Convergence of a Nonlocal to a Local Diffuse Interface Model for Two-Phase Flow with Unmatched Densities

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Dedicated to Maurizio Grasselli on the occasion of his 60th birthday

Abstract

We prove convergence of suitable subsequences of weak solutions of a diffuse interface model for the two-phase flow of incompressible fluids with different densities with a nonlocal Cahn-Hilliard equation to weak solutions of the corresponding system with a standard "local" Cahn-Hilliard equation. The analysis is done in the case of a sufficiently smooth bounded domain with no-slip boundary condition for the velocity and Neumann boundary conditions for the Cahn-Hilliard equation. The proof is based on the corresponding result in the case of a single Cahn-Hilliard equation and compactness arguments used in the proof of existence of weak solutions for the diffuse interface model.

Key words: Two-phase flow, Navier-Stokes equation, diffuse interface model, mixtures of viscous fluids, Cahn-Hilliard equation, non-local operators
AMS-Classification: Primary: 76T99; Secondary: 35Q30, 35Q35, 76D03, 76D05, 76D27, 76D45

1 Introduction

In this paper, we consider the convergence of a non-local diffuse interface model for the two-phase flows of two incompressible fluids with unmatched densities to the corresponding "local" system. More precisely, we consider the non-local Navier-Stokes/Cahn-Hilliard system

$$\partial_t(\rho_\varepsilon \mathbf{v}_\varepsilon) + \operatorname{div}(\mathbf{v} \otimes (\rho \mathbf{v}_\varepsilon + \widetilde{\mathbf{J}}_\varepsilon)) - \operatorname{div}(2\nu(\varphi_\varepsilon) D\mathbf{v}_\varepsilon) + \nabla p_\varepsilon = \mu_\varepsilon \nabla \varphi_\varepsilon \quad \text{in } Q_T, \tag{1.1}$$

$$\operatorname{div} \mathbf{v}_\varepsilon = 0 \quad \text{in } Q_T, \tag{1.2}$$

$$\partial_t \varphi_\varepsilon + \mathbf{v}_\varepsilon \cdot \nabla \varphi_\varepsilon = \operatorname{div}(m(\varphi_\varepsilon)\nabla \mu_\varepsilon) \quad \text{in } Q_T, \tag{1.3}$$

$$F'(\varphi_\varepsilon) + a_\varepsilon(x)\varphi_\varepsilon - J_\varepsilon * \varphi_\varepsilon = \mu_\varepsilon \quad \text{in } Q_T, \tag{1.4}$$

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where $\rho_\varepsilon = \rho(\varphi_\varepsilon) := \frac{\tilde\rho_1+\tilde\rho_2}{2} + \frac{\tilde\rho_2-\tilde\rho_1}{2}\varphi_\varepsilon$ is the density of the mixture of the two fluids, $\tilde\rho_1, \tilde\rho_2 > 0$ are the specific constant mass densities of the unmixed fluids,

$$\widetilde{\mathbf{J}}_\varepsilon = -\frac{\tilde\rho_2 - \tilde\rho_1}{2} m(\varphi_\varepsilon)\nabla\mu$$

is a relative mass flux, $Q_T = \Omega \times (0,T)$, where $T \in (0,\infty)$ is arbitrary. We assume that $\Omega \subset \mathbb{R}^d$, $d = 2,3$, is a bounded domain with C^2-boundary. Here $\mathbf{v}_\varepsilon\colon Q_T \to \mathbb{R}^d$ is the (mean) velocity of the fluid mixture, $p_\varepsilon\colon Q_T \to \mathbb{R}$ is its pressure, $\varphi_\varepsilon\colon Q_T \to \mathbb{R}$ is the difference of volume fractions of the fluids, and $\mu_\varepsilon\colon Q_T \to \mathbb{R}$ is the chemical potential related to φ_ε, which is the first variation of the (non-local) free energy

$$E_\varepsilon(\varphi) = \frac{1}{4}\int_\Omega\int_\Omega J_\varepsilon(x-y)|\varphi(x)-\varphi(y)|^2\,dx\,dy + \int_\Omega F(\varphi(x))\,dx.$$

Moreover, J_ε is a nonnegative function on $\mathbb{R}^d$, F is a homogeneous free energy density and

$$J_\varepsilon * \varphi(x) := \int_\Omega J_\varepsilon(x-y)\varphi(y)\,dy, \quad a_\varepsilon(x) := \int_\Omega J_\varepsilon(x-y)\,dy \qquad \text{for all } x\in\Omega.$$

More precisely, we assume that $J_\varepsilon(x) = \frac{\eta_\varepsilon(|x|)}{|x|^2}$ for all $x \in \mathbb{R}^d$ and $J_\varepsilon \in W^{1,1}(\mathbb{R}^d)$ for $\varepsilon > 0$ and $(\eta_\varepsilon)_{\varepsilon>0}$ is a family of molifiers with the following properties:

$$\eta_\varepsilon\colon \mathbb{R} \longrightarrow [0,+\infty), \quad \eta_\varepsilon \in L^1_{loc}(\mathbb{R}), \qquad \eta_\varepsilon(r) = \eta_\varepsilon(-r) \quad \forall r\in\mathbb{R}, \varepsilon>0;$$
$$\int_0^{+\infty} \eta_\varepsilon(r) r^{d-1}\,dr = \frac{2}{C_d} \quad \forall \varepsilon > 0;$$
$$\lim_{\varepsilon\to 0+}\int_\delta^{+\infty} \rho_\varepsilon(r) r^{d-1}\,dr = 0 \quad \forall \delta > 0,$$

where $C_d := \int_{S^{d-1}} |e_1\cdot\sigma|^2 d\mathcal{H}^{d-1}(\sigma)$. Moreover, we assume that $F\colon [-1,1] \to \mathbb{R}$ is given by

$$F(s) = \frac{\theta}{2}((1+s)\log(1+s) + (1-s)\log(1-s)) - \frac{\theta_c}{2}s^2 \qquad \text{for all } s\in[-1,1]$$

for some $0 < \theta_c < \theta$ for simplicity. But every F satisfying the assumptions in [8] and [10] can be treated as well. Finally, $\nu\colon [-1,1] \to (0,\infty)$ and $m\colon [-1,1] \to (0,\infty)$ are viscosity and mobility coefficients, which are assumed to be sufficiently smooth.

