sum_{M \in D_{S_q}} \nu_\gamma(M) &= q(q+1) \sum_{x \in S_q \setminus \{1\}} \gamma(x) = q(q+1) \left(\sum_{x \in S_q} \gamma(x) - \gamma(1) \right) = -q(q+1), \\ \sum_{M \in D_{\overline{S_q}}} \nu_\gamma(M) &= q(q+1) \sum_{x \in \overline{S_q}} \gamma(x) = 0. \end{aligned}$$

Take a morphism $\beta : \mathbb{F}_{q^2}^* / \mathbb{F}_q^* \rightarrow \mathbb{C}^*$ of order greater than 2.

$$\begin{aligned} \sum_{M \in V_{S_{q^2}}} \eta_\beta(M) &= q(q-1) \sum_{\substack{r \in \mathbb{F}_{q^2}^* / \mathbb{F}_q^* \\ r \in S_{q^2} \setminus \mathbb{F}_q^*}} \beta(r) = -q(q-1), \\ \sum_{M \in V_{\overline{S_{q^2}}}} \eta_\beta(M) &= q(q-1) \sum_{\substack{r \in \mathbb{F}_{q^2}^* / \mathbb{F}_q^* \\ r \in \overline{S_{q^2}}}} \beta(r) = 0. \end{aligned}$$

We can now easily calculate the eigenvalue matrix of this subscheme from the formula for the eigenvalues of the conjugacy class scheme. This yields

$$P = \begin{pmatrix} 1 & q^2 - 1 & q \frac{(q-3)(q+1)}{4} & q \frac{q^2-1}{4} & q \frac{(q-1)^2}{4} & q \frac{q^2-1}{4} \\ 1 & q^2 - 1 & q \frac{(q-3)(q+1)}{4} & -q \frac{q^2-1}{4} & q \frac{(q-1)^2}{4} & -q \frac{q^2-1}{4} \\ 1 & 0 & \frac{(q-3)(q+1)}{4} & \frac{q^2-1}{4} & -\frac{(q-1)^2}{4} & -\frac{q^2-1}{4} \\ 1 & 0 & \frac{(q-3)(q+1)}{4} & -\frac{q^2-1}{4} & -\frac{(q-1)^2}{4} & \frac{q^2-1}{4} \\ 1 & -(q+1) & 0 & 0 & q & 0 \\ 1 & q-1 & -q & 0 & 0 & 0 \end{pmatrix}$$

Remark 3.4. One can also deduce from the character table that the dimensions of the eigenspaces are $1, 1, q^2, q^2, \frac{(q-1)^3}{2}, (q+1)^2 \frac{q-3}{2}$. If $q = 3$, the last eigenspace has dimension 0, since $S_3 \setminus \{1\} = \emptyset$. On the other hand, relation R_2 is empty, since there are no elements of $\text{PGL}(3, q)$ conjugate to D_x , $x \in S_3 \setminus \{1\}$. Thus, if $q = 3$ we get a 4-class association scheme.

4 A point not lying on few circles

As a first step to characterise large intersecting families \mathcal{F} , we prove that some point lies on “not few” circles of \mathcal{F} .

Definition 4.1. For an incidence structure $(\mathcal{P}, \mathcal{B})$, define the i -*intersecting graph* as the graph with vertices \mathcal{B} , where two blocks are adjacent if and only if they intersect in exactly i points. We will denote this graph by G_i .

The key ingredient in this section is that the G_1 graphs for the known circle geometries have good expanding properties. By this we mean that given two vertices, the number of their common neighbours can not deviate too much from the average number of common neighbours.

In an ovoidal Laguerre plane of even order q , the graph G_1 is a union of q disjoint copies of the complete graph K_{q^2} . This graph does not have good expanding properties. This issue can be resolved by switching to the extended Laguerre plane, for which G_1 is the empty graph on q^3 vertices. However, this example illustrates that we cannot prove for general circle geometries that G_1 has good expanding properties. Therefore, we will restrict ourselves to the known circle geometries, where we have more control over G_1 .

Lemma 4.2. *The 1-intersecting graph G_1 of a $\text{CM}(\rho, q)$ is $(q^2 - 1)$ -regular.*

Proof. Take a circle c . For any point $P \in c$, the number of circles that intersect c exactly in P equals the number of lines parallel to (but distinct from) $c \setminus \{P\}$ in the affine residue at P . This number is $q - 1$. Since there are $q + 1$ choices for P , there are $(q + 1)(q - 1)$ circles intersecting c in exactly one point. \square

The main idea behind this section is the following elementary counting argument.

Lemma 4.3. *Let \mathcal{F} be an intersecting family in a $\text{CM}(\rho, q)$ of size f . Consider the induced subgraph $G_1[\mathcal{F}]$ of G_1 on \mathcal{F} . Suppose that $G_1[\mathcal{F}]$ has at most E edges. Then there exists a point that lies on at least*

$$\frac{2f + (q - 1) - \frac{2E}{f}}{q + 1}$$

circles of \mathcal{F} .

Proof. The sum of the degrees in $G_1[\mathcal{F}]$ equals at most $2E$, thus there exists a circle c that has degree at most $\frac{2E}{f}$ in $G_1[\mathcal{F}]$. Let n_i denote the number of circles of \mathcal{F} intersecting c in exactly i points. Then $n_{q+1} = 1$, $n_2 = f - 1 - n_1$, and $n_1 \leq \frac{2E}{f}$. By performing a double count, we see that

$$|\{(P, c') \in c \times \mathcal{F} \mid P \in c'\}| = \sum_i i n_i = (q + 1) + n_1 + 2(f - 1 - n_1) = 2f + (q - 1) - n_1.$$

Thus, some point of c lies on at least

$$\frac{2f + (q - 1) - n_1}{q + 1} \geq \frac{2f + (q - 1) - \frac{2E}{f}}{q + 1}$$

circles of \mathcal{F} . \square

We will use the notation from Lemma 4.3 throughout this section.

