The split Casimir operator and solutions of the Yang-Baxter equation for the osp(M|N) and $s\ell(M|N)$ Lie superalgebras, higher Casimir operators, and the Vogel parameters

A. P. Isaev a,b , A. A. Provorov a,c

^a Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna, Moscow region, Russia

^b Faculty of Physics, M. V. Lomonosov Moscow State University, Moscow, Russia

^c Moscow Institute of Physics and Technology (National Research University), Dolgoprudny, Moscow Region, Russia

isaevap@theor.jinr.ru, aleksanderprovorov@gmail.com

Abstract

We find the characteristic identities for the split Casimir operator in the defining and adjoint representations of the osp(M|N) and $s\ell(M|N)$ Lie superalgebras. These identities are used to build the projectors onto invariant subspaces of the representation $T^{\otimes 2}$ of the osp(M|N) and $s\ell(M|N)$ Lie superalgebras in the cases when T is the defining and adjoint representations. For defining representations, the osp(M|N)- and $s\ell(M|N)$ -invariant solutions of the Yang-Baxter equation are expressed as rational functions of the split Casimir operator. For the adjoint representation, the characteristic identities and invariant projectors obtained are considered from the viewpoint of a universal description of Lie superalgebras by means of the Vogel parametrization. We also construct a universal generating function for higher Casimir operators of the osp(M|N) and $s\ell(M|N)$ Lie superalgebras in the adjoint representation.

1 Introduction

It is known that the split Casimir operator \widehat{C} (see definition in Sec. 2; also see [1]) plays an important role in the description of Lie algebras and superalgebras as well as in the study of their representations. Furthermore, the operator \widehat{C} is used for constructing solutions of the semiclassical and quantum Yang-Baxter equations that are invariant under the action of Lie algebras and superalgebras in various representations (see, e.g., [2], [3]).

In the present paper, we use the operator \widehat{C} to construct a system of projectors onto invariant subspaces of the representations $T \otimes T$ of the complex Lie superalgebras osp(M|N) and $s\ell(M|N)$ in the cases when $T = T_f$ is the defining representation and when $T = \operatorname{ad}$ is the adjoint representation.

The idea to construct projectors onto invariant supspaces of representations of Lie algebras and superalgebras by means of invariant operators is not new. For example, the invariant projectors that act on the tensor product of the $s\ell(N)$ Lie algebra defining representations are called the Young symmetrizers and are constructed as images of specific elements of the group algebra $\mathbb{C}[S_r]$ of the symmetric group S_r . The algebra $\mathbb{C}[S_r]$ centralises the action of the algebra sl(N) in the representation $T_f^{\otimes r}$. For the so(N) and sp(N) Lie algebras (where N=2n is even) there exists an analogous statement: the action of those algebras in the representation $T_f^{\otimes r}$ is centralized by the Brauer algebra $B_r(N)$ (see e.g. [26]). The aforementioned properties of the $s\ell(N)$, so(N), and sp(N) Lie algebras are carried over to the case of Lie superalgebras: in [4] and [5] the method of describing subrepresentations of $T_f^{\otimes r}$ by means of the Young symmetrizers was generalized to encompass the $s\ell(M|N)$ Lie superalgebras, and in [6] an analogous result was obtained for the osp(M|N) Lie superalgebras. In our work, we consider a decomposition of the representation $T\otimes T$ into subrepresentations by using the operator \widehat{C} , that is defined uniformly for all Lie superalgebras with the non-degenerate Cartan-Killing metric. Within this approach the $s\ell(M|N)$ - and osp(M|N)-Lie superalgebras are described in a similar fashion.

In the case where T= ad is the adjoint representation, the construction of projectors onto invariant subspaces of the representation $T\otimes T=$ ad \otimes ad by using \widehat{C} has one more significance. It is related to the notion of the Universal Lie algebra, which was introduced by Vogel in [7] (see also [8], [9]). The Universal Lie algebra was supposed to be a model of all complex simple Lie algebras, embracing some Lie superalgebras additionally. For example, many quantities that characterize the Lie algebra \mathfrak{g} in different representations T_{λ} (possibly reducible) that participate in the decomposition $\mathrm{ad}^{\otimes k} = \sum_{\lambda} T_{\lambda}$ where $k \geq 1$ are expressed as rational functions of the three Vogel parameters (see their definition in Sec. 5). These parameters take specific values for all complex simple Lie algebras as well as for all basic classical Lie superalgebras (see, e.g., [10], and Sec.5 below). In particular, it was shown for Lie algebras that using the Vogel parameters one can express the dimensions of the representations T_{λ} , when k=2,3 [7], the dimensions of the ad-series representations, i.e., the representations $T_{\lambda'}$ with the highest weight $\lambda'=k\lambda_{\mathrm{ad}}$ where λ_{ad} is the highest weight of the given Lie algebra [11], the dimensions of the representations of X_2 -series [12] as well as the values of higher Casimir operators in the adjoint representation of the given Lie algebra [10]. Furthermore, in [13], [14] it was shown that the universal description of complex simple Lie algebras allows formulating some types of knot polynomials via characters simultaneously for all types of quantum simple Lie groups.

The paper is organized as follows. In Section 2, we recall the main notions of Lie superalgebra theory and introduce some necessary conventions that we use throughout the work. Sections 3 and 4 are dedicated to calculating the characteristic identity for the split Casimir operator in the defining T_f and adjoint ad representations of the osp(M|N) and $s\ell(M|N)$ Lie superalgebras and to constructing projectors onto invariant subspaces of the representations $T_f^{\otimes 2}$ and $ad^{\otimes 2}$. We also show that the results obtained are in full correspondence with the conclusions of [15] and [16] where analogous calculations were carried out for Lie algebras. In Section 5, we write characteristic identities for the symmetric part of the split Casimir operator as well as the corresponding projectors onto symmetric invariant subspaces, uniformly (in a universal way) for both the osp(M|N) and $s\ell(M|N)$ Lie algebras by using the Vogel parameters. In Section 6, following the approach of [1] and [10], we find a universal form of the generating function of the higher Casimir operators of the osp(M|N) and $s\ell(M|N)$ Lie superalgebras in the adjoint representation.

2 General information on Lie superalgebras

In this section, we briefly discuss the main definitions and conventions from the theory of Lie superalgebras(see, e.g., [17], [18]) and introduce the notation to be used in what follows.

2.1 Lie superalgebras and associative algebras

A linear superspace (or \mathbb{Z}_2 -graded space) over the field \mathbb{C} is a linear space $V = V_{\overline{0}} \oplus V_{\overline{1}}$, which is a direct sum of the linear spaces $V_{\overline{0}}$ and $V_{\overline{1}}$ over the field \mathbb{C} . The spaces $V_{\overline{0}}$ and $V_{\overline{1}}$ are called even and odd, respectively. The vectors from V that lie in the even subspace $V_{\overline{0}}$ are called even, and those lying in the odd space $V_{\overline{1}}$ are called odd. Those vectors that are either even or odd are called homogeneous. The grading of an arbitrary homogeneous vector $v \in V$ is denoted by $\deg(v) \equiv [v] \in \mathbb{Z}_2$, i.e., $[v] = 0, 1 \mod (2)$. The linear superspace $V = V_{\overline{0}} \oplus V_{\overline{1}}$, where $\dim V_{\overline{0}} = M$ and $\dim V_{\overline{1}} = N$, will be written as $V_{(M|N)}$. The superdimension of the space $V_{(M|N)}$ is defined by $\mathrm{sdim}(V_{(M|N)}) \equiv M - N$. Throughout the rest of this paper we always assume the basis $\{e_a\}_{a=1}^{M+N}$ of $V_{(M|N)}$ to be homogeneous, with the first M of its elements being even and the last N of them being odd. The grading of the basis element e_a will be written as [a]. Note that in our convention the grading is carried by the basis vectors of the space $V_{(M|N)}$. For example, there is another (but equivalent) convention whereby the grading is carried by the coordinates (see, e.g., [19] and [20]).

Let \mathfrak{g} be a Lie superalgebra over the field \mathbb{C} with a Lie superbracket $[\ ,\]: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. For arbitrary homogeneous vectors $X, Y, Z \in \mathfrak{g}$ the following two properties must be satisfied (see, e.g., [17]):

$$[X,Y] \in \mathfrak{g}_{\overline{[X]+[Y]}}, \qquad [X,Y] = -(-1)^{[X][Y]}[Y,X],$$
 (2.1)

$$(-1)^{[X][Z]}[X, [Y, Z]] + (-1)^{[Y][X]}[Y, [Z, X]] + (-1)^{[Z][Y]}[Z, [X, Y]] = 0,$$
(2.2)

Let $\{X_i\}$ $(i = 1, ..., \dim \mathfrak{g})$ be a homogeneous basis of \mathfrak{g} . Then

$$[X_i, X_j] = X_k X_{ij}^k, (2.3)$$

where the numbers $X^k_{\ ij}$ are the structure constants of the algebra \mathfrak{g} . Clearly, $X^k_{\ ij} = 0$ as long as ([i] + [j] + [k]) mod $(2) \neq 0$, and $X^k_{\ ij} = -(-1)^{[i][j]}X^k_{\ ji}$.

Consider an associative superalgebra $\mathcal{A} = \mathcal{A}_{\overline{0}} \oplus \mathcal{A}_{\overline{1}}$. For any two homogeneous elements $A, B \in \mathcal{A}$ we can define the bracket $[\ ,\]$ as follows:

$$[A, B] := AB - (-1)^{[A][B]} BA.$$
(2.4)

It is easy to check that this bracket satisfies (2.1) and (2.2). Thus, the algebra \mathcal{A} can be viewed as a Lie superalgebra with respect to the bracket (2.4). Following [17], we denote this Lie superalgebra by $(\mathcal{A})_L$.

A representation of the Lie superalgebra \mathfrak{g} is a homomorphism $T:\mathfrak{g}\to (\operatorname{End}(V_{(M|N)}))_L$, such that for any $X,Y\in\mathfrak{g}$ we must have

$$T([X,Y]) = [T(X), T(Y)],$$
 (2.5)

where the Lie superbracket in the right hand-side of (2.5) is defined in (2.4). In this paper, the key role is played by the adjoint representation ad : $\mathfrak{g} \to (\operatorname{End}(\mathfrak{g}))_L$, which is defined by the formula

$$ad(X) \cdot Y = [X, Y] \tag{2.6}$$

for arbitrary vectors $X, Y \in \mathfrak{g} \equiv V_{\text{ad}}$. From (2.3) and (2.6) it follows that the entries of the matrix of the operators $\text{ad}(X_i)$ in the homogeneous basis $\{X_i\}$ are equal to the structure constants of \mathfrak{g} :

$$\operatorname{ad}(X_i)^k{}_j = X^k{}_{ij}. (2.7)$$

Consider a linear superspace $V_{(M|N)}$ and an operator $A:V_{(M|N)}\to V_{(M|N)}$ with the matrix $||A^a{}_b||$ in some homogeneous basis $\{e_a\}_{a=1}^{M+N}$ of $V_{(M|N)}$. Recall that the supertrace of A is the quantity str $A=(-1)^{[a]}A^a{}_a$, which has the following important property:

$$str([A, B]) = 0 (2.8)$$

for any A and B acting in $V_{(M|N)}$.

The Cartan-Killing metric g of the Lie superalgebra g is defined in the standard way:

$$g_{ij} = \text{str}(\text{ad}(X_i) \text{ ad}(X_j)) = (-1)^{[m]} X^m_{ik} X^k_{jm}.$$
 (2.9)

Note that g has the following properties:

$$g(X,Y) = (-1)^{[X][Y]}g(Y,X) \qquad \forall X \in \mathfrak{g}_{\overline{\alpha}}, \forall Y \in \mathfrak{g}_{\overline{\beta}}, \ \overline{\alpha}, \overline{\beta} \in \mathbb{Z}_2, \tag{2.10}$$

$$g([X,Y],Z) = g(X,[Y,Z]) \qquad \forall X,Y,Z \in \mathfrak{g}, \tag{2.11}$$

and

$$\mathbf{g}_{ij} = (-1)^{[i][j]} \mathbf{g}_{ji} = (-1)^{[i]} \mathbf{g}_{ji} = (-1)^{[j]} \mathbf{g}_{ji}, \tag{2.12}$$

$$g_{ij} = 0$$
, if $[i] + [j] \neq 0 \mod (2)$. (2.13)

In the case of the nondegenerate Cartan-Killing metric, we also introduce the inverse Cartan-Killing metric with the components $\overline{\mathbf{g}}^{ij}$ given by the relations

$$\overline{g}^{ij}g_{jk} = \delta^i_k, \qquad g_{ij}\overline{g}^{jk} = \delta^k_i.$$
(2.14)

The metric $\overline{\mathbf{g}}^{ij}$ has the same properties (2.12) with respect to index permutations. One can use the metrics \mathbf{g}_{ij} and $\overline{\mathbf{g}}^{ij}$ to lower and raise the indices of the vectors and covectors in \mathfrak{g} . Note that the raising of the indices in \mathbf{g}_{ij} yields $\mathbf{g}^{ij} = \overline{\mathbf{g}}^{ik}\overline{\mathbf{g}}^{jm}\mathbf{g}_{km} = \overline{\mathbf{g}}^{ji}$, so the metric tensor with the upper indices \mathbf{g}^{ij} does not coincide with the inverse matrix $\overline{\mathbf{g}}^{ij}$. From now on we only use $\overline{\mathbf{g}}^{ij}$.

Let us now introduce the structure constants of \mathfrak{g} with the lower indices:

$$X_{kij} \equiv \mathsf{g}_{km} X^m_{\ ij} \ . \tag{2.15}$$

From (2.7), (2.11) and (2.12) we deduce the following properties of X_{ijk} with respect to index permutation:

$$X_{kji} = -(-1)^{[i][j]} X_{kij} , \quad X_{jik} = -(-1)^{[k]+[j]+[k][j]} X_{kij} , \quad X_{ikj} = -(-1)^{[i][k]} X_{kij}.$$
 (2.16)

2.2 The split Casimir operator and comultiplication for Lie superalgebras

Let $\mathcal{U}(\mathfrak{g})$ denote the enveloping algebra of the Lie superalgebra \mathfrak{g} . Consider the quadratic Casimir operator:

$$C_2 = \mathsf{g}^{ij} X_i X_j \in \mathcal{U}(\mathfrak{g}). \tag{2.17}$$

In view of (2.13), the operator C_2 is even and commutes with all generators X_k of $\mathcal{U}(\mathfrak{g})$ with respect to the bracket (2.4). Therefore, C_2 belongs to the centre of $\mathcal{U}(\mathfrak{g})$.

Consider two associative superalgebras \mathcal{A} and \mathcal{B} . The graded tensor product of \mathcal{A} and \mathcal{B} is the associative superalgebra $\mathcal{A} \otimes \mathcal{B}$ that coincides as a linear space with the tensor product of the spaces \mathcal{A} and \mathcal{B} , and the multiplication in $\mathcal{A} \otimes \mathcal{B}$ is defined for arbitrary homogeneous vectors $A, A' \in \mathcal{A}$ and $B, B' \in \mathcal{B}$ as

$$(A \otimes B) \cdot (A' \otimes B') = (-1)^{[A'][B]} A A' \otimes B B'. \tag{2.18}$$

Here and below, by 'tensor product' we will always mean the 'graded tensor product', for which (2.18) holds. The matrix of the operator $A \otimes B$ in the basis $\{e_i \otimes \varepsilon_\alpha\}$ of the space $V \otimes V'$ is given by

$$(A \otimes B)(e_i \otimes \varepsilon_{\alpha}) = \begin{cases} e_k \otimes \varepsilon_{\beta} (A \otimes B)^{k\beta}{}_{i\alpha} \\ (-1)^{[B][i]} (Ae_i) \otimes (B\varepsilon_{\alpha}) = (-1)^{[B][i]} (e_k A^k{}_i) \otimes (\varepsilon_{\beta} B^{\beta}{}_{\alpha}) \end{cases},$$

from where for the case of a homogeneous B we get:

$$(A \otimes B)^{k\beta}{}_{i\alpha} = (-1)^{[B][i]} A^k{}_i B^\beta{}_\alpha. \tag{2.19}$$

The formula (2.19) can be generalized to arbitrary (not necessarily homogeneous) operators $B \in \text{End}(V')$ as

$$(A \otimes B)^{k\beta}{}_{i\alpha} = (-1)^{([\alpha] + [\beta])[i]} A^{k}{}_{i} B^{\beta}{}_{\alpha}. \tag{2.20}$$

Generally, for the matrix of the operator $(A \otimes B \otimes C \otimes \cdots \otimes E)$ acting in $V_1 \otimes V_2 \otimes V_3 \otimes \cdots \otimes V_n$, we have

$$(A \otimes B \otimes C \otimes \cdots \otimes E)_{i_{1}...i_{n}}^{k_{1}...k_{n}} = (A)_{i_{1}}^{k_{1}}(-1)^{[i_{1}]([k_{2}]+[i_{2}])}(B)_{i_{2}}^{k_{2}}(-1)^{([i_{1}]+[i_{2}])([k_{3}]+[i_{3}])}(C)_{i_{3}}^{k_{3}} \cdots \\ \cdots (-1)^{([i_{1}]+[i_{2}]+...+[i_{n-1}])([k_{n}]+[i_{n}])}(E)_{i_{n}}^{k_{n}}.$$

For an arbitrary $A:V\to V$ we define the operators $A_1,A_2:V^{\otimes 2}\to V^{\otimes 2}$ by

$$A_1 \equiv A \otimes I, \qquad A_2 \equiv I \otimes A, \tag{2.21}$$

where $I: V \to V$ is the identity operator. Applying (2.21), we get:

$$A_1B_2 = (A \otimes I)(I \otimes B) = A \otimes B$$
 and $B_2A_1 = (I \otimes B)(A \otimes I) = (-1)^{[A][B]}A \otimes B$,

i.e., generally,

$$A_1B_2 \neq B_2A_1$$
.

The notation introduced above can be generalized to the case of any operator

$$A = \hat{A}^{i_1 \dots i_r}{}_{j_1 \dots j_r} e_{i_1}{}^{j_1} \otimes \dots e_{i_r}{}^{j_r} \in \operatorname{End}(V^{\otimes r}), \qquad (2.22)$$

where e_i^j are the matrix identities which are operators that act on the space V with the basis $\{e_a\}$ as

$$e_i{}^j \cdot e_a = e_b \left(e_i{}^j \right)^b{}_a = e_i \, \delta^j_a \quad \Leftrightarrow \quad \left(e_i{}^j \right)^b{}_a = \delta^b_i \delta^j_a \,. \tag{2.23}$$

Let s > r and $1 \le \alpha_1 < \cdots < \alpha_r \le s$. Define $A_{\alpha_1 \dots \alpha_r} \in \text{End}(V^{\otimes s})$ as

$$A_{\alpha_1...\alpha_r} = \hat{A}^{i_1...i_r}{}_{j_1...j_r} \underbrace{I \otimes \cdots \otimes I \otimes e_{i_1}{}^{j_1} \otimes I \otimes \cdots \otimes I \otimes e_{i_r}{}^{j_r} \otimes I \otimes \cdots \otimes I}_{s}, \tag{2.24}$$

where each of the matrix identities $e_{i_k}{}^{j_k}$ $(k=1,\ldots r)$ stands at the α_k -th place in the tensor product in the right-hand side of (2.24), while at all the other places there are identity operators $I:V\to V$. For instance, given $A=\hat{A}^{ij}{}_{km}e_i{}^k\otimes e_i{}^m$, the operators $A_{13},A_{12},A_{23}:V^{\otimes 4}\to V^{\otimes 4}$ are defined by

$$A_{13} = \hat{A}^{ij}{}_{km} e_i{}^k \otimes I \otimes e_j{}^m \otimes I , \quad A_{12} = \hat{A}^{ij}{}_{km} (e_i{}^k \otimes e_j{}^m \otimes I \otimes I) ,$$

$$A_{23} = \hat{A}^{ij}{}_{km} (I \otimes e_i{}^k \otimes e_j{}^m \otimes I) .$$

$$(2.25)$$

In what follows we will need the superpermutation operator

$$\mathcal{P} = (-1)^{[j]} e_i{}^j \otimes e_j{}^i \qquad \Longrightarrow \qquad \mathcal{P}^{k_1 k_2}{}_{m_1 m_2} = (-1)^{[k_1][k_2]} \delta_{m_2}^{k_1} \delta_{m_2}^{k_2}. \tag{2.26}$$

Note that the operators $\mathcal{P}_{\alpha,\alpha+1}$ ($\alpha=1,\ldots,s-1$) given in accordance with (2.24) define the representation $\tau: S_s \to \operatorname{End}(V^{\otimes s})$ of the symmetric group S_s with generators σ_{α} :

$$\sigma_{\alpha}\sigma_{\alpha+1}\sigma_{\alpha} = \sigma_{\alpha+1}\sigma_{\alpha}\sigma_{\alpha+1} \qquad \forall \alpha = 1, \dots, s-2,$$

$$\sigma_{\alpha}\sigma_{\beta} = \sigma_{\beta}\sigma_{\alpha} \qquad \forall \alpha, \beta = 1, \dots, s-1, \quad |\alpha - \beta| > 1,$$

$$\sigma_{\alpha}^{2} = e \qquad \forall \alpha = 1, \dots, s-1,$$

$$(2.27)$$

where e is the identity of S_s , and

$$\tau(\sigma_{\alpha}) = \mathcal{P}_{\alpha,\alpha+1} , \quad \tau(e) = I^{\otimes s} .$$
(2.28)

Using direct calculations and the definition of $\mathcal{P}_{\alpha,\alpha+1}$, one can prove the following statement.

Proposition 1. Let A be an operator (2.22) acting in $V^{\otimes r}$, and $A_{\alpha_1...\alpha_r}$ be an operator (2.24) that acts on $V^{\otimes s}$ (s > r). If $\alpha_p + 1 < \alpha_{p+1}$ for p < r, or $\alpha_p + 1 \le s$ for p = r, then

$$\mathcal{P}_{\alpha_p,\alpha_p+1}A_{\alpha_1...\alpha_p...\alpha_r}\mathcal{P}_{\alpha_p,\alpha_p+1} = A_{\alpha_1...\alpha_p+1...\alpha_r}.$$
(2.29)

That is, if the conditions are satisfied, the superpermutation $\mathcal{P}_{\alpha_p,\alpha_p+1}$ moves the nontrivial factor at the α_p -th position in $A_{\alpha_1...\alpha_r}$ to the adjacent place α_p+1 on the right, while the identity I standing at the α_p+1 th place in $A_{\alpha_1...\alpha_r}$ is moved to the α_p -th position.

