

# INTEGRAL MEANS OF DERIVATIVES OF UNIVALENT FUNCTIONS IN HARDY SPACES

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ABSTRACT. We show that the norm in the Hardy space  $H^p$  satisfies

$$\|f\|_{H^p}^p \asymp \int_0^1 M_q^p(r, f')(1-r)^{p(1-\frac{1}{q})} dr + |f(0)|^p \quad (\dagger)$$

for all univalent functions provided that either  $q \geq 2$  or  $\frac{2p}{2+p} < q < 2$ . This asymptotic was previously known in the cases  $0 < p \leq q < \infty$  and  $\frac{p}{1+p} < q < p < 2 + \frac{2}{157}$  by results due to Pommerenke (1962), Baernstein, Girela and Peláez (2004) and González and Peláez (2009). It is also shown that  $(\dagger)$  is satisfied for all close-to-convex functions if  $1 \leq q < \infty$ . A counterpart of  $(\dagger)$  in the setting of weighted Bergman spaces is also briefly discussed.

## 1. INTRODUCTION AND RESULTS

Let  $\mathcal{H}(\mathbb{D})$  denote the space of analytic functions in the unit disc  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ . For  $0 < p \leq \infty$ , the Hardy space  $H^p$  consists of those  $f \in \mathcal{H}(\mathbb{D})$  such that

$$\|f\|_{H^p} = \sup_{0 < r < 1} M_p(r, f) < \infty,$$

where

$$M_p(r, f) = \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}}, \quad 0 < r < 1,$$

is the  $L^p$ -mean of the restriction of  $f$  to the circle of radius  $r$ , and  $M_\infty(r, f) = \max_{|z|=r} |f(z)|$  is the maximum modulus function. The monographs [2] and [5] are excellent sources for the theory of the Hardy spaces.

An injective function in  $\mathcal{H}(\mathbb{D})$  is called a conformal map or univalent, and the class of all such functions is denoted by  $\mathcal{U}$ . Let  $\mathcal{S}$  denote the set of  $f \in \mathcal{U}$  normalized such that  $f(0) = 0$  and  $f'(0) = 1$ . If  $f \in \mathcal{U}$ , then  $(f - f(0))/f'(0)$  belongs to  $\mathcal{S}$ . We refer to [3], [11] and [12] for the theory of univalent functions.

In 1927 Prawitz [13] showed that

$$M_p^p(r, f) \leq p \int_0^r M_\infty^p(t, f) \frac{dt}{t}, \quad 0 < r < 1, \quad 0 < p < \infty, \quad f \in \mathcal{S}. \quad (1.1)$$

This combined with the Hardy-Littlewood [7, p. 411] inequality

$$\int_0^r M_\infty^p(t, f) dt \leq \pi r M_p^p(r, f), \quad 0 < r < 1, \quad 0 < p < \infty, \quad f \in \mathcal{H}(\mathbb{D}), \quad (1.2)$$

the proof of which can be found in [10, Hilfssatz 1] and [1, p. 841], shows that for each  $0 < p < \infty$  we have the well-known asymptotic equality

$$\|f\|_{H^p}^p \asymp \int_0^1 M_\infty^p(t, f) dt, \quad f \in \mathcal{U}. \quad (1.3)$$

Therefore the containment of  $f \in \mathcal{U}$  in the Hardy space  $H^p$  is neatly characterized by the behavior of its maximum modulus. Another well-known characterization is given in terms of

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the arc-length. Namely, the image of the circle of radius  $r$  centered at the origin under  $f \in \mathcal{U}$  with  $f(0) = 0$  is a Jordan curve with zero in its inner domain. The length of this image is  $2\pi r M_1(r, f')$ , and therefore

$$M_\infty(r, f) \leq \pi r M_1(r, f'), \quad 0 < r < 1, \quad f \in \mathcal{U}, \quad f(0) = 0. \quad (1.4)$$

This inequality is in a sense sharp for conformal maps as its proof shows, and it is actually valid for all analytic functions. Namely, (1.2) applied to  $f'$  yields

$$|f(re^{i\theta})| \leq \int_0^r |f'(te^{i\theta})| dt \leq \pi r M_1(r, f'), \quad 0 < r < 1, \quad f \in \mathcal{H}(\mathbb{D}), \quad f(0) = 0.$$

This combined with the Prawitz' inequality (1.1) shows that

$$M_p^p(r, f) \leq p\pi^p \int_0^r M_1^p(t, f') t^{1-p} dt, \quad 0 < r < 1, \quad 0 < p < \infty, \quad f \in \mathcal{S}.$$

A kind of converse of this inequality is also valid for some  $p$ . Indeed, in 1962 Pommerenke [10, Satz 4] showed that for  $0 < p < 2$  it holds that

$$r^p \int_0^r M_1^p(t, f') dt \lesssim M_p^p(r, f), \quad 0 < r < 1, \quad f \in \mathcal{U}, \quad f(0) = 0.$$

This asymptotic inequality, (1.3) and (1.4) show that for each  $0 < p < 2$  we have

$$\|f\|_{H^p}^p \asymp \int_0^1 M_1^p(t, f') dt + |f(0)|^p, \quad f \in \mathcal{U}. \quad (1.5)$$