Extended Laguerre planes.

In an extended Laguerre plane, $E = 0$.

Ovoidal circle geometries with 3-class association schemes.

There are three types of ovoidal circle geometries that give rise to 3-class association schemes, namely Möbius and Minkowski planes of even order and Laguerre planes of odd order. For each of these circle geometries, as can be seen from the previous section, $\lambda_2(G_1) = q - 1$. By the non-bipartite expander mixing lemma,

$$2E \leq \frac{q^2 - 1}{b} |\mathcal{F}|^2 + (q - 1) |\mathcal{F}| \left(1 - \frac{|\mathcal{F}|}{b}\right) = q \frac{q - 1}{b} |\mathcal{F}|^2 + (q - 1) |\mathcal{F}|,$$

with $b = q^3 + (1 - \rho)q$.

Möbius and Minkowski planes of odd order.

First, we prove that it suffices to check the ovoidal circle geometries. Recall Construction 2.1.

Lemma 4.4. *Let q be an odd prime power. The 1-intersecting graphs of $\text{CM}(2, q, \varphi)$ are isomorphic for all field automorphisms φ .*

Proof. Take a field automorphism φ . We prove that the G_1 graph of $\text{CM}(\rho, q, \varphi)$ is isomorphic to the G_1 graph of $\text{CM}(\rho, q, \text{id})$. Take $M \in \text{PGL}(2, q)$. Let f_M be as in Construction 2.1 with field automorphism φ . If M and N are in $\text{PSL}(2, q)$, then f_M and f_N are adjacent in G_1 if and only if MN^{-1} has a unique fixed point. If M and N are both not in $\text{PSL}(2, q)$, then f_M and f_N are adjacent in G_1 if and only if there is a unique projective point P with $MP^\varphi = NP^\varphi$. This is also equivalent to MN^{-1} having a unique fixed point. So the map $M \mapsto f_M$ is an embedding of the 1-intersecting graph of $\text{CM}(2, q, \text{id})$ into the 1-intersecting graph of $\text{CM}(2, q, \varphi)$. Since these graphs have the same valency, it is an isomorphism. \square

Now consider the ovoidal Möbius and Minkowski planes of odd order. It is known, see e.g. Hartmann [Har04], that the automorphism group of such a circle geometry works transitively on the circles. Thus, G_1 is vertex transitive. Furthermore, we know that G_1 has (at least) two connected components, namely the points P with $\kappa(P) = S_q$ and $\kappa(P) = \overline{S_q}$. That there are no more than two connected components, can be seen as follows. In a regular graph, the number of connected components equals the multiplicity

of the valency as eigenvalue. For the ovoidal Minkowski plane of odd order, consider the matrix P of eigenvalues of its 5-class association scheme. We see that the column of P corresponding to G_1 has $q^2 - 1$ in its first two rows, and both rows correspond to an eigenspace with dimension 1. For the ovoidal Möbius plane of order q , the eigenvalues of the G_1 graph restricted to the planes P^\perp with $\kappa(P) = S_q$ can be found in [BHS90, Table VIII and IX], in the unique column with valency $q^2 - 1$ (with $m = 2$ in the notation of the paper).

Thus, in both cases, G_1 consists of two isomorphic connected components, which means its adjacency matrix (after reordering if necessary) is of the form $A(G_1) = \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}$, where M is the adjacency matrix of a connected component. Moreover, the eigenvalues of $A(G_1)$ are the eigenvalues of M with their multiplicities doubled. Let C_1 denote a connected component of G_1 . Then $\lambda_2(C_1) = q - 1$ (see the matrix of eigenvalues in the previous section for Minkowski planes or [BHS90] for the Möbius planes). Now suppose that $|\mathcal{F}| = f$, and that there are s circles of \mathcal{F} in C_1 , then we apply the expander mixing lemma to both components of G_1 and find $2E$ is bounded by

$$\begin{aligned} & \max_{0 \leq s \leq f} \left[\left(\frac{2(q^2 - 1)}{b} s^2 + (q - 1)s \left(1 - \frac{2s}{b} \right) \right) + \left(\frac{2(q^2 - 1)}{b} (f - s)^2 + (q - 1)(f - s) \left(1 - \frac{2(f - s)}{b} \right) \right) \right] \\ & = 2q \frac{q - 1}{b} |\mathcal{F}|^2 + (q - 1) |\mathcal{F}|. \end{aligned}$$

This leaves us with the following proposition.

Proposition 4.5. *Consider a $\text{CM}(\rho, q)$ that is either ovoidal or from Construction 2.1. Let \mathcal{F} be an intersecting family. Define*

$$a = \begin{cases} 0 & \text{if } \rho = 1, \text{ and } q \text{ is even,} \\ 1 & \text{if } \rho = 0, 2 \text{ and } q \text{ is even, or } \rho = 1 \text{ and } q \text{ odd,} \\ 2 & \text{if } \rho = 0, 2 \text{ and } q \text{ is odd.} \end{cases}$$

Then some point lies on at least

$$\left(2 - a \frac{q - 1}{q^2 + 1 - \rho} \right) \frac{|\mathcal{F}|}{q + 1}$$

circles of \mathcal{F} .