In particular, Statement 1 implies for the operators $A_{\alpha_1\alpha_2}: V^{\otimes 4} \to V^{\otimes 4}$ defined in (2.25) the relations

$$A_{13} = \mathcal{P}_{23} A_{12} \mathcal{P}_{23} = \mathcal{P}_{12} A_{23} \mathcal{P}_{12} \quad \Rightarrow \quad \mathcal{P}_{13} = \mathcal{P}_{23} \mathcal{P}_{12} \mathcal{P}_{23} = \mathcal{P}_{12} \mathcal{P}_{23} \mathcal{P}_{12} , \qquad (2.30)$$

where the chain of equalities on the right is in accordance with (2.27). Besides, it follows from (2.29) that for any $A: V^{\otimes r} \to V^{\otimes r}$ we have

$$A_{\alpha_1...\alpha_r} = \tau(\sigma)A_{1...r}\tau(\sigma)^{-1},\tag{2.31}$$

where $\sigma \in S_s$, $s \ge r$ and $\tau(\sigma)$ is its image in the representation (2.28) constructed as a product of $P_{\alpha,\alpha+1}$. For $\mathcal{U}(\mathfrak{g})$, define a homomorphic map $\Delta : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$. It acts on the generators X_i of $\mathcal{U}(\mathfrak{g})$ by

$$\Delta X_i = X_i \otimes I + I \otimes X_i. \tag{2.32}$$

The map Δ is called the comultiplication of $\mathcal{U}(\mathfrak{g})$. Acting by Δ on the quadratic Casimir operator C_2 (2.17) yields:

$$\Delta(C_2) = C_2 \otimes I + I \otimes C_2 + 2\widehat{C}, \tag{2.33}$$

where $\widehat{C} \in \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ is called the split Casimir operator. Explicitly,

$$\widehat{C} = \overline{\mathbf{g}}^{ij} X_i \otimes X_j. \tag{2.34}$$

The operator \widehat{C} has the property of ad-invariance, i.e., for all generators $X_i \in \mathcal{U}(\mathfrak{g})$ we have, according to (2.33):

$$[\widehat{C}, \Delta X_i] = \frac{1}{2} \Delta([C_2, X_i]) - \frac{1}{2} [C_2, X_i] \otimes I - \frac{1}{2} I \otimes [C_2, X_i] = 0.$$
 (2.35)

It is convenient to use the notion of tensor product of the enveloping superalgebras $\mathcal{U}(\mathfrak{g})$ and comultiplication in $\mathcal{U}(\mathfrak{g})$ to define the tensor product $T \otimes T'$ of the representations $T : \mathfrak{g} \to (\operatorname{End}(V))_L$ and $T' : \mathfrak{g} \to (\operatorname{End}(V'))_L$ of the Lie superalgebra \mathfrak{g} . For an arbitrary homogeneous vector $X \in \mathfrak{g}$ we define $(T \otimes T')(X)$ as

$$(T \otimes T')(X) \equiv (T \otimes T')(\Delta(X)) = T(X) \otimes T'(I) + T(I) \otimes T'(X), \qquad (2.36)$$

so for any homogeneous vectors $v \in V$ and $u \in V'$ we have

$$(T \otimes T')(X) \cdot (v \otimes u) = (T(X) \cdot v) \otimes u + (-1)^{[X][v]} v \otimes (T'(X) \cdot u). \tag{2.37}$$

The map thus defined $(T \otimes T')$: $\mathcal{U}(\mathfrak{g}) \to \operatorname{End}(V \otimes V')$ is indeed homomorphic and therefore is a representation of $\mathcal{U}(\mathfrak{g})$ on $V \otimes V'$.

From (2.35) and (2.36) one can infer that for any representations T and T' the operator $(T \otimes T')(\widehat{C})$ commutes with $(T \otimes T')(X)$ for all $X \in \mathfrak{g}$. Recall that by Schur's lemma (more precisely, by its generalization to the case of Lie superalgebras) for each *irreducible* representation \widetilde{T} of any complex Lie superalgebra \mathcal{A} , an even operator A that commutes with all the elements of \mathcal{A} in the representation \widetilde{T} must be proportional to the identity operator, that is $A = \lambda I$, where $\lambda \in \mathbb{C}$. Thus, if an irreducible representation \widetilde{T} of $\mathcal{U}(\mathfrak{g})$ is contained in $(T \otimes T')$, we have $\widetilde{T}(\widehat{C}) \sim I_{\widetilde{T}}$. The following corollary of Schur's lemma is central for our work: if a representation $T \otimes T$ of a Lie superalgebra \mathfrak{g} in the space $V_T \otimes V_T$ is completely reducible, then $V_T \otimes V_T$ can be expanded as a direct sum of invariant eigenspaces of the operator $(T \otimes T)(\widehat{C})$. From now on, we denote this operator by \widehat{C}_T . If the operator \widehat{C}_T satisfies a characteristic identity

$$(\widehat{C}_T - a_1 I_T^{\otimes 2})(\widehat{C}_T - a_2 I_T^{\otimes 2}) \dots (\widehat{C}_T - a_p I_T^{\otimes 2}) = 0, \tag{2.38}$$

where $I_T^{\otimes 2}$ is the identity operator on $V_T \otimes V_T$, all the complex numbers a_1, a_2, \ldots, a_p are different, and crossing out any of the parentheses on the left of (2.38) breaks the identity, then the numbers a_1, a_2, \ldots, a_p are the eigenvalues of \widehat{C}_T , and a projector onto the eigenspace of \widehat{C}_T corresponding to the eigenvalue a_j is given by the formula

$$P_{j} \equiv P_{a_{j}} = \prod_{\substack{i=1\\i\neq j}}^{p} \frac{\widehat{C}_{T} - a_{i} I_{T}^{\otimes 2}}{a_{j} - a_{i}}.$$
(2.39)

Moreover, $P_k P_j = \delta_{kj} P_j$ and $\sum_{i=1}^p P_i = I_T^{\otimes 2}$. We emphasize that the spaces extracted by the projectors P_j are not necessarily spaces of irreducible representations of \mathfrak{g} , as there may exist other nontrivial invariant operators in $V_T \otimes V_T$ that cannot be expressed as polynomials in \widehat{C}_T . Therefore, the spaces $P_j(V_T \otimes V_T)$ can in principle be further expanded into a direct sum of nontrivial invariant subspaces.

Unlike the case of Lie algebras, reducible representations of simple Lie superalgebras are not always completely reducible. Correspondingly, the Casimir operators in such representations are not always diagonilizable: this situation is present in our paper. The operator \widehat{C}_T is not diagonilizable if and only if it does not satisfy any identity of the form (2.38) with pairwise different a_i . In this case, \widehat{C}_T must satisfy

$$(\widehat{C}_T - a_1 I_T^{\otimes 2})^{k_1} (\widehat{C}_T - a_2 I_T^{\otimes 2})^{k_2} \dots (\widehat{C}_T - a_p I_T^{\otimes 2})^{k_p} = 0, \tag{2.40}$$

where all $k_i \in \mathbb{Z}_{\geq 1}$ are minimal, i.e., subtracting 1 from any of them breaks the identity and, as earlier, a_i are pairwise different. Instead of projectors onto eigenspaces of \widehat{C}_T , we can construct projectors onto its generalised

eigenspaces (weight spaces):1:

$$P_{j} \equiv P_{a_{j}} = I_{T}^{\otimes 2} - \left(I_{T}^{\otimes 2} - \prod_{\substack{i=1\\i \neq j}}^{p} \left(\frac{\widehat{C}_{T} - a_{i}I_{T}^{\otimes 2}}{a_{j} - a_{i}}\right)^{k_{i}}\right)^{k_{j}}.$$
(2.41)

If $k_i = 1$ for some i, then the image of P_i is an eigenspace of \widehat{C}_T . If $k_i > 1$, then P_i projects onto a generalized eigenspace. Note that for $k_1 = k_2 = \cdots = k_p = 1$ (2.40) and (2.41) turn into (2.38) and (2.39), respectively.

2.3 The split Casimir operator for simple complex Lie superalgebras with nondegenerate Cartan-Killing metric

Hereinafter, we consider only those simple Lie superalgebras, for which the Cartan-Killing metric \mathbf{g}_{ab} is nondegenerate, i.e., there must be an inverse metric $\overline{\mathbf{g}}^{ab}$ satisfying (2.14). It also implies that the structure constants X_{ij}^k satisfy the relation

$$str(ad(X_i)) = (-1)^k X_{ik}^k = 0, \quad \forall i.$$
 (2.42)

Using (2.19), (2.7) and (2.34), we can find the components of the split Casimir operator in the adjoint representation with respect to the basis $\{X_a \otimes X_b\}$ of $V_{\rm ad} \otimes V_{\rm ad}$ ($V_{\rm ad}$ coincides with $\mathfrak g$ as a vector space and is defined to be the space of the adjoint representation):

$$(\widehat{C}_{\mathrm{ad}})^{i_1 i_2}{}_{j_1 j_2} = (-1)^{[a_2][j_1]} \overline{\mathsf{g}}^{a_1 a_2} X^{i_1}{}_{a_1 j_1} X^{i_2}{}_{a_2 j_2}. \tag{2.43}$$

Here we need other three ad-invariant operators in $V_{\rm ad} \otimes V_{\rm ad}$ with the components

$$(\mathbf{I})^{i_1 i_2}{}_{j_1 j_2} = \delta^{i_1}_{j_1} \delta^{i_2}_{j_2}, \qquad (\mathbf{P})^{i_1 i_2}{}_{j_1 j_2} = (-1)^{[i_1][i_2]} \delta^{i_1}_{j_2} \delta^{i_2}_{j_1}, \qquad (\mathbf{K})^{i_1 i_2}{}_{j_1 j_2} = \overline{\mathsf{g}}^{i_1 i_2} \mathsf{g}_{j_1 j_2}.$$
 (2.44)

Therefore, **P** is the operator of superpermutation of the two spaces $V_{\rm ad}$. One can easily check that for any $X \in \mathfrak{g}$,

$$[\operatorname{ad}^{\otimes 2}(\Delta X), \mathbf{P}] = 0, \tag{2.45}$$

As **P** is invariant under the adjoint action of \mathfrak{g} , then so are its eigenspaces, which are extracted by the projectors $\frac{1}{2}(\mathbf{I} + \mathbf{P})$ and $\frac{1}{2}(\mathbf{I} - \mathbf{P})$.

Using the component form (2.44) and (2.43) of I, P, K and \widehat{C}_{ad} , one can check the identities

$$\mathbf{P} \cdot \mathbf{P} = \mathbf{P}, \qquad \mathbf{P} \hat{C}_{ad} \mathbf{P} = \hat{C}_{ad}, \qquad \mathbf{P} \cdot \mathbf{K} = \mathbf{K} \cdot \mathbf{P} = \mathbf{K},$$
 (2.46)

$$\mathbf{K} \cdot \mathbf{K} = \operatorname{sdim} \mathfrak{g} \cdot \mathbf{K}. \tag{2.47}$$

Applying the projectors

$$\mathbf{P}_{\pm}^{(\mathrm{ad})} = \frac{1}{2}(\mathbf{I} \pm \mathbf{P}) \tag{2.48}$$

to \widehat{C}_{ad} , we define the symmetric and antisymmetric parts of the split Casimir operator:

$$\widehat{C}_{\pm} = \mathbf{P}_{\pm}^{(\mathrm{ad})} \widehat{C}_{\mathrm{ad}} = \widehat{C}_{\mathrm{ad}} \mathbf{P}_{\pm}^{(\mathrm{ad})}$$
(2.49)

where the last equality is verified by using (2.46). From (2.49) and the relations

$$\mathbf{P}_{+}^{(\mathrm{ad})} + \mathbf{P}_{-}^{(\mathrm{ad})} = \mathbf{I}, \quad \mathbf{P}_{+}^{(\mathrm{ad})} \mathbf{P}_{-}^{(\mathrm{ad})} = \mathbf{P}_{-}^{(\mathrm{ad})} \mathbf{P}_{+}^{(\mathrm{ad})} = 0$$
 (2.50)

that follow from (2.49), we instantly get:

$$\hat{C}_{ad} = \hat{C}_{+} + \hat{C}_{-}, \qquad \hat{C}_{+}\hat{C}_{-} = \hat{C}_{-}\hat{C}_{+} = 0.$$
 (2.51)

Utilizing the symmetry properties (2.16) of the structure constants of \mathfrak{g} with respect to index permutation, the graded Jacobi identity (2.2), and the identity

$$(-1)^{[i_1][i_2]} X^{i_1 i_2}{}_a X^b{}_{i_1 i_2} = -\delta^b_a \quad \Leftrightarrow \quad (-1)^{[i_1][i_2]} X^{i_1 i_2}{}_a X_{i_1 i_2 b} = -\mathsf{g}_{ab} \;, \tag{2.52}$$

¹Here by weight spaces we mean spaces in which \hat{C}_T acts as a Jordan cell with the corresponding eigenvalue

where $X^{i_1i_2}{}_a=\mathsf{g}^{i_2j_2}X^{i_1}{}_{j_2a},$ we get a convenient form of \widehat{C}_- :

$$(\widehat{C}_{-})^{i_1 i_2}{}_{j_1 j_2} = \frac{1}{2} (-1)^{[i_1][i_2]} X^{i_1 i_2}{}_a X^a{}_{j_1 j_2}. \tag{2.53}$$

By (2.53), (2.16), (2.9) and identities (2.52), one can derive the following equality:

$$\hat{C}_{-}^{2} = -\frac{1}{2}\hat{C}_{-}.\tag{2.54}$$

Besides, from (2.53), (2.16), (2.12) we get:

$$\widehat{C}_{ad}\mathbf{K} = \mathbf{K}\widehat{C}_{ad} = -\mathbf{K}, \qquad \widehat{C}_{-}\mathbf{K} = \mathbf{K}\widehat{C}_{-} = 0, \qquad \widehat{C}_{+}\mathbf{K} = \mathbf{K}\widehat{C}_{+} = -\mathbf{K},$$
 (2.55)

where the last equality in (2.55) is a consequence of the other two and of the first relation in (2.51).

Supertraces of the operators \mathbf{I} , \mathbf{P} , \mathbf{K} , $\widehat{C}_{\mathrm{ad}}$, \widehat{C}_{+} , and \widehat{C}_{-} , as well as of some of their powers can be obtained by using (2.42), (2.43), (2.44), (2.53), (2.54):

$$\mathbf{str}(\widehat{C}_{\mathrm{ad}}) = 0, \qquad \mathbf{str}(\widehat{C}_{\pm}) = \pm \frac{1}{2} \operatorname{sdim} \mathfrak{g}, \qquad \mathbf{str}(\widehat{C}_{\mathrm{ad}}^2) = \operatorname{sdim} \mathfrak{g},$$

$$\mathbf{str}(\widehat{C}_{-}^2) = -\frac{1}{2} \mathbf{str}(\widehat{C}_{-}) = \frac{1}{4} \operatorname{sdim} \mathfrak{g}, \qquad \mathbf{str}(\widehat{C}_{+}^2) = \mathbf{str}(\widehat{C}_{\mathrm{ad}}^2 - \widehat{C}_{-}^2) = \frac{3}{4} \operatorname{sdim} \mathfrak{g},$$

$$\mathbf{str}(\widehat{C}_{-}^3) = \frac{1}{4} \mathbf{str}(\widehat{C}_{-}) = -\frac{1}{8} \operatorname{sdim} \mathfrak{g}, \qquad \mathbf{str}(\widehat{C}_{+}^3) = \mathbf{str}(\widehat{C}_{\mathrm{ad}}^3 - \widehat{C}_{-}^3) = -\frac{1}{8} \operatorname{sdim} \mathfrak{g},$$

$$\mathbf{str}(\mathbf{K}) = \operatorname{sdim} \mathfrak{g}, \qquad \mathbf{str}(\mathbf{I}) = (\operatorname{sdim} \mathfrak{g})^2, \qquad \mathbf{str}(\mathbf{P}) = \operatorname{sdim} \mathfrak{g}.$$

$$(2.56)$$

Here $\mathbf{str} = \operatorname{str}_1 \operatorname{str}_2$ is the trace in $V_{\operatorname{ad}} \otimes V_{\operatorname{ad}}$, and the indices 1 and 2 in str_1 and str_2 refer to the tensor components of $V_{\operatorname{ad}} \otimes V_{\operatorname{ad}}$.

3 The osp(M|N) Lie superalgebra

There exist various conventions on how to define the orthosymplectic Lie superalgebras osp(M|N) (see e.g.[17] – [24]), all of which are equivalent. In this section, we fix the definition of these algebras that was formulated in [19], [20], as it is the most convenient for our purpose. For this, we introduce a scalar product ε in the space ε $V_{(M|N)}$ (here N=2n is even). Its components $\varepsilon_{ab} \equiv \varepsilon(e_a,e_b)$ in some homogeneous basis $\{e_a\}_{a=1}^{M+N}$ of $V_{(M|N)}$ are given by the matrix:

$$\varepsilon = \left(\begin{array}{c|c} I_M & 0 \\ \hline 0 & J_N \end{array}\right). \tag{3.1}$$

Here I_M is the $(M \times M)$ identity matrix. The antisymmetric matrix J_N is

$$J_N = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}. \tag{3.2}$$

The definition (3.1) of ε implies the following relations on its components:

$$\varepsilon_{ba} = (-1)^{[a]} \varepsilon_{ab} = (-1)^{[b]} \varepsilon_{ab} = (-1)^{[a][b]} \varepsilon_{ab} . \tag{3.3}$$

The metric ε can be used to raise and lower indices:

$$z_{a...} = \varepsilon_{ab} z^b_{...}, \qquad z^{a...} = \overline{\varepsilon}^{ab} z_b^{...}, \qquad (3.4)$$

where $\overline{\varepsilon}^{ab}$ are the components of the matrix ε^{-1} :

$$\varepsilon^{ab}\varepsilon_{bc} = \delta^a_c, \qquad \varepsilon_{ab}\varepsilon^{bc} = \delta^c_a.$$
 (3.5)

By (3.4), $\varepsilon^{ab} = \overline{\varepsilon}^{ac} \overline{\varepsilon}^{bd} \varepsilon_{cd} = \overline{\varepsilon}^{ba}$, i.e., the matrix of the metric tensor with the upper indices ε^{ab} does not coincide with the inverse matrix $\overline{\varepsilon}^{ab}$. From now on, we only use the matrix $\overline{\varepsilon}^{ab}$.

Define the osp(M|N) Lie superalgebra as the algebra of operators $A:V_{(M|N)}\to V_{(M|N)}$ that leave the metric ε invariant. For homogeneous A this means:

$$\varepsilon(A(u), v) + (-1)^{[A][u]} \varepsilon(u, A(v)) = 0,$$
(3.6)

for any homogeneous $u, v \in V_{(M|N)}$. If $A = A_{\overline{0}} + A_{\overline{1}}$ is not homogeneous, with $A_{\overline{0}}$ and $A_{\overline{1}}$ being its even and odd parts, respectively, we require equation (3.6) to hold simultaneously for $A_{\overline{0}}$ and $A_{\overline{1}}$. Expanding $u = u^a e_a$ and $v = v^b e_b$ over the basis $\{e_a\}$, using the definitions of the components $\varepsilon_{ab} = \varepsilon(e_a, e_b)$ of the metric ε and of the operator $A(e_a) = e_b A^b{}_a$, as well as the fact that the grading of a homogeneous operator A with nonzero components $A^b{}_a$ equals [a] + [b], we can rewrite (3.6) in the component form [19], [20]:

$$A^{c}{}_{a}\varepsilon_{cb} + (-1)^{[a]+[a][b]}\varepsilon_{ac}A^{c}{}_{b} = 0. \tag{3.7}$$

Note that for the left-hand side of (3.7) to be well-defined, A needs not be homogeneous; thus, (3.7) can be viewed as the definition of invariance of a metric with respect to the action of an arbitrary operator A. Multiplying (3.7) by $(-1)^{[a]+[a][b]}$ yields:

$$(-1)^{[a]+[a][b]}A^c{}_a\varepsilon_{cb} + \varepsilon_{ac}A^c{}_b = 0.$$

$$(3.8)$$

To draw an analogy between (3.8) and the definition of the matrix Lie algebras so(M) and sp(N), we introduce the operation of supertransposition of any $A: V_{(M|N)} \to V_{(M|N)}$, the application of which results in an operator $A^T: \overline{V}_{(M|N)} \to \overline{V}_{(M|N)}$ where $\overline{V}_{(M|N)}$ is the dual space of $V_{(M|N)}$. The components $(A^T)_b{}^a$ of A^T , which are defined by $A^T(\epsilon^a) = \epsilon^b(A^T)_b{}^a$ with $\{\epsilon^a\}$ being the dual basis of $\{e_a\}$ in $\overline{V}_{(M|N)}$, are given explicitly by

$$(A^T)_a{}^b = (-1)^{[a]+[a][b]} A^b{}_a. (3.9)$$

Thus, for the $(M+N)\times (M+N)$ matrix of A^T we have:

$$A = \begin{pmatrix} X & Y \\ \hline Z & W \end{pmatrix} \implies A^T = \begin{pmatrix} X^t & Z^t \\ \hline -Y^t & W^t \end{pmatrix}. \tag{3.10}$$

Here X, Y, Z, W are $M \times M, M \times N, N \times M, N \times N$ matrices, respectively, and t denotes the usual matrix transposition.

Using (3.9) and (3.3), we can rewrite (3.8) as

$$(A^T)_a{}^c \varepsilon_{cb} + \varepsilon_{ac} A^c{}_b = 0. (3.11)$$

The matrix form of (3.11) is:

$$A^T \varepsilon + \varepsilon A = 0. \tag{3.12}$$

The form of (3.12) coincides with that of an analogous expression used in the definitions of the matrix algebras so(M) and sp(N) (N=2n) with the appropriate choice of the metric ε and supertransposition substituted with the regular transposition. Using (3.10), where Y, Z and W are viewed as block matrices, and (3.12) we infer the explicit form of the matrix of A:

$$A = \begin{pmatrix} X & Q & S \\ -S^t & E & F \\ Q^t & G & -E^t \end{pmatrix} . \tag{3.13}$$

Here X is a matrix of dimension $M \times M$ (as in (3.10)), $M \times n$ blocks Q and S form the $M \times 2n$ matrix Y = (Q S), and the $n \times n$ matrices E, F, G comprise W. Furthermore, X, F and G satisfy $X^t = -X$, $F^t = F$ and $G^t = G$.