We complement the system (1.1)-(1.4) with the following boundary and initial conditions:

$$\mathbf{v}_\varepsilon|_{\partial\Omega} = 0, \qquad \left.\frac{\partial\mu_\varepsilon}{\partial\mathbf{n}}\right|_{\partial\Omega} = 0 \qquad \text{on } \partial\Omega\times(0,T), \tag{1.5}$$

$$\mathbf{v}_\varepsilon|_{t=0} = \mathbf{v}_{0,\varepsilon}, \qquad \varphi_\varepsilon|_{t=0} = \varphi_{0,\varepsilon} \quad \text{in } \Omega. \tag{1.6}$$

The system (1.1)-(1.4) is a variant of the following diffuse interface model for the two-phase flows of two incompressible fluids with unmatched densities, which was derived in [4]:

$$\partial_t(\rho\mathbf{v}) + \operatorname{div}(\mathbf{v}\otimes(\rho\mathbf{v}+\widetilde{\mathbf{J}})) - \operatorname{div}(2\nu(\varphi)D\mathbf{v}) + \nabla p = \mu\nabla\varphi \qquad \text{in } Q_T, \tag{1.7}$$

$$\operatorname{div}\mathbf{v} = 0 \qquad \text{in } Q_T, \tag{1.8}$$

$$\partial_t\varphi + \mathbf{v}\cdot\nabla\varphi = \operatorname{div}(m(\varphi)\nabla\mu) \quad \text{in } Q_T, \tag{1.9}$$

$$\mu = F'(\varphi) - \Delta\varphi \qquad \text{in } Q_T, \tag{1.10}$$

where $\rho = \rho(\varphi) := \frac{\tilde{\rho}_1+\tilde{\rho}_2}{2} + \frac{\tilde{\rho}_2-\tilde{\rho}_1}{2}\varphi$ is the density of the mixture of the two fluids and $\widetilde{\mathbf{J}} = -\frac{\tilde{\rho}_2-\tilde{\rho}_1}{2} m(\varphi)\nabla\mu$ is a relative mass flux as before. This system is complemented by the boundary and initial conditions

$$\mathbf{v}|_{\partial\Omega} = 0, \qquad \left.\frac{\partial\mu}{\partial\mathbf{n}}\right|_{\partial\Omega} = \left.\frac{\partial\varphi}{\partial\mathbf{n}}\right|_{\partial\Omega} = 0 \quad \text{on } \partial\Omega\times(0,T), \tag{1.11}$$

$$\mathbf{v}|_{t=0} = \mathbf{v}_0, \qquad \varphi|_{t=0} = \varphi_0 \qquad\qquad \text{in } \Omega. \tag{1.12}$$

We note that (1.1)-(1.4) is obtained by the latter system by replacing the standard "local" Cahn-Hilliard equation (1.9)-(1.10) (with an additional convection term $\mathbf{v}\cdot\nabla\varphi$) by its non-local variant (1.3)-(1.4). Moreover, note that in (1.11) an additional Neumann boundary condition for φ is present, which is not posed for the non-local system.

It is the goal of the present contribution to show convergence of weak solutions of (1.1)-(1.4) together with (1.5)-(1.6) to a weak solution of (1.7)-(1.10) together with (1.11)-(1.12) for a suitable subsequence and under suitable conditions on the initial values. Existence of weak solutions of (1.7)-(1.12) was first proven by A., Depner, and Garcke in [3]. Existence of strong solutions for small times was proved by Weber [18], cf. also A. and Weber [6]. A result on well-posedness of this system in two-space dimensions and further references can be found in the recent contribution by Giorgini [14]. The existence of weak solutions to the non-local model (1.1)-(1.4) together with (1.5)-(1.6) was proved by Frigeri [10] for suitable integrable kernels J_ε and by the authors in [5] for singular kernels. We refer to Frigeri [11] for a recent overview of the literature for these non-local models, to Gal, Grasselli, and Wu [13] for a recent result on the (local) Navier-Stokes/Cahn-Hilliard system with different densities and further references, and to Frigeri, Gal, and Grasselli [12] for a recent result on the nonlocal Cahn-Hilliard equation with singular potentials and degenerate mobility and further references.

Convergence of solutions of the nonlocal Cahn-Hilliard equation, i.e., (1.3)-(1.4) with $\mathbf{v}_\varepsilon \equiv 0$, to the local Cahn-Hilliard equation, i.e., (1.9)-(1.10) with $\mathbf{v} \equiv 0$, was proved by Melchionna et al. [15] in the case of periodic boundary conditions and a regular free energy density F, by Davoli et al. [7] in the case of periodic boundary conditions and singular free energy densities, by Davoli et al. [9] in the case of Neumann boundary conditions with an additional viscosity term in the nonlocal Cahn-Hilliard equation and in [8] in the case of Neumann boundary conditions and $W^{1,1}$-kernels. We note that these results are based on the results of Ponce [16, 17], which in particular yield Γ-convergence of the non-local free energy of E_ε to the corresponding local free energy

$$E(\varphi) := \int_\Omega \frac{|\nabla\varphi|^2}{2}\,dx + \int_\Omega F(\varphi)\,dx.$$

We refer to [8] for further references.

In this contribution we combine the arguments from [8] for the convergence of the nonlocal to the local Cahn-Hilliard equation and [3] for the existence of weak solutions to the limit system (1.7)-(1.12) to show convergence of weak solutions of (1.1)-(1.4) together with (1.5)-(1.6) to a weak solution of (1.7)-(1.10) together with (1.11)-(1.12) for a suitable subsequence. The structure of this contribution is as follows: In Section 2 we recall some preliminary results and basic definition. Then the main result is proved in Section 3.

2 Preliminaries

In the following $D\mathbf{v} = \frac{1}{2}(\nabla\mathbf{v} + \nabla\mathbf{v}^T)$ denotes the symmetric part of the gradient of a vector field $\mathbf{v}$. The tensor product $a \otimes b$ of the vectors a and b is $(a \otimes b)_{i,j} = a_i b_j$ for $i, j = 1, \cdots, d$. For a normed space X we denote by

$$\langle x', x \rangle_X := x'(x) \qquad \text{for all } x' \in X', x \in X$$

its duality product. $C([0,T];X)$ denotes the space of all strongly continuous $f\colon [0,T] \to X$ equipped with the supremum-norm. $BC_w([0,T];X)$ denotes the spaces of all bounded and weakly continuous $f\colon [0,T] \to X$ equipped with the supremum-norm. If $M \subset \mathbb{R}^d$ is measurable, $L^q(M)$ denotes the usual Lebesgue space and $\|\cdot\|_q$ its norm. Moreover, $L^q(M;X)$ denotes the set of all strongly measurable q-integrable functions/essentially bounded functions, where X is a Banach space. If $M = (a,b)$, we write for simplicity $L^q(a,b)$ and $L^q(a,b;X)$.

Let $\Omega \subset \mathbb{R}^d$ be a domain. Then $W_q^m(\Omega)$, $m \in \mathbb{N}_0$, $1 \le q \le \infty$, denotes the usual L^q-Sobolev space, $W_{q,0}^m(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W_q^m(\Omega)$ and $W_q^{-m}(\Omega) = (W_{q',0}^m)'$. $H^s(\Omega)$, $s \in \mathbb{R}$, denotes the usual L^2-Bessel potential spaces and $H_0^s(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $H^s(\Omega)$ when s is positive.