In 2009 González and Peláez [6, Theorem 1] generalized this results to the range  $0 < p < 2 + \frac{2}{157}$ , showed that it fails for  $p \geq \frac{100}{17} \approx 5.88$ , and also observed by using 1967-results due to Thomas [14] that (1.5) is valid for all  $0 < p < \infty$  if  $\mathcal{U}$  is replaced by its proper subclass of all close-to-convex (univalent) functions [6, Proposition 1]. Recall that  $f \in \mathcal{H}(\mathbb{D})$  is close-to-convex if there exists a convex function  $g$  such that the real part of the quotient  $f'/g'$  is strictly positive on  $\mathbb{D}$ . The class of close-to-convex functions  $f$  normalized such that  $f(0) = 0$  and  $f'(0) = 1$  is denoted by  $K$  and it was introduced by Kaplan in 1952, see [3, Chapter 2] and [11, Chapter 2] for further information. At this point we only mention that an important subclass of close-to-convex functions is the class of starlike functions. Starlike functions are conformal maps which map  $\mathbb{D}$  onto a domain starlike with respect to the origin.

Integral means of derivatives different from  $M_1(r, f')$  appearing in (1.5) can also be used to characterize univalent functions in  $H^p$ . Namely, by combining the 2004-result by Baernstein, Girela and Peláez [1, Theorem 1] and [6, Theorem 2] due to González and Peláez we deduce

$$\|f\|_{H^p}^p \asymp \int_0^1 M_q^p(r, f') (1-r)^{p(1-\frac{1}{q})} dr + |f(0)|^p, \quad f \in \mathcal{U}, \quad (1.6)$$

if either  $0 < p \leq q < \infty$  or  $\frac{p}{1+p} < q < p < 2 + \frac{2}{157}$ . The main result of this note shows that these hypotheses can be significantly relaxed in a certain sense.

**Theorem 1.** *Let  $0 < p, q < \infty$  such that either  $\frac{2p}{2+p} < q < 2$  or  $q \geq 2$ . Then (1.6) is valid. Moreover, if  $0 < p < \infty$  and  $1 \leq q < \infty$ , then (1.6) is valid for all close-to-convex functions  $f$ .*

On one hand, Theorem 1 shows that for  $q \geq 2$  there is no restriction on  $p$ . On the other hand,  $\frac{2p}{2+p} \in (0, 2)$  for all  $0 < p < \infty$ , and hence the range  $\frac{2p}{2+p} < q < 2$  covers many cases previously excluded by the requirement  $p < 2 + \frac{2}{157}$ . However, the hypothesis  $\frac{2p}{2+p} < q$  is obviously strictly stronger than  $\frac{p}{1+p} < q$  for each  $0 < p < \infty$ . The statement on close-to-convex functions is a generalization of [6, Proposition 1] concerning the case  $q = 1$ .

The proof of Theorem 1 occupies most of the remaining part of the paper, and it is given in Section 2. At this point we only mention that we offer two proofs concerning the case  $q \geq 2$ ,

and one of them reveals that for  $p \geq q$  we have the asymptotic equality

$$\|f\|_{H^p}^p \asymp \int_0^1 \left( \int_{D(0,r)} \Delta |f'|^q(z) dA(z) \right)^{\frac{p}{q}} (1-r)^p dr, \quad f \in \mathcal{U},$$

where, as usual,  $\Delta$  stands for the Laplacian. We have not found this asymptotic in the existing literature and believe that it is of interest.

We next shortly discuss an application of Theorem 1 to Bergman spaces. Let  $\omega : \mathbb{D} \rightarrow [0, \infty)$  such that  $\omega(z) = \omega(|z|)$  for all  $z \in \mathbb{D}$ , and  $\int_{\mathbb{D}} \omega(z) dA(z) < \infty$ , where  $dA(z)$  denotes the element of the Lebesgue area measure on  $\mathbb{D}$ . For  $0 < p < \infty$  and such an  $\omega$ , the weighted Bergman space  $A_{\omega}^p$  consists of  $f \in \mathcal{H}(\mathbb{D})$  such that

$$\|f\|_{A_{\omega}^p}^p = \int_{\mathbb{D}} |f(z)|^p \omega(z) dA(z) < \infty.$$

By combining Prawitz' result (1.1) and the Hardy-Littlewood inequality (1.2), and then integrating over  $[0, 1)$  with respect to  $\omega(r)r dr$ , we obtain through Fubini's theorem the chain of inequalities

$$2 \int_0^1 M_{\infty}^p(r, f) \left( \int_r^1 \omega(t) dt \right) dr \leq \|f\|_{A_{\omega}^p}^p \leq 2\pi p \int_0^1 M_{\infty}^p(r, f) \left( \int_r^1 \omega(t)t dt \right) \frac{dr}{r},$$

valid for all  $f \in \mathcal{U}$  with  $f(0) = 0$ . Standard arguments then show that

$$\|f\|_{A_{\omega}^p}^p \asymp \int_0^1 M_{\infty}^p(r, f) \left( \int_r^1 \omega(t)t dt \right) dr, \quad f \in \mathcal{U}. \quad (1.7)$$

We may also transfer (1.6) to the setting of the weighted Bergman spaces as the following result shows.