Note that supertransposition does not possess some properties intrinsic to the usual transposition. In general, for some $A, B: V_{(M|N)} \to V_{(M|N)}$ it may be that $(A^T)^T \neq A$ and $(AB)^T \neq B^T A^T$. Nevertheless, by (3.9) and (3.10):

$$[A, B]^T = -[A^T, B^T],$$
 (3.14)

from where it follows that the vector space of all operators A, satisfying (3.11) and (3.12), is closed under the Lie bracket (2.4). Therefore, it forms a Lie superalgebra.

To find all solutions of (3.7), we introduce the following operators in the space $\operatorname{End}(V_{(M|N)})$:

$$P_{\pm}(E)^{a}{}_{c} = \frac{1}{2} \left(E^{a}{}_{c} \pm (-1)^{[c]+[c][a]} \varepsilon_{cb} E^{b}{}_{d} \overline{\varepsilon}^{da} \right), \qquad (3.15)$$

where $E \in \text{End}(V_{(M|N)})$, and $E^a{}_b$ is its matrix. It is easy to check that the operators P_{\pm} satisfy

$$P_A P_B = P_A \delta_{AB} \quad (A, B = +, -), \quad P_+ + P_- = I_{M+N}.$$
 (3.16)

and therefore constitute a full system of mutually orthogonal projectors in $\operatorname{End}(V_{(M|N)})$.

In terms of P_{\pm} , equation (3.7) is rewritten as:

$$(P_{+}A)^{a}_{b} = 0 \iff P_{+}A = 0 \iff P_{-}A = A.$$
 (3.17)

The latter condition is satisfied if and only if A lies in the image of P_- . Hence, the matrix of any operator $A \in osp(M|N)$ is of the form:

$$A^{a}{}_{b} = E^{a}{}_{b} - (-1)^{[b]+[b][a]} \varepsilon_{bc} E^{c}{}_{d} \overline{\varepsilon}^{da}, \tag{3.18}$$

where $||E^a{}_b|| \in \operatorname{Mat}_{M+N}(\mathbb{C})$ is an arbitrary matrix.

The basis elements $M_i{}^j \in osp(M|N)$ in the defining representation are realized as matrices $(M_i{}^j)^a{}_b$ obtained from (3.18) by the substitution $E^a{}_b \to (e_i{}^j)^a{}_b$:

$$(M_i{}^j)^a{}_b = (e_i{}^j)^a{}_b - (-1)^{[b]+[b][a]} \varepsilon_{bc} (e_i{}^j)^c{}_d \overline{\varepsilon}^{da} , \qquad (3.19)$$

where $e_i{}^j$ are the matrix identities in $V_{(M|N)}$, $(e_i{}^j)^a{}_b = \delta^j_b \delta^a_i$, see (2.23). Lowering the index j in (3.19) via the metric ε given in (3.1), we get the final form of the matrices of the osp(M|N) Lie superalgebra basis elements M_{ij} :

$$(M_{ij})^a{}_b = \varepsilon_{jb}\delta^a_i - (-1)^{[a][b]}\varepsilon_{ib}\delta^a_j = \varepsilon_{jb}\delta^a_i - (-1)^{[i][j]}\varepsilon_{ib}\delta^a_j. \tag{3.20}$$

The degree of M_{ij} is [i] + [j]. Moreover, M_{ij} satisfy:

$$M_{ij} = -(-1)^{[i][j]} M_{ji}. (3.21)$$

Taking this condition into account, we require the components of any vector (or covector) from the osp(M|N) Lie superalgebra to satisfy the same index permutation symmetry, that is: $X^{ij} = -(-1)^{[i][j]}X^{ji}$, $Y_{ij} = -(-1)^{[i][j]}Y_{ji}$, where X^{ij} are the coordinates of an arbitrary vector from osp(M|N), and Y_{ij} are the coordinates of an arbitrary covector. This requirement allows us to achieve uniqueness in assignment of coordinates to vectors of osp(M|N) in the basis(3.20) (and to covectors in the dual basis of (3.20)).

The Lie superbracket (2.4) of M_{ij} is

$$[M_{ij}, M_{km}] = \varepsilon_{jk} M_{im} - (-1)^{[k][m]} \varepsilon_{jm} M_{ik} - (-1)^{[i][j]} \varepsilon_{ik} M_{jm} + (-1)^{[i][j] + [k][m]} \varepsilon_{im} M_{jk}$$
(3.22)

The structure constants $X_{i_1i_2,j_1j_2}^{k_1k_2}$ of osp(M|N) are defined by

$$[M_{i_1i_2}, M_{j_1j_2}] = X_{i_1i_2, j_1j_2}^{k_1k_2} M_{k_1k_2}, (3.23)$$

and have the following explicit form:

$$\begin{split} X^{k_1k_2}{}_{i_1i_2,j_1j_2} &= \varepsilon_{i_2j_1} \delta^{(k_1}_{i_1} \delta^{k_2)}_{j_2} - (-1)^{[j_1][j_2]} \varepsilon_{i_2j_2} \delta^{(k_1}_{i_1} \delta^{k_2)}_{j_1} - \\ &- (-1)^{[i_1][i_2]} \varepsilon_{i_1j_1} \delta^{(k_1}_{i_2} \delta^{k_2)}_{j_2} + (-1)^{[i_1][i_2] + [j_1][j_2]} \varepsilon_{i_1j_2} \delta^{(k_1}_{i_2} \delta^{k_2)}_{j_1} \;, \end{split} \tag{3.24}$$

where $A^{(k_1k_2)}$ stands for:

$$A^{(k_1k_2)} = \frac{1}{2} (A^{k_1k_2} - (-1)^{[k_1][k_2]} A^{k_2k_1}). \tag{3.25}$$

By (2.9), the Cartan-Killing metric (2.9) of osp(M|N) in the basis (3.20) equals

$$\mathbf{g}_{i_1 i_2, j_1 j_2} = (-1)^{[m_1] + [m_2]} X^{m_1 m_2}{}_{i_1 i_2, k_1 k_2} X^{k_1 k_2}{}_{j_1 j_2, m_1 m_2}
= 2(\omega - 2) [\varepsilon_{i_1 j_2} \varepsilon_{i_2 j_1} - (-1)^{[j_1] [j_2]} \varepsilon_{i_1 j_1} \varepsilon_{i_2 j_2}],$$
(3.26)

where $\omega \equiv M-N$. For $\omega=2$ the metric (3.26) is degenerate and thus this case is omitted in what follows. The identity operator \widehat{I} that acts on the algebra osp(M|N) (which is considered here as a vector space embedded into $V_{(M|N)}^{\otimes 2}$) has the following components in the basis (3.20)

$$\widehat{I}^{i_1 i_2}{}_{j_1 j_2} = \frac{1}{2} \left(\delta^{i_1}_{j_1} \delta^{i_2}_{j_2} - (-1)^{[i_1][i_2]} \delta^{i_1}_{j_2} \delta^{i_2}_{j_1} \right) , \tag{3.27}$$

and appears to be a projector onto (super)antisymmetric second-rank tensors (see Section 3.1). The components of the inverse Cartan-Killing metric are defined by (2.14):

$$\mathsf{g}^{i_1 i_2, j_1 j_2} \mathsf{g}_{j_1 j_2, k_1 k_2} = \widehat{I}^{i_1 i_2}{}_{k_1 k_2}, \qquad \mathsf{g}_{i_1 i_2, j_1 j_2} \mathsf{g}^{j_1 j_2, k_1 k_2} = \widehat{I}^{k_1 k_2}{}_{i_1 i_2}. \tag{3.28}$$

Direct calculations yield their explicit form:

$$\mathsf{g}^{i_1 i_2, j_1 j_2} = \frac{1}{8(\omega - 2)} \left(\overline{\varepsilon}^{i_1 j_2} \overline{\varepsilon}^{i_2 j_1} - (-1)^{[i_1][i_2]} \overline{\varepsilon}^{i_1 j_1} \overline{\varepsilon}^{i_2 j_2} \right). \tag{3.29}$$

3.1 Projectors onto invariant subspaces of tensor product of two defining representations

Using (3.20), (3.29) and (2.34), we write the split Casimir operator in the tensor product of two defining representations [19], [20]:

$$\widehat{C}_{f}^{k_{1}k_{2}}{}_{m_{1}m_{2}} = \frac{1}{2(\omega - 2)} \left((-1)^{[k_{1}][k_{2}]} \delta_{m_{2}}^{k_{1}} \delta_{m_{1}}^{k_{2}} - \overline{\varepsilon}^{k_{1}k_{2}} \varepsilon_{m_{1}m_{2}} \right). \tag{3.30}$$

Define the operators $\mathbf{1}, \mathcal{P}, \mathcal{K}: V_{(M|N)}^{\otimes 2} \to V_{(M|N)}^{\otimes 2}$: (the matrix forms of these operators are given on the right of these formulas):

$$\mathbf{1} = e_i^{\ i} \otimes e_j^{\ j} \qquad \Longrightarrow \qquad \mathbf{1}^{k_1 k_2}_{m_1 m_2} = \delta_{m_1}^{k_1} \delta_{m_2}^{k_2} \,, \tag{3.31}$$

$$\mathcal{P} = (-1)^{[j]} e_i{}^j \otimes e_j{}^i \qquad \Longrightarrow \qquad \mathcal{P}^{k_1 k_2}{}_{m_1 m_2} = (-1)^{[k_1][k_2]} \delta^{k_1}_{m_2} \delta^{k_2}_{m_2} , \qquad (3.32)$$

$$\mathcal{K} = (-1)^{[i]+[i][k]} \varepsilon^{ij} \varepsilon_{km} e_i^{\ k} \otimes e_j^{\ m} \qquad \Longrightarrow \qquad \mathcal{K}^{k_1 k_2}{}_{m_1 m_2} = \varepsilon^{k_1 k_2} \varepsilon_{m_1 m_2} . \tag{3.33}$$

Here **1** is the identity operator, \mathcal{P} is the superpermutation, and e_i^j are the defined in (2.23) matrix identities that act on $V_{(M|N)}$. **1**, \mathcal{P} and \mathcal{K} have the following properties:

$$\mathcal{P}^2 = 1, \quad \mathcal{K}^2 = \omega \mathcal{K}, \quad \mathcal{P}\mathcal{K} = \mathcal{K}\mathcal{P} = \mathcal{K}.$$
 (3.34)

In terms of \mathcal{P} and \mathcal{K} , the split Casimir operator in the defining representation can be written as [19], [20]

$$\widehat{C}_f = \frac{1}{2(\omega - 2)} (\mathcal{P} - \mathcal{K}) . \tag{3.35}$$

The characteristic identity for \widehat{C}_f has degree three:

$$\widehat{C}_f^3 + \frac{\omega - 1}{2(\omega - 2)}\widehat{C}_f^2 - \frac{1}{4(\omega - 2)^2}\widehat{C}_f - \frac{\omega - 1}{8(\omega - 2)^3} = 0.$$
(3.36)

Formula (3.36) is easily verified by using (3.35) and the relations

$$\widehat{C}_f^2 = \frac{1}{4(\omega - 2)^2} \mathbf{1} + \frac{1}{4(\omega - 2)} \mathcal{K} , \qquad \widehat{C}_f^3 = \frac{1}{8(\omega - 2)^3} \left[\mathcal{P} - (\omega^2 - 3\omega + 3) \mathcal{K} \right] . \tag{3.37}$$

The left-hand side of (3.36) can be factorized, which results in:

$$\left(\widehat{C}_f - \frac{1}{2(\omega - 2)}\right) \left(\widehat{C}_f + \frac{1}{2(\omega - 2)}\right) \left(\widehat{C}_f + \frac{\omega - 1}{2(\omega - 2)}\right) = 0.$$
 (3.38)

Using this factorized form of the characteristic identity for \widehat{C}_f and (2.39), where we put p=3, fix the roots

$$a_1 = \frac{1}{2(\omega - 2)}$$
, $a_2 = -\frac{1}{2(\omega - 2)}$, $a_3 = \frac{1 - \omega}{2(\omega - 2)}$,

and utilize (3.35), and the left hand-side of (3.37) for \widehat{C}_f and \widehat{C}_f^2 gives us three projectors onto invariant subspaces of $V_{(M|N)}^{\otimes 2}$:

$$P_1 = \frac{1}{2}(1+\mathcal{P}) - \frac{1}{\omega}\mathcal{K}, \quad P_2 = \frac{1}{2}(1-\mathcal{P}), \quad P_3 = \frac{1}{\omega}\mathcal{K}.$$
 (3.39)

Note that the substitutions $\omega \to M$ and $\omega \to -N$ turn the projectors (3.39) into the corresponding projectors onto invariant subspaces of the representation $T_f^{\otimes 2}$ of the so(M) and sp(N) Lie algebras (for the explicit formulas see, e.g. [25] and [26]).

To conclude this subsection, we show that the solution $R_{k_1k_2}^{i_1i_2}(u)$ of the graded Yang-Baxter equation [27] (see also [19], [28]):

$$R^{i_1i_2}_{j_1j_2}(u)(-1)^{[j_1][j_2]}R^{j_1i_3}_{k_1j_3}(u+v)(-1)^{[k_1][j_2]}R^{j_2j_3}_{k_2k_3}(v) = = R^{i_2i_3}_{j_3j_2}(v)(-1)^{[i_1][j_2]}R^{i_1j_3}_{j_1k_2}(u+v)(-1)^{[j_1][j_2]}R^{j_1j_2}_{k_1k_2}(u),$$
(3.40)

which is invariant under the action of osp(M|N) in the defining representation, can be written in terms of the split Casimir operator (3.35). Recall that this solution can be written in several equaivalent ways [19], [20], [28]:

$$R(u) = \frac{1}{1-u} \left(u + \mathcal{P} - \frac{u}{u+\omega/2-1} \mathcal{K} \right) = \frac{1+u}{1-u} P_1 - P_2 + \frac{\omega/2-1-u}{\omega/2-1+u} P_3.$$
 (3.41)

The important point to note here is that this solution can be written as a rational function of C_f :

$$R(u) = \frac{(\omega - 2)\hat{C}_f + 1/2 + u}{(\omega - 2)\hat{C}_f + 1/2 - u}.$$
(3.42)

This form of the R-matrix generalizes that obtained in [15] for the so(M) and sp(2n) Lie superalgebras in the defining representation. Note that the solution (3.41), (3.42) is unitary, i.e., $\mathcal{P}R(u)\mathcal{P}R(-u) = 1$.

3.2Projectors onto invariant subspaces of the tensor product of two adjoint representations

In order to find a characteristic identity for the split Casimir operator in the adjoint representation \widehat{C}_{ad} , we first write the components of the basis elements of osp(M|N) in this representation. By (2.7), they coincide with the structure constants (3.24) of osp(M|N):

$$(M_{i_1i_2})^{k_1k_2}{}_{j_1j_2} = \varepsilon_{i_2j_1} \delta_{i_1}^{(k_1} \delta_{j_2}^{k_2} - (-1)^{[j_1][j_2]} \varepsilon_{i_2j_2} \delta_{i_1}^{(k_1} \delta_{j_1}^{k_2} - (-1)^{[i_1][i_2]} \varepsilon_{i_1j_1} \delta_{i_2}^{(k_1} \delta_{j_2}^{k_2} + (-1)^{[i_1][i_2] + [j_1][j_2]} \varepsilon_{i_1j_2} \delta_{i_2}^{(k_1} \delta_{j_1}^{k_2}) .$$

$$(3.43)$$

A comparison of (3.20) and (3.43) suggests a convenient relation between the components of the basis elements M_{ij} of osp(M|N) in the adjoint and defining representations:

$$(M_{i_1 i_2})^{k_1 k_2}{}_{j_1 j_2} = 2(M_{i_1 i_2})^{(k_1}{}_{(j_1} \delta^{k_2}{}_{j_2}) = 4 \operatorname{Sym}_{1 \leftrightarrow 2} \varepsilon_{i_2 j_1} \delta^{k_1}{}_{i_1} \delta^{k_2}{}_{j_2},$$
(3.44)

where $\operatorname{Sym}_{1\leftrightarrow 2}$ denotes (anti)symmetrisation over the pairs of indices $(i_1,i_2), (j_1,j_2)$ and (k_1,k_2) . Using (3.44), we deduce the following form of the components of $C_{\rm ad}$:

$$(\widehat{C}_{ad})^{k_1 k_2 k_3 k_4} {}_{m_1 m_2 m_3 m_4} = g^{i_1 i_2 j_1 j_2} (M_{i_1 i_2} \otimes M_{j_1 j_2})^{k_1 k_2 k_3 k_4} {}_{m_1 m_2 m_3 m_4}$$

$$= (-1)^{([j_1] + [j_2])([m_1] + [m_2])} g^{i_1 i_2 j_1 j_2} (M_{i_1 i_2})^{k_1 k_2} {}_{m_1 m_2} (M_{j_1 j_2})^{k_3 k_4} {}_{m_3 m_4}$$

$$= 4(-1)^{([j_1] + [j_2])([m_1] + [m_2])} g^{i_1 i_2 j_1 j_2} (M_{i_1 i_2})^{(k_1} {}_{m_1} \delta^{k_2}_{m_2}) (M_{j_1 j_2})^{(k_3} {}_{m_3} \delta^{k_4}_{m_4}),$$

$$(3.45)$$

As a result, we find a connection between the components of the split Casimir operator in the adjoint and defining representations:

$$(\widehat{C}_{ad})^{k_1 k_2 k_3 k_4}{}_{m_1 m_2 m_3 m_4} = 4(\widehat{C}_f)^{(k_1 k_2)(k_3 k_4)}_{13}{}_{(m_1 m_2)(m_3 m_4)}, \qquad (3.46)$$

where

$$(\widehat{C}_f)_{13}^{k_1 k_2 k_3 k_4}{}_{m_1 m_2 m_3 m_4} = g^{i_1 i_2 j_1 j_2} (M_{i_1 i_2} \otimes I \otimes M_{j_1 j_2} \otimes I)^{k_1 k_2 k_3 k_4}{}_{m_1 m_2 m_3 m_4}$$

$$= (-1)^{([j_1] + [j_2])([m_1] + [m_2])} g^{i_1 i_2 j_1 j_2} (M_{i_1 i_2})^{k_1}{}_{m_1} \delta_{m_2}^{k_2} (M_{j_1 j_2})^{k_3}{}_{m_3} \delta_{m_4}^{k_4}.$$

$$(3.47)$$

The lower index "13" of \widehat{C}_f is defined in accordance with (2.25). In what follows, we need the operators $\mathcal{P}_{\alpha\beta}$ and $\mathcal{K}_{\alpha\beta}: V_{(M|N)}^{\otimes 4} \to V_{(M|N)}^{\otimes 4}$ ($\alpha, \beta = 1, \ldots, 4, \alpha \neq \beta$), which are built from (3.32) and (3.33) by (2.24). Direct calculations show that $\mathcal{P}_{\alpha,\alpha+1}$ and $\mathcal{K}_{\alpha,\alpha+1}$ satisfy the relations between the generators σ_{α} and κ_{α} of the Brauer algebra $\mathcal{B}r_4(\omega)$:

$$\sigma_{\alpha}^{2} = I , \ \kappa_{\alpha}^{2} = \omega \kappa_{\alpha} , \ \sigma_{\alpha} \kappa_{\alpha} = \kappa_{\alpha} \sigma_{\alpha} = \kappa_{\alpha} , \ \sigma_{\alpha} \kappa_{\alpha} = \kappa_{\alpha} \sigma_{\alpha} = \kappa_{\alpha} , \ \alpha = 1, \dots, 3 ,$$

$$\sigma_{\alpha} \sigma_{\beta} = \sigma_{\beta} \sigma_{\alpha} , \quad \kappa_{\alpha} \kappa_{\beta} = \kappa_{\beta} \kappa_{\alpha} , \quad \sigma_{\alpha} \kappa_{\beta} = \kappa_{\beta} \sigma_{\alpha} , \quad |\alpha - \beta| > 1 ,$$

$$\sigma_{\alpha} \sigma_{\alpha+1} \sigma_{\alpha} = \sigma_{\alpha+1} \sigma_{\alpha} \sigma_{\alpha+1} \quad \kappa_{\alpha} \kappa_{\alpha+1} \kappa_{\alpha} = \kappa_{\alpha} , \quad \kappa_{\alpha+1} \kappa_{\alpha} \kappa_{\alpha+1} = \kappa_{\alpha+1} ,$$

$$\sigma_{\alpha} \kappa_{\alpha+1} \kappa_{\alpha} = \sigma_{\alpha+1} \kappa_{\alpha} , \quad \kappa_{\alpha+1} \kappa_{\alpha} \sigma_{\alpha+1} = \kappa_{\alpha+1} \sigma_{\alpha} , \quad \alpha = 1, \dots, 3 .$$

$$(3.48)$$

Thus, $\mathcal{P}_{\alpha,\alpha+1} = \tau(\sigma_{\alpha})$ and $\mathcal{K}_{\alpha,\alpha+1} = \tau(\kappa_{\alpha})$, where τ is a representation of $\mathcal{B}r_4(\omega)$ in the space $V_{(M|N)}^{\otimes 4}$. Recall that, by convention (2.31), the operators $\mathcal{P}_{\alpha\beta}$ for $\beta > \alpha + 1$ can be obtained from $\mathcal{P}_{\alpha'\beta'}$ by a consequtive action of adjacent transpositions $\mathcal{P}_{\gamma,\gamma+1}$. For instance,

$$\mathcal{P}_{14} = \mathcal{P}_{34}\mathcal{P}_{13}\mathcal{P}_{34} = \mathcal{P}_{34}\mathcal{P}_{23}\mathcal{P}_{12}\mathcal{P}_{23}\mathcal{P}_{34} = \tau(\sigma_3\sigma_2\sigma_1\sigma_2\sigma_3) \ . \tag{3.49}$$

Besides, being even (or, alternatively by (3.48)), the operators $\mathcal{P}_{\alpha_1\alpha_2}$ and $\mathcal{P}_{\beta_1\beta_2}$ commute for $\alpha_1 \neq \beta_1, \beta_2$ and $\alpha_2 \neq \beta_1, \beta_2$, and so do $\mathcal{K}_{\alpha_1\alpha_2}$ and $\mathcal{K}_{\beta_1\beta_2}$

$$\mathcal{P}_{\alpha_1 \alpha_2} \mathcal{P}_{\beta_1 \beta_2} = \mathcal{P}_{\beta_1 \beta_2} \mathcal{P}_{\alpha_1 \alpha_2}, \quad \mathcal{K}_{\alpha_1 \alpha_2} \mathcal{K}_{\beta_1 \beta_2} = \mathcal{K}_{\beta_1 \beta_2} \mathcal{K}_{\alpha_1 \alpha_2}. \tag{3.50}$$

Define the antisymmetrizer $\mathcal{P}^-: V_{(M|N)}^{\otimes 2} \to V_{(M|N)}^{\otimes 2}$ by

$$\mathcal{P}^{-} = \frac{1}{2}(\mathbf{1} - \mathcal{P}) , \qquad (3.51)$$

where 1 and \mathcal{P} are given in (3.31) and (3.33). Then, by (3.35) and (3.46),

$$\widehat{C}_{ad} = 4\mathcal{P}_{12}^{-}\mathcal{P}_{34}^{-}(\widehat{C}_f)_{13}\mathcal{P}_{12}^{-}\mathcal{P}_{34}^{-} = \frac{2}{\omega - 2}\mathcal{P}_{12}^{-}\mathcal{P}_{34}^{-}(\mathcal{P}_{13} - \mathcal{K}_{13})\mathcal{P}_{12}^{-}\mathcal{P}_{34}^{-}.$$
(3.52)

Define the space $V_{\rm ad}$ of the adjoint representation of the osp(M|N) Lie superalgebra by $V_{\rm ad} = \mathcal{P}^-V_{(M|N)}^{\otimes 2}$. The algebra osp(M|N) coincides with $V_{\rm ad}$ as a vector space. Now introduce the following operators that act on $V_{\rm ad}^{\otimes 2} \subset V_{(M|N)}^{\otimes 4}$:

$$\mathbf{I} = \mathcal{P}_{12}^{-} \mathcal{P}_{34}^{-} \equiv \mathcal{P}_{12.34}^{-} , \qquad \mathbf{P} = \mathcal{P}_{12.34}^{-} \mathcal{P}_{13} \mathcal{P}_{24} \mathcal{P}_{12.34}^{-} , \qquad \mathbf{K} = \mathcal{P}_{12.34}^{-} \mathcal{K}_{13} \mathcal{K}_{24} \mathcal{P}_{12.34}^{-} , \qquad (3.53)$$

where we denoted $\mathcal{P}_{12,34}^- = \mathcal{P}_{12}^- \mathcal{P}_{34}^-$. Note that by (2.24), the operators (3.53) (in a way similar to \mathcal{P} and \mathcal{K}) define a Brauer algebra $\mathcal{B}r_s(\omega)$ representation in the space $V_{\rm ad}^{\otimes s}$.