Furthermore,

$$\mathcal{D}(A) = H^2(\Omega)^d \cap H_0^1(\Omega)^d \cap L_\sigma^2(\Omega)$$

denotes the domain of the Stokes operator on $L_\sigma^2(\Omega) := \overline{\{\boldsymbol{\psi} \in C_0^\infty(\Omega)^d : \operatorname{div} \boldsymbol{\psi} = 0\}}^{L^2(\Omega)}$.

We consider weak solutions of (1.1)-(1.4) together with (1.5)-(1.6) in the following sense:

Definition 2.1. *Let* $\mathbf{v}_0 \in L_\sigma^2(\Omega)$, $\varphi_0 \in L^\infty(\Omega)$ *with* $|\varphi_0| \le 1$ *almost everywhere and* $T \in (0,\infty)$, $\varepsilon > 0$ *be given. Then* $(\mathbf{v}, \varphi, \mu)$ *is a weak solution of* (1.1)-(1.6) *if*

$$\begin{aligned}
\mathbf{v} &\in BC_w([0,T]; L_\sigma^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega)^d),\\
\varphi &\in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega)),\\
\mu = a_\varepsilon\varphi - J_\varepsilon * \varphi + F'(\varphi) &\in L^2(0,T; H^1(\Omega)),\\
\partial_t(\rho\mathbf{v}) &\in L^{4/3}(0,T; \mathcal{D}(A)'), \quad \partial_t\varphi \in L^2(0,T; (H^1(\Omega))'),
\end{aligned}$$

$|\varphi(x,t)| < 1$ *almost everywhere in* Q_T, $\mathbf{v}|_{t=0} = \mathbf{v}_0$, $\varphi|_{t=0} = \varphi_0$ *and the following holds true:*

(i) For every $\psi \in H^1(\Omega)$ *and* $\boldsymbol{\psi} \in \mathcal{D}(A)$ *and almost every* $t \in (0,T)$ *we have*

$$\langle \partial_t(\rho\mathbf{v})(t), \boldsymbol{\psi} \rangle_{\mathcal{D}(A)} - \int_\Omega ((\mathbf{v} + \widetilde{\mathbf{J}}) \otimes \rho\mathbf{v} : D\boldsymbol{\psi}\, dx + \int_\Omega 2\nu(\varphi) D\mathbf{v} : D\boldsymbol{\psi}\, dx = -\int_\Omega \varphi\nabla\mu \cdot \boldsymbol{\psi}\, dx,$$

$$\langle \partial_t\varphi(t), \psi \rangle_{H^1(\Omega)} + \int_\Omega m(\varphi)\nabla\mu \cdot \nabla\psi = \int_\Omega \mathbf{v}\varphi \cdot \nabla\psi\, dx,$$

where $\widetilde{\mathbf{J}} = -\frac{\tilde\rho_2 - \tilde\rho_1}{2} m(\varphi)\nabla\mu$.

(ii) The energy inequality

$$\mathcal{E}_\varepsilon(\mathbf{v}(t), \varphi(t)) + \int_0^t \int_\Omega (2\nu(\varphi)|D\mathbf{v}|^2 + m(\varphi)|\nabla\mu|^2)\, dx\, d\tau \le \mathcal{E}_\varepsilon(\mathbf{v}(0), \varphi(0)) \tag{2.1}$$

holds true for all $t \in [0,T]$, where

$$\mathcal{E}_\varepsilon(\mathbf{v},\varphi) := \frac{1}{2}\int_\Omega \rho(\varphi)|\mathbf{v}|^2\,dx + E_\varepsilon(\varphi),$$
$$E_\varepsilon(\varphi) := \frac{1}{4}\int_\Omega\int_\Omega J_\varepsilon(x-y)(\varphi(x)-\varphi(y))^2\,dx\,dy + \int_\Omega F(\varphi(x))\,dx.$$

Existence of weak solutions for any $\mathbf{v}_0 \in L^2_\sigma(\Omega)$, $\varphi_0 \in L^\infty(\Omega)$ with $|\varphi_0| \leq 1$ almost everywhere and $T \in (0,\infty)$, $\varepsilon > 0$ follows from [10, Theorem 1].

For the following we denote by

$$E^0_\varepsilon(\varphi) := \frac{1}{4}\int_\Omega\int_\Omega J_\varepsilon(x-y)(\varphi(x)-\varphi(y))^2\,dx\,dy \quad \text{for } \varphi \in L^2(\Omega),$$
$$E^0(\varphi) := \frac{1}{2}\int_\Omega |\nabla\varphi(x)|^2\,dx \quad \text{for } \varphi \in H^1(\Omega)$$

the first parts of the free energies in the nonlocal and local case. We note that

$$E^0_\varepsilon(\varphi) \leq E_\varepsilon(\varphi) + C, \qquad E^0(\varphi) \leq E(\varphi) + C$$

for some $C > 0$ independent of $\varepsilon \in (0,1)$ since $F\colon [-1,1] \to \mathbb{R}$ is bounded below. The following two lemmas will be important to obtain compactness as $\varepsilon \to 0$:

Lemma 2.2. *For every $\varphi, \zeta \in H^1(\Omega)$ it holds that*

$$\lim_{\varepsilon\to 0} E^0_\varepsilon(\varphi) = E^0(\varphi),$$
$$\lim_{\varepsilon\to 0}\int_\Omega (a_\varepsilon\varphi - J_\varepsilon * \varphi)(x)\zeta(x)\,dx = \int_\Omega \nabla\varphi(x)\cdot\nabla\zeta(x)\,dx.$$

Moreover, for every sequence $(\varphi_\varepsilon)_{\varepsilon>0} \subseteq L^2(\Omega)$ and $\varphi \in L^2(\Omega)$ it holds that

$$\sup_{\varepsilon>0} E^0_\varepsilon(\varphi) < +\infty \quad \Rightarrow \quad (\varphi_\varepsilon)_{\varepsilon>0} \text{ is relatively compact in } L^2(\Omega),$$
$$\varphi_\varepsilon \to_{\varepsilon\to 0} \varphi \quad \text{in } L^2(\Omega) \quad \Rightarrow \quad E^0(\varphi) \leq \liminf_{\varepsilon\to 0} E^0_\varepsilon(\varphi_\varepsilon).$$

We refer to [8, Lemma 3.3] for the proof of this lemma.

Lemma 2.3. *For any $\delta > 0$, there exists some $C_\delta > 0$ and $\varepsilon_\delta > 0$ such that for any $(\varphi_\varepsilon)_{\varepsilon>0} \subset L^2(\Omega)$*

$$\|\varphi_{\varepsilon_1} - \varphi_{\varepsilon_2}\|^2_{L^2(\Omega)} \leq \delta(E^0_{\varepsilon_1}(\varphi_{\varepsilon_1}) + E^0_{\varepsilon_2}(\varphi_{\varepsilon_2})) + C_\delta\|\varphi_{\varepsilon_1} - \varphi_{\varepsilon_2}\|^2_{(H^1(\Omega))'} \tag{2.2}$$

holds for any $\varepsilon_1, \varepsilon_2 \in (0, \varepsilon_\delta)$.