**Corollary 2.** *Let  $0 < p, q < \infty$  and let  $\omega : \mathbb{D} \rightarrow [0, \infty)$  such that  $\omega(z) = \omega(|z|)$  for all  $z \in \mathbb{D}$ . Further, assume that one of the following conditions is satisfied:*

- (i)  $0 < p \leq q < \infty$ ;
- (ii)  $\frac{p}{1+p} < q < p < 2 + \frac{2}{157}$ ;
- (iii)  $q \geq 2$ ;
- (iv)  $\frac{2p}{2+p} < q < 2$ .

Then

$$\|f\|_{A_{\omega}^p}^p \asymp \int_0^1 M_q^p(r, f')(1-r)^{p(1-\frac{1}{q})} \left( \int_r^1 \omega(t)t dt \right) dr + |f(0)|^p \int_0^1 \omega(r)r dr, \quad f \in \mathcal{U}. \quad (1.8)$$

The natural approach that we adopt to obtain (1.8) consists of first applying (1.6) to the univalent dilatation  $f_r(z) = f(rz)$  appearing in the Bergman space norm of  $f$ , and then changing the order of radial integrations. The problem then no longer involves the weight  $\omega$  and the final step is managed by using the fact  $|f'(\rho\xi)| \asymp |f'(r\xi)|$  for all  $\xi$  on the boundary of  $\mathbb{D}$  and  $0 \leq r \leq \rho < 1$  such that  $1-r \asymp 1-\rho$ . Corollary 2 is proved in Section 3.

The case  $q = p$  of Corollary 2 is of special interest. It states that

$$\|f\|_{A_{\omega}^p}^p \asymp \int_{\mathbb{D}} |f'(z)|^p (1-|z|)^{p-1} \left( \int_{|z|}^1 \omega(r)r dr \right) dA(z) + |f(0)|^p \int_0^1 \omega(r)r dr, \quad f \in \mathcal{U}. \quad (1.9)$$

It is well known that this asymptotic equality is valid for all  $f \in \mathcal{H}(\mathbb{D})$  if  $\omega$  is the standard radial weight  $(1-|z|^2)^{\alpha}$  with  $-1 < \alpha < \infty$ . These kind of asymptotic equalities are known as Littlewood-Paley formulas. The rough idea behind these asymptotics is that  $f'$  behaves in a somewhat similar way as  $f$  divided by the distance from the boundary. However, it is known that all Bergman spaces do not admit this property. Namely, there exist radial weights  $\omega$  such that

$$\|f\|_{A_{\omega}^p}^p \asymp \int_{\mathbb{D}} |f'(z)|^p (1-|z|)^p W(z) dA(z) + |f(0)|^p, \quad f \in \mathcal{H}(\mathbb{D}), \quad (1.10)$$

fails to be true for each non-negative radial function  $W$  on  $\mathbb{D}$  unless  $p = 2$  [9, Proposition 4.3]. However, it was recently discovered in [8, Theorem 5] that (1.10) with  $W = \omega$  is valid if and only if  $\omega$  satisfies a certain two-sided doubling condition which imposes severe restrictions to the growth, the decay and the oscillation of the weight. Nevertheless, the asymptotic (1.9) shows that if we restrict our consideration to univalent functions, then a Littlewood-Paley formula exists for all weighted Bergman spaces induced by radial weights.

To this end a couple of words about the notation already used. If there exists a constant  $C > 0$  such that  $A(x) \leq CB(x)$  for all  $x$  in some set  $I$ , then we write either  $A(x) \lesssim B(x)$ ,  $x \in I$ , or  $B(x) \gtrsim A(x)$ ,  $x \in I$ , and the notation  $A(x) \asymp B(x)$ ,  $x \in I$ , stands for  $A(x) \lesssim B(x) \lesssim A(x)$  for all  $x \in I$ .

## 2. PROOF OF THEOREM 1

First observe that [4, Theorems 6 and 7] imply

$$\int_0^1 M_\infty^p(r, f) dr \asymp \int_0^1 M_\infty^p(r, f')(1-r)^p dr + |f(0)|^p, \quad f \in \mathcal{H}(\mathbb{D}). \quad (2.1)$$

Moreover, a careful inspection of the proof of [2, Theorem 5.9] shows that for  $0 < \alpha < \beta \leq \infty$  there exists a constant  $C = C(\alpha, \beta) > 0$  such that

$$M_\beta(r, g) \leq CM_\alpha\left(\frac{1+r}{2}, g\right) (1-r)^{\frac{1}{\beta}-\frac{1}{\alpha}}, \quad 0 \leq r < 1, \quad g \in \mathcal{H}(\mathbb{D}). \quad (2.2)$$

By combining (2.1) and (2.2), with  $\beta = \infty$  and  $\alpha = q$ , we deduce

$$\int_0^1 M_\infty^p(r, f) dr \lesssim \int_0^1 M_q^p(r, f')(1-r)^{p(1-\frac{1}{q})} dr + |f(0)|^p = I_{p,q}(f) + |f(0)|^p, \quad f \in \mathcal{H}(\mathbb{D}).$$

This together with (1.3) yields  $\|f\|_{H^p}^p \lesssim I_{p,q}(f) + |f(0)|^p$  for all  $f \in \mathcal{U}$ . Observe that this part of the proof is valid for all  $0 < p, q < \infty$ .