The following relations hold for I, P and K introduced in (3.53):

$$\mathbf{I} = \mathbf{I}\mathcal{P}_{12}\mathcal{P}_{34} = \mathcal{P}_{12}\mathcal{P}_{34}\mathbf{I}, \qquad \mathbf{P} = \mathcal{P}_{12}\mathcal{P}_{34}\mathbf{P} = \mathbf{P}\mathcal{P}_{12}\mathcal{P}_{34}, \tag{3.54}$$

$$\mathbf{P}^2 = \mathbf{I}, \quad \mathbf{K}\mathbf{P} = \mathbf{P}\mathbf{K} = \mathbf{K}, \quad \mathbf{K}^2 = \frac{\omega(\omega - 1)}{2}\mathbf{K},$$
 (3.55)

$$\widehat{C}_{ad}\mathbf{P} = \mathbf{P}\widehat{C}_{ad}, \qquad \widehat{C}_{ad}\mathbf{K} = \mathbf{K}\widehat{C}_{ad} = -\mathbf{K}.$$
 (3.56)

The operators (3.53) are invariant with respect to the osp(M|N) Lie superalgebra in the adjoint representation (the definition of ad-invariance is given in (2.35)). Comparing the last formula in (3.55) and (2.47), we get

$$sdim \mathfrak{g} = \frac{\omega(\omega - 1)}{2} . \tag{3.57}$$

To find the characteristic identity for \widehat{C}_{ad} , it is convenient to introduce the symmetric \widehat{C}_{+} and antisymmetric \widehat{C}_{-} projections of \widehat{C}_{ad} :

$$\widehat{C}_{\pm} = \frac{1}{2} (\mathbf{I} \pm \mathbf{P}) \widehat{C}_{\mathrm{ad}}, \tag{3.58}$$

which satisfy:

$$\widehat{C}_{\pm}\widehat{C}_{\mp} = 0, \quad \mathbf{P}\widehat{C}_{\pm} = \pm \widehat{C}_{\pm}, \quad \mathbf{K}\widehat{C}_{-} = \widehat{C}_{-}\mathbf{K} = 0, \quad \mathbf{K}\widehat{C}_{+} = \widehat{C}_{+}\mathbf{K} = -\mathbf{K}.$$
 (3.59)

Note that formulas (3.55), (3.56) and (3.59) were derived for all Lie superalgebras with the nondegenerate Cartan-Killing metric in Section 2.3. Substitution of (3.52) and (3.53) into (3.58) gives explicit formulas for the antisymmetric and symmetric parts of \widehat{C}_{ad} :

$$\widehat{C}_{-} = \frac{1}{\omega - 2} \mathcal{P}_{12,34}^{-} (\mathcal{K}_{13} \mathcal{P}_{24} - \mathcal{K}_{13}) \mathcal{P}_{12,34}^{-} , \qquad (3.60)$$

$$\widehat{C}_{+} = \frac{1}{\omega - 2} \mathcal{P}_{12,34}^{-} (2\mathcal{P}_{24} - \mathcal{K}_{13} - \mathcal{K}_{13}\mathcal{P}_{24}) \mathcal{P}_{12,34}^{-} . \tag{3.61}$$

Proposition 2. The antisymmetric \widehat{C}_{-} and symmetric \widehat{C}_{+} parts of the split Casimir operator of the osp(M|N) Lie superalgebra for $M-N \equiv \omega \neq 0,1,2,4,8$ satisfies:

$$\hat{C}_{-}^2 = -\frac{1}{2}\hat{C}_{-} \iff \hat{C}_{-}(\hat{C}_{-} + \frac{1}{2}) = 0,$$
(3.62)

$$\widehat{C}_{+}^{3} = -\frac{1}{2}\widehat{C}_{+}^{2} - \frac{\omega - 8}{2(\omega - 2)^{2}}\widehat{C}_{+} + \frac{\omega - 4}{2(\omega - 2)^{3}}(\mathbf{I} + \mathbf{P} - 2\mathbf{K}), \qquad (3.63)$$

$$\widehat{C}_{+}^{4} + \frac{3}{2}\widehat{C}_{+}^{3} + \frac{(\omega+1)(\omega-4)}{2(\omega-2)^{2}}\widehat{C}_{+}^{2} + \frac{\omega^{2}-12\omega+24}{2(\omega-2)^{3}}\widehat{C}_{+} - \frac{(\omega-4)}{2(\omega-2)^{3}}(\mathbf{I} + \mathbf{P}) = 0,$$
(3.64)

$$\widehat{C}_{+}\left(\widehat{C}_{+}+\mathbf{I}\right)\left(\widehat{C}_{+}-\frac{\mathbf{I}}{\omega-2}\right)\left(\widehat{C}_{+}+\frac{2\mathbf{I}}{\omega-2}\right)\left(\widehat{C}_{+}+\frac{(\omega-4)\mathbf{I}}{2(\omega-2)}\right)=0. \tag{3.65}$$

The split Casimir operator $\hat{C}_{ad} = \hat{C}_{-} + \hat{C}_{+}$ for $\omega \neq 0, 1, 2, 4, 6, 8$ satisfies:

$$\widehat{C}_{ad}(\widehat{C}_{ad} + \frac{1}{2})(\widehat{C}_{ad} + 1)(\widehat{C}_{ad} - \frac{1}{\omega - 2})(\widehat{C}_{ad} + \frac{2}{\omega - 2})(\widehat{C}_{ad} + \frac{\omega - 4}{2(\omega - 2)}) = 0.$$
(3.66)

Proof. For $M-N\equiv\omega=2$ the Cartan-Killing metric (3.26) of osp(M|N) is degenerate, so this case is excluded from consideration. The special cases of $\omega=0,1,4,6,8$ are considered later.

Identity (3.62) for osp(M|N) is a special case of (2.54), which holds for all Lie superalgebras with the nondegenerate Cartan-Killing metric. Note also a useful consequence of (3.62):

$$\widehat{C}_{-}^{k} = \left(-\frac{1}{2}\right)^{k-1} \widehat{C}_{-}, \quad k \ge 1.$$
 (3.67)

Using the explicit formula (3.61) for \widehat{C}_+ , one can directly calculate an expression for \widehat{C}_+^2 :

$$\widehat{C}_{+}^{2} = \frac{1}{(\omega - 2)^{2}} (\mathbf{I} + \mathbf{P} + \mathbf{K}) - \frac{1}{\omega - 2} \widehat{C}_{+} + \frac{\omega - 8}{2(\omega - 2)^{2}} \mathcal{P}_{12,34}^{-} (\mathcal{K}_{13} \mathcal{P}_{24} + \mathcal{K}_{13}) \mathcal{P}_{12,34}^{-} . \tag{3.68}$$

If $\omega = 8$, then the last term in (3.68) is nullified and the characteristic identity for \hat{C}_+ takes the form:

$$\widehat{C}_{+}^{2} = -\frac{1}{6}\widehat{C}_{+} + \frac{1}{36}(\mathbf{I} + \mathbf{P} + \mathbf{K})$$
(3.69)

If $\omega \neq 2, 8$, then multiplication of (3.68) by \widehat{C}_+ yields the third-degree identity (3.63). Note that for $\omega = 4$ the last term in (3.63) is zero, hence in this case (3.63) is the characteristic identity for \widehat{C}_+ that has the following explicit form:

$$\widehat{C}_{+}^{3} = -\frac{1}{2}\widehat{C}_{+}^{2} + \frac{1}{2}\widehat{C}_{+} . \tag{3.70}$$

To obtain a characteristic identity for \widehat{C}_+ when $\omega \neq 2, 4, 8$, we get rid of **P** and **K** in (3.63). Multiplying (3.63) by \widehat{C}_+ and using (3.59), we can express **K** in terms of \widehat{C}_+ :

$$\frac{\omega - 4}{(\omega - 2)^3} \mathbf{K} = \widehat{C}_+^4 + \frac{1}{2} \widehat{C}_+^3 + \frac{\omega - 8}{2(\omega - 2)^2} \widehat{C}_+^2 - \frac{\omega - 4}{(\omega - 2)^3} \widehat{C}_+ . \tag{3.71}$$

Substitution of **K** from (3.71) into (3.63) gives (3.64). Multiplying both sides of (3.71) by $(\widehat{C}_+ + \mathbf{I})$ and using the last relation in (3.59), we get the characteristic identity for \widehat{C}_+ :

$$\widehat{C}_{+}^{5} + \frac{3}{2}\widehat{C}_{+}^{4} + \frac{(\omega+1)(\omega-4)}{2(\omega-2)^{2}}\widehat{C}_{+}^{3} + \frac{\omega^{2}-12\omega+24}{2(\omega-2)^{3}}\widehat{C}_{+}^{2} - \frac{\omega-4}{(\omega-2)^{3}}\widehat{C}_{+} = 0, \tag{3.72}$$

which can be rewritten in the following form:

$$\widehat{C}_{+}^{5} = -\frac{3}{2}\widehat{C}_{+}^{4} - \frac{(\omega+1)(\omega-4)}{2(\omega-2)^{2}}\widehat{C}_{+}^{3} - \frac{\omega^{2}-12\omega+24}{2(\omega-2)^{3}}\widehat{C}_{+}^{2} + \frac{\omega-4}{(\omega-2)^{3}}\widehat{C}_{+}.$$
(3.73)

Now, (3.65) is the result of factorizing (3.72). For further calculations, we also need an expression for \widehat{C}_{+}^{6} , which can be derived by multiplying (3.73) by \widehat{C}_{+} and using the known polynomial for \widehat{C}_{+}^{5} from (3.73):

$$\widehat{C}_{+}^{6} = \frac{(7\omega^{2} - 30\omega + 44)}{4(\omega - 2)^{2}} \widehat{C}_{+}^{4} + \frac{3\omega^{3} - 17\omega^{2} + 30\omega - 24}{4(\omega - 2)^{3}} \widehat{C}_{+}^{3} + \frac{3\omega^{2} - 32\omega + 56}{4(\omega - 2)^{3}} \widehat{C}_{+}^{2} - \frac{3(\omega - 4)}{2(\omega - 2)^{3}} \widehat{C}_{+}.$$
(3.74)

Our next goal is to find a characteristic polynomial for the split Casimir operator $\hat{C}_{ad} = \hat{C}_+ + \hat{C}_-$ by using the expressions obtained. We look for such an expression in the form of a polynomial in \hat{C}_{ad} of degree six with arbitrary coefficients α_i :

$$\widehat{C}_{\mathrm{ad}}^6 + \alpha_5 \widehat{C}_{\mathrm{ad}}^5 + \alpha_4 \widehat{C}_{\mathrm{ad}}^4 + \alpha_3 \widehat{C}_{\mathrm{ad}}^3 + \alpha_2 \widehat{C}_{\mathrm{ad}}^2 + \alpha_1 \widehat{C}_{\mathrm{ad}} + \alpha_0 . \tag{3.75}$$

We need to find α_i that nullify (3.75). The first formula in (3.59) implies $\widehat{C}^k = \widehat{C}_+^k + \widehat{C}_-^k$, thus equating the polynomial (3.75) to zero yields the equation

$$\hat{C}_{+}^{6} + \alpha_{5}\hat{C}_{+}^{5} + \alpha_{4}\hat{C}_{+}^{4} + \alpha_{3}\hat{C}_{+}^{3} + \alpha_{2}\hat{C}_{+}^{2} + \alpha_{1}\hat{C}_{+}$$

$$+\hat{C}_{-}^{6} + \alpha_{5}\hat{C}_{-}^{5} + \alpha_{4}\hat{C}_{-}^{4} + \alpha_{3}\hat{C}_{-}^{3} + \alpha_{2}\hat{C}_{-}^{2} + \alpha_{1}\hat{C}_{-} + \alpha_{0} = 0.$$

Using the expessions for $\widehat{C}_{+}^{5,6}$ in terms of $\widehat{C}_{+}^{4,3,2,1}$ given by (3.73) and (3.74) as well as the expressions for $\widehat{C}_{-}^{6,5,4,3,2}$ in terms of \widehat{C}_{-} from (3.67) and setting the coefficients of those operators to zero, we get the values of α_i :

$$\alpha_0 = 0 , \qquad \alpha_1 = -\frac{\omega - 4}{2(\omega - 2)^3} , \qquad \alpha_2 = \frac{\omega^2 - 16\omega + 40}{4(\omega - 2)^3} ,$$

$$\alpha_3 = \frac{\omega^3 - 3\omega^2 - 22\omega + 56}{4(\omega - 2)^3} , \qquad \alpha_4 = \frac{5\omega^2 - 18\omega + 4}{4(\omega - 2)^2} , \qquad \alpha_5 = 2 .$$

Thus, the characteristic identity for \widehat{C}_{ad} takes the form:

$$\widehat{C}_{\text{ad}}^{6} + 2\widehat{C}_{\text{ad}}^{5} + \frac{5\omega^{2} - 18\omega + 4}{4(\omega - 2)^{2}}\widehat{C}_{\text{ad}}^{4} + \frac{\omega^{3} - 3\omega^{2} - 22\omega + 56}{4(\omega - 2)^{3}}\widehat{C}_{\text{ad}}^{3} + \frac{\omega^{2} - 16\omega + 40}{4(\omega - 2)^{3}}\widehat{C}_{\text{ad}}^{2} - \frac{\omega - 4}{2(\omega - 2)^{3}}\widehat{C}_{\text{ad}} = 0.$$
(3.76)

The roots of the polynomial on the left of (3.76) may be found explicitly:

$$a_1 = 0$$
, $a_2 = -\frac{1}{2}$, $a_3 = -1$, $a_4 = \frac{1}{\omega - 2}$, $a_5 = \frac{-2}{\omega - 2}$, $a_6 = \frac{4 - \omega}{2(\omega - 2)}$. (3.77)

Note that degenerate roots appear for the following values of ω :

$$\omega = 0 \implies a_2 = a_4 = -\frac{1}{2}, \quad a_3 = a_6 = -1, \qquad \omega = 1 \implies a_3 = a_4 = -1,$$

$$\omega = 4 \implies a_1 = a_6 = 0, \quad a_3 = a_5 = -1, \qquad \omega = 6 \implies a_2 = a_5 = -\frac{1}{2},$$

$$\omega = 8 \implies a_5 = a_6 = -\frac{1}{3}.$$
(3.78)

Therefore, the cases $\omega = 0, 1, 4, 6, 8$ are to be considered separately (see Remark below).

Using the roots (3.77) of the characteristic polynomial in the left-hand side of (3.76), we can rewrite the characteristic identity (3.76) for $\omega \neq 0, 1, 4, 6, 8$ as (3.66).

Remark. In order to get characteristic identities for \widehat{C}_{ad} when $\omega = 0, 1, 4, 6, 8$, we need for all the degenerate roots in (3.78) to leave in (3.66) only one of the parentheses corresponding to such a root. This operation turns the left hand side of (3.66) into the following polynomials in \widehat{C}_{ad} :

$$\omega = 0: \quad \widehat{C}_{ad}(\widehat{C}_{ad} + \frac{1}{2})(\widehat{C}_{ad} + 1)(\widehat{C}_{ad} - 1) = \frac{1}{2}\mathbf{K} \neq 0,
\omega = 1: \quad \widehat{C}_{ad}(\widehat{C}_{ad} + \frac{1}{2})(\widehat{C}_{ad} + 1)(\widehat{C}_{ad} - 2)(\widehat{C}_{ad} + \frac{3}{2}) = -\frac{3}{2}\mathbf{K} \neq 0,
\omega = 4: \quad \widehat{C}_{ad}(\widehat{C}_{ad} + \frac{1}{2})(\widehat{C}_{ad} + 1)(\widehat{C}_{ad} - \frac{1}{2}) = 0,
\omega = 6: \quad \widehat{C}_{ad}(\widehat{C}_{ad} + \frac{1}{2})(\widehat{C}_{ad} + 1)(\widehat{C}_{ad} - \frac{1}{4})(\widehat{C}_{ad} + \frac{1}{4}) = 0,
\omega = 8: \quad \widehat{C}_{ad}(\widehat{C}_{ad} + \frac{1}{2})(\widehat{C}_{ad} + 1)(\widehat{C}_{ad} - \frac{1}{6})(\widehat{C}_{ad} + \frac{1}{3}) = 0.$$
(3.79)

The equalities in the right-hand side of (3.79) are obtained by substitution $\widehat{C}_{ad} = \widehat{C}_+ + \widehat{C}_-$ and using (3.67). From (3.79), we see that for $\omega = 0, 1$ the characteristic polynomial for \widehat{C}_{ad} of the form (2.38) does not exist; hence \widehat{C}_{ad} is not diagonalizable. Correspondingly, the representation $\mathrm{ad}^{\otimes 2}$ of osp(M|N) in this case is not completely reducible. It can also be deduced from the fact that for $\omega = 0, 1$ the ad-invariant operator \mathbf{K} is nilpotent (as $\mathbf{K}^2 = 0$) and, therefore, not diagonalizable.

The form (3.66) of the characteristic identity of \widehat{C}_{ad} for $\omega \neq 0, 1, 4, 6, 8$ allows us to construct projectors onto invariant subspaces of $V_{ad} \otimes V_{ad}$ by (2.39), where p = 6 and a_i are the roots (3.77) of the characteristic equation (3.66). Using (3.59),(3.63), (3.67), and (3.73), we find explicit expressions for the projectors (2.39) in terms of \mathbf{I} , \mathbf{P} , \mathbf{K} , \widehat{C}_+ , \widehat{C}_- :

$$P_{1} = \frac{1}{2}(\mathbf{I} - \mathbf{P}) + 2\hat{C}_{-}, \qquad P_{2} = -2\hat{C}_{-}, \qquad P_{3} = \frac{2\mathbf{K}}{(\omega - 1)\omega},$$

$$P_{4} = \frac{2}{3}(\omega - 2)\hat{C}_{+}^{2} + \frac{\omega}{3}\hat{C}_{+} + \frac{(\omega - 4)(\mathbf{I} + \mathbf{P})}{3(\omega - 2)} - \frac{2(\omega - 4)\mathbf{K}}{3(\omega - 2)(\omega - 1)},$$

$$P_{5} = -\frac{2(\omega - 2)^{2}}{3(\omega - 8)}\hat{C}_{+}^{2} - \frac{(\omega - 2)(\omega - 6)}{3(\omega - 8)}\hat{C}_{+} + \frac{(\omega - 4)(\mathbf{I} + \mathbf{P})}{6(\omega - 8)} + \frac{2\mathbf{K}}{3(\omega - 8)},$$

$$P_{6} = \frac{4(\omega - 2)}{\omega - 8}\hat{C}_{+}^{2} + \frac{4}{\omega - 8}\hat{C}_{+} - \frac{4(\mathbf{I} + \mathbf{P})}{(\omega - 2)(\omega - 8)} - \frac{8(\omega - 4)\mathbf{K}}{\omega(\omega - 2)(\omega - 8)}.$$

$$(3.80)$$

The images of P_1 and P_2 are contained in the antisymmetric part $\mathbf{P}_-(V_{\mathrm{ad}}^{\otimes 2})$, while the images of P_i , (i=3,...,6), lie within the symmetric part $\mathbf{P}_+(V_{\mathrm{ad}}^{\otimes 2})$ of $V_{\mathrm{ad}}^{\otimes 2}$ where $\mathbf{P}_\pm = \frac{1}{2}(\mathbf{I} \pm \mathbf{P})$. Note that for $\omega=4,6$ all the projectors (3.80) are well defined and constructed from the ad-invariant operators \mathbf{I} , \mathbf{P} , \mathbf{K} , \widehat{C}_- , \widehat{C}_+ and \widehat{C}_+^2 ; hence they are projectors onto the invariant subspaces of the representation $\mathrm{ad}^{\otimes 2}$ of osp(M|N). Besides, although P_5 and P_6 are not formally defined for $\omega=8$, substitution of the explicit expressions for \mathbf{I} , \mathbf{P} , \mathbf{K} , \widehat{C}_+ and \widehat{C}_+^2 into (3.80) and the subsequent cancellation of the pole at $\omega=8$ yield:

$$P_5 = \frac{1}{6} (1 - (\mathcal{P}_{14} + \mathcal{P}_{23} + \mathcal{P}_{13} + \mathcal{P}_{24}) + \mathcal{P}_{13}\mathcal{P}_{24})\mathcal{P}_{12,34}, \tag{3.81}$$

$$P_6 = \frac{4}{\omega - 2} \mathcal{P}_{12,34} \mathcal{K}_{13} \left[\frac{1}{2} (1 + \mathcal{P}_{24}) - \frac{1}{\omega} \mathcal{K}_{24} \right] \mathcal{P}_{12,34}. \tag{3.82}$$

It is worth pointing out that expression (3.81) for the projector P_5 does not depend on ω and coincides with the full (anti)symmetrizer of $V_{(M|N)}^{\otimes 4}$.