We end with a little note on some of the G_1 graphs.

Lemma 4.6. *Let q be odd. Consider a connected component C_1 of the 1-intersecting graph of an ovoidal Möbius plane, or a $\text{CM}(2, q, \varphi)$, of odd order q . Take two distinct circles c_1 and c_2 , which are vertices in C_1 . Then the number of common neighbours of c_1 and c_2 equals*

$$\begin{cases} 2(q - 1) & \text{if } |c_1 \cap c_2| \in \{1, 2\}, \\ 2(q + 1) & \text{if } |c_1 \cap c_2| = 0. \end{cases}$$

Therefore, C_1 is a Deza graph¹.

Proof. First consider an ovoidal Möbius or Minkowski plane. Let P^\perp and R^\perp be oval planes to \mathcal{Q}^ε with $\kappa(P) = \kappa(R)$. Denote $l = (PR)^\perp$. We count the number of planes intersecting P^\perp and R^\perp in a tangent line to \mathcal{Q}^ε . Since a tangent plane Q^\perp contains $1 + \varepsilon \in \{0, 2\}$ $(q + 1)$ -secants to \mathcal{Q}^ε , Q is the only point in Q^\perp on more than one 1-secant. Thus, the only tangent plane that we will count is Q^\perp when $l \cap \mathcal{Q}^\varepsilon = \{Q\}$. First suppose that $|l \cap \mathcal{Q}^\varepsilon| = 1$. Then every point of $l \setminus \mathcal{Q}^\varepsilon$ lies on a unique tangent line distinct from l in P^\perp and R^\perp . Also, there are $q - 2$ oval planes through l distinct from P^\perp and R^\perp . This yields a total of $2(q - 1)$.

Now suppose that $|l \cap \mathcal{Q}^\varepsilon| = i \in \{0, 2\}$. There are $\frac{q+1-i}{2}$ points on l which are external in P^\perp , and therefore also external in R^\perp . Each of these points gives 2 tangent lines in both planes, so 4 planes intersecting both P^\perp and R^\perp in a 1-secant.

Secondly, consider a $\text{CM}(2, q, \varphi)$. Since, we are working in a connected component of G_1 , the vertices correspond to graphs of f_M with M the elements of a coset of $\text{PSL}(2, q)$. Thus, the size of the intersection of the circles corresponding to f_M and f_N equals the number of fixed points of MN^{-1} . Hence, it is independent of the choice of φ . Now apply the isomorphism to the G_1 graph of $\text{CM}(2, q, \text{id})$. \square

¹A Deza graph is a graph in which the number of common neighbours of distinct vertices can only take two values, see [EFH⁺99].

5 A point lies on either few or many circles of a large intersecting family

Throughout this section, let $D = (\mathcal{P}, \mathcal{B})$ be a $\text{CM}(\rho, q)$ with $q > 2$. We will prove that if \mathcal{F} is a large intersecting family in this circle geometry, any point lies on either few or many circles of \mathcal{F} . We will do this using the following graph.

Definition 5.1. Take a point $P \in \mathcal{P}$. Define the sets $L = \{c \in \mathcal{B} \mid P \in c\}$, and $R = \mathcal{B} \setminus L$. We define a graph G_P with bipartition (L, R) where $c_l \in L$ and $c_r \in R$ are adjacent if and only if $c_l \cap c_r = \emptyset$.

Lemma 5.2. Let G_P be as in Definition 5.1, defined on a $\text{CM}(\rho, q)$.

(1) Every vertex in L has degree $\delta = (q + \rho - 2) \binom{q}{2}$.

(2) Every vertex in R has degree $\frac{(q + \rho - 2)(q + 1 - \rho)}{2}$.

Proof. (1) Take a circle $c \in L$. Let n_i denote the number of circles intersecting c in exactly i points. Since all circles in L contain P , $\delta = n_0$. By Lemma 4.2, $n_1 = q^2 - 1$. Through 2 non-parallel points, there are $q + 1 - \rho$ circles, so $n_2 = \binom{q+1}{2}(q - \rho)$. Obviously, $n_{q+1} = 1$ and $\sum_i n_i = b = q^3 + (1 - \rho)q$, so

$$\delta = n_0 = q^3 + (1 - \rho)q - (q^2 - 1) - \binom{q+1}{2}(q - \rho) - 1 = (q + \rho - 2) \binom{q}{2}.$$

(2) Take a circle $c \in R$. Let n_i denote the number of circles through P intersecting c in exactly i points. Every point Q of c not parallel to P determines a unique tangent circle to c through P , so $n_1 = q + 1 - \rho$. There are $q - \rho$ other circles through Q and P , each intersecting c in 2 points, hence $n_2 = \frac{1}{2}(q + 1 - \rho)(q - \rho)$. The total number of circles through P equals $q(q + 1 - \rho)$. Therefore,

$$n_0 = q(q + 1 - \rho) - \frac{1}{2}(q + 1 - \rho)(q - \rho) - (q + 1 - \rho) = \frac{(q + \rho - 2)(q + 1 - \rho)}{2}. \quad \square$$

We want to determine the second largest eigenvalue of G_P . The following lemma is well-known, but we include it for completeness sake.

Lemma 5.3. Let L and G_P be as in Definition 5.1. Let N denote the square matrix labelled by L , whose (c_1, c_2) entry equals the number of common neighbours of c_1 and c_2 in G_P . Then $\lambda_2(G_P) = \sqrt{\lambda_2(N)}$.