For the converse implication assume first that  $0 < q < 2$ , and write  $q = \alpha + \beta$ , where  $0 < \alpha, \beta < q$ . By [12, Proposition 8.1], for each fixed  $0 < p < \infty$ , we have

$$\int_0^{2\pi} \Delta |f|^p(re^{i\theta}) d\theta \lesssim \frac{M_\infty^p(r, f)}{1-r}, \quad \frac{1}{2} \leq r < 1, \quad f \in \mathcal{S}. \quad (2.3)$$

Hölder's inequality and (2.3) yield

$$\begin{aligned} 2\pi M_q^q(r, f') &\leq \left( \int_0^{2\pi} \left| \frac{f'(re^{i\theta})}{f(re^{i\theta})} \right|^2 |f(re^{i\theta})|^{\alpha \frac{2}{q}} d\theta \right)^{\frac{q}{2}} \left( \int_0^{2\pi} |f(re^{i\theta})|^{\beta \frac{2}{2-q}} d\theta \right)^{\frac{2-q}{2}} \\ &= \left( \int_0^{2\pi} |f(re^{i\theta})|^{\alpha \frac{2}{q}-2} |f'(re^{i\theta})|^2 d\theta \right)^{\frac{q}{2}} \left( \int_0^{2\pi} |f(re^{i\theta})|^{\frac{2\beta}{2-q}} d\theta \right)^{\frac{2-q}{2}} \\ &\lesssim \frac{M_\infty^\alpha(r, f)}{(1-r)^{\frac{q}{2}}} \left( \int_0^{2\pi} |f(re^{i\theta})|^{\frac{2\beta}{2-q}} d\theta \right)^{\frac{2-q}{2}}, \quad \frac{1}{2} \leq r < 1, \quad f \in \mathcal{S}. \end{aligned}$$

Another application of Hölder's inequality gives

$$\begin{aligned}
 I_{p,q}(f) &\lesssim \int_{\frac{1}{2}}^1 M_q^p(r, f') (1-r)^{p(1-\frac{1}{q})} dr \\
 &\lesssim \int_{\frac{1}{2}}^1 M_\infty^{\frac{p}{q}}(r, f) \left( \int_0^{2\pi} |f(re^{i\theta})|^{\frac{2\beta}{2-q}} d\theta \right)^{\frac{2-q}{2} \frac{p}{q}} (1-r)^{p\frac{q-2}{2q}} dr \\
 &\lesssim \left( \int_0^1 M_\infty^p(r, f) dr \right)^{\frac{q}{q}} \left( \int_0^1 \left( \int_0^{2\pi} |f(re^{i\theta})|^{\frac{2\beta}{2-q}} d\theta \right)^{p\frac{2-q}{2\beta}} (1-r)^{-p\frac{2-q}{2\beta}} dr \right)^{\frac{\beta}{q}} \\
 &\asymp \left( \int_0^1 M_\infty^p(r, f) dr \right)^{\frac{q}{q}} \left( \int_0^1 M_s^p(r, f) (1-r)^{-\frac{p}{s}} dr \right)^{\frac{\beta}{q}},
 \end{aligned} \tag{2.4}$$

where  $s = \frac{2\beta}{2-q}$ . Our hypothesis  $q > \frac{2p}{2+p}$  allows us to choose  $\frac{(2-q)p}{2} < \beta < q$  which guarantees  $s > p$ . Now [4, Theorems 6 and 7] imply

$$\int_0^1 M_s^p(r, f) (1-r)^{-\frac{p}{s}} dr \asymp \int_0^1 M_s^p(r, f') (1-r)^{p(1-\frac{1}{s})} dr + |f(0)|^p, \quad f \in \mathcal{H}(\mathbb{D}), \tag{2.5}$$

where the right-hand side is comparable to  $\|f\|_{H^p}^p$  for all  $f \in \mathcal{U}$  by (1.6). Therefore (2.4), (2.5), (1.6) and (1.3) give  $I_{p,q}(f) \lesssim \|f\|_{H^p}^p$  for all  $f \in \mathcal{S}$ . An application of this to  $(f - f(0))/f'(0)$  gives the assertion for  $f \in \mathcal{U}$ .