To build projectors onto the generalized eigenspaces of \widehat{C}_{ad} for $\omega = 0, 1$, we use the characteristic identity (2.40). For $\omega = 0$ it takes the form:

$$\widehat{C}_{\rm ad}(\widehat{C}_{\rm ad} + \frac{1}{2})(\widehat{C}_{\rm ad} + 1)^2(\widehat{C}_{\rm ad} - 1) = 0,$$

so one needs to put $a_1 = 1$, $a_2 = -\frac{1}{2}$, $a_3 = -1$, $a_4 = 1$, $k_1 = k_2 = k_4 = 1$, $k_3 = 1$ in (2.40). Then (2.41) gives the projectors onto the generalized eigenspaces of \hat{C}_{ad} :

$$P_{1} = \frac{1}{2}(\mathbf{I} - \mathbf{P}) + 2\hat{C}_{-}, \qquad P_{2} = -2\hat{C}_{-} + \frac{2}{3}(\mathbf{I} + \mathbf{P}) + \frac{4}{3}\mathbf{K} - \frac{4}{3}\hat{C}_{+}^{2},$$

$$P_{3} = -\frac{1}{4}(\mathbf{I} + \mathbf{P}) - \frac{5}{4}\mathbf{K} - \frac{1}{2}\hat{C}_{+} + \hat{C}_{+}^{2}, \qquad P_{4} = \frac{1}{12}(\mathbf{I} + \mathbf{P}) - \frac{1}{12}\mathbf{K} + \frac{1}{2}\hat{C}_{+} + \frac{1}{3}\hat{C}_{+}^{2}.$$
(3.83)

Here the operators P_1 , P_2 and P_4 extract eigenspaces of \widehat{C}_{ad} , while P_3 projects $V_{ad}^{\otimes 2}$ onto the generalized eigenspace of \widehat{C}_{ad} . We thus conclude that the restriction of the representation $\operatorname{ad}^{\otimes 2}$ to $P_3(V_{ad}^{\otimes 2})$ is reducible but not completely reducible. It is also worth noting that P_2 given in (3.83) is a linear combination of symmetric operators $(\mathbf{I} + \mathbf{P})$, \mathbf{K} , \widehat{C}_+ and an antisymmetric operator \widehat{C}_- . As pointed out above (see Section 2.3), the symmetric and antisymmetric parts of $V_{ad}^{\otimes 2}$ are invariant under the action of any Lie superalgebra \mathfrak{g} in the representation $\operatorname{ad}^{\otimes 2}$, so P_2 may be invariantly split into its symmetric and antisymmetric parts:

$$P_2^{(+)} = \frac{2}{3}(\mathbf{I} + \mathbf{P}) + \frac{4}{3}\mathbf{K} - \frac{4}{3}\widehat{C}_+, \qquad P_2^{(-)} = -2\widehat{C}_-.$$
 (3.84)

Finally, for osp(M|N) in the case of $\omega \equiv M-N=0$, the needed projectors are:

$$\begin{split} P_1 &\equiv P_1^{(-)} = \frac{1}{2} (\mathbf{I} - \mathbf{P}) + 2 \widehat{C}_-, \qquad \qquad P_2^{(+)} = \frac{2}{3} (\mathbf{I} + \mathbf{P}) + \frac{4}{3} \mathbf{K} - \frac{4}{3} \widehat{C}_+^2, \\ P_2^{(-)} &= -2 \widehat{C}_-, \qquad \qquad P_3 \equiv P_3^{(+)} = -\frac{1}{4} (\mathbf{I} + \mathbf{P}) - \frac{5}{4} \mathbf{K} - \frac{1}{2} \widehat{C}_+ + \widehat{C}_+^2, \\ P_4 &\equiv P_4^{(+)} = \frac{1}{12} (\mathbf{I} + \mathbf{P}) - \frac{1}{12} \mathbf{K} + \frac{1}{2} \widehat{C}_+ + \frac{1}{3} \widehat{C}_+^2. \end{split}$$

If $\omega = 1$, then the characteristic identity (2.40) for \widehat{C}_{ad} is:

$$\widehat{C}_{\rm ad}(\widehat{C}_{\rm ad} + \frac{1}{2})(\widehat{C}_{\rm ad} + 1)^2(\widehat{C}_{\rm ad} - 2)(\widehat{C}_{\rm ad} + \frac{3}{2}) = 0,$$

which implies $a_1=0$, $a_2=-\frac{1}{2}$, $a_3=-1$, $a_4=2$, $a_5=-\frac{3}{2}$, $k_1=k_2=k_4=k_5=1$, $k_3=2$ in (2.40). The projectors onto the generalised eigenspaces of $\widehat{C}_{\rm ad}$ are then specified by (2.41):

$$P_{1} = \frac{1}{2}(\mathbf{I} - \mathbf{P}) + 2\hat{C}_{-}, \qquad P_{3} = \mathbf{I} + \mathbf{P} - \frac{10}{3}\mathbf{K} + \frac{1}{3}\hat{C}_{+} - \frac{2}{3}\hat{C}_{+}^{2},$$

$$P_{2} = -2\hat{C}_{-}, \qquad P_{4} = \frac{1}{14}(\mathbf{I} + \mathbf{P}) - \frac{2}{21}\mathbf{K} + \frac{5}{21}\hat{C}_{+} + \frac{2}{21}\hat{C}_{+}^{2},$$

$$P_{5} = -\frac{4}{7}(\mathbf{I} + \mathbf{P}) + \frac{24}{7}\mathbf{K} - \frac{4}{7}\hat{C}_{+} + \frac{4}{7}\hat{C}_{+}^{2}.$$

Here the operators P_1 , P_2 , P_4 and P_5 extract eigenspaces of \widehat{C}_{ad} , while P_3 projects $V_{ad}^{\otimes 2}$ onto the generalized eigenspace of \widehat{C}_{ad} . It can be then concluded that $P_3(V_{ad}^{\otimes 2})$ is not a space of an irreducible or completely reducible representation of osp(M|N) for $\omega = 1$.

In order to find the dimensions of the invariant subspaces, we need to calculate the traces and supertraces of P_1, \ldots, P_6 . First, we calculate the following auxiliary traces and supertraces (here we also use the notation $\xi \equiv M + N$):

$$\operatorname{tr} \mathbf{I} = \frac{1}{4} (\xi^{2} - \omega)^{2} , \qquad \operatorname{str} \mathbf{I} = \frac{1}{4} \omega^{2} (\omega - 1)^{2} ,$$

$$\operatorname{tr} \mathbf{P} = \frac{1}{2} \omega (\omega - 1) , \qquad \operatorname{str} \mathbf{P} = \frac{1}{2} \omega (\omega - 1) ,$$

$$\operatorname{tr} \mathbf{K} = \frac{1}{2} \omega (\omega - 1) , \qquad \operatorname{str} \mathbf{K} = \frac{1}{2} \omega (\omega - 1) ,$$

$$\operatorname{tr} \widehat{C}_{-} = -\frac{1}{4} (\xi^{2} - \omega) , \qquad \operatorname{str} \widehat{C}_{-} = -\frac{1}{4} \omega (\omega - 1) ,$$

$$\operatorname{tr} \widehat{C}_{+} = \frac{1}{4} (\xi^{2} - \omega) , \qquad \operatorname{str} \widehat{C}_{+} = \frac{1}{4} \omega (\omega - 1) ,$$

$$\operatorname{tr} \widehat{C}_{+}^{2} = \frac{2\xi^{4} + (\omega^{2} - 16\omega + 20)\xi^{2} + \omega^{3} + 4\omega^{2} - 12\omega}{8(\omega - 2)^{2}} , \quad \operatorname{str} \widehat{C}_{+}^{2} = \frac{3}{8} \omega (\omega - 1) .$$

$$(3.85)$$

Here the formulas for str in the second column correspond to the general formulas (2.56) in view of (3.57). From (3.85) and (3.80), we get the traces

$$\operatorname{tr} P_{1} = \frac{1}{8} (\xi^{4} - 2\xi^{2}(\omega + 2) - \omega(\omega - 6)) , \qquad \operatorname{tr} P_{4} = \frac{1}{12} (\xi^{4} - 10\xi^{2} + 3\omega(\omega - 2)) ,$$

$$\operatorname{tr} P_{2} = \frac{1}{2} (\xi^{2} - \omega) , \qquad \operatorname{tr} P_{5} = \frac{1}{24} (\xi^{4} + 2\xi^{2}(3\omega - 4) + 3\omega(\omega - 2) ,$$

$$\operatorname{tr} P_{3} = 1 , \qquad \operatorname{tr} P_{6} = \frac{1}{2} (\xi^{2} + \omega - 2)$$

$$(3.86)$$

snd supertraces of P_i :

$$str P_{1} = \frac{1}{8}\omega(\omega - 1)(\omega + 2)(\omega - 3) , \qquad str P_{4} = \frac{1}{12}\omega(\omega + 1)(\omega + 2)(\omega - 3) ,
str P_{2} = \frac{1}{2}\omega(\omega - 1) , \qquad str P_{5} = \frac{1}{24}\omega(\omega - 1)(\omega - 2)(\omega - 3) ,
str P_{3} = 1 , \qquad str P_{6} = \frac{1}{2}(\omega - 1)(\omega + 2)$$
(3.87)

for $\omega \neq 0, 1$. The dimension $\dim_{\overline{0}} V_i$ of the even part of the invariant subspace $V_i = V_{i\overline{0}} \oplus V_{i\overline{1}} \subseteq V_{(M|N)}^{\otimes 4}$ extracted by P_i is $\dim_{\overline{0}} V_i = \frac{1}{2}(\operatorname{tr} P_i + \operatorname{str} P_i)$, while the dimension of the odd part $V_{i\overline{1}}$ of V_i is $\dim_{\overline{1}} V_i = \frac{1}{2}(\operatorname{tr} P_i - \operatorname{str} P_i)$. Using (3.86) and (3.87), and substituting $\omega = M - N$ and $\xi = M + N$, we obtain the following values for dimensions of the invariant subspaces:

$$\dim_{\overline{0}} V_{1} = \frac{1}{8}M(M-1)(M+2)(M-3) + \frac{1}{8}N(N+1)(N-2)(N+3) + \frac{1}{4}MN(3MN+M-N+1),$$

$$\dim_{\overline{0}} V_{2} = \frac{1}{2}M(M-1) + \frac{1}{2}N(N+1), \qquad \dim_{\overline{0}} V_{3} = 1,$$

$$\dim_{\overline{0}} V_{4} = \frac{1}{12}M(M+1)(M+2)(M-3) + \frac{1}{12}N(N-1)(N-2)(N+3) + \frac{1}{2}MN(MN-1),$$

$$\dim_{\overline{0}} V_{5} = \frac{1}{24}M(M-1)(M-2)(M-3) + \frac{1}{24}N(N+1)(N+2)(N+3) + \frac{1}{4}MN(M-1)(N+1),$$

$$\dim_{\overline{0}} V_{6} = \frac{1}{2}(M-1)(M+2) + \frac{1}{2}N(N-1),$$

$$\dim_{\overline{1}} V_{1} = \frac{1}{2}MN(M(M-1) + (N-1)(N+2)), \qquad \dim_{\overline{1}} V_{2} = MN,$$

$$\dim_{\overline{1}} V_{3} = 0, \qquad \dim_{\overline{1}} V_{4} = \frac{1}{3}MN(M^{2} + N^{2} - 5),$$

$$\dim_{\overline{1}} V_{5} = \frac{1}{6}MN((M-1)(M-2) + (N+1)(N+2)), \qquad \dim_{\overline{1}} V_{6} = MN.$$

$$(3.89)$$

Note that the substitution $\omega = M$ (which implies N = 0) into the identites (3.62)–(3.66) yields analogous identities for the so(M) Lie algebra that are given in [15] and [16], and the dimensions (3.88) of the invariant subspaces of the osp(M|N) Lie superalgebra representation $\operatorname{ad}^{\otimes 2}$ transform into the corresponding dimensions of the invariant subspaces of so(M). Analogously, the substitution $\omega = -N$ (which means M = 0) transforms identities (3.62)–(3.66) into analogous identities for the algebra sp(N) while (3.88) gives the dimensions of the invariant subspaces of the sp(N) Lie algebra representation $\operatorname{ad}^{\otimes 2}$. Indeed, the substitutions M = 0 and N = 0 nullify the dimensions of the odd parts of the invariant subspaces (3.89), which corresponds to the transition from the osp(M|N) Lie superalgebra to the so(M) or sp(N) Lie algebras.

4 The $s\ell(M|N)$ Lie superalgebra

The $s\ell(M|N)$ Lie superalgebra (where $M \neq N$) is defined as the algebra of the operators $A: V_{(M|N)} \to V_{(M|N)}$ that satisfy:

$$str A = 0, (4.1)$$

and the Lie superbracket of which is given by (2.4). It is known that $s\ell(N,N)$ is not simple, and $s\ell(M|N) \cong s\ell(N|M)$, so we restrict ourselves to $\omega = M - N > 0$.

To build a basis of $s\ell(M|N)$, we use the same method as in the osp(M|N) case, that is, we build a projector onto the space of solutions of (4.1). Consider the operators P_0 and P_I that act on $\operatorname{End}(V_{(M|N)})$ by

$$P_0(E) = E - \frac{\operatorname{str} E}{M - N} I,$$

$$P_I(E) = \frac{\operatorname{str} E}{M - N} I,$$
(4.2)

for any $E \in \text{End}(V_{(M|N)})$ and I being the identity operator acting in $V_{(M|N)}$. Clearly,

$$str P_0(E) = 0,
str P_I(E) = str E,$$
(4.3)

so $P_A P_B = \delta_{AB} P_A$ (A, B = 0, I). Therefore, P_0 and P_I comprise a full system of projectors in $\operatorname{End}(V_{(M|N)})$. Analogously to the osp(M|N) case, (4.1) can be rewritten in terms of P_A as $P_I(A) = 0$ or $P_0(A) = A$, which implies the following general solution of (4.1):

$$A^{a}{}_{c} = E^{a}{}_{c} - \frac{(-1)^{[b]} E^{b}{}_{b}}{M - N} \delta^{a}_{c}, \tag{4.4}$$

where E is an arbitrary element of $\operatorname{End}(V_{(M|N)})$.

Consider the matrix identities $e_{ij}: V_{(M|N)} \to V_{(M|N)}$ with the components

$$(e_{ij})^a{}_b = \delta^a_i \delta_{jb} . (4.5)$$

Note that unlike the matrix identites e_i^j of the osp(M|N) case, both indices of e_{ij} are lower. The matrices of the basis elements $(T_{ij})^a{}_b$ of $s\ell(M|N)$ in the defining representation are obtained by substituting $E^a{}_b \to (e_{ij})^a{}_b$ into (4.4):

$$(T_{ij})^a{}_b = (e_{ij})^a{}_b - \frac{(-1)^{[i][j]}}{M - N} \delta_{ij} \delta^a_b. \tag{4.6}$$

The degree of T_{ij} coincides with that of e_{ij} and equals [i] + [j]. Besides, we have the following equality for T_{ij} :

$$Tr(T) \equiv T_{ii} = 0. \tag{4.7}$$

In order for the vectors $X = X^{ij}T_{ij}$ of the algebra $s\ell(M|N)$ to correspond uniquely to their coordinates X^{ij} , we require the numbers X^{ij} to satisfy the condition

$$(-1)^{[i]}X^{ii} = 0, (4.8)$$

which has the following advantage: for any $X = X^{ij}T_{ij} \in s\ell(M|N)$, in the defining representation $X = X^{ij}T_{ij} = X^{ij}e_{ij} \in s\ell(M|N)$. If the vector $X \in \operatorname{Mat}_{M+N}(\mathbb{C})$ is expanded over the basis $\{e_{ij}\}_{i,j=1}^{M+N}$, (4.8) means that X lies in $s\ell(M|N)$. We also impose the conditions analogous to (4.7) on the coordinates Y_{ij} of the covectors Y in the dual basis of (4.6):

$$Tr(Y) = Y_{ii} = 0. (4.9)$$

Calculating the Lie superbracket (2.4) of $T_{ij} \in s\ell(M|N)$ defined in (4.6), we obtain:

$$[T_{i_1i_2}, T_{j_1j_2}] = \delta_{j_1i_2} T_{i_1j_2} - (-1)^{([i_1] + [i_2])([j_1] + [j_2])} \delta_{i_1j_2} T_{j_1i_2} = T_{k_1k_2} X^{k_1k_2}_{i_1i_2, j_1j_2}, \tag{4.10}$$

where the structure constants $X^{k_1k_2}{}_{i_1i_2,j_1j_2}$ are written explicitly as

$$X^{k_1 k_2}{}_{i_1 i_2, j_1 j_2} = \delta_{j_1 i_2} \delta_{i_1}^{k_1} \delta_{j_2}^{k_2} - (-1)^{([i_1] + [i_2])([j_1] + [j_2])} \delta_{i_1 j_2} \delta_{j_1}^{k_1} \delta_{i_2}^{k_2}. \tag{4.11}$$

One can check that the pairs of indices (i_1, i_2) , (j_1, j_2) and (k_1, k_2) in (4.11) satisfy (4.7), (4.9) and (4.8), respectively.

The Cartan-Killing metric (2.9) of $s\ell(M|N)$ in the basis (4.6) is calculated by (2.9):

$$\mathsf{g}_{i_1 i_2, j_1 j_2} = 2\omega \left((-1)^{[i_1][j_2]} \delta_{j_1 i_2} \delta_{i_1 j_2} - \frac{(-1)^{[i_1] + [j_2]}}{\omega} \delta_{i_1 i_2} \delta_{j_1 j_2} \right),\tag{4.12}$$

where $\omega = M - N$, as in the osp(M|N) case. For $\omega = 0$, the metric (4.12) is degenerate: $\mathsf{g}_{i_1i_2,j_1j_2} = -2(-1)^{[i_1]+[j_2]}\delta_{i_1i_2}\delta_{j_1j_2}$. However, as we have mentioned, this case is omitted in our paper. Note also that for (i_1,i_2) and (j_1,j_2) in (4.12) the condition (4.9) holds.

Let us introduce the projector \overline{I} that acts on $V_{(M|N)}^{\otimes 2}$ with the components

$$\overline{I}^{i_1 i_2}{}_{j_1 j_2} = \delta^{i_1}_{j_1} \delta^{i_2}_{j_2} - \frac{(-1)^{[j_1][j_2]}}{\omega} \delta^{i_1 i_2} \delta_{j_1 j_2} \quad \Rightarrow \quad \overline{I}^2 = \overline{I} . \tag{4.13}$$

It maps an arbitrary $Y \in V_{(M|N)}^{\otimes 2}$ with the components Y^{ik} to $X \in V_{(M|N)}^{\otimes 2}$ with the components $X^{ik} = \overline{I}^{ik}_{\ j\ell}Y^{j\ell}$ that satisfy (4.8). If we identify the space $s\ell(M|N)$ with $\overline{I}(V_{(M|N)}^{\otimes 2}) \subset V_{(M|N)}^{\otimes 2}$, then \overline{I} can be viewed as the identity operator \overline{I} acting in the algebra $s\ell(M|N)$.

The components of the inverse Cartan-Killing metric in the basis (4.6) are defined by (2.14):

$$\mathsf{g}^{i_1 i_2, j_1 j_2} \mathsf{g}_{j_1 j_2, k_1 k_2} = \overline{I}^{i_1 i_2}_{k_1 k_2}, \qquad \mathsf{g}_{i_1 i_2, j_1 j_2} \mathsf{g}^{j_1 j_2, k_1 k_2} = \overline{I}^{k_1 k_2}_{i_1 i_2}. \tag{4.14}$$

Direct calculations yield

$$\mathsf{g}^{i_1 i_2, j_1 j_2} = \frac{1}{2\omega} \left((-1)^{[j_1][i_2]} \delta^{j_1 i_2} \delta^{i_1 j_2} - \frac{1}{\omega} \delta^{i_1 i_2} \delta^{j_1 j_2} \right) \,, \tag{4.15}$$

where the pairs of indices (i_1, i_2) and (j_1, j_2) satisfy (4.9), as expected.

4.1 Projectors onto invariant subspaces of the tensor product of two defining representations

Using (2.34), (2.20), (4.6) and (4.15), we can write the matrix of the split Casimir operator of $s\ell(M|N)$ in the defining representation:

$$\widehat{C}_{f}^{k_1 k_2}{}_{m_1 m_2} = \frac{1}{2\omega} \left((-1)^{[m_1][m_2]} \delta_{m_2}^{k_1} \delta_{m_1}^{k_2} - \frac{1}{\omega} \delta_{m_1}^{k_1} \delta_{m_2}^{k_2} \right). \tag{4.16}$$

Define the identity operator 1 and the superpermutation \mathcal{P} acting in $V_{(M|N)}^{\otimes 2}$:

$$\mathbf{1} = e_{ii} \otimes e_{jj} \qquad \Longrightarrow \qquad \mathbf{1}^{k_1 k_2}{}_{m_1 m_2} = \delta^{k_1}_{m_1} \delta^{k_2}_{m_2} \,, \tag{4.17}$$

$$\mathcal{P} = (-1)^{[j]} e_{ij} \otimes e_{ji} \qquad \Longrightarrow \qquad \mathcal{P}^{k_1 k_2}{}_{m_1 m_2} = (-1)^{[k_1][k_2]} \delta^{k_1}_{m_2} \delta^{k_2}_{m_2} , \qquad (4.18)$$

where e_{ij} are the matrix identities (4.5) on $V_{(M|N)}$. The matrices of **1** and \mathcal{P} are presented in the right-hand sides of these equalities. By (4.18), $\mathcal{P}^2 = \mathbf{1}$. A comparison of (4.16) and (4.17), (4.18) shows that \widehat{C}_f can be written in terms of **1** and \mathcal{P} as

$$\widehat{C}_f = \frac{1}{2\omega} \left(\mathcal{P} - \frac{1}{\omega} \mathbf{1} \right). \tag{4.19}$$

Using this formula, we obtain the second-degree characteristic identity for \widehat{C}_f :

$$\widehat{C}_f^2 + \frac{1}{\omega^2} \widehat{C}_f - \frac{\omega^2 - 1}{4\omega^4} \mathbf{1} = 0.$$
 (4.20)

The left-hand side of this equality can be factorized, which yields

$$\left(\widehat{C}_f - \frac{\omega - 1}{2\omega^2} \mathbf{1}\right) \left(\widehat{C}_f + \frac{\omega + 1}{2\omega^2} \mathbf{1}\right) = 0. \tag{4.21}$$

The projectors P_+ and P_- onto the eigenspaces of \widehat{C}_f that correspond to the roots $a_+ = \frac{\omega - 1}{2\omega^2}$ and $a_- = -\frac{\omega + 1}{2\omega^2}$ of equation (4.21) are built by (2.39) with p = 2:

$$P_{\pm} = \pm \left(\omega \widehat{C}_f + \frac{1 \pm \omega}{2\omega}\right) = \frac{1}{2} (\mathbf{1} \pm \mathcal{P}). \tag{4.22}$$

Thus, P_+ and P_- turn out to be the super-symmetrizer and the super-antisymmetriser in $V_{(M|N)}^{\otimes 2}$

The solution of the graded Yang-Baxter equation (3.40) that is invariant under the action of $s\ell(M|N)$ in the defining representation can be written in several equivalent ways:

$$R(u) = \frac{u + \mathcal{P}}{1 - u} = \frac{1 + u}{1 - u} P_{+} - P_{-}$$
(4.23)

where u is the spectral parameter. Analogously to the osp(M|N) case, this solution can be expressed as a rational function of \widehat{C}_f :

$$R(u) = \frac{P_{+} + u}{P_{+} - u} = \frac{\omega \hat{C}_{f} + \frac{1 + \omega}{2\omega} + u}{\omega \hat{C}_{f} + \frac{1 + \omega}{2\omega} - u}.$$
 (4.24)

Note that the solution R(u) is unitary: $\mathcal{P}R(u)\mathcal{P}R(-u) = 1$. It is defined up to multiplication by an arbitrary function f(u) for which f(u)f(-u) = 1.