The lemma is proved in [8, Lemma 3.4].

3 Main Result

Theorem 3.1. *For any* $\varepsilon \in (0,1)$ *let* $\mathbf{v}_{0,\varepsilon} \in L^2_\sigma(\Omega)$ *and* $\varphi_{0,\varepsilon} \in L^\infty(\Omega)$ *with* $|\varphi_{0,\varepsilon}(x)| \leq 1$ *almost everywhere,* $\frac{1}{|\Omega|}\int_\Omega \varphi_{0,\varepsilon}(x)\,dx = m_\Omega$ *for all* $\varepsilon \in (0,1)$ *and some* $m_\Omega \in (-1,1)$*. Moreover, we assume that there are* $\mathbf{v}_0 \in L^2_\sigma(\Omega)$ *and* $\varphi_0 \in H^1(\Omega)$ *such that* $\mathbf{v}_{0,\varepsilon} \to_{\varepsilon\to 0} \mathbf{v}_0$ *in* $L^2_\sigma(\Omega)$*,* $\varphi_{0,\varepsilon} \to_{\varepsilon \to 0} \varphi_0$ *in* $L^2(\Omega)$*, and*

$$\mathcal{E}_\varepsilon(\mathbf{v}_{0,\varepsilon}, \varphi_{0,\varepsilon}) \to_{\varepsilon\to 0} \mathcal{E}(\mathbf{v}_0, \varphi_0),$$

where

$$\mathcal{E}(\mathbf{v},\varphi) := \frac{1}{2}\int_\Omega \rho(\varphi)|\mathbf{v}|^2\,dx + E(\varphi), \quad E(\varphi) := \frac{1}{2}\int_\Omega |\nabla\varphi(x)|^2\,dx + \int_\Omega F(\varphi(x))\,dx.$$

If $\mathbf{v}_\varepsilon$*,* φ_ε *and* μ_ε *are weak solutions of* (1.1)-(1.6) *with initial values* $(\mathbf{v}_{0,\varepsilon}, \varphi_{0,\varepsilon})$*, then*

$$\mathbf{v}_\varepsilon \rightharpoonup \mathbf{v} \qquad \text{weakly-}* \text{ in } L^\infty(0,T; L^2_\sigma(\Omega)), \tag{3.1}$$

$$\mathbf{v}_\varepsilon \rightharpoonup \mathbf{v} \qquad \text{weakly in } L^2(0,T; H^1_0(\Omega)^d), \tag{3.2}$$

$$\mu_\varepsilon \rightharpoonup \mu \qquad \text{weakly in } L^2(0,T;V), \tag{3.3}$$

$$\mathbf{v}_\varepsilon \to \mathbf{v} \qquad \text{strongly in } L^2(0,T;L^2(\Omega)) \text{ and almost everywhere,} \tag{3.4}$$

$$\varphi_\varepsilon \to \varphi \qquad \text{strongly in } C([0,T];L^2(\Omega)) \text{ and almost everywhere} \tag{3.5}$$

for a suitable subsequence $\varepsilon = \varepsilon_k \to_{k\to\infty} 0$*, where* $(\mathbf{v},\varphi,\mu)$ *is a weak solution* (1.7)-(1.12) *in the sense that*

$$\begin{aligned}
&\mathbf{v} \in BC_w([0,T]; L^2_\sigma(\Omega)) \cap L^2(0,T; H^1_0(\Omega)^d),\\
&\varphi \in C([0,T];L^2(\Omega)) \cap BC_w([0,T];H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)), F'(\varphi) \in L^2(0,T;L^2(\Omega)),\\
&\mu \in L^2(0,T;H^1(\Omega)),
\end{aligned}$$

$|\varphi(x,t)| < 1$ *almost everywhere in* Q*, and the following holds true:*

(*i*) $\mu = -\Delta\varphi + F'(\varphi)$ *almost everywhere in* Q_T*.*

(*ii*) *For every* $\psi \in H^1_0(\Omega)^d \cap L^2_\sigma(\Omega)$ *and* $\boldsymbol{\psi} \in \mathcal{D}(A)$ *and almost every* $t \in (0,T)$ *we have*

$$\langle \partial_t(\rho\mathbf{v}), \boldsymbol{\psi}\rangle_{\mathcal{D}(A)} - \int_\Omega ((\mathbf{v} + \widetilde{\mathbf{J}}) \otimes \rho\mathbf{v} : D\boldsymbol{\psi}\,dx + \int_\Omega 2\nu(\varphi) D\mathbf{v} : D\boldsymbol{\psi}\,dx = -\int_\Omega \varphi\nabla\mu\cdot\boldsymbol{\psi}\,dx,$$

$$\langle \partial_t\varphi, \psi\rangle_{H^1(\Omega)} + \int_\Omega m(\varphi)\nabla\mu\cdot\nabla\psi = \int_\Omega \mathbf{v}\varphi\cdot\nabla\psi\,dx,$$

where $\widetilde{\mathbf{J}} = -\frac{\tilde\rho_2 - \tilde\rho_1}{2} m(\varphi)\nabla\mu$*.*

(*iii*) *The energy inequality*

$$\mathcal{E}(\mathbf{v}(t),\varphi(t)) + \int_0^t \int_\Omega (2\nu(\varphi)|D\mathbf{v}|^2 + m(\varphi)|\nabla\mu|^2\,dx\,d\tau \leq \mathcal{E}(\mathbf{v}_0,\varphi_0) \tag{3.6}$$

holds true for all $t \in [0,T]$*.*

Proof. From the energy inequality (2.1), we see that $(\mathbf{v}_\varepsilon)_{\varepsilon\in(0,1)}$ is bounded in $L^\infty(0,T;L^2_\sigma(\Omega))$ and $L^2(0,T;H^1_0(\Omega)^d)$. Hence one can find a subsequence such that (3.1) and (3.2) hold. Moreover, since $|\varphi_\varepsilon(x,t)| < 1$ almost everywhere, $(\varphi_\varepsilon)_{\varepsilon\in(0,1)}$ is obviously bounded in $L^\infty(0,T;L^2(\Omega))$. We also see from the energy inequality that $(\nabla\mu_\varepsilon)_{\varepsilon\in(0,1)}$ is bounded in $L^2(0,T;L^2(\Omega))$. To see that $(\mu_\varepsilon)_{\varepsilon(0,1)}$ is bounded in $L^2(0,T;H^1(\Omega))$, we know from the Poincaré-Wirtinger inequality that it is enough to show that $(\mu)_\Omega := \frac{1}{|\Omega|}\int_\Omega \mu\,dx \in L^2(0,T)$. The argument below for showing this are an adaptation of the arguements in Section 4.1 of [8]. We include it for the reader's convenience.