Assume now that  $2 \leq q < \infty$ . We offer two different proofs of which the first one is valid for all  $0 < p < \infty$  and the second one only for  $q \leq p$ . The importance of the second proof lies in the fact that it allows us to characterize univalent functions in  $H^p$  in terms of the Laplacian of  $|f'|^q$ . The first proof is pretty straightforward and reads as follows. It is well-known [2, Chapter 5] that for each fixed  $0 < p \leq \infty$  we have

$$M_p(r, f') \lesssim \frac{M_p(\frac{1+r}{2}, f)}{1-r}, \quad 0 < r < 1, \quad f \in \mathcal{H}(\mathbb{D}). \tag{2.6}$$

By applying (2.6), with  $p = \infty$ , and (2.3) we deduce

$$\begin{aligned}
 I_{p,q}(f) &\asymp \int_{\frac{1}{2}}^1 M_q^p(r, f') (1-r)^{p(1-\frac{1}{q})} dr \\
 &\lesssim \int_{\frac{1}{2}}^1 (M_\infty(r, f'))^{(q-2)\frac{p}{q}} \left( \int_0^{2\pi} |f'(re^{i\theta})|^2 d\theta \right)^{\frac{p}{q}} (1-r)^{p(1-\frac{1}{q})} dr \\
 &\lesssim \int_0^1 \frac{(M_\infty(\frac{1+r}{2}, f))^{(q-2)\frac{p}{q}}}{(1-r)^{(q-2)\frac{p}{q}}} \left( \frac{M_\infty^2(r, f)}{1-r} \right)^{\frac{p}{q}} (1-r)^{p(1-\frac{1}{q})} dr \\
 &\lesssim \int_0^1 M_\infty^p\left(\frac{1+r}{2}, f\right) dr \lesssim \int_0^1 M_\infty^p(r, f) dr, \quad f \in \mathcal{S}.
 \end{aligned}$$

This together with (1.3) yields  $I_{p,q}(f) + |f(0)|^p \lesssim \|f\|_{H^p}^p$  for all  $f \in \mathcal{U}$ .

The first step towards the second proof is to estimate  $I_{p,q}(f)$  upwards for all  $f \in \mathcal{H}(\mathbb{D})$ , and it is valid on the range  $0 < \frac{p}{p+1} < q \leq p < \infty$ , that is,  $0 < p(1 - \frac{1}{q}) + 1 \leq p$ . An integration

by parts and Hölder's inequality show that

$$\begin{aligned} I_{p,q}(f) &\lesssim \int_0^1 \frac{\partial}{\partial r} M_q^p(r, f') (1-r)^{p(1-\frac{1}{q})+1} dr + |f'(0)|^p \\ &= \frac{p}{q} \int_0^1 M_q^{p-q}(r, f') \left( \frac{\partial}{\partial r} M_q^q(r, f') \right) (1-r)^{p(1-\frac{1}{q})+1} dr + |f'(0)|^p \\ &\lesssim (I_{p,q}(f))^{\frac{p-q}{p}} \left( \int_0^1 \left( \frac{\partial}{\partial r} M_q^q(r, f') \right)^{\frac{p}{q}} (1-r)^p dr \right)^{\frac{q}{p}} + |f'(0)|^p, \quad f \in \mathcal{H}(\mathbb{D}), \end{aligned}$$

and it follows that

$$I_{p,q}(f) \lesssim \int_0^1 \left( \frac{\partial}{\partial r} M_q^q(r, f') \right)^{\frac{p}{q}} (1-r)^p dr + |f'(0)|^p, \quad f \in \mathcal{H}(\mathbb{D}), \quad (2.7)$$

provided  $0 < p(1 - \frac{1}{q}) + 1 < p$ . The next step is to estimate  $\frac{\partial}{\partial r} M_q^q(r, f')$  for  $f \in \mathcal{U}$ , and this is done in three separate cases. If  $q \geq 4$ , then (2.6), with  $p = \infty$ , yields

$$\begin{aligned} 2\pi r \frac{\partial}{\partial r} M_q^q(r, f') &= q^2 \int_{D(0,r)} |f'(z)|^{q-2} |f''(z)|^2 dA(z) \\ &\leq q^2 M_\infty^2(r, f'') M_\infty^{q-4}(r, f') \int_{D(0,r)} |f'(z)|^2 dA(z) \\ &\lesssim \frac{M_\infty^{q-2}(\frac{1+r}{2}, f')}{(1-r)^2} M_\infty^2(f, r) \lesssim \frac{M_\infty^q(\frac{3+r}{4}, f)}{(1-r)^q}, \quad 0 < r < 1, \quad f \in \mathcal{U}. \end{aligned}$$

If  $2 < q < 4$ , then Hölder's inequality and (2.6), first with  $p = \frac{4}{4-q}$  and then for  $p = \infty$ , yield

$$\begin{aligned} 2\pi r \frac{\partial}{\partial r} M_q^q(r, f') &\leq q^2 \left( \int_{D(0,r)} |f'(z)|^2 dA(z) \right)^{\frac{q-2}{2}} \left( \int_{D(0,r)} |f''(z)|^{\frac{4}{4-q}} dA(z) \right)^{\frac{4-q}{2}} \\ &\lesssim M_\infty^{q-2}(r, f) \left( \int_0^r \frac{M_{\frac{4-q}{4-q}}^{\frac{4}{4-q}}(\frac{1+s}{2}, f')}{(1-s)^{\frac{4}{4-q}}} ds \right)^{\frac{4-q}{2}} \\ &\lesssim \frac{M_\infty^{q-2}(r, f)}{(1-r)^2} \left( M_\infty \left( \frac{1+r}{2}, f' \right) \right)^{\frac{4-q}{2} \left( \frac{4}{4-q} - 2 \right)} M_\infty^{4-q} \left( \frac{1+r}{2}, f \right) \\ &\leq \frac{M_\infty^2(\frac{1+r}{2}, f)}{(1-r)^2} M_\infty^{q-2} \left( \frac{1+r}{2}, f' \right) \\ &\lesssim \frac{M_\infty^q(\frac{3+r}{4}, f)}{(1-r)^q}, \quad 0 < r < 1, \quad f \in \mathcal{U}. \end{aligned}$$