4.2 Projectors onto invariant subspaces of the tensor product of two adjoint representations

In order to make the following calculations more concise, we introduce the operator $\mathcal{K}: V_{(M|N)}^{\otimes 2} \to V_{(M|N)}^{\otimes 2}$ the components of which in the homogeneous basis $\{e_{i_1} \otimes e_{i_2}\}$ of this space are

$$\mathcal{K}^{i_1 i_2}{}_{j_1 j_2} = (-1)^{[j_1][j_2]} \delta^{i_1 i_2} \delta_{j_1 j_2} \,. \tag{4.25}$$

One can derive the following identities for \mathcal{P}_{ab} , $\mathcal{K}_{ab}: V_{(M|N)}^{\otimes 4} \to V_{(M|N)}^{\otimes 4}$, $(a, b = 1, \dots, 4)$:

$$\mathcal{K}_{ab}\mathcal{K}_{ab} = \omega \mathcal{K}_{ab}, \quad \mathcal{P}_{ab}\mathcal{K}_{ad}\mathcal{K}_{bc} = \mathcal{P}_{cd}\mathcal{K}_{ad}\mathcal{K}_{bc}, \quad \mathcal{K}_{ad}\mathcal{K}_{bc}\mathcal{P}_{ab} = \mathcal{K}_{ad}\mathcal{K}_{bc}\mathcal{P}_{cd},
\mathcal{K}_{ab}\mathcal{P}_{ab}\mathcal{K}_{bc} = \mathcal{K}_{ab}\mathcal{P}_{ab}\mathcal{P}_{ac} = \mathcal{P}_{ac}\mathcal{P}_{bc}\mathcal{K}_{bc}, \quad \mathcal{K}_{ab}\mathcal{P}_{bc}\mathcal{K}_{bc} = \mathcal{K}_{ab}\mathcal{P}_{ab}\mathcal{P}_{ac} = \mathcal{P}_{ac}\mathcal{P}_{bc}\mathcal{K}_{bc},
\mathcal{K}_{ab}\mathcal{P}_{ab}\mathcal{K}_{bc} = \mathcal{K}_{ab}\mathcal{P}_{ab}\mathcal{P}_{ac} = \mathcal{P}_{ac}\mathcal{P}_{bc}\mathcal{K}_{bc}, \quad \mathcal{K}_{ab}\mathcal{P}_{bc}\mathcal{K}_{bc} = \mathcal{K}_{ab}\mathcal{P}_{ab}\mathcal{P}_{ac} = \mathcal{P}_{ac}\mathcal{P}_{bc}\mathcal{K}_{bc},$$

$$(4.26)$$

 $(\mathcal{P}_{ab} \text{ and } \mathcal{K}_{ab} \text{ are defined in (2.24)})$. It is worth pointing out that \overline{I} , defined in (4.13), can be written as $\overline{I} = \mathbf{1} - \frac{1}{\omega} \mathcal{K}$ where $\mathbf{1}$ is given in (4.17).

The components of the basis elements $T_{i_1i_2} \in s\ell(M|N)$ in the adjoint representation are precisely the structure constants (4.11):

$$(T_{i_1 i_2})^{k_1 k_2}{}_{j_1 j_2} = \delta_{j_1 i_2} \delta_{i_1}^{k_1} \delta_{j_2}^{k_2} - (-1)^{([i_1] + [i_2])([j_1] + [j_2])} \delta_{i_1 j_2} \delta_{j_1}^{k_1} \delta_{i_2}^{k_2}. \tag{4.27}$$

Now using (2.34), (4.27) and (4.15), we find an explicit form of \widehat{C}_{ad} . In terms of the operators (4.18) and (4.25), it can be written as follows:

$$\widehat{C}_{ad} = \frac{1}{2\omega} (\mathcal{P}_{13} + \mathcal{P}_{24} - \mathcal{K}_{32} - \mathcal{K}_{14}). \tag{4.28}$$

In what follows, we need three more operators: \mathbf{K} , $\mathbf{P}^{\mathrm{ad}}:V_{\mathrm{ad}}^{\otimes 2}\to V_{\mathrm{ad}}^{\otimes 2}$ and $\mathbf{P}:V_{(M|N)}^{\otimes 4}\to V_{(M|N)}^{\otimes 4}$, \mathbf{K} is defined by

$$\mathbf{K}^{i_1 i_2 i_3 i_4}{}_{j_1 j_2 j_3 j_4} = \overline{\mathbf{g}}^{i_1 i_2 i_3 i_4} \mathbf{g}_{j_1 j_2 j_3 j_4}, \tag{4.29}$$

and expressing it in terms of the operators (4.18), (4.25), we obtain

$$\mathbf{K} = \mathcal{K}_{32}\mathcal{K}_{14} - \frac{1}{\omega}\mathcal{P}_{24}\mathcal{K}_{12}\mathcal{K}_{34} - \frac{1}{\omega}\mathcal{P}_{13}\mathcal{K}_{32}\mathcal{K}_{14} + \frac{1}{\omega^2}\mathcal{K}_{12}\mathcal{K}_{34}.$$
 (4.30)

Furthermore, $\mathbf{K}^2 = (\omega^2 - 1)\mathbf{K}$. Recall that $\mathcal{K}_{32} = \mathcal{P}_{23}\mathcal{K}_{23}\mathcal{P}_{23}$, $\mathcal{K}_{14} = \mathcal{P}_{34}\mathcal{P}_{23}\mathcal{K}_{12}\mathcal{P}_{23}\mathcal{P}_{34}$ etc. The operator **P** permutes the first and third, second and fourth factors in $V_{(M|N)}^{\otimes 4}$ and is defined by

$$\mathbf{P} = \mathcal{P}_{13}\mathcal{P}_{24} \implies (\mathbf{P})^{i_1 i_2 i_3 i_4}_{j_1 j_2 j_3 j_4} = (-1)^{[i_1][i_3] + [i_1][i_4] + [i_2][i_3] + [i_2][i_4]}_{\delta_{j_3}^{i_1}} \delta_{j_4}^{i_2} \delta_{j_1}^{i_3} \delta_{j_2}^{i_4}, \tag{4.31}$$

so we have $\mathbf{P}^2 = I$ where I is the identity operator acting in $V_{(M|N)}^{\otimes 4}$. Finally, the operator

$$\mathbf{P}^{(\mathrm{ad})} = \overline{I}_{12}\overline{I}_{34}\mathbf{P} , \qquad (4.32)$$

where \overline{I} is given in (4.13), is the permutation operator in the space $V_{ad} \otimes V_{ad} \subset V_{(M|N)}^{\otimes 4}$.

Note that **P** commutes with both \widehat{C}_{ad} and **K**:

$$\mathbf{P}\hat{C}_{\mathrm{ad}} = \hat{C}_{\mathrm{ad}}\mathbf{P}, \qquad \mathbf{P}\mathbf{K} = \mathbf{K} = \mathbf{K}\mathbf{P}.$$
 (4.33)

Using **P** and $\mathbf{P}^{(\mathrm{ad})}$ we define the symmetrizer $\mathbf{P}_{+}^{(\mathrm{ad})}$ and antisymmetrizer $\mathbf{P}_{-}^{(\mathrm{ad})}$ in $V_{\mathrm{ad}}^{\otimes 2}$:

$$\mathbf{P}_{\pm}^{(\mathrm{ad})} = \frac{1}{2} (\mathbf{I} \pm \mathbf{P}^{(\mathrm{ad})}) = \frac{1}{2} (I \pm \mathbf{P}) \overline{I}_{12} \overline{I}_{34} = \frac{1}{2} \overline{I}_{12} \overline{I}_{34} (I \pm \mathbf{P}) \implies \\
\mathbf{P}_{-}^{(\mathrm{ad})} = \frac{1}{2} (I - \mathcal{P}_{13} \mathcal{P}_{24}) (I - \frac{1}{\omega} (\mathcal{K}_{12} + \mathcal{K}_{34})), \qquad (4.34) \\
\mathbf{P}_{+}^{(\mathrm{ad})} = \frac{1}{2} (I + \mathcal{P}_{13} \mathcal{P}_{24}) (I - \frac{1}{\omega} (\mathcal{K}_{12} + \mathcal{K}_{34}) + \frac{1}{\omega^{2}} \mathcal{K}_{12} \mathcal{K}_{34}),$$

where $\mathbf{I} = \overline{I}_{12}\overline{I}_{34}$ is the identity operator acting in $V_{\mathrm{ad}}^{\otimes 2}$ that satisfies

$$\mathbf{IP}^{(\mathrm{ad})} = \mathbf{P}^{(\mathrm{ad})} = \mathbf{P}^{(\mathrm{ad})}\mathbf{I}, \quad \mathbf{IK} = \mathbf{K} = \mathbf{KI}, \quad \mathbf{I}\widehat{C}_{\mathrm{ad}} = \widehat{C}_{\mathrm{ad}} = \widehat{C}_{\mathrm{ad}}\mathbf{I}.$$
 (4.35)

Define now the symmetric and antisymmetric parts of the split Casimir operator (4.28):

$$\widehat{C}_{+} = \mathbf{P}_{+}^{(\mathrm{ad})} \widehat{C}_{\mathrm{ad}} = \frac{1}{2} (I + \mathbf{P}) \widehat{C}_{\mathrm{ad}}
= \frac{1}{4\omega} (2\mathcal{P}_{13} + 2\mathcal{P}_{24} - (I + \mathcal{P}_{13}\mathcal{P}_{24}) \mathcal{K}_{32} - (I + \mathcal{P}_{13}\mathcal{P}_{24}) \mathcal{K}_{14}),
\widehat{C}_{-} = \mathbf{P}_{-}^{(\mathrm{ad})} \widehat{C}_{\mathrm{ad}} = \frac{1}{2} (I - \mathbf{P}) \widehat{C}_{\mathrm{ad}}
= \frac{1}{4\omega} (\mathcal{P}_{13}\mathcal{P}_{24} - I) (\mathcal{K}_{14} + \mathcal{K}_{32}) = \frac{1}{4\omega} (\mathcal{K}_{14} + \mathcal{K}_{32}) (\mathcal{P}_{13}\mathcal{P}_{24} - I).$$
(4.36)

By (4.26) and (4.28), the following relations hold for \widehat{C}_- , \widehat{C}_+ and \mathbf{K} :

$$\widehat{C}_{+} + \widehat{C}_{-} = \widehat{C}_{ad}, \qquad \mathbf{P}\widehat{C}_{\pm} = \widehat{C}_{\pm}\mathbf{P} = \pm \widehat{C}_{\pm}, \qquad \widehat{C}_{+}\widehat{C}_{-} = \widehat{C}_{-}\widehat{C}_{+} = 0,
\mathbf{K}\widehat{C}_{-} = \widehat{C}_{-}\mathbf{K} = 0, \qquad \mathbf{K}\widehat{C}_{+} = \widehat{C}_{+}\mathbf{K} = -\mathbf{K},
\mathbf{K}\widehat{C}_{ad} = \widehat{C}_{ad}\mathbf{K} = -\mathbf{K}.$$
(4.37)

Proposition 3. The antisymmetric \widehat{C}_{-} and symmetric \widehat{C}_{+} parts of the split Casimir operator of the $s\ell(M|N)$ Lie superalgebra for $\omega \neq 0, 1, 2$ satisfy

$$\hat{C}_{-}^{2} = -\frac{1}{2}\hat{C}_{-} \iff \hat{C}_{-}(\hat{C}_{-} + \frac{1}{2}\mathbf{I}) = 0.$$
(4.38)

$$\widehat{C}_{+}^{3} = -\frac{1}{2}\widehat{C}_{+}^{2} + \frac{1}{\omega^{2}}\widehat{C}_{+} + \frac{1}{4\omega^{2}}(\mathbf{I} + \mathbf{P}^{(ad)} - 2\mathbf{K})$$
(4.39)

$$\widehat{C}_{+}^{4} = -\frac{3}{2}\widehat{C}_{+}^{3} - \frac{\omega^{2} - 2}{2\omega^{2}}\widehat{C}_{+}^{2} + \frac{3}{2\omega^{2}}\widehat{C}_{+} + \frac{1}{4\omega^{2}}(\mathbf{I} + \mathbf{P}^{(ad)}). \tag{4.40}$$

$$\widehat{C}_{+}(\widehat{C}_{+} + \mathbf{I})(\widehat{C}_{+} - \frac{1}{\omega}\mathbf{I})(\widehat{C}_{+} + \frac{1}{\omega}\mathbf{I})(\widehat{C}_{+} + \frac{1}{2}\mathbf{I}) = 0.$$
(4.41)

The split Casimir operator $\hat{C}_{ad} = \hat{C}_{-} + \hat{C}_{+}$ for $\omega \neq 0, 1, 2$ satisfies

$$\widehat{C}_{\mathrm{ad}}(\widehat{C}_{\mathrm{ad}} + \mathbf{I})(\widehat{C}_{\mathrm{ad}} - \frac{1}{\omega}\mathbf{I})(\widehat{C}_{\mathrm{ad}} + \frac{1}{\omega}\mathbf{I})(\widehat{C}_{\mathrm{ad}} + \frac{1}{2}\mathbf{I}) = 0.$$

$$(4.42)$$

Proof. Identity (4.38) for $s\ell(M|N)$ is a special case of (2.54) that holds for all Lie superalgebras with the nondegenerate Cartan-Killing metric.

By (4.28) and (4.26) we obtain for \widehat{C}_{+}^{2}

$$\widehat{C}_{+}^{2} = \frac{1}{8\omega^{2}} (I + \mathcal{P}_{13}\mathcal{P}_{24}) (4I + 2\mathcal{K}_{32}\mathcal{K}_{14} + \omega\mathcal{K}_{32} + \omega\mathcal{K}_{14}) - \frac{1}{4\omega^{2}} (\mathcal{P}_{13} + \mathcal{P}_{24}) (\mathcal{K}_{12} + \mathcal{K}_{34} + \mathcal{K}_{32} + \mathcal{K}_{14}).$$

$$(4.43)$$

Multiplying (4.43) by \hat{C}_+ yields (4.39). Multiplying once more (4.39) by \hat{C}_+ and using (4.37), we obtain:

$$\widehat{C}_{+}^{4} = -\frac{1}{2}\widehat{C}_{+}^{3} + \frac{1}{\omega^{2}}\widehat{C}_{+}^{2} + \frac{1}{2\omega^{2}}\widehat{C}_{+} + \frac{1}{2\omega^{2}}\mathbf{K}.$$
(4.44)

Now we express **K** from (4.44) and substitute the result into (4.39), which gives (4.40). Multiplying both sides of (4.44) by $(\hat{C}_+ + \mathbf{I})$ and using the last identity from (4.37), or multiplying both sides of (4.40) by \hat{C}_+ and using the second identity from (4.37) yields

$$\widehat{C}_{+}^{5} = -\frac{3}{2}\widehat{C}_{+}^{4} - \frac{\omega^{2} - 2}{2\omega^{2}}\widehat{C}_{+}^{3} - \frac{3}{2\omega^{2}}\widehat{C}_{+}^{2} - \frac{1}{2\omega^{2}}\widehat{C}_{+}, \tag{4.45}$$

which can be rewritten as

$$\widehat{C}_{+}^{5} + \frac{3}{2}\widehat{C}_{+}^{4} + \frac{\omega^{2} - 2}{2\omega^{2}}\widehat{C}_{+}^{3} - \frac{3}{2\omega^{2}}\widehat{C}_{+}^{2} - \frac{1}{2\omega^{2}}\widehat{C}_{+} = 0.$$
(4.46)

Identity (4.46) is characteristic for \widehat{C}_{+} . The roots of the polynomial on the left-hand side of (4.46) are

$$a_1 = 0, \quad a_2 = -1, \quad a_3 = \frac{1}{\omega}, \quad a_4 = -\frac{1}{\omega}, \quad a_5 = -\frac{1}{2},$$
 (4.47)

hence the characteristic identity (4.46) takes the form (4.41). Note that for $\omega = 1, 2$ we have degenerate roots, and these cases are considered separately (see below).

Since $\widehat{C}_{-} = \widehat{C}_{-} \mathbf{P}_{-}^{(\mathrm{ad})}$ and $\widehat{C}_{+} = \widehat{C}_{+} \mathbf{P}_{+}^{(\mathrm{ad})}$, from (4.38) and (4.41) one infers

$$\widehat{C}_{-}(\widehat{C}_{-} + \frac{1}{2}\mathbf{P}_{-}^{(ad)}) = 0,$$
 (4.48)

$$\widehat{C}_{+}(\widehat{C}_{+} + \mathbf{P}_{+}^{(\mathrm{ad})})(\widehat{C}_{+} - \frac{1}{\omega}\mathbf{P}_{+}^{(\mathrm{ad})})(\widehat{C}_{+} + \frac{1}{\omega}\mathbf{P}_{+}^{(\mathrm{ad})})(\widehat{C}_{+} + \frac{1}{2}\mathbf{P}_{+}^{(\mathrm{ad})}) = 0. \tag{4.49}$$

These identities can be viewed as characteristic for \widehat{C}_{-} and \widehat{C}_{+} that are restricted to $\mathbf{P}_{-}^{(\mathrm{ad})}(V_{\mathrm{ad}}^{\otimes 2})$ and $\mathbf{P}_{+}^{(\mathrm{ad})}(V_{\mathrm{ad}}^{\otimes 2})$, respectively. In this sense, the roots of the characteristic polynomial on the right of (4.49) are

$$a'_1 = -1, \quad a'_2 = \frac{1}{\omega}, \quad a'_3 = -\frac{1}{\omega}, \quad a'_4 = -\frac{1}{2}.$$
 (4.50)

To find the characteristic polynomial of \widehat{C}_{ad} , we substitute \widehat{C}_{ad} for \widehat{C}_{+} in (4.41) and use $\widehat{C}_{ad} = \widehat{C}_{-} + \widehat{C}_{+}$ and $\widehat{C}_{+}\widehat{C}_{-} = 0$:

$$\widehat{C}_{ad}(\widehat{C}_{ad} + \mathbf{I})(\widehat{C}_{ad} - \frac{1}{\omega}\mathbf{I})(\widehat{C}_{ad} + \frac{1}{\omega}\mathbf{I})(\widehat{C}_{ad} + \frac{1}{2}\mathbf{I}) =
= \widehat{C}_{+}(\widehat{C}_{+} + \mathbf{I})(\widehat{C}_{+} - \frac{1}{\omega}\mathbf{I})(\widehat{C}_{+} + \frac{1}{\omega}\mathbf{I})(\widehat{C}_{+} + \frac{1}{2}\mathbf{I}) +
+ \widehat{C}_{-}(\widehat{C}_{-} + \mathbf{I})(\widehat{C}_{-} - \frac{1}{\omega}\mathbf{I})(\widehat{C}_{-} + \frac{1}{\omega}\mathbf{I})(\widehat{C}_{-} + \frac{1}{2}\mathbf{I}) = 0,$$
(4.51)

where the last relation holds by (4.38) and (4.41). Therefore, the characteristic identity for \widehat{C}_{ad} is given by (4.42).

Note that the roots of the polynomial on the left of (4.42) coincide with (4.47), and for $\omega = 1, 2$ we have degenerate roots:

$$\omega = 1 \implies a_2 = a_4 = -1, \qquad \omega = 2 \implies a_4 = a_5 = -\frac{1}{2}$$
 (4.52)

Leaving in the polynomial on the left of (4.42) only one of the parentheses that correspond to the roots (4.52), we get for $\omega = 1$:

$$\widehat{C}_{ad}(\widehat{C}_{ad} + \mathbf{I})(\widehat{C}_{ad} - \mathbf{I})(\widehat{C}_{ad} + \frac{1}{2}\mathbf{I}) = \frac{1}{2}\mathbf{K} \neq 0.$$
(4.53)

Therefore, for $\omega = 1$ the $s\ell(M|N)$ representation ad^{$\otimes 2$} is not completely reducible. For $\omega = 2$, we analogously get

$$\widehat{C}_{ad}(\widehat{C}_{ad} + \mathbf{I})(\widehat{C}_{ad} - \frac{1}{2}\mathbf{I})(\widehat{C}_{ad} + \frac{1}{2}\mathbf{I}) = \frac{1}{16}(\mathbf{P}_{+}^{(ad)} + \mathbf{K}) - \frac{1}{4}\widehat{C}_{+}^{2}.$$
(4.54)

Using the component form of (4.54), one can check that for M=2, N=0 this expression is nullified, while for M=3, N=1 it does not. Therefore, the representation $\mathrm{ad}^{\otimes 2}$ of $s\ell(M|N)$ for $\omega=2$ in general is not completely reducible. The exceptional cases $\omega\neq 1,2$ are to be considered later in this section.