For the following we define

$$\mathcal{N}(\varphi_\varepsilon(t))\colon (H^1_{(0)}(\Omega))' \to H^1_{(0)}(\Omega) := \left\{u \in H^1(\Omega) : \int_\Omega u\,dx = 0\right\} : f \mapsto u,$$

where $u \in H^1_{(0)}(\Omega)$ is the solution of

$$\int_\Omega m(\varphi_\varepsilon(t))\nabla u \cdot \nabla\psi\,dx = \langle f, \psi\rangle \qquad \text{for all } \psi \in H^1_{(0)}(\Omega).$$

Since m is strictly bounded below (independent of $\varphi_\varepsilon(t)$), there is some constant C, independent of $\varphi_\varepsilon(t)$, such that

$$\|\mathcal{N}(\varphi_\varepsilon(t))f\|_{H^1(\Omega)} \le C\|f\|_{(H^1_{(0)}(\Omega))'} \qquad \text{for all } f \in (H^1_{(0)}(\Omega))'.$$

Then testing (1.3) by $\mathcal{N}(\varphi_\varepsilon(t))(\varphi_\varepsilon(t) - m_\Omega)$ (in the weak sense), (1.4) with $\varphi_\varepsilon(t) - m_\Omega$ and taking the sum yields

$$\begin{aligned}\langle \partial_t\varphi_\varepsilon(t), \mathcal{N}(\varphi_\varepsilon(t))(\varphi_\varepsilon(t) - m_\Omega)\rangle_{H^1_{(0)}(\Omega)} + 2E^0_\varepsilon(\varphi_\varepsilon(t)) + \int_\Omega F_0'(\varphi_\varepsilon(x,t))(\varphi_\varepsilon(x,t) - m_\Omega)\,dx \\ = \int_\Omega \varphi_\varepsilon(x,t)\mathbf{v}_\varepsilon(x,t)\cdot\nabla\mathcal{N}(\varphi_\varepsilon(t))(\varphi_\varepsilon(x,t) - m_\Omega)\,dx - \int_\Omega \theta_c\varphi_\varepsilon(x,t)(\varphi_\varepsilon(x,t) - m_\Omega)\,dx, \end{aligned} \tag{3.7}$$

where

$$F_0(s) := F(s) + \theta_c\frac{s^2}{2} \qquad \text{for } s \in [-1,1]$$

is the "convex part" of F. Using the weak form of (1.3) it is easy to see that $\partial_t\varphi_\varepsilon$ is bounded in $L^2(0,T;(H^1(\Omega))')$ since $\nabla\mu_\varepsilon$ and $\varphi_\varepsilon\mathbf{v}_\varepsilon$ are bounded in $L^2(0,T;L^2(\Omega)^d)$. Using this and the energy inequality, we observe that the first term in the left-hand side of (3.7) is bounded in $L^2(0,T)$ independently of ε and the second is non-negative. Using the properties of $\mathcal{N}$ and the Hölder inequality, the right-hand side of (3.7) can be estimated from above by a constant multiple of

$$\|\varphi_\varepsilon\|_{L^\infty(\Omega)}\|\mathbf{v}_\varepsilon\|_{L^2(\Omega)}\|\varphi_\varepsilon - m_\Omega\|_{(H^1_{(0)}(\Omega))'} + \theta_c\|\varphi_\varepsilon\|_{L^2(\Omega)}\|\varphi_\varepsilon - m_\Omega\|_{L^2(\Omega)}. \tag{3.8}$$

Hence they are bounded in $L^\infty(0,T)$. Moreover, using the estimate in the last line of p. 462 in [3], there exist constants c_1 and c_2 such that

$$\int_\Omega F_0'(\varphi_\varepsilon(x,t))(\varphi_\varepsilon(x,t) - m_\Omega)dx \ge c_1\|F_0'(\varphi_\varepsilon)\|_{L^1(\Omega)} - c_2. \tag{3.9}$$

Combining these estimates and (3.7), we have that $F_0'(\varphi_\varepsilon)$ is bounded in $L^2(0,T;L^1(\Omega))$. Moreover, integrating $\mu_\varepsilon = a_\varepsilon\varphi_\varepsilon - J_\varepsilon * \varphi_\varepsilon + F'(\varphi_\varepsilon)$ in Ω, yields that

$$(\mu_\varepsilon)_\Omega = \frac{1}{|\Omega|}\int_\Omega (F_0'(\varphi_\varepsilon) - \theta_c\varphi_\varepsilon)\, dx$$

is bounded in $L^2(0,T)$. Hence $(\mu_\varepsilon)_{\varepsilon\in(0,1)}$ is bounded in $L^2(0,T;H^1(\Omega))$ and we can choose a subsequence such that (3.3) holds.

Next we show (3.5) for a suitable subsequence. As seen before $(\partial_t\varphi_\varepsilon)_{\varepsilon\in(0,1)} \subseteq L^2(0,T;(H^1(\Omega))')$ is bounded. Furthermore $(\varphi_\varepsilon)_{\varepsilon\in(0,1)}$ is bounded in $L^\infty(0,T;L^2(\Omega))$ since $|\varphi_\varepsilon(x,t)| < 1$ almost everywhere in Q_T. Since $L^2(\Omega)$ is compactly embedded in $(H^1(\Omega))'$, we have $\varphi_\varepsilon \to \varphi$ in $C([0,T];(H^1(\Omega))')$ for a suitable subsequence by the Aubin-Lions lemma. Using Lemma 2.3 and the bounds on the energies, we have $\varphi_\varepsilon \to \varphi$ in $C([0,T];L^2(\Omega))$ and almost everywhere for a suitable subsequence. Since the function $\rho(\varphi_\varepsilon)$ is bounded and depends continuously on φ_ε, using Lebesgue's dominated convergence theorem, we have $\rho(\varphi_\varepsilon) \to \rho(\varphi)$ strongly in $L^q(\Omega)$ for any $1 \le q < \infty$. Using the energy inequality, $\mathbf{v}_\varepsilon$ is uniformly bounded in $L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;L^6(\Omega)^d)$ and hence also in $L^{\frac{10}{3}}(Q_T)^d$. Thus $\mathbf{v}_\varepsilon \rightharpoonup \mathbf{v}$ weakly in $L^{\frac{10}{3}}(Q_T)^d$. Combining these convergence result, we derive $\rho(\varphi_\varepsilon)\mathbf{v}_\varepsilon \rightharpoonup \rho(\varphi)\mathbf{v}$ weakly in $L^{\frac{10}{3}-\gamma}(Q_T)^d$ for any $0<\gamma<\frac{10}{3}$ and a suitable subsequence. This implies

$$\rho(\varphi_\varepsilon)\mathbf{v}_\varepsilon \rightharpoonup \rho(\varphi)\mathbf{v} \qquad \text{weakly in } L^2(0,T;L^2(\Omega)^d).$$

Since the Helmholtz projection $\mathbb{P}_\sigma$ is weakly continuous in $L^2(0,T;L^2(\Omega)^d)$, we obtain

$$\mathbb{P}_\sigma(\rho(\varphi_\varepsilon)\mathbf{v}_\varepsilon) \rightharpoonup \mathbb{P}_\sigma(\rho(\varphi)\mathbf{v}) \qquad \text{weakly in } L^2(0,T;L^2(\Omega)^d).$$