In the case  $q = 2$ , (2.6), with  $p = 2$ , gives

$$\begin{aligned} 2\pi r \frac{\partial}{\partial r} M_2^2(r, f') &= 4 \int_{D(0,r)} |f''(z)|^2 dA(z) \lesssim \int_0^r \frac{M_2^2(\frac{1+s}{2}, f')}{(1-s)^2} ds \\ &\lesssim \frac{M_\infty^2(\frac{1+r}{2}, f)}{(1-r)^2}, \quad 0 < r < 1, \quad f \in \mathcal{U}. \end{aligned}$$

Therefore we have shown that, for each fixed  $2 \leq q < \infty$ , we have

$$2\pi r \frac{\partial}{\partial r} M_q^q(r, f') = \int_{D(0,r)} \Delta |f'|^q(z) dA(z) \lesssim \frac{M_\infty^q(\frac{3+r}{4}, f)}{(1-r)^q}, \quad 0 < r < 1, \quad f \in \mathcal{U}. \quad (2.8)$$

This estimate together with (2.7) and (1.3) yields

$$\begin{aligned} I_{p,q}(f) &\lesssim \int_{\frac{1}{2}}^1 \left( \frac{\partial}{\partial r} M_q^q(r, f') \right)^{\frac{p}{q}} (1-r)^p dr + |f'(0)|^p \lesssim \int_0^1 M_\infty^p \left( \frac{3+r}{4}, f \right) dr + |f'(0)|^p \\ &\lesssim \int_0^1 M_\infty^p(r, f) dr + |f'(0)|^p \lesssim \|f\|_{H^p}^p, \quad f \in \mathcal{U}. \end{aligned}$$

This finishes the proof of (1.6) for  $q \geq 2$ .

It remains to prove the assertion for close-to-convex functions. Since we have already shown that  $\|f\|_{H^p}^p \lesssim I_{p,q}(f) + |f(0)|^p$  for all  $f \in \mathcal{U}$ , and this is valid for all  $0 < p, q < \infty$ , it remains to estimate  $I_{p,q}(f)$  upwards to  $\|f\|_{H^p}^p$ . We claim that for  $1 \leq q < \infty$ ,  $\varepsilon > 0$  and for all close-to-convex functions  $f$  we have

$$M_q^q(r, f') \lesssim \frac{1}{(1-r)^\varepsilon} \int_0^r \frac{M_\infty^q(t, f)}{(1-t)^{q-\varepsilon}} dt, \quad \frac{1}{2} \leq r < 1, \quad (2.9)$$

the proof of which is postponed for a moment. Let  $\frac{p-q}{q} < x < p$ , and pick up  $\varepsilon = \varepsilon(p, q) > 0$  such that  $x < p(1 - \frac{\varepsilon}{q})$ . Then (2.9), Hölder's inequality and Fubini's theorem yield

$$\begin{aligned} I_{p,q}(f) &\lesssim \int_{\frac{1}{2}}^1 \left( \int_0^r \frac{M_\infty^q(t, f)}{(1-t)^{q-\varepsilon-\frac{qx}{p}}} (1-t)^{-\frac{qx}{p}} dt \right)^{\frac{p}{q}} (1-r)^{p(1-\frac{1+\varepsilon}{q})} dr \\ &\leq \int_0^1 \frac{M_\infty^p(t, f)}{(1-t)^{p-\frac{\varepsilon p}{q}-x}} \left( \int_t^1 \left( \int_0^r \frac{ds}{(1-s)^{\frac{qx}{p-q}}} \right)^{\frac{p-q}{q}} (1-r)^{p(1-\frac{1+\varepsilon}{q})} dr \right) dt \\ &= \int_0^1 \frac{M_\infty^p(t, f)}{(1-t)^{p-\frac{\varepsilon p}{q}-x}} \left( \int_t^1 \frac{dr}{(1-r)^{x+1-p+\frac{p\varepsilon}{q}}} \right) dt \lesssim \int_0^1 M_\infty^p(t, f) dt, \end{aligned}$$

and we are done by (1.3).