Proof. Since G_P is bipartite, $A(G_P)$ is of the form $\begin{pmatrix} 0 & M \\ M^t & 0 \end{pmatrix}$ for some matrix M whose rows and columns are labelled by L and R respectively. Then $A(G_P)^2 = \begin{pmatrix} MM^t & 0 \\ 0 & M^tM \end{pmatrix}$. The matrices MM^t and M^tM have the same non-zero eigenvalues with the same multiplicity. Furthermore, the spectrum of $A(G_P)$ is symmetric around 0. These two properties imply that any non-zero eigenvalue λ of MM^t gives non-zero eigenvalues $\pm\sqrt{\lambda}$ of $A(G_P)$ with the same multiplicity. Therefore, $\lambda_2(G_P) = \sqrt{\lambda_2(MM^t)}$. To end the proof, one just has to note that MM^t is labelled by the vertices of L , and gives the number of common neighbours in G_P . \square

From now on, let N be as defined in the previous lemma. We need to compute its spectrum.

Lemma 5.4. If D is an extended Laguerre plane of even order, then $\lambda_2(G_P) = \frac{1}{2}q\sqrt{q-1}$.

Proof. Let \mathcal{Q}_+ denote the hyperoval cone used to construct D , and denote its vertex by U . Suppose that c_1 and c_2 are distinct circles through P . Then they intersect in a second point P_2 . Let π_1 and π_2 be the planes in $\text{PG}(3, q)$ spanned by c_1 and c_2 respectively. Denote the intersection line of these planes as l . There are $q - 1$ points Q on $l \setminus \{P, P_2\}$. Through Q there are $\frac{q}{2}$ skew lines l_1 to c_1 in π_1 . There are equally many skew lines l_2 to c_2 in π_2 through Q . However, if $l_2 = \langle l_1, U \rangle \cap \pi_2$, then $\langle l_1, l_2 \rangle$ is not an (hyper)oval plane. Thus, there are $(q - 1)\frac{q}{2}(\frac{q}{2} - 1)$ circles disjoint to c_1 and c_2 . Therefore,

$$N = \delta I + \frac{q-2}{4}(q-1)q(J-I)$$

Its eigenspaces are $\langle \mathbf{1} \rangle$ and $\langle \mathbf{1} \rangle^\perp$. It follows that the second largest eigenvalue of N equals $\delta - \frac{q-2}{4}(q-1)q = \frac{q-1}{4}q^2$. \square

Lemma 5.5. *Let D be a Möbius or Minkowski plane of even order, or an ovoidal Laguerre plane of odd order. Then $\lambda_2(G_P) = \frac{1}{2}\sqrt{q(q+(2-\rho))(q-(2-\rho))}$.*

Proof. Consider the 3-class association scheme as described in §3 linked to the circle geometries mentioned in the lemma. Denote the intersection numbers of these association schemes as p_{ij}^k . These numbers can be found in [Adr21]. The relevant ones are

$$p_{33}^1 = \frac{q}{4}(q-2+\rho)(q-4+\rho), \quad p_{33}^2 = \frac{q-1}{4}(q-2+\rho)^2.$$

To construct N , assume that the circles of L are ordered in $q+1-\rho$ blocks of size q , each representing a parallel class in the affine residue at P . Then

$$N = \delta I + p_{33}^1 I_{q+1-\rho} \otimes (J-I)_q + p_{33}^2 (J-I)_{q+1-\rho} \otimes J_q.$$

The eigenspaces of N are spanned by the following vectors. Here y_d denotes any vector in \mathbb{R}^d , and x_d denotes a vector in $\langle \mathbf{1}_d \rangle^\perp$.

1. The all-one vector with eigenvalue $\delta + p_{33}^1(q-1) + p_{33}^2(q-\rho)q$,
2. vectors $y_{q+1-\rho} \otimes x_q$ with eigenvalue $\delta - p_{33}^1$,
3. vectors $x_{q+1-\rho} \otimes \mathbf{1}_q$ with eigenvalue $\delta + p_{33}^1(q-1) - p_{33}^2q = 0$.

Therefore, $\lambda_2(N) = \delta - p_{33}^1$. Now apply Lemma 5.3. □

Lastly, we deal with the Möbius and Minkowski planes of odd order. We can deduce the entries of N from the structure of the G_1 graph. This is a bit convoluted for the ovoidal circle geometries, but allows us to use a unified proof strategy, which also works for the $\text{CM}(2, q, \varphi)$ geometries.

Lemma 5.6. *Let L and R be as in Definition 5.1. Take two distinct circles c_1 and c_2 in L . Let n_{11} denote the number of circles in R that intersect c_1 and c_2 in exactly one point, and let s denote $|c_1 \cap c_2|$. Then the number of common neighbours in G_P of c_1 and c_2 equals*

$$\frac{q-s+1}{4}(q-3+\rho+s)(q-5+\rho+s) + \frac{n_{11}}{4}.$$

Proof. Let n_{ij} denote the number of circles $c \in R$ with $|c_1 \cap c| = i$ and $|c_2 \cap c| = j$, where $i, j \in \{0, 1, 2\}$. We want to calculate the number n_{00} . Note that $\sum_{i,j=0}^2 n_{ij} = |R| = q^2(q-1)$, and that $\sum_{i=0}^2 n_{i0} = \sum_{j=0}^2 n_{0j} = \delta$. Now define the following sets.

$$N = \{(Q_1, Q_2, c) \in c_1 \times c_2 \times R \mid Q_1, Q_2 \in c\},$$

$$T_a = \{(Q_1, Q_2, c) \in N \mid c_a \cap c = \{Q_a\}\} \text{ for } a = 1, 2.$$