Using the weak form of (1.1), we have

$$\begin{aligned}\langle \partial_t(\mathbb{P}_\sigma(\rho_\varepsilon\mathbf{v}_\varepsilon)(t),\boldsymbol{\psi}\rangle_{\mathcal{D}(A)} - \int_\Omega \rho_\varepsilon\mathbf{v}_\varepsilon\otimes(\mathbf{v}_\varepsilon + \widetilde{\mathbf{J}}_\varepsilon)) : D\boldsymbol{\psi}\, dx \\ - \int_\Omega 2\nu(\varphi_\varepsilon)D\mathbf{v}_\varepsilon : D\boldsymbol{\psi}\, dx = -\int_\Omega \varphi_\varepsilon\nabla\mu_\varepsilon\cdot\boldsymbol{\psi}\, dx \qquad (3.10)\end{aligned}$$

for all $\boldsymbol{\psi} \in \mathcal{D}(A)$ and almost every $t\in(0,T)$. Since $\mathbb{P}_\sigma$ is bounded in $L^2(\Omega)^d$, $\mathbb{P}_\sigma(\rho_\varepsilon\mathbf{v}_\varepsilon)$ is bounded in $L^2(0,T;L^2(\Omega)^d)$. Moreover, $\rho_\varepsilon\mathbf{v}_\varepsilon\otimes\mathbf{v}_\varepsilon$ is bounded in $L^2(0,T;L^{\frac{3}{2}}(\Omega)^{d\times d})$ and $\mathbf{v}_\varepsilon\otimes\widetilde{\mathbf{J}}_\varepsilon$ is bounded in $L^{\frac{8}{7}}(0,T;L^{\frac{4}{3}}(\Omega)^{d\times d})$ since $\widetilde{\mathbf{J}}_\varepsilon = -\frac{\tilde\rho_2-\tilde\rho_1}{2}m(\varphi_\varepsilon)\nabla\varphi_\varepsilon$ is bounded in $L^2(0,T;L^2(\Omega)^d)$ and $\mathbf{v}_\varepsilon$ is bounded in $L^{\frac{8}{3}}(0,T;L^4(\Omega)^d)$. Using these bounds and the boundedness of $2\eta(\varphi_\varepsilon)D\mathbf{v}_\varepsilon$ in $L^2(0,T;L^2(\Omega)^{d\times d})$, we have that $\partial_t(\mathbb{P}_\sigma(\rho_\varepsilon\mathbf{v}_\varepsilon))$ is bounded in $L^{\frac{8}{7}}(0,T;W^{-1}_{\frac{4}{3},\sigma}(\Omega))$ because of (3.10), where $W^{-1}_{\frac{4}{3},\sigma}(\Omega) = (W^1_{4,0}(\Omega)\cap L^2_\sigma(\Omega))'$. Since $L^2_\sigma(\Omega)$ is compactly embedded in $H^{-1}_\sigma(\Omega) := (H^1_0(\Omega)^d\cap L^2_\sigma(\Omega))'$ and $H^{-1}_\sigma(\Omega)$ is continuously embedded in $W^{-1}_{\frac{4}{3},\sigma}(\Omega)$, the Aubin-Lions' lemma yields that

$$\mathbb{P}_\sigma(\rho_\varepsilon\mathbf{v}_\varepsilon) \to \mathbf{w}_1 \qquad \text{in } L^2(0,T;H^{-1}_\sigma(\Omega))$$

for some $\mathbf{w}_1$ in $L^2(0,T;H^{-1}_\sigma(\Omega))$ and a suitable subsequence. Since $\mathbb{P}_\sigma(\rho(\varphi_\varepsilon)\mathbf{v}_\varepsilon) \rightharpoonup \mathbb{P}_\sigma(\rho(\varphi)\mathbf{v})$ weakly in $L^2(0,T;L^2(\Omega))$, $\mathbf{w}_1 = \mathbb{P}_\sigma(\rho(\varphi)\mathbf{v})$. Hence we have

$$\mathbb{P}_\sigma(\rho_\varepsilon\mathbf{v}_\varepsilon) \to \mathbb{P}_\sigma(\rho\mathbf{v}) \qquad \text{in } L^2(0,T;H^{-1}_\sigma(\Omega)) \qquad (3.11)$$

Because of the boundedness of $\partial_t\left(\mathbb{P}_\sigma(\rho_\varepsilon \mathbf{v}_\varepsilon)\right)$ in $L^{\frac{8}{7}}(0,T;W^{-1}_{\frac{4}{3},\sigma}(\Omega))$ and $\mathbb{P}_\sigma(\rho(\varphi_\varepsilon)\mathbf{v}_\varepsilon) \rightharpoonup \mathbb{P}_\sigma(\rho(\varphi)\mathbf{v})$ weakly in $L^2(0,T;L^2(\Omega))$, we have

$$\partial_t\left(\mathbb{P}_\sigma(\rho_\varepsilon \mathbf{v}_\varepsilon)\right) \rightharpoonup \partial_t\left(\mathbb{P}_\sigma(\rho \mathbf{v})\right) \qquad \text{weakly in } L^{\frac{8}{7}}(0,T;W^{-1}_{\frac{4}{3},\sigma}(\Omega)).$$

Using also the boundedness of $\mathbf{v}_\varepsilon$ in $L^2(0,T;H^1_0(\Omega)^d)$, we conclude that $\mathbf{v}_\varepsilon$ converges weakly to $\mathbf{v}$ in $L^2(0,T;H^1_0(\Omega)^d)$ for some subsequence. Combining this with (3.11), we obtain

$$\int_{Q_T} \rho_\varepsilon |\mathbf{v}_\varepsilon|^2\, d(x,t) = \int_{Q_T} \mathbb{P}_\sigma(\rho_\varepsilon \mathbf{v}_\varepsilon)\cdot \mathbf{v}_\varepsilon\, d(x,t) \to \int_{Q_T} \mathbb{P}_\sigma(\rho\mathbf{v})\cdot \mathbf{v}\, d(x,t) = \int_{Q_T} \rho|\mathbf{v}|^2\, d(x,t).$$

Together with the weak convergence of $\mathbf{v}_\varepsilon$ and $\rho_\varepsilon^{\frac{1}{2}}\mathbf{v}_\varepsilon$ in $L^2(Q_T)^d$, we conclude that $\rho_\varepsilon^{\frac{1}{2}}\mathbf{v}_\varepsilon \to \rho^{\frac{1}{2}}\mathbf{v}$ strongly in $L^2(0,T;L^2(\Omega)^d)$. Moreover, since $\rho(\varphi_\varepsilon) \to \rho(\varphi)$ almost everywhere, $\rho_\varepsilon \geq c$ for some $c > 0$, the convergence $\rho_\varepsilon^{\frac{1}{2}}\mathbf{v}_\varepsilon \to \rho^{\frac{1}{2}}\mathbf{v}$ in $L^2(0,T;L^2(\Omega)^d)$ implies $\mathbf{v}_\varepsilon \to \mathbf{v}$ in $L^2(0,T;L^2(\Omega)^d)$, i.e., (3.4) holds true.