It remains to prove (2.9). If  $2 \leq q < \infty$ , this, with  $\varepsilon = 0$ , follows from the estimate (2.8) by integrating. Namely, [12, Corollary 1.6] shows that  $|f'(\rho\xi)| = |f'(r\xi)|$  for all  $\xi$  on the boundary of  $\mathbb{D}$  and  $0 \leq r \leq \rho < 1$  such that  $1-r = 1-\rho$ , provided  $f \in \mathcal{U}$ , and hence  $M_q(r, f') = M_q(\rho, f')$ . This together with (2.8) and (2.6) imply

$$\begin{aligned} M_q^q \left( \frac{3+r}{4}, f' \right) &= M_q^q(r, f') \lesssim \int_0^r \frac{M_\infty^q \left( \frac{3+t}{4}, f \right)}{(1-t)^q} dt + |f'(0)|^q \\ &\lesssim \int_{\frac{3}{4}}^{\frac{3+r}{4}} \frac{M_\infty^q(t, f)}{(1-t)^q} dt + M_q^q \left( \frac{1}{2}, f \right) \lesssim \int_0^{\frac{3+r}{4}} \frac{M_\infty^q(t, f)}{(1-t)^q} dt, \quad 0 < r < 1. \end{aligned}$$

Further, by the proofs of [14, Theorems 2 and 3] we have (2.9) for  $q = 1$  with  $\varepsilon = 0$ . Since the right-hand side of (2.9) is increasing, it remains to consider the case  $1 < q < 2$ . To do this, we use ideas from [14]. Since  $f$  is close-to-convex, Alexander's theorem [3, Theorem 2.12] implies that there exists a starlike function  $g$  such that  $\Re \frac{zf'(z)}{g(z)} > 0$  for all  $z \in \mathbb{D}$ . Write  $h(z) = \frac{zf'(z)}{g(z)}$  for all  $z \in \mathbb{D}$ . Then

$$\begin{aligned} r^q \int_0^{2\pi} |f'(re^{i\theta})|^q d\theta &= \int_0^{2\pi} |g(re^{i\theta})h(re^{i\theta})|^q d\theta \\ &\lesssim \int_0^{2\pi} \left( \int_0^r |g'(te^{i\theta})h(te^{i\theta})| dt \right)^q d\theta + \int_0^{2\pi} \left( \int_0^r |g(te^{i\theta})h'(te^{i\theta})| dt \right)^q d\theta \\ &= I_1(r) + I_2(r). \end{aligned}$$

Since  $g$  is starlike, there exists  $\varphi$  such that  $\Re \varphi > 0$  and  $zg'(z) = g(z)\varphi(z)$  for all  $z \in \mathbb{D}$ . Hence

$$I_1(r) = \int_0^{2\pi} \left( \int_0^r |g'(te^{i\theta})h(te^{i\theta})| dt \right)^q d\theta = \int_0^{2\pi} \left( \int_0^r |f'(te^{i\theta})\varphi(te^{i\theta})| dt \right)^q d\theta.$$

Let  $x = \frac{\varepsilon+q-1}{q} > 0$ . Observe that  $|\widehat{\varphi}(n)| \lesssim 1$  because  $\Re\varphi > 0$ . This together with Hölder's inequality, Fubini's theorem, (2.3), (2.6) and Parseval's identity yields

$$\begin{aligned}
I_1(r) &= \int_0^{2\pi} \left( \int_0^r |f'(te^{i\theta})\varphi(te^{i\theta})|(1-t)^x(1-t)^{-x} dt \right)^q d\theta \\
&\lesssim \frac{1}{(1-r)^\varepsilon} \int_0^r \left( \int_0^{2\pi} |f'(te^{i\theta})\varphi(te^{i\theta})|^q d\theta \right) (1-t)^{q-1+\varepsilon} dt \\
&\lesssim \frac{1}{(1-r)^\varepsilon} \int_0^r M_\infty^{q-1}(t, f') M_\infty^{q-1}(t, \varphi) \\
&\quad \cdot \left( \int_0^{2\pi} |f'(te^{i\theta})|^2 d\theta \right)^{\frac{1}{2}} \left( \int_0^{2\pi} |\varphi(te^{i\theta})|^2 d\theta \right)^{\frac{1}{2}} (1-t)^{q-1+\varepsilon} dt \\
&\lesssim \frac{1}{(1-r)^\varepsilon} \int_0^r \frac{M_\infty^{q-1}(\frac{1+t}{2}, f)}{(1-t)^{q-1}} \left( \sum_n |\widehat{\varphi}(n)| t^n \right)^{q-1} \\
&\quad \cdot \left( \frac{M_\infty^2(t, f)}{1-t} \right)^{\frac{1}{2}} \left( \sum_n |\widehat{\varphi}(n)|^2 t^{2n} \right)^{\frac{1}{2}} (1-t)^{q-1+\varepsilon} dt \\
&\lesssim \frac{1}{(1-r)^\varepsilon} \int_0^r \frac{M_\infty^q(\frac{1+t}{2}, f)}{(1-t)^{q-\varepsilon}} dt \lesssim \frac{1}{(1-r)^\varepsilon} \int_0^{\frac{1+r}{2}} \frac{M_\infty^q(t, f)}{(1-t)^{q-\varepsilon}} dt.
\end{aligned}$$

Since  $\Re h > 0$ , there exists an increasing function  $\mu$  such that

$$h(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 + ze^{-i\alpha}}{1 - ze^{-i\alpha}} d\mu(\alpha), \quad \frac{1}{2\pi} \int_0^{2\pi} d\mu(\alpha) = 1.$$

Hence

$$h'(z) = \frac{1}{\pi} \int_0^{2\pi} \frac{e^{-i\alpha}}{(1 - ze^{-i\alpha})^2} d\mu(\alpha).$$