To construct projectors onto invariant subspaces of the representation $\operatorname{ad}^{\otimes 2}$ of $s\ell(M|N)$, we utilize the fact that for an arbitrary Lie superalgebra the symmetric and antisymmetric parts of $V_{\operatorname{ad}}^{\otimes 2}$ are invariant (see Section 2.3 and (2.45)). The symmetric invariant subspaces $\mathbf{P}_{+}^{(\operatorname{ad})}(V_{\operatorname{ad}}^{\otimes 2})$ of $V_{\operatorname{ad}}^{\otimes 2}$ can be expressed as eigenspaces of \widehat{C}_{+} , which can be viewed as acting in $\mathbf{P}_{+}^{(\operatorname{ad})}(V_{\operatorname{ad}}^{\otimes 2})$. The role of the identity operator here is played by $\mathbf{P}_{+}^{(\operatorname{ad})}$. By (2.39), where we suppose p=5, $\widehat{C}_{T}=\widehat{C}_{+}$, $I_{T}^{\otimes 2}=\mathbf{P}_{+}^{(\operatorname{ad})}$, and a_{i} are given in (4.50),

$$P_{a'_{1}}^{(+)} \equiv P_{1}^{(+)} = \frac{1}{\omega^{2} - 1} \mathbf{K},$$

$$P_{a'_{2}}^{(+)} \equiv P_{2}^{(+)} = -\frac{\omega}{2(\omega + 1)(\omega + 2)} \mathbf{K} + \frac{\omega^{2}}{\omega + 2} \widehat{C}_{+}^{2} + \frac{\omega}{2} \widehat{C}_{+} + \frac{\omega}{2(\omega + 2)} \mathbf{P}_{+}^{(\text{ad})},$$

$$P_{a'_{3}}^{(+)} \equiv P_{3}^{(+)} = \frac{\omega}{2(\omega - 1)(\omega - 2)} \mathbf{K} - \frac{\omega^{2}}{\omega - 2} \widehat{C}_{+}^{2} - \frac{\omega}{2} \widehat{C}_{+} + \frac{\omega}{2(\omega - 2)} \mathbf{P}_{+}^{(\text{ad})},$$

$$P_{a'_{4}}^{(+)} \equiv P_{4}^{(+)} = \frac{4}{\omega^{2} - 4} (\omega^{2} \widehat{C}_{+}^{2} - \mathbf{P}_{+}^{(\text{ad})} - \mathbf{K}).$$

$$(4.55)$$

One can easily check that $P_1^{(+)} + P_2^{(+)} + P_3^{(+)} + P_4^{(+)} = \mathbf{P}_+^{(ad)}$

For $\omega = 1$, the characteristic identity for \widehat{C}_{+} is

$$(\widehat{C}_{+} + \mathbf{I})^{2}(\widehat{C}_{+} - \mathbf{I})(\widehat{C}_{+} + \frac{1}{2}\mathbf{I})\mathbf{P}_{+}^{(\mathrm{ad})} = 0.$$

$$(4.56)$$

Therefore, in (2.40) one needs to put $a'_1 = -1$, $a'_2 = 1$, $a'_3 = -\frac{1}{2}$, $k_1 = 2$, $k_2 = k_3 = 1$. Similarly to the considered above general case, we build projectors onto symmetric generalized eigenspaces of \widehat{C}_+ . By (2.41), where we substitute $\mathbf{P}_+^{(\mathrm{ad})}$ for $\mathbf{I}_T^{\otimes 2}$,

$$P_{a'_{1}}^{(+)} \equiv P_{1}^{(+)} = -\frac{1}{2} \mathbf{P}_{+}^{(\mathrm{ad})} - \frac{5}{4} \mathbf{K} - \frac{1}{2} \widehat{C}_{+} + \widehat{C}_{+}^{2},$$

$$P_{a'_{2}}^{(+)} \equiv P_{2}^{(+)} = \frac{1}{6} \mathbf{P}_{+}^{(\mathrm{ad})} - \frac{1}{12} \mathbf{K} + \frac{1}{2} \widehat{C}_{+} + \frac{1}{3} \widehat{C}_{+}^{2},$$

$$P_{a'_{3}}^{(+)} \equiv P_{3}^{(+)} = \frac{4}{3} \mathbf{P}_{+}^{(\mathrm{ad})} + \frac{4}{3} \mathbf{K} - \frac{4}{3} \widehat{C}_{+}^{2}.$$

$$(4.57)$$

Of these operators, $P_2^{(+)}$ and $P_3^{(+)}$ project onto the eigenspaces of \widehat{C}_+ , while $P_1^{(+)}$ extracts its generalized eigenspace. Thus, the action of $s\ell(M|N)$ in $P_1^{(+)}(V_{ad}^{\otimes 2})$ for $\omega=1$ is reducible but not completely reducible. It is related to the fact that \mathbf{K} , satisfying $\mathbf{K}^2=0$, is nilpotent and thus not diagonalizable in this case.

For $\omega = 2$, the characteristic identity for \widehat{C}_{+} is

$$(\widehat{C}_{+} + \mathbf{I})(\widehat{C}_{+} - \frac{1}{2}\mathbf{I})(\widehat{C}_{+} + \frac{1}{2}\mathbf{I})^{2}\mathbf{P}_{+}^{(ad)} = 0,$$
 (4.58)

so in (2.40) we put $a_1' = -1$, $a_2' = \frac{1}{2}$, $a_3' = -\frac{1}{2}$, $k_1 = k_2 = 1$, $k_3 = 2$. The projectors onto the generalized eigenspaces of \widehat{C}_+ are given by (2.41):

$$P_{a'_{1}}^{(+)} \equiv P_{1}^{(+)} = \frac{1}{3}\mathbf{K},$$

$$P_{a'_{2}}^{(+)} \equiv P_{2}^{(+)} = \frac{1}{4}\mathbf{P}_{+}^{(\mathrm{ad})} - \frac{1}{12}\mathbf{K} + \hat{C}_{+} + \hat{C}_{+}^{2},$$

$$P_{a'_{3}}^{(+)} \equiv P_{3}^{(+)} = \frac{3}{4}\mathbf{P}_{+}^{(\mathrm{ad})} - \frac{1}{4}\mathbf{K} - \hat{C}_{+} - \hat{C}_{+}^{2}.$$
(4.59)

Of these operators, $P_1^{(+)}$ and $P_2^{(+)}$ project onto the eigenspaces of \widehat{C}_+ while the image of $P_3^{(+)}$ is a generalized eigenspace of \widehat{C}_+ . Thus, the restriction of the representation $\operatorname{ad}^{\otimes 2}$ to $P_3^{(+)}(V_{\operatorname{ad}}^{\otimes 2})$ is neither irreducible nor completely reducible for $\omega \equiv M - N = 2$. For simplicity, in what follows we assume $P_4^{(+)} = 0$ for $\omega = 1, 2$.

completely reducible for $\omega \equiv M-N=2$. For simplicity, in what follows we assume $\mathbf{P}_4^{(+)}=0$ for $\omega=1,2$. To find projectors onto antisymmetric invariant subspaces of the representation $\mathrm{ad}^{\otimes 2}$ of $s\ell(M|N)$ that can be expressed as eigenspaces of \widehat{C}_- (which is viewed here as acting in $\mathbf{P}_-^{(\mathrm{ad})}(V_{\mathrm{ad}}^{\otimes 2})$, where the role of the identity operator is played by $\mathbf{P}_-^{(\mathrm{ad})}$), we use (2.39) where, in accordance with (4.48), we put p=2, $\widehat{C}_T=\widehat{C}_-$, $I_T^{\otimes 2}=\mathbf{P}_-^{(\mathrm{ad})}$, $a_1=0$ and $a_2=-\frac{1}{2}$:

$$P_{a_1}^{(-)} \equiv P_1^{(-)} = 2\hat{C}_- + \mathbf{P}_-^{(ad)}, \qquad P_{a_2}^{(-)} \equiv P_2^{(-)} = -2\hat{C}_-.$$
 (4.60)

Apparently, $P_1^{(-)} + P_2^{(-)} = \mathbf{P}_-^{(ad)}$, i.e. $P_1^{(-)} + P_2^{(-)} + P_2^{(+)} + P_3^{(+)} + P_4^{(+)} = \mathbf{P}_-^{(ad)} + \mathbf{P}_+^{(ad)} = \mathbf{I}$.

As a result, we have the following full system of projectors: $P_1^{(-)}$, $P_2^{(-)}$, $P_2^{(+)}$, $P_3^{(+)}$, $P_4^{(+)}$, $P_5^{(+)}$. However, not all of those projectors are primitive. In order to show this, we use the following relation, which holds for the generators T_{ij} of $s\ell(M|N)$ in the defining representation T_f (we denote $T_f(T_{ij}) = T_{ij}$):

$$T_{ij}T_{km} + (-1)^{([i]+[j])([k]+[m])}T_{km}T_{ij} \equiv [T_{ij}, T_{km}]_{+} = D^{rs}_{ij,km}T_{rs} + \alpha g_{ij,km}. \tag{4.61}$$

Here $\alpha = \frac{2c_2(T_f)}{\operatorname{sdim}(s\ell(M|N))} = \frac{1}{\omega^2}$ where $c_2(T_f) = \frac{\omega^2 - 1}{2\omega^2}$ is the value of the quadratic Casimir operator (2.17) in the defining representation T_f , and the numbers $D^{rs}_{ij,km}$ are called the symmetric in pairs of indices (ij) and (km) structure constants of $s\ell(M|N)$. Define the operators \widetilde{C} and \widetilde{C}_- , that act on $V_{ad}^{\otimes 2}$:

$$\widetilde{C}^{i_1 i_2 i_3 i_4}{}_{j_1 j_2 j_3 j_4} = (-1)^{([j_1] + [j_2])([b_1] + [b_2])} \overline{\mathbf{g}}^{a_1 a_2, b_1 b_2} X^{i_1 i_2}{}_{a_1 a_2, j_1 j_2} D^{i_3 i_4}{}_{b_1 b_2, j_3 j_4},
\widetilde{C}_{-} = \frac{\omega}{4} (\mathbf{I} - \mathbf{P}^{(\text{ad})}) \widetilde{C} (\mathbf{I} - \mathbf{P}^{(\text{ad})}),$$
(4.62)

where $X^{i_1i_2}{}_{a_1a_2,j_1j_2}$ are the structure constants of $s\ell(M|N)$, given in (4.11), and $\overline{\mathsf{g}}^{a_1a_2,b_1b_2}$ is the inverse Cartan-Killing metric (4.15).

Proposition 4. The explicit form of \widetilde{C}_- defined in (4.62) and acting in $V_{\rm ad}^{\otimes 2}$, is

$$\widetilde{C}_{-} = \frac{1}{2} (\mathcal{P}_{13} - \mathcal{P}_{24}) \left(I - \frac{1}{\omega} (\mathcal{K}_{12} + \mathcal{K}_{34} + \mathcal{K}_{32} + \mathcal{K}_{14}) \right). \tag{4.63}$$

Besides, \tilde{C}_{-} satisfies

$$\mathbf{P}_{+}^{(\mathrm{ad})}\widetilde{C}_{-} = \widetilde{C}_{-}\mathbf{P}_{+}^{(\mathrm{ad})} = 0, \qquad \widetilde{C}_{-}\widehat{C}_{-} = \widehat{C}_{-}\widetilde{C}_{-} = 0,
\widetilde{C}_{-}^{2} = 2\widehat{C}_{-} + \mathbf{P}_{-}^{(\mathrm{ad})}, \qquad \widetilde{C}_{-}(\widetilde{C}_{-} + \mathbf{I})(\widetilde{C}_{-} - \mathbf{I}) = 0.$$
(4.64)

The last identity in the second row in (4.64) is characteristic for \hat{C}_{-} .

Proof. Direct calculations show that the symmetric structure constants $D^{rs}_{ij,km}$ of $s\ell(M|N)$ satisfy

$$D^{rs}{}_{ij,km} = (\delta^r_l \delta^s_n - \frac{1}{\omega} \mathcal{K}^{rs}{}_{ln}) \bar{D}_{ij,km}{}^{ln}, \qquad (4.65)$$

$$\bar{D}^{rs}{}_{ij,km} = \delta^r_i \delta^s_m \delta_{jk} + (-1)^{([i]+[j])([k]+[m])} \delta^r_k \delta^s_j \delta_{im} - \frac{2}{\omega} ((-1)^{[i]} \delta^r_k \delta^s_m \delta_{ij} + (-1)^{[m]} \delta^r_i \delta^s_j \delta_{km}).$$

Thus, the operator \widetilde{C} given in (4.62) equals

$$\widetilde{C} = \left(I - \frac{1}{\omega} \mathcal{K}_{34}\right) \left(\mathcal{P}_{13} - \mathcal{P}_{24} + \mathcal{K}_{14} - \mathcal{K}_{32} + \frac{2}{\omega} (\mathcal{P}_{24} - \mathcal{P}_{13}) \mathcal{K}_{34}\right) \tag{4.66}$$

Using (4.62) and the equalities $(\mathbf{I} - \mathbf{P}^{(\mathrm{ad})}) = (I - \mathbf{P})\overline{I}_{12}\overline{I}_{34}$, $\mathbf{P} = \mathcal{P}_{13}\mathcal{P}_{24}$, we get (4.63). One can check (4.64) by direct calculations using (4.63), (4.36), (4.34).

Since $\widetilde{C}_{-} = \widetilde{C}_{-} \mathbf{P}_{-}^{(\mathrm{ad})} = \mathbf{P}_{-}^{(\mathrm{ad})} \widetilde{C}_{-}$, the last equality in (4.64) can be rewritten as

$$\widetilde{C}_{-}(\widetilde{C}_{-} + \mathbf{P}_{-}^{(\mathrm{ad})})(\widetilde{C}_{-} - \mathbf{P}_{-}^{(\mathrm{ad})}) = 0. \tag{4.67}$$

It is the characteristic identity for \widetilde{C}_- , which is restricted to the antisymmetric part $\mathbf{P}_-^{(\mathrm{ad})}(V_{\mathrm{ad}}^{\otimes 2})$ of $V_{\mathrm{ad}}^{\otimes 2}$. Using (4.67) and (2.39), one can obtain projectors onto the eigenspaces of \widetilde{C}_{-} . The explicit formulas are

$$\widetilde{P}_{0}^{(-)} = -2\widehat{C}_{-}, \qquad \widetilde{P}_{-1}^{(-)} = \widehat{C}_{-} + \frac{1}{2}\mathbf{P}_{-}^{(ad)} - \frac{1}{2}\widetilde{C}_{-}, \qquad \widetilde{P}_{1}^{(-)} = \widehat{C}_{-} + \frac{1}{2}\mathbf{P}_{-}^{(ad)} + \frac{1}{2}\widetilde{C}_{-}. \tag{4.68}$$

In (4.68), the lower indices of the projectors equal the eigenvalues of \widetilde{C}_{-} in the corresponding eigenspaces.

Note that $\widetilde{P}_{-1}^{(-)} + \widetilde{P}_{1}^{(-)} = P_{1}^{(-)}$ where $P_{1}^{(-)}$ is given in (4.60), and $\widetilde{P}_{0}^{(-)} = P_{2}^{(-)}$ where $P_{2}^{(-)}$ is defined in (4.60). Thus, we get the following full system of mutually orthogonal projectors for $\omega \neq 0, 1, 2$:

$$\widetilde{P}_{-1}^{(-)} = \widehat{C}_{-} + \frac{1}{2} \mathbf{P}_{-}^{(ad)} - \frac{1}{2} \widetilde{C}_{-}, \qquad \widetilde{P}_{1}^{(-)} = \widehat{C}_{-} + \frac{1}{2} \mathbf{P}_{-}^{(ad)} + \frac{1}{2} \widetilde{C}_{-}, \qquad P_{2}^{(-)} = -2 \widehat{C}_{-},
P_{1}^{(+)} = \frac{1}{\omega^{2} - 1} \mathbf{K},
P_{2}^{(+)} = -\frac{\omega}{2(\omega + 1)(\omega + 2)} \mathbf{K} + \frac{\omega^{2}}{\omega + 2} \widehat{C}_{+}^{2} + \frac{\omega}{2} \widehat{C}_{+} + \frac{\omega}{2(\omega + 2)} \mathbf{P}_{+}^{(ad)},
P_{3}^{(+)} = \frac{\omega}{2(\omega - 1)(\omega - 2)} \mathbf{K} - \frac{\omega^{2}}{\omega - 2} \widehat{C}_{+}^{2} - \frac{\omega}{2} \widehat{C}_{+} + \frac{\omega}{2(\omega - 2)} \mathbf{P}_{+}^{(ad)},
P_{4}^{(+)} = \frac{4}{\omega^{2} - 4} (\omega^{2} \widehat{C}_{+}^{2} - \mathbf{P}_{+}^{(ad)} - \mathbf{K}).$$
(4.69)

The images of $\widetilde{\mathbf{P}}_{\pm 1}^{(-)}$ and $\mathbf{P}_{2}^{(-)}$ lie within the antisymmetric part $\mathbf{P}_{-}^{(\mathrm{ad})}(V_{\mathrm{ad}}^{\otimes 2})$ of the space $V_{\mathrm{ad}}^{\otimes 2}$, while the images of $\mathbf{P}_{i}^{(+)}$, (i=2,...,5) belong to its symmetric part $\mathbf{P}_{+}^{(\mathrm{ad})}(V_{\mathrm{ad}}^{\otimes 2})$. For $\omega=1,2$ the projectors are given by (the left and right columns correspond to $\omega=1$ and $\omega=2$, respectively)

$$\begin{split} \widetilde{P}_{-1}^{(-)} &= \widehat{C}_{-} + \frac{1}{2} \mathbf{P}_{-}^{(\mathrm{ad})} - \frac{1}{2} \widetilde{C}_{-}, \\ \widetilde{P}_{1}^{(-)} &= \widehat{C}_{-} + \frac{1}{2} \mathbf{P}_{-}^{(\mathrm{ad})} + \frac{1}{2} \widetilde{C}_{-}, \\ \widetilde{P}_{1}^{(-)} &= \widehat{C}_{-} + \frac{1}{2} \mathbf{P}_{-}^{(\mathrm{ad})} + \frac{1}{2} \widetilde{C}_{-}, \\ P_{2}^{(-)} &= -2 \widehat{C}_{-}, \\ P_{1}^{(+)} &= -\frac{1}{2} \mathbf{P}_{+}^{(\mathrm{ad})} - \frac{5}{4} \mathbf{K} - \frac{1}{2} \widehat{C}_{+} + \widehat{C}_{+}^{2}, \\ P_{2}^{(+)} &= \frac{1}{6} \mathbf{P}_{+}^{(\mathrm{ad})} - \frac{1}{12} \mathbf{K} + \frac{1}{2} \widehat{C}_{+} + \frac{1}{3} \widehat{C}_{+}^{2}, \\ P_{2}^{(+)} &= \frac{4}{3} \mathbf{P}_{+}^{(\mathrm{ad})} + \frac{4}{3} \mathbf{K} - \frac{4}{3} \widehat{C}_{+}^{2}, \\ P_{3}^{(+)} &= \frac{4}{3} \mathbf{P}_{+}^{(\mathrm{ad})} + \frac{4}{3} \mathbf{K} - \frac{4}{3} \widehat{C}_{+}^{2}, \\ P_{3}^{(+)} &= \frac{3}{4} \mathbf{P}_{+}^{(\mathrm{ad})} - \frac{1}{4} \mathbf{K} - \widehat{C}_{+} - \widehat{C}_{+}^{2}. \end{split}$$

In order to find the dimensions of the invariant subspaces extracted by the projectors (4.69), we calculate the traces and supertraces of those projectors. First, we compute some auxiliary traces and supertraces (recall that

$$\operatorname{tr} \mathbf{I} = (\xi^{2} - 1)^{2}, \qquad \operatorname{str} \mathbf{I} = (\omega^{2} - 1)^{2}, \\
\operatorname{tr} \mathbf{P}_{+}^{(\mathrm{ad})} = \frac{1}{2}(\xi^{2} - 1)^{2} + \frac{1}{2}(\omega^{2} - 1), \qquad \operatorname{str} \mathbf{P}_{+}^{(\mathrm{ad})} = \frac{1}{2}\omega^{2}(\omega^{2} - 1), \\
\operatorname{tr} \mathbf{P}_{-}^{(\mathrm{ad})} = \frac{1}{2}(\xi^{2} - 1)^{2} - \frac{1}{2}(\omega^{2} - 1), \qquad \operatorname{str} \mathbf{P}_{-}^{(\mathrm{ad})} = \frac{1}{2}(\omega^{2} - 1)(\omega^{2} - 2), \\
\operatorname{tr} \mathbf{K} = \omega^{2} - 1, \qquad \operatorname{str} \mathbf{K} = \omega^{2} - 1, \\
\operatorname{tr} \widehat{C}_{+} = \frac{1}{2}(\xi^{2} - 1), \qquad \operatorname{str} \widehat{C}_{+} = \frac{1}{2}(\omega^{2} - 1), \\
\operatorname{tr} \widehat{C}_{+} = \frac{\xi^{4}}{2\omega^{2}} + \frac{\xi^{2}}{4} - 2\frac{\xi^{2}}{\omega^{2}} + \frac{5}{4}, \qquad \operatorname{str} \widehat{C}_{+}^{2} = \frac{3}{4}(\omega^{2} - 1), \\
\operatorname{tr} \widehat{C}_{-} = -\frac{1}{2}(\xi^{2} - 1), \qquad \operatorname{str} \widehat{C}_{-} = -\frac{1}{2}(\omega^{2} - 1), \\
\operatorname{tr} \widehat{C}_{-} = 0, \qquad \operatorname{str} \widehat{C}_{-} = 0. \\
\end{cases}$$
(4.70)

Using (4.69) and (4.70), we obtain the traces

$$\operatorname{tr} \widetilde{P}_{-1}^{(-)} = \frac{1}{4} ((\xi^2 - 2)^2 - \omega^2), \qquad \operatorname{tr} P_1^{(+)} = 1, \qquad (4.71)$$

$$\operatorname{tr} \widetilde{P}_1^{(-)} = \frac{1}{4} ((\xi^2 - 2)^2 - \omega^2), \qquad \operatorname{tr} P_2^{(+)} = \frac{1}{4} ((\xi^2 - 1)^2 + 2(\xi^2 + 1)(\omega - 1) + (\omega - 1)^2),$$

$$\operatorname{tr} P_2^{(-)} = \xi^2 - 1, \qquad \operatorname{tr} P_3^{(+)} = \frac{1}{4} ((\xi^2 - 1)^2 - 2(\xi^2 + 1)(\omega + 1) + (\omega + 1)^2),$$

$$\operatorname{tr} P_4^{(+)} = \xi^2 - 1$$

and supertraces of the projectors (4.69):

$$str \widetilde{P}_{-1}^{(-)} = \frac{1}{4}(\omega^{2} - 1)(\omega^{2} - 4), \qquad str P_{1}^{(+)} = 1,
str \widetilde{P}_{1}^{(-)} = \frac{1}{4}(\omega^{2} - 1)(\omega^{2} - 4), \qquad str P_{2}^{(+)} = \frac{1}{4}\omega^{2}(\omega - 1)(\omega + 3),
str P_{2}^{(-)} = \omega^{2} - 1, \qquad str P_{3}^{(+)} = \frac{1}{4}\omega^{2}(\omega + 1)(\omega - 3),
str P_{4}^{(+)} = \omega^{2} - 1$$
(4.72)

for $\omega \neq 0, 1, 2$. Analogously to the osp(M|N) case, we find the dimensions of even and odd parts of the invariant subspaces:

$$\dim_{\overline{0}} \widetilde{V}_{-1}^{(-)} = \frac{1}{4} (M^2 - 1)(M^2 - 4) + \frac{1}{4} (N^2 - 1)(N^2 - 4) + \frac{1}{2} (MN + 1)(3MN - 2),$$

$$\dim_{\overline{0}} \widetilde{V}_{1}^{(-)} = \frac{1}{4} (M^2 - 1)(M^2 - 4) + \frac{1}{4} (N^2 - 1)(N^2 - 4) + \frac{1}{2} (MN + 1)(3MN - 2),$$

$$\dim_{\overline{0}} V_{2}^{(-)} = M^2 + N^2 - 1,$$

$$\dim_{\overline{0}} V_{1}^{(+)} = 1,$$

$$\dim_{\overline{0}} V_{2}^{(+)} = \frac{1}{4} M^2 (M - 1)(M + 3) + \frac{1}{4} N^2 (N + 1)(N - 3) + \frac{1}{2} MN(3MN - M + N - 1),$$

$$\dim_{\overline{0}} V_{3}^{(+)} = \frac{1}{4} M^2 (M + 1)(M - 3) + \frac{1}{4} N^2 (N - 1)(N + 3) + \frac{1}{2} MN(3MN + M - N - 1),$$

$$\dim_{\overline{0}} V_{4}^{(+)} = M^2 + N^2 - 1,$$

$$(4.74)$$

$$\dim_{\overline{1}} \widetilde{V}_{-1}^{(-)} = MN(M^{2} + N^{2} - 2), \qquad \dim_{\overline{1}} V_{1}^{(+)} = 0, \qquad (4.75)$$

$$\dim_{\overline{1}} \widetilde{V}_{1}^{(-)} = MN(M^{2} + N^{2} - 2), \qquad \dim_{\overline{1}} V_{2}^{(+)} = MN(M(M+1) + N(N-1) - 2), \qquad \dim_{\overline{1}} V_{2}^{(-)} = 2MN, \qquad \dim_{\overline{1}} V_{3}^{(+)} = MN(M(M-1) + N(N+1) - 2), \qquad \dim_{\overline{1}} V_{4}^{(+)} = 2MN.$$

Note that substitutions M=0 and N=0 nullify the dimensions of the odd parts of $\mathrm{ad}^{\otimes 2}$ -invariant subspaces of $s\ell(M|N)$, as it should be. Besides, the mentioned substitutions turn (4.73) into the corresponding expressions for the dimensions of the invariant subspaces of the $s\ell(N)$ (or $s\ell(M)$) Lie algebra, which are given in [15].