Then $|N| = \sum_{i,j} ijn_{ij}$, $|T_1| = n_{11} + 2n_{12}$, $|T_2| = n_{11} + 2n_{21}$. These equations imply that

$$4 \sum_{i,j=1}^2 n_{ij} = |N| + |T_1| + |T_2| + n_{11}.$$

Taking the sum over all n_{ij} where i or j is zero, equals $2\delta - n_{00}$. Therefore,

$$2\delta - n_{00} + \frac{1}{4}(|N| + |T_1| + |T_2| + n_{11}) = \sum_{i,j=0}^2 n_{ij} = q^2(q-1)$$

$$\implies n_{00} = 2(q+\rho-2) \binom{q}{2} + \frac{1}{4}(|N| + |T_1| + |T_2| + n_{11}) - q^2(q-1).$$

To finish the proof, we will calculate $|N|$, $|T_1|$, and $|T_2|$ depending on the size of $c_1 \cap c_2$.

Case 1: $c_1 \cap c_2 = \{P\}$.

1. $|N| = q(q-\rho)^2$,

2. $|T_a| = q(q - \rho)$ for $a = 1, 2$.

This can be seen as follows: there are q choices for $Q_a \in c_a \setminus \{P\}$, and $q - \rho$ choices for a point $Q_{3-a} \in c \setminus \{P\}$, not parallel with Q_a . There are $q - \rho$ circles c through Q_1 and Q_2 , not containing P , which yields $|N|$. There is a unique circle through Q_{3-a} which intersects c_a exactly in Q_a , which yields $|T_a|$.

Case 2: $c_1 \cap c_2 = \{P, P_2\}$.

1. $|N| = (q - 1)(q + 1 - \rho)^2$: We can choose $q - 1$ points Q_1 on $c_1 \setminus \{P, P_2\}$, $q - 1 - \rho$ points Q_2 on $c_2 \setminus \{P, P_2\}$ not parallel to Q_2 , and $q - \rho$ circles through Q_1, Q_2 and not through P . If we choose exactly one of the Q_a 's equal to P_2 , then we can choose Q_{3-a} to be any point of $c_{3-a} \setminus \{P, P_2\}$, which also give us $q - \rho$ circles missing P . We can also choose $Q_1 = Q_2 = P_2$, in which case there are r circles through Q_1, Q_2 , of which $q + 1 - \rho$ contain P . Hence, $|N| = (q - 1)(q - 1 - \rho)(q - \rho) + 2(q - 1)(q - \rho) + (q^2 + (1 - \rho)q - (q + 1 - \rho))$.
2. $|T_a| = (q - 1)(q + 1 - \rho)$ for $a = 1, 2$: There are $q - 1$ ways to choose $Q_a \in c_a \setminus \{P, P_2\}$, $q - \rho - 1$ ways to choose a non-parallel point $Q_{3-a} \in c_{3-a} \setminus \{P, P_2\}$, and one circle through Q_{3-a} , which intersects c_a exactly in Q_a . If we choose $Q_a = P_2$, then we can either choose $Q_{3-a} \neq P_2$, which leaves $q - 1$ choices for Q_{3-a} , each on a unique circle touching c_a in Q_a . Or we can choose $Q_1 = Q_2 = P_2$, then there are $q - 1$ circles touching c_a in P_2 (look in the affine residue at P_2), none of which can contain P . Thus, $|T_a| = (q - 1)(q - \rho - 1) + (q - 1) + (q - 1)$. \square

Definition 5.7. Let q be an odd prime power. First consider an ovoidal Möbius plane of order q , with κ and \perp as before. Call a circle *of square type* if its corresponding oval plane is P^\perp with $\kappa(P) = S_q$. Now consider a $\text{CM}(2, q, \varphi)$. Call a circle *of square type* if it is the graph of f_A with $A \in \text{PSL}(2, q)$.

Note that in the geometric description of an ovoidal Minkowski plane, circles of square type also correspond to the oval plane sections with planes P^\perp for which $\kappa(P) = S_q$.

Lemma 5.8. *Half of the circles through any point of an ovoidal Möbius plane, or a $\text{CM}(2, q, \varphi)$, of odd order are of square type.*

Proof. We first prove this for ovoidal Möbius and Minkowski planes. Let κ , \perp , and \mathcal{Q}^ε be as before. Take a line l and an oval plane P^\perp through l . If $|l \cap \mathcal{Q}^\varepsilon| = 1$, then all points of $l \setminus \mathcal{Q}^\varepsilon$ are external to $P^\perp \cap \mathcal{Q}^\varepsilon$, and thus of the same type (i.e. square or non-square). If $|l \cap \mathcal{Q}^\varepsilon|$ equals 0 or 2, then half of the points $R \in l \setminus \mathcal{Q}^\varepsilon$ satisfy $\kappa(R) = S_q$. Now take a point $Q \in \mathcal{Q}^\varepsilon$, and a line l in Q^\perp not through Q . Then $|l \cap \mathcal{Q}^\varepsilon| = 1 + \varepsilon$. This way we see that \mathcal{Q}^ε contains $\frac{q-\varepsilon}{2}$ 1-secants through Q on which all points $R \neq Q$ satisfy $\kappa(R) = S_q$, and equally many 1-secants with $\kappa(R) = \overline{S}_q$. The lemma now follows by applying the polarity.