Since $\mathbf{v}_\varepsilon \otimes \mathbf{v}_\varepsilon$ is bounded in $L^{\frac{5}{3}}(Q_T)^{d\times d}$, it converges weakly to some $\mathbf{w}_2$ in $L^{\frac{5}{3}}(Q_T)^{d\times d}$. On the other hand, since $\mathbf{v}_\varepsilon \to \mathbf{v}$ in $L^2(Q_T)^d$, $\mathbf{v}_\varepsilon \otimes \mathbf{v}_\varepsilon \to \mathbf{v}\otimes\mathbf{v}$ in $L^1(Q_T)^{d\times d}$. Hence $\mathbf{w}_2 = \mathbf{v}\otimes\mathbf{v}$. This means $\mathbf{v}_\varepsilon \otimes \mathbf{v}_\varepsilon$ converges weakly to $\mathbf{v}\otimes\mathbf{v}$ in $L^{\frac{5}{3}}(Q)^{d\times d}$. Since $\rho(\varphi_\varepsilon)$ converges strongly to $\rho(\varphi)$ in $L^p(Q_T)$ for any $1 \leq p < \infty$, we conclude that $\rho(\varphi_\varepsilon)\mathbf{v}_\varepsilon\otimes\mathbf{v}_\varepsilon$ converges weakly to $\rho(\varphi)\mathbf{v}\otimes\mathbf{v}$ in $L^{\frac{5}{3}-\gamma}(Q_T)^{d\times d}$ for any $\gamma \in (0,\frac{5}{3})$. Furthermore, since $\nu(\varphi_\varepsilon)D\mathbf{v}_\varepsilon$ is bounded in $L^2(Q_T)^{d\times d}$, it converges weakly to some $\mathbf{w}_3$ in $L^2(Q_T)^{d\times d}$. On the other hand, since $D\mathbf{v}_\varepsilon$ converges weakly to $D\mathbf{v}$ in $L^2(Q_T)^{d\times d}$ and $\nu(\varphi_\varepsilon)$ converges strongly to $\nu(\varphi)$ in $L^p(Q)$ for any $1 \leq p < \infty$, $\nu(\varphi_\varepsilon)D\mathbf{v}_\varepsilon$ converges weakly to $\nu(\varphi)D\mathbf{v}$ in $L^{2-\gamma}(Q_T)^{d\times d}$ for any $\gamma \in (0,2)$. Hence $\mathbf{w}_3 = \nu(\varphi)D\mathbf{v}$. Similarly as above, one shows $\widetilde{\mathbf{J}}_\varepsilon = -\beta m(\varphi_\varepsilon)\nabla\mu_\varepsilon \rightharpoonup \widetilde{\mathbf{J}} = -\beta m(\varphi)\nabla\mu$ in $L^2(Q_T)^d$. Using this together with $\mathbf{v}_\varepsilon \to \mathbf{v}$ in $L^2(0,T;L^2(\Omega)^d)$, we have that $\mathbf{v}_\varepsilon \otimes \widetilde{\mathbf{J}}_\varepsilon \rightharpoonup \mathbf{v}\otimes\widetilde{\mathbf{J}}$ in $L^1(Q_T)^{d\times d}$. Similarly as above, we see $\varphi_\varepsilon\nabla\mu_\varepsilon \rightharpoonup \varphi\nabla\mu$ in $L^2(Q)^d$. Hence we can pass to the limit in the weak form of (1.1).

Since $\partial_t\varphi_\varepsilon$ is bounded in $L^2(0,T;(H^1(\Omega))')$, $\partial_t\varphi_\varepsilon$ converges weakly to $\partial_t\varphi$ in $L^2(0,T;(H^1(\Omega))')$. Since $\mathbf{v}_\varepsilon \to \mathbf{v}$ strongly in $L^2(Q_T)^d$ and $\varphi_\varepsilon \to \varphi$ almost everywhere and is uniformly bounded, we have that $\mathbf{v}_\varepsilon\varphi_\varepsilon \to \mathbf{v}\varphi$ strongly in $L^2(Q_T)^d$ by Lebesgue's dominated convergence theorem. We also have $m(\varphi_\varepsilon)\nabla\mu_\varepsilon$ converges weakly to $m(\varphi)\nabla\mu$ in $L^2(Q)^d$. Thus we can pass to the limit in the weak form of (1.3).

The following argument is from Chapter 5 in [8]. We repeat the argument for the convenience of the reader. Testing (1.4) by $F_0'(\varphi_\varepsilon)$, taking into account that $(\mu_\varepsilon)_{\varepsilon\in(0,1)}$ is bounded in $L^2(0,T;L^2(\Omega))$ and using the monotonicity of F_0', we derive that

$$\|F_0'(\varphi_\varepsilon)\|_{L^2(0,T;L^2(\Omega))} \leq M \tag{3.12}$$

for some $M > 0$ independent of $\varepsilon \in (0,1)$. From this and (1.4), we have

$$\|a_\varepsilon\varphi_\varepsilon - J_\varepsilon * \varphi_\varepsilon\|_{L^2(0,T;L^2(\Omega))} \leq M. \tag{3.13}$$

Because of (3.12) and (3.13), there exist $\xi, \eta \in L^2(0,T;L^2(\Omega))$ such that

$$\begin{aligned} F_0'(\varphi_\varepsilon) &\rightharpoonup \xi \qquad \text{in } L^2(0,T;L^2(\Omega)),\\ a_\varepsilon\varphi_\varepsilon - J_\varepsilon * \varphi_\varepsilon &\rightharpoonup \eta \qquad \text{in } L^2(0,T;L^2(\Omega)) \end{aligned}$$

for a suitable subsequence. Using $\varphi_\varepsilon \to \varphi$ in $C([0,T];L^2(\Omega))$ one can deduce $F_0'(\varphi_\varepsilon) \to F_0'(\varphi)$ almost everywhere, cf. e.g. [2, page 1093], and therefore in $L^q(Q_T)$ for every $1 \le q < 2$.

Passing to the limit in the weak formulation of (1.3) we have

$$\langle \partial_t \varphi(t), \psi(t)\rangle_{H^1(\Omega)} + \int_\Omega m(\varphi(x,t))\nabla\mu(x,t)\cdot\nabla\psi(x)dx = \int_\Omega \varphi(x,t)\mathbf{v}(x,t)\cdot\nabla\psi(x)dx$$

for every $\psi \in H^1(\Omega)$ and for almost every $t \in (0,T)$ and that $\mu = \eta + \xi - \theta_c\varphi$. It only remains to show that $\varphi \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$ and $\eta = -\Delta\varphi$.