By using [14, Lemma 2] and (2.6) it follows that

$$\begin{aligned}
I_2(r) &\lesssim \frac{1}{(1-r)^\varepsilon} \int_0^r \left( \int_0^{2\pi} |g(te^{i\theta})h'(te^{i\theta})|^q d\theta \right) (1-t)^{q-1+\varepsilon} dt \\
&\leq \frac{1}{(1-r)^\varepsilon} \int_0^r \left( \int_0^{2\pi} \left( \int_0^{2\pi} \frac{1-t^2}{|1-te^{i\theta}e^{-i\alpha}|^2} d\mu(\alpha) \frac{|g(te^{i\theta})|}{1-t^2} \right)^q d\theta \right) (1-t)^{q-1+\varepsilon} dt \\
&\leq \frac{1}{(1-r)^\varepsilon} \int_0^r \left( \int_0^{2\pi} (\Re h(te^{i\theta})|g(te^{i\theta})|)^q d\theta \right) (1-t)^{\varepsilon-1} dt \\
&\lesssim \frac{1}{(1-r)^\varepsilon} \int_0^r M_\infty^{q-1}(t, f') \left( \int_0^{2\pi} \Re h(te^{i\theta})|g(te^{i\theta})| d\theta \right) (1-t)^{\varepsilon-1} dt \\
&\lesssim \frac{1}{(1-r)^\varepsilon} \int_0^r \frac{M_\infty^{q-1}(\frac{1+t}{2}, f)}{(1-t)^{q-1}} M_\infty(r, f) (1-t)^{\varepsilon-1} dt \lesssim \frac{1}{(1-r)^\varepsilon} \int_0^{\frac{1+r}{2}} \frac{M_\infty^q(t, f)}{(1-t)^{q-\varepsilon}} dt.
\end{aligned}$$

Therefore we have shown that

$$r^q M_q^q(r, f') \lesssim I_1(r) + I_2(r) \lesssim \frac{1}{(1-r)^\varepsilon} \int_0^{\frac{1+r}{2}} \frac{M_\infty^q(t, f)}{(1-t)^{q-\varepsilon}} dt, \quad 0 < r < 1,$$

provided  $1 < q < 2$ . This together with [12, Corollary 1.6] yields (2.9).

### 3. PROOF OF COROLLARY 2

If the integral  $\int_0^1 \omega(r)r dr$  vanishes or diverges then there is nothing to prove, so assume  $\int_0^1 \omega(r)r dr \in (0, \infty)$ . Then there exists  $R = R(\omega) \in (0, 1)$  such that  $\int_R^1 \omega(r)r dr > 0$ . By (1.6)

and Theorem 1 we have

$$\begin{aligned}
 \|f\|_{A_{\omega}^p}^p &\asymp \int_R^1 \|f_r\|_{H^p}^p \omega(r) r \, dr \asymp \int_R^1 \left( \int_0^1 M_q^p(s, f'_r) (1-s)^{p(1-\frac{1}{q})} r^p \, ds + |f_r(0)|^p \right) \omega(r) r \, dr \\
 &= \int_R^1 \left( \int_0^r M_q^p(t, f') \left(1 - \frac{t}{r}\right)^{p(1-\frac{1}{q})} r^{p-1} \, dt + |f(0)|^p \right) \omega(r) r \, dr \\
 &\asymp \int_R^1 \left( \int_0^r M_q^p(t, f') (r-t)^{p(1-\frac{1}{q})} \, dt \right) \omega(r) r \, dr + |f(0)|^p \int_0^1 \omega(r) r \, dr, \quad f \in \mathcal{U}.
 \end{aligned}$$

Moreover, Fubini's theorem yields

$$\begin{aligned}
 &\int_0^1 M_q^p(t, f') (1-t)^{p(1-\frac{1}{q})} \left( \int_t^1 \omega(r) r \, dr \right) dt \\
 &= \int_0^1 \left( \int_0^r M_q^p(t, f') (1-t)^{p(1-\frac{1}{q})} \, dt \right) \omega(r) r \, dr \\
 &\asymp \int_R^1 \left( \int_0^r M_q^p(t, f') (1-t)^{p(1-\frac{1}{q})} \, dt \right) \omega(r) r \, dr.
 \end{aligned}$$

Therefore it suffices to show that

$$\int_0^r M_q^p(t, f') (1-t)^\alpha \, dt \asymp \int_0^r M_q^p(t, f') (r-t)^\alpha \, dt, \quad R \leq r < 1, \quad (3.1)$$

where  $\alpha = \alpha(p, q) = p \left(1 - \frac{1}{q}\right)$ . To see this, fix  $M = M(\omega) > 0$  such that  $M > 1/R$ . If  $t \leq \frac{Mr-1}{M-1} \in (0, r)$ , then  $1-t \leq M(r-t) \leq M(1-t)$ . Moreover, [12, Corollary 1.6] yields

$$\int_{\frac{Mr-1}{M-1}}^r M_q^p(t, f') (1-t)^\alpha \, dt \asymp M_q^p(r, f') (1-r)^{\alpha+1} \asymp \int_{\frac{Mr-1}{M-1}}^r M_q^p(t, f') (r-t)^\alpha \, dt,$$

from which (3.1) follows.

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