Next, consider a $\text{CM}(2, q, \varphi)$. Take a point, which is of the form $(P, Q) \in \text{PG}(1, q)^2$. Since the lemma holds in the ovoidal Minkowski plane of order q , the number of elements $A \in \text{PSL}(2, q)$ with $AP = Q$ equals half of the number of circles through (P, Q) . \square

Lemma 5.9. *If D is an ovoidal Möbius plane, or a $\text{CM}(2, q, \varphi)$, of odd order q , then $\lambda_2(G_P) = \frac{1}{2}\sqrt{q(q^2 - 1)}$.*

Proof. Take two distinct circles c_1 and c_2 through the point P . First we compute the number n_{11} from Lemma 5.6. If c_1 and c_2 are of a different type (i.e. exactly one of them is of square type), then $n_{11} = 0$. If $|c_1 \cap c_2| = 1$, then c_1 and c_2 have $2(q - 1)$ common neighbours in G_1 by Lemma 4.6. However, in the affine residue through P , we find $q - 2$ other circles in the same parallel class as c_1 and c_2 . Hence, $n_{11} = q$. If $|c_1 \cap c_2| = 2$ and they are of the same type, $n_{11} = 2(q - 1)$, again by Lemma 4.6. Using Lemma 5.6, we see that the number of common neighbours of c_1 and c_2 equals

$$\begin{cases} m_1 = \frac{q}{4}(q - 3 + \rho)^2 & \text{if } |c_1 \cap c_2| = 1, \\ m_2 = \frac{q-1}{4}(q^2 + 2(\rho - 2)(q - 1) + 1) & \text{if } |c_1 \cap c_2| = 2 \text{ and } c_1 \text{ and } c_2 \text{ are of the same type,} \\ m_3 = \frac{(q-1)^2}{4}(q - 3 + 2\rho) & \text{otherwise.} \end{cases}$$

Half of the parallel classes in the affine residue through P consist of square type circles. Arrange the circles through P such that they consist of consecutive blocks of the circles in a parallel class in the affine residue. Put the circles of square type before the circles of non-square type. Then

$$N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \left[I_{\frac{q+1-\rho}{2}} \otimes (\delta I_q + m_1(J - I)_q) + (J - I)_{\frac{q+1-\rho}{2}} \otimes m_2 J_q \right] + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes m_3 J_{q \frac{q+1-\rho}{2}}.$$

The eigenspaces are spanned by the following eigenvectors. Here y_d denotes a vector from \mathbb{R}^d , and x_d denotes a vector from $\langle \mathbf{1}_d \rangle^\perp$.

1. The all-one vector with eigenvalue $\delta \frac{(q+\rho-2)(q+1-\rho)}{2}$,
2. vectors $y_{q+1-\rho} \otimes x_q$, with eigenvalue $\delta - m_1$,
3. vectors $y_2 \otimes x_{\frac{q+1-\rho}{2}} \otimes \mathbf{1}_q$ with eigenvalue $\delta + (q-1)m_1 - qm_2 = 0$,
4. vectors $x_2 \otimes \mathbf{1}_{q\frac{q+1-\rho}{2}}$ with eigenvalue $\delta + (q-1)m_1 + q\frac{q-1-\rho}{2}m_2 - q\frac{q+1-\rho}{2}m_3$.

From this, the spectrum of N can be calculated. Then apply Lemma 5.3. \square

Proposition 5.10. *Let \mathcal{F} be an intersecting family in an ovoidal $\text{CM}(\rho, q)$ or a $\text{CM}(2, q, \varphi)$, and let P be a point. In case of a Laguerre plane of even order, switch to the extended Laguerre plane. Define $S = \{c \in \mathcal{F} \mid P \in c\}$ and $T = \mathcal{F} \setminus S$, and define*

$$\lambda = \begin{cases} q(q^2 - 1) & \text{if } q \text{ is odd,} \\ q(q-2+\rho)(q+(1-\rho)(2-\rho)) & \text{if } q \text{ is even.} \end{cases}$$

Then

$$|S||T| \leq \left(\frac{q}{q+\rho-2} \right)^2 \lambda \left(1 - \frac{|S|}{q(q+1-\rho)} \right) \left(1 - \frac{|T|}{q^2(q-1)} \right).$$

Proof. Consider the graph G_P from Definition 5.1. Since \mathcal{F} is an intersecting family, $e(S, T) = 0$. Now apply the bipartite expander mixing lemma to G_P on the sets S and T . The eigenvalue $\lambda_2(G_P)$ can be found in the above lemmata. \square

Remark 5.11. The constant $\left(\frac{q}{q+\rho-2} \right)^2 \lambda$ above is at most $\frac{q^3(q^2-1)}{(q-2)^2}$, which occurs for Möbius planes of odd order.

6 The main theorems

First we prove Erdős-Ko-Rado type results for the known circle geometries, with very mild conditions on the order. Then we prove a stability result in these circle geometries. We will use the following lemma.