Because of $\varphi_\varepsilon \to \varphi$ in $C([0,T];L^2(\Omega))$, Lemma 2.2, and the energy estimate, we conclude

$$\|E^0(\varphi)\|_{L^\infty(0,T)} \le \liminf_{\varepsilon\to 0} \|E^0_\varepsilon(\varphi_\varepsilon)\|_{L^\infty(0,T)} \le M$$

Hence $\varphi \in L^\infty(0,T;H^1(\Omega))$. Since $E^0_\varepsilon(\varphi_\varepsilon)$ is quadratic in φ_ε, we have

$$\int_0^T E^0_\varepsilon(\varphi_\varepsilon(t))\,dt + \int_{Q_T} (a_\varepsilon\varphi_\varepsilon - J_\varepsilon * \varphi_\varepsilon)(t,x)(\psi - \varphi_\varepsilon)\,dx\,dt \le \int_0^T E^0_\varepsilon(\psi(t))\,dt$$

for any $\psi \in H^1(\Omega)$ with $(\psi)_\Omega = m$. Since $\varphi_\varepsilon \to \varphi$ in $C([0,T];L^2(\Omega))$, using Lemma 2.2 and Fatou's lemma, we derive

$$\frac{1}{2}\int_{Q_T} |\nabla\varphi(t,x)|^2\,dx\,dt + \int_{Q_T} \eta(t,x)(\psi-\varphi)(t,x)d(x,t) \le \frac{1}{2}\int_{Q_T} |\nabla\psi(t,x)|^2\,d(x,t)$$

for every $\psi \in L^2(0,T;H^1(\Omega))$ with $(\psi(t))_\Omega = m$ for almost every $t \in (0,T)$.

If we take $\psi(t,x) = \varphi(t,x) + h\chi(t)\tau(x)$, where $h \in \mathbb{R}$, $\chi \in C([0,T])$ and $\tau \in H^1_{(0)}(\Omega)$, and passing to the limit $h \to 0$, we obtain

$$\int_\Omega \eta(x,t)\tau(x)dx = \int_\Omega \nabla\varphi(x,t)\cdot\nabla\tau(x)dx$$

for a.e. $t \in (0,T)$ and for all $\tau \in H^1_{(0)}(\Omega)$. By classical elliptic regularity theory, we conclude that $\varphi \in L^2(0,T;H^2(\Omega))$ and $\eta = -\Delta\varphi$ and $\frac{\partial\varphi}{\partial\mathbf{n}}|_{\partial\Omega} = 0$. Furthermore, since $\varphi \in L^\infty(0,T;H^1(\Omega)) \cap H^1(0,T;(H^1(\Omega))')$, we have $\varphi \in BC_w([0,T];H^1(\Omega))$, cf. e.g. [1, Lemma 4.1]. Moreover, using the same arguments as in [3, Section 5.2] one shows $\mathbf{v} \in BC_w([0,T];L^2_\sigma(\Omega))$ and $\mathbf{v}|_{t=0} = \mathbf{v}_0$.

Finally, we prove the energy inequality for the limit $(\mathbf{v},\varphi,\mu)$. Using (2.1), we obtain

$$\mathcal{E}_\varepsilon(\mathbf{v}_\varepsilon(t),\varphi_\varepsilon(t)) + \int_{Q_t} (2\nu(\varphi_\varepsilon)|D\mathbf{v}_\varepsilon|^2 + m(\varphi_\varepsilon)|\nabla\mu_\varepsilon|^2)\,d(x,\tau) \le \mathcal{E}_\varepsilon(\mathbf{v}_\varepsilon(0),\varphi_\varepsilon(0)) \tag{3.14}$$

If we take liminf of both sides of (3.14) as $\varepsilon \searrow 0$, we have

$$\liminf_{\varepsilon\searrow 0} \mathcal{E}_\varepsilon(\mathbf{v}_\varepsilon(t),\varphi_\varepsilon(t)) + \liminf_{\varepsilon\searrow 0} \int_{Q_t} (2\nu(\varphi_\varepsilon)|D\mathbf{v}_\varepsilon|^2 + m(\varphi_\varepsilon)|\nabla\mu_\varepsilon|^2)\,d(x,\tau) \le \lim_{\varepsilon\searrow 0} \mathcal{E}_\varepsilon(\mathbf{v}_\varepsilon(0),\varphi_\varepsilon(0)),$$

where from our assumption on the sequence of the initial data, we conclude

$$\liminf_{\varepsilon\searrow 0} \mathcal{E}_\varepsilon(\mathbf{v}_\varepsilon(0),\varphi_\varepsilon(0)) = \lim_{\varepsilon\searrow 0} \mathcal{E}_\varepsilon(\mathbf{v}_{0,\varepsilon},\varphi_{0,\varepsilon}) = \mathcal{E}(\mathbf{v}_0,\varphi_0).$$

For almost all $t \in (0,T)$, we have

$$\mathcal{E}(\mathbf{v}(t), \varphi(t)) \leq \liminf_{\varepsilon \searrow 0} \mathcal{E}_\varepsilon(\mathbf{v}_\varepsilon(t), \varphi_\varepsilon(t))$$

because of $\mathbf{v}_\varepsilon(t) \to \mathbf{v}(t)$ in $L^2(\Omega)^d$, $\varphi_\varepsilon(t) \to_{\varepsilon\to 0} \varphi(t)$ in $L^2(\Omega)$ for almost every $t \in (0,T)$, and Lemma 2.2. Furthermore, for any $t \in (0,T)$ we obtain

$$\int_{Q_t} (2\nu(\varphi)|D\mathbf{v}|^2 + m(\varphi)|\nabla\mu|^2)\, d(x,\tau) \leq \liminf_{\varepsilon \searrow 0} \int_{Q_t} (2\nu(\varphi_\varepsilon)|D\mathbf{v}_\varepsilon|^2 + m(\varphi_\varepsilon)|\nabla\mu_\varepsilon|^2)\, d(x,\tau)$$

using weak lower semicontinuity of norms and $\nu(\varphi_\varepsilon)^{\frac{1}{2}} D\mathbf{v}_\varepsilon \rightharpoonup \nu(\varphi)^{\frac{1}{2}} D\mathbf{v}$ weakly in $L^2(0,T;L^2(\Omega))$ and $m(\varphi_\varepsilon)^{\frac{1}{2}}\nabla\mu_\varepsilon \rightharpoonup m(\varphi)^{\frac{1}{2}}\nabla\mu$ weakly in $L^2(0,T;L^2(\Omega))$. In summary, we have shown (3.6) for almost every $t \in (0,T)$. But using $\mathbf{v} \in BC_w([0,T];L^2_\sigma(\Omega))$, $\varphi \in C([0,T];L^2(\Omega)) \cap BC_w([0,T];H^1(\Omega))$, hence $\rho^{\frac{1}{2}}\mathbf{v} \in BC_w([0,T];L^2(\Omega))$, and suitable properties of $\mathcal{E}$ which concerns continuity or weak lower semi-continuity of each terms, we finally obtain (3.6) for every $t \in [0,T]$ by a density argument. This completes the proof of Theorem 3.1. $\square$

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