Lemma 6.1. *Suppose that \mathcal{F} is an intersecting family in a $\text{CM}(\rho, q)$. If a point P lies on more than $\binom{q+2-\rho}{2}$ circles of \mathcal{F} , then P lies on all circles of \mathcal{F} .*

Proof. Suppose that \mathcal{F} contains a circle c not through P . Then every circle through P in \mathcal{F} must intersect c . Every two points of c not parallel to P determine a unique circle through P , and every point Q of c not parallel to P determines a unique circle through P tangent to c in Q . Therefore, the number of circles through P intersecting c equals $\binom{q+1-\rho}{2} + q + 1 - \rho = \binom{q+2-\rho}{2}$. \square

6.1 Characterisation of the largest intersecting families

Theorem 6.2. *Let \mathcal{F} be an intersecting family in an ovoidal Möbius plane of odd order $q > 3$. Then $|\mathcal{F}| \leq q(q+1)$, with equality if and only if \mathcal{F} consists of all circles through a fixed point.*

Proof. Let \mathcal{F} be an intersecting family of size $q(q+1)$. By Proposition 4.5, there exists a point P that lies on $s \geq 2q-1$ circles of \mathcal{F} . Apply Proposition 5.10 to this point P . This yields

$$\begin{aligned} s(q(q+1) - s) &\leq \frac{q^2}{(q-2)^2} q(q^2-1) \left(1 - \frac{s}{q(q+1)} \right) \left(1 - \frac{q(q+1)-s}{q^2(q-1)} \right) \\ &= \frac{1}{(q-2)^2} (q(q+1) - s) (q^2(q-1) - q(q+1) + s) \end{aligned}$$

Suppose that $s < q(q+1)$. Then this inequality can be rewritten as

$$s((q-2)^2 - 1) \leq q^3 - 2q^2 - q.$$

Since $q \geq 5$, this implies that $s < q + 4$, contradicting $s \geq 2q - 1$. Therefore, s must equal $q(q + 1)$, or in other words \mathcal{F} must consist of all circles through P .

It now follows that if \mathcal{F} is an intersecting family of size at least $q(q + 1)$, \mathcal{F} contains a subfamily of size $q(q + 1)$ which must consist of all circles through a fixed point. By Lemma 6.1, \mathcal{F} must equal this subfamily. \square

Remark 6.3. There is a unique Möbius plane of order 3, see e.g. [Tha94]. This Möbius plane is ovoidal, and constructed from plane sections with $\mathcal{Q}^-(3, 3)$. Take two oval planes P^\perp and R^\perp . Suppose that P and R have coordinate vectors x and y respectively. Then $(PR)^\perp$ is a 0-secant to \mathcal{Q}^- if and only if PR is a 2-secant, by Lemma 3.1. This means that the points with coefficients $x \pm y$ must both be in \mathcal{Q}^- or equivalently that for $\alpha = \pm 1$

$$\kappa(x + \alpha y) = \kappa(x) + 2\alpha b(x, y) + \alpha^2 \kappa(y) = \kappa(x) + 2\alpha b(x, y) + \kappa(y) = 0.$$

This happens if and only if $\kappa(x) + \kappa(y) = 0$ and $b(x, y) = 0$. Therefore, the sets $\{P^\perp \mid \kappa(P) = S_3\}$ and $\{P^\perp \mid \kappa(P) = \overline{S_3}\}$ are intersecting families of size $\frac{q(q^2+1)}{2} = 15$. Note that $q(q + 1) = 12$. Furthermore, it is well-known that the only cocliques in a connected regular bipartite graph of maximal size are the bipartition classes. Thus, these two families are the only cocliques of size 15 in the G_0 graph, proving they are the only two intersecting families of size 15 in the $\text{CM}(0, 3)$.

Theorem 6.4. *Let \mathcal{F} be an intersecting family in a $\text{CM}(2, q, \varphi)$ from Construction 2.1. Then $|\mathcal{F}| \leq q(q - 1)$, with equality if and only if \mathcal{F} consists of all circles through a fixed point.*

Proof. If $\varphi = \text{id}$, the corresponding Minkowski plane is ovoidal, and the theorem has been proven by Meagher and Spiga [MS11]. So suppose that $\varphi \neq \text{id}$. Then q cannot be prime, hence $q \geq 9$. Consider an intersecting family \mathcal{F} of size $q(q - 1)$. By Proposition 4.5, there exists a point P that lies on $s \geq 2q - 5$ circles of \mathcal{F} . Then by Proposition 5.10,

$$s(q(q - 1) - s) \leq q(q^2 - 1) \left(1 - \frac{s}{q(q - 1)}\right) \left(1 - \frac{q(q - 1) - s}{q^2(q - 1)}\right) < (q + 1)(q(q - 1) - s).$$

Thus, if $s < q(q - 1)$, then this inequality implies that $s < q + 1$, contradicting $s \geq 2q - 5$ since $q \geq 9$. Therefore, the only intersecting families of size $q(q - 1)$ are the ones containing all circles through a fixed point, and the theorem follows. \square

Together with the main theorems of [Adr21], this proves Theorem 1.2.

6.2 Stability result

Theorem 6.5. *Consider an intersecting family \mathcal{F} in one of the known finite circle geometries, i.e. an ovoidal circle geometry or Construction 2.1, of order q . In case of an ovoidal Laguerre plane of even order, switch to the extended Laguerre plane. If $|\mathcal{F}| \geq \frac{1}{\sqrt{2}}q^2 + 2\sqrt{2}q + 8$, then \mathcal{F} consists of circles through a common point.*

Proof. Denote the size of \mathcal{F} by f . We may assume that $q \geq 11$, otherwise f exceeds the size of the largest intersecting family. This implies that the largest intersecting families are the sets of all circles through a fixed point. Then by Proposition 4.5, some point P lies on $s \geq 2\frac{q}{(q+1)^2}f$ circles of \mathcal{F} . By Proposition 5.10 and Remark 5.11, we obtain the inequality

$$s(f - s) \leq q^3 \frac{q^2 - 1}{(q - 2)^2}.$$

The inequality does not hold for $s = 2\frac{q}{(q+1)^2}f$ and for $s = \binom{q+2}{2}$ given the assumptions on f and q . Since it is quadratic in s , this implies that it doesn't hold for any intermediary values. Hence, P lies on more than $\binom{q+2}{2}$ circles of \mathcal{F} , and by Lemma 6.1 on all circles of \mathcal{F} . \square

6.3 Corollaries in other incidence structures

An intersecting family in $\text{PGL}(2, q)$ is a set \mathcal{F} of elements of $\text{PGL}(2, q)$ such that for each $M, N \in \mathcal{F}$ there exists a point P of $\text{PG}(1, q)$ with $MP = NP$. We know that this is equivalent to an intersecting family in the ovoidal Minkowski plane of order q .

Corollary 6.6. *An intersecting family in $\text{PGL}(2, q)$ of size at least $\frac{1}{\sqrt{2}}q^2 + 2\sqrt{2}q + 8$ lies in a coset of the stabiliser of some point of $\text{PG}(1, q)$.*

Let $U_{2,q}$ denote the set of all polynomials of degree at most 2 over \mathbb{F}_q . For $f(X) = aX^2 + bX + c$, we define $f(\infty) = a$. For each $f \in U_{2,q}$, define its graph $\{(x, f(x)) \mid f \in \mathbb{F}_q \cup \{\infty\}\}$. Consider the incidence structure $(\mathcal{P}, \mathcal{B})$ with $\mathcal{P} = (\mathbb{F}_q \cup \{\infty\}) \times \mathbb{F}_q$ and \mathcal{B} the set of graphs of $f \in U_{2,q}$. This incidence structure is an ovoidal Laguerre plane. If q is even, we can go to the extended Laguerre plane by adding the points $\{-\infty\} \times \mathbb{F}_q$, and adding $(-\infty, b)$ to the graph of $aX^2 + bX + c$.

We call $\mathcal{F} \subseteq U_{q,2}$ an intersecting family if for every $f, g \in \mathcal{F}$ there exists an $x \in \mathbb{F}_q \cup \{\pm\infty\}$ with $f(x) = g(x)$.

Corollary 6.7. *An intersecting family in $U_{2,q}$ of size at least $\frac{1}{\sqrt{2}}q^2 + 2\sqrt{2}q + 8$ consists of functions f with $f(x) = y$ for some fixed x and y in \mathbb{F}_q .*

Proof. Analogous to the first part of the proof of [Adr21, Theorem 6.2]. □

7 Concluding remarks

In this paper, we characterised large intersecting families in the known finite $\text{CM}(\rho, q)$ as sets of circles through a common point or nucleus. One could wonder whether the bound can't be pushed further. If an Erdős-Ko-Rado result holds in an incidence structure $(\mathcal{P}, \mathcal{B})$, such as a $\text{CM}(\rho, q)$, then often the largest intersecting families \mathcal{F} with $\bigcap \mathcal{F} = \emptyset$ are of the following form, where P and B are a non-incident point and block respectively.

$$\{B' \in \mathcal{B} \mid P \in B', B' \cap B \neq \emptyset\} \cup \{B\}$$

If this is true in some incidence structure, it is often referred to as a Hilton-Milner type result, after the paper by Hilton and Milner [HM67], which proves such a result where \mathcal{B} is the set of all subsets of \mathcal{P} of size k . In a circle geometry, such a Hilton-Milner type family would be of size $\binom{q+2-\rho}{2} + 1$, as can be seen from Lemma 6.1. To prove a Hilton-Milner result, we would need to improve the constant $\frac{1}{\sqrt{2}}$ in the current bound to $\frac{1}{2}$.

A set of blocks \mathcal{F} is called a t -intersecting family if any two elements of \mathcal{F} intersect in at least t elements. In general, the bounds on the size of a 2-intersecting family in a $\text{CM}(\rho, q)$ are of the form $\frac{1}{2}q^2 + \mathcal{O}(q)$, but no 2-intersecting family with size larger than $2q$ is known. To the author's best knowledge, ovoidal Laguerre planes of even order are the only type of circle geometry for which better bounds are known. There, the largest 2-intersecting families have size q . Thus, proving a Hilton-Milner type result for circle geometries could have interesting consequences for the size of the largest 2-intersecting families.

Recently, Maleki and Razafimahatratra [MR21] proved that the only intersecting families in $\text{GL}(2, q)$ acting on $\mathbb{F}_q^2 \setminus \{\mathbf{0}\}$ are stabilisers of a point or a hyperplane. This is in contrast to the existence of Hilton-Milner type families in $\text{PGL}(2, q)$ acting on $\text{PG}(1, q)$, or equivalently in the ovoidal Minkowski planes.

One could also wonder whether the main theorem of this paper can be proven purely from the combinatorial definition of circle geometries, instead of having to use the structure of the known examples. As was noted in §4, a key ingredient in the proof is that the 1-intersecting graphs of the known circle geometries have good expanding properties, except for ovoidal Laguerre planes of even order. The fact that this doesn't hold for these Laguerre planes, might suggest that some extra structure needs to be imposed on the circle geometries.

It would also be nice to show that the relations as defined in the second column of Table 2 yield a 5-class association scheme for ovoidal Minkowski planes of odd order (or a 4-class scheme if $q = 3$). It has been checked that this is the case for $q \leq 13$ by computer.

Lastly, we remark that it might be interesting to explore further applications of the technique of this paper to prove (stability of) Erdős-Ko-Rado results in other incidence structures. The scope of our technique to find a point on "not few" circles of the intersecting family is probably limited to incidence structures in which the size of the intersection of two blocks can only take a very limited number of values.

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