5 Universal characteristic identities for \widehat{C}_+ in the cases of the osp(M|N) and $s\ell(M|N)$ Lie superalgebras

For the osp(M|N) and $s\ell(M|N)$ Lie superalgebras (which are denoted by \mathfrak{g} in this section), the characteristic identities (3.63) and (4.39) for \widehat{C}_+ in the adjoint representation can be written in the following general form:

$$\widehat{C}_{+}^{3} + \frac{1}{2}\widehat{C}_{+}^{2} = \mu_{1}\widehat{C}_{+} + \mu_{2}(\mathbf{I} + \mathbf{P}^{(ad)} - 2\mathbf{K}), \qquad (5.1)$$

which coincides precisely with the form of the universal identity for \widehat{C}_+ in the case of the classical Lie algebras [15]. The parameters μ_1 and μ_2 corresponding to the algebras osp(M|N) and $s\ell(M|N)$ are given in Table 1.

Table 1: The values of μ_1 and μ_2 for the osp(M|N) and $s\ell(M|N)$ Lie superalgebras

	μ_1	μ_2
osp(M N)	$-\frac{\omega-8}{2(\omega-2)^2}$	$\frac{\omega-4}{2(\omega-2)^3}$
$s\ell(M N)$	$\frac{1}{\omega^2}$	$\frac{1}{4\omega^2}$

The subsequent analysis of (5.1) mostly follows the consideration [15] of an analogous identity for the classical Lie algebras. Multiplying both sides of (5.1) by **K** and using

$$\mathbf{K}(\mathbf{I} + \mathbf{P}^{(\mathrm{ad})}) = 2\mathbf{K}, \qquad \mathbf{K}\widehat{C}_{+} = -\mathbf{K}, \qquad \mathbf{K} \cdot \mathbf{K} = \mathrm{sdim}\,\mathfrak{g}\mathbf{K},$$
 (5.2)

one can express the superdimension of \mathfrak{g} as a function of μ_1 and μ_2 :

$$\operatorname{sdim} \mathfrak{g} = \frac{2\mu_2 - \mu_1 + \frac{1}{2}}{2\mu_2}.$$
 (5.3)

Multiplying then both sides of (5.1) by $\widehat{C}_{+}(\widehat{C}_{+}+\mathbf{I})$, we obtain the characteristic identity for \widehat{C}_{+} :

$$\widehat{C}_{+}(\widehat{C}_{+} + \mathbf{I})(\widehat{C}_{+}^{3} + \frac{1}{2}\widehat{C}_{+}^{2} - \mu_{1}\widehat{C}_{+} - 2\mu_{2}\mathbf{I}) = 0,$$
(5.4)

the factorized form of which is

$$\widehat{C}_{+}(\widehat{C}_{+} + \mathbf{I})(\widehat{C}_{+} + \frac{\alpha}{2t}\mathbf{I})(\widehat{C}_{+} + \frac{\beta}{2t}\mathbf{I})(\widehat{C}_{+} + \frac{\gamma}{2t}\mathbf{I}) = 0.$$
(5.5)

Thus, the roots of the polynomial on the left of (5.4) are

$$a_1 = 0,$$
 $a_2 = -1,$ $a_3 = -\frac{\alpha}{2t},$ $a_4 = -\frac{\beta}{2t},$ $a_5 = -\frac{\gamma}{2t},$ (5.6)

where the normalisation parameter $t = \alpha + \beta + \gamma$, as by (5.4) and (5.5),

$$\frac{\alpha}{2t} + \frac{\beta}{2t} + \frac{\gamma}{2t} = \frac{1}{2},\tag{5.7}$$

We choose $t = h^{\vee}$ where h^{\vee} is the dual Coxeter number of \mathfrak{g} . The values of h^{\vee} (see, e.g., [29]) for $s\ell(M|N)$ and osp(M|N) are presented in Table 2. As usual, $\omega = 2m+1-N$ and $\omega = 2m-N$ for $\mathfrak{g} = osp(2m+1|N)$ and $\mathfrak{g} = osp(2m,N)$, respectively.

Table 2: The dual Coxeter numbers for the Lie superalgebras $s\ell(M|N)$ and osp(M|N)

	$s\ell(M N)$	$osp(2m+1 N), \ \omega > 1$ $osp(2m N), \ \omega > 0$	$osp(2m+1 N), \ \omega \le 1$ $osp(2m N), \ \omega \le 0$
h^{\vee}	ω	$\omega - 2$	$-\frac{1}{2}(\omega-2)$

The parameters α , β and γ were introduced by Vogel in [7]. The values of these parameters for the algebras osp(M|N) and $s\ell(M|N)$ can be found from (3.65) and (4.41) and are given in Table 3.

Table 3: The Vogel parameters for the osp(M|N) and $s\ell(M|N)$ Lie superalgebras

	$s\ell(M N)$	$osp(2m+1 N), \ \omega > 1$ $osp(2m N), \ \omega > 0$	$osp(2m+1 N), \ \omega \le 1$ $osp(2m N), \ \omega \le 0$
α	-2	-2	1
β	2	4	-2
γ	ω	$\omega - 4$	$-\frac{1}{2}(\omega-4)$
t	ω	$\omega - 2$	$-\frac{1}{2}(\omega-2)$

A comparison of (5.4) and (5.5) shows that μ_1 and μ_2 can be expressed in terms of the Vogel parameters:

$$\mu_1 = -\frac{\alpha\beta + \alpha\gamma + \beta\gamma}{4t^2}, \qquad \mu_2 = -\frac{\alpha\beta\gamma}{16t^3}, \tag{5.8}$$

while the superdimension (5.3) of \mathfrak{g} acquires the universal form

$$\operatorname{sdim} \mathfrak{g} = \frac{(\alpha - 2t)(\beta - 2t)(\gamma - 2t)}{\alpha\beta\gamma}.$$
(5.9)

Now using (5.5), we can obtain a universal form of the projectors $P_{(a_i)}^{(+)}$ onto the invariant subspaces $V_{(a_i)}$ of the symmetric space $\mathbf{P}_+(V_{\mathrm{ad}}^{\otimes 2})$:

$$P_{(-\frac{\alpha}{2t})}^{(+)} = P^{(+)}(\alpha|\beta,\gamma), \qquad P_{(-\frac{\beta}{2t})}^{(+)} = P^{(+)}(\beta|\alpha,\gamma), \qquad P_{(-\frac{\gamma}{2t})}^{(+)} = P^{(+)}(\gamma|\alpha,\beta)$$

$$P_{(-1)}^{(+)} = \frac{1}{\operatorname{sdim}\mathfrak{q}}\mathbf{K}, \tag{5.10}$$

where we denoted

$$P^{(+)}(\alpha|\beta,\gamma) = \frac{4t^2}{(\beta-\alpha)(\gamma-\alpha)} \Big(\widehat{C}_+^2 + \Big(\frac{1}{2} - \frac{\alpha}{2t}\Big) \widehat{C}_+ + \frac{\beta\gamma}{8t^2} \Big(\mathbf{I} + \mathbf{P}^{(ad)} - \frac{2\alpha}{\alpha - 2t} \mathbf{K} \Big) \Big).$$
 (5.11)

By (2.56) and (5.9), the supertrace of $P^{(+)}(\alpha|\beta,\gamma)$ is

$$\operatorname{str} \mathbf{P}^{(+)}(\alpha|\beta,\gamma) = -\frac{(3\alpha - 2t)(\beta - 2t)(\gamma - 2t)(\beta + t)(\gamma + t)t}{\alpha^2(\alpha - \beta)(\alpha - \gamma)\beta\gamma}.$$
 (5.12)

From (5.12) and (5.9) we get the superdimensions of the invariant subspaces $V_{(-1)}, V_{(-\frac{\alpha}{2t})}, V_{(-\frac{\beta}{2t})}$ and $V_{(-\frac{\gamma}{2t})}$

extracted by the projectors (5.10):

$$sdim V_{(-1)} = str P_{(-1)}^{(+)} = 1,$$

$$sdim V_{(-\frac{\alpha}{2t})} = str P_{(-1)}^{(+)} = -\frac{(3\alpha - 2t)(\beta - 2t)(\gamma - 2t)(\beta + t)(\gamma + t)t}{\alpha^2(\alpha - \beta)(\alpha - \gamma)\beta\gamma},$$

$$sdim V_{(-\frac{\beta}{2t})} = str P_{(-1)}^{(+)} = -\frac{(3\beta - 2t)(\alpha - 2t)(\gamma - 2t)(\alpha + t)(\gamma + t)t}{\beta^2(\beta - \alpha)(\beta - \gamma)\alpha\gamma},$$

$$sdim V_{(-\frac{\gamma}{2t})} = str P_{(-1)}^{(+)} = -\frac{(3\gamma - 2t)(\beta - 2t)(\alpha - 2t)(\beta + t)(\alpha + t)t}{\gamma^2(\gamma - \beta)(\gamma - \alpha)\beta\alpha}.$$
(5.13)

It is worth noting that for the case of Lie algebras nullification of either $3\alpha - 2t$, $3\beta - 2t$, or $3\gamma - 2t$ corresponds to the exceptional Lie algebras g_2 , f_4 , e_6 , e_7 , e_8 as well as $s\ell(3)$ and so(8). In this sense, the "exceptional" basic classical Lie superalgebras (see their definition, e.g., in [24]) are $s\ell(M|N)$ for $M - N = 0, \pm 3$, osp(M|N) for M - N = -1, 8 and F(4).

6 Eigenvalues of higher Casimir operators in the adjoint representation

In this section, we derive a formula that expresses the eigenvalues of the higher Casimir operators in the adjoint representation in terms of the Vogel parameters. Our method of constructing higher Casimir operators for Lie superalgebras is based on the method proposed in [1] for Lie algebras (see also [30]).

Let Y_A be a homogeneous basis of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ of the Lie superalgebra \mathfrak{g} . If $\widetilde{C} = D^{AB}Y_A \otimes Y_B \in \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ is an ad-invariant operator, then for an arbitrary representation $T: \mathfrak{g} \to (\operatorname{End}(V))_L$ of \mathfrak{g} the operator

$$C = \operatorname{str}_2((\operatorname{id} \otimes T) \widetilde{C}) = D^{AB} Y_A \operatorname{str}(T(Y_B))$$
(6.1)

lies in the centre of $\mathcal{U}(\mathfrak{g})$. Here id is the identity operator and str_2 denotes the supertrace in the second factor in $\mathcal{U}(\mathfrak{g}) \otimes ((\operatorname{End}(V))_L)$. In what follows, we are only interested in the operator C in a particular representation T'. For T'(C) we get:

$$T'(C) = \operatorname{str}_2((T' \otimes T) \widetilde{C}) = D^{AB}T'(Y_A)\operatorname{str}(T(Y_B)). \tag{6.2}$$

Apparently, for the simple Lie superalgebra \mathfrak{g} where $\dim(\mathfrak{g}) = n$ and the Cartan-Killing metric \mathfrak{g}_{ab} of which is nondegenerate, an arbitrary power \widehat{C}^k of \widehat{C} defined in (2.34) is ad-invariant. The explicit form of \widehat{C}^k is:

$$\widehat{C}^k = (-1)^{\sum_{i>j}^n [a_i][a_j]} \mathbf{g}^{a_1 b_1} \dots \mathbf{g}^{a_n b_n} X_{a_1} \dots X_{a_n} \otimes X_{b_1} \dots X_{b_n}.$$
(6.3)

Substituting $\widetilde{C} = \widehat{C}^k$ and T' = T = ad into (6.2) yields a relation for the k-th Casimir operator $\text{ad}(C_k)$ in the adjoint representation:

$$\operatorname{ad}(C_k) = \operatorname{str}_2(\operatorname{ad}^{\otimes 2}(\widehat{C}^k)) \equiv \operatorname{str}_2(\widehat{C}^k_{\operatorname{ad}}) = \operatorname{\mathsf{g}}^{a_1 \dots a_n} \operatorname{ad}(X_{a_1}) \dots \operatorname{ad}(X_{a_n}), \tag{6.4}$$

where we denoted

$$\mathsf{g}^{a_1...a_2} = (-1)^{\sum_{i>j}^n [a_i][a_j]} \mathsf{g}^{a_1b_1} \cdots \mathsf{g}^{a_nb_n} \operatorname{str} \left(\operatorname{ad}(X_{b_1} \cdots \operatorname{ad}(X_{b_n})) \right)$$
(6.5)

and $\widehat{C}_{ad} = ad^{\otimes 2}(\widehat{C})$. As the adjoint representation of a simple Lie superalgebra is irreducible, then, by Schur's lemma, $ad(C_k)$ is the scalar operator with the eigenvalue c_k , i.e. $ad(C_k) = c_k I$ where I is the identity operator acting in V_{ad} .

Let us introduce the generating function for c_k :

$$c(z) = \sum_{p=0}^{\infty} c_p z^p. \tag{6.6}$$

By (2.51) and (6.4),

$$c(z) \cdot I = \operatorname{str}_2\left(\sum_{p=0}^{\infty} \widehat{C}_{\mathrm{ad}}^p z^p\right) = \operatorname{str}_2\left(\sum_{p=0}^{\infty} \widehat{C}_+^p z^p\right) + \operatorname{str}_2\left(\sum_{p=0}^{\infty} \widehat{C}_-^p z^p\right),\tag{6.7}$$

where we assume $\widehat{C}^0_{\pm}=\mathbf{P}^{\mathrm{ad}}_{\pm}$, and $\widehat{C}^0_{\mathrm{ad}}=\mathbf{I}$. By (2.54), $\widehat{C}^p_{-}=\left(-\frac{1}{2}\right)^{p-1}\widehat{C}_{-}$, so

$$\sum_{p=0}^{\infty} \widehat{C}_{-}^{p} z^{p} = \mathbf{P}_{-}^{\mathrm{ad}} + \sum_{p=1}^{\infty} \left(-\frac{1}{2} \right)^{p-1} \widehat{C}_{-} z^{p} = \mathbf{P}_{-}^{(\mathrm{ad})} + \frac{z}{1 + \frac{z}{2}} \widehat{C}_{-}.$$
 (6.8)

Now we express \widehat{C}_{+}^{p} in terms of $P^{(+)}(\alpha|\beta,\gamma)$, $P^{(+)}(\beta|\gamma,\alpha)$, $P^{(+)}(\gamma|\alpha,\beta)$ and $P^{(+)}_{(-1)}$, that were defined in (5.10) and (5.11). Using the condition

$$\mathbf{P}_{+}^{(\mathrm{ad})} = \mathbf{P}^{(+)}(\alpha|\beta,\gamma) + \mathbf{P}^{(+)}(\beta|\gamma,\alpha) + \mathbf{P}^{(+)}(\gamma|\alpha,\beta) + \mathbf{P}_{(-1)}^{(+)}$$
(6.9)

yields

$$\begin{split} \widehat{C}_{+}^{p} &= \widehat{C}_{+}^{p} \left(\mathbf{P}^{(+)}(\alpha | \beta, \gamma) + \mathbf{P}^{(+)}(\beta | \gamma, \alpha) + \mathbf{P}^{(+)}(\gamma | \alpha, \beta) + \mathbf{P}_{(-1)}^{(+)} \right) \\ &= \left(-\frac{\alpha}{2t} \right)^{p} \mathbf{P}^{(+)}(\alpha | \beta, \gamma) + \left(-\frac{\beta}{2t} \right)^{p} \mathbf{P}^{(+)}(\beta | \gamma, \alpha) + \left(-\frac{\gamma}{2t} \right)^{p} \mathbf{P}^{(+)}(\gamma | \alpha, \beta) + (-1)^{p} \mathbf{P}_{(-1)}^{(+)} , \end{split}$$

where $\left(-\frac{\alpha}{2t}\right)$, $\left(-\frac{\beta}{2t}\right)$, $\left(-\frac{\gamma}{2t}\right)$ and $\left(-1\right)$ are the eigenvalues of \widehat{C}_{+} corresponding to the projectors mentioned. Therefore,

$$\sum_{p=0}^{\infty} \widehat{C}_{+}^{p} z^{p} = \sum_{p=0}^{\infty} \left(\left(-\frac{\alpha z}{2t} \right)^{p} P^{(+)}(\alpha | \beta, \gamma) + \left(-\frac{\beta z}{2t} \right)^{p} P^{(+)}(\beta | \gamma, \alpha) \right)
+ \left(-\frac{\gamma z}{2t} \right)^{p} P^{(+)}(\gamma | \alpha, \beta) + (-z)^{p} P^{(+)}_{(-1)} \right)
= \frac{1}{1 + \frac{\alpha z}{2t}} P^{(+)}(\alpha | \beta, \gamma) + \frac{1}{1 + \frac{\beta z}{2t}} P^{(+)}(\beta | \gamma, \alpha) + \frac{1}{1 + \frac{\gamma z}{2t}} P^{(+)}(\gamma | \alpha, \beta) +
+ \frac{1}{1 + z} P^{(+)}_{(-1)}.$$
(6.10)

By (6.9), (6.10) can be rewritten as

$$\sum_{p=0}^{\infty} \widehat{C}_{+}^{p} z^{p} = -\frac{\alpha z}{2t + \alpha z} P^{(+)}(\alpha | \beta, \gamma) - \frac{\beta z}{2t + \beta z} P^{(+)}(\beta | \gamma, \alpha) - \frac{\gamma z}{2t + \gamma z} P^{(+)}(\gamma | \alpha, \beta) - \frac{z}{1 + z} P^{(+)}_{(-1)} + \mathbf{P}^{(\text{ad})}_{+}.$$
(6.11)

Summing (6.8) and (6.11) leads to

$$\sum_{p=0}^{\infty} \widehat{C}_{ad}^{p} z^{p} = -\frac{\alpha z}{2t + \alpha z} P^{(+)}(\alpha | \beta, \gamma) - \frac{\beta z}{2t + \beta z} P^{(+)}(\beta | \gamma, \alpha) - \frac{\gamma z}{2t + \gamma z} P^{(+)}(\gamma | \alpha, \beta) - \frac{z}{1+z} P^{(+)}_{(-1)} + \frac{2z}{2+z} \widehat{C}_{-} + \mathbf{I}.$$
(6.12)

To find c(z) by (6.7), we need to calculate the supertrace str₂ of the right hand side of (6.12). Using (2.14), (2.43), (2.44) and (2.49), we get the auxiliary supertraces

$$\operatorname{str}_{2}(\mathbf{I}) = \operatorname{sdim} \mathfrak{g} \cdot I, \quad \operatorname{str}_{2}(\widehat{C}_{-}) = -\frac{1}{2}I, \quad \operatorname{str}_{2}(\mathbf{P}^{(\operatorname{ad})}) = I,$$

$$\operatorname{str}_{2}(\widehat{C}_{+}) = \frac{1}{2}I, \quad \operatorname{str}_{2}(\mathbf{K}) = I, \quad \operatorname{str}_{2}(\widehat{C}_{+}^{2}) = \frac{3}{4}I,$$

$$(6.13)$$

which are in accordance with (2.56). Then by (5.9) and (6.13),

$$\operatorname{str}_{2} \mathbf{P}^{(+)}(\alpha|\beta,\gamma) = -\frac{(3\alpha - 2t)t(\beta + t)(\gamma + t)}{\alpha(\alpha - \beta)(\alpha - \gamma)(\alpha - 2t)} I. \tag{6.14}$$

Using (5.9), (5.10) and (6.13) shows that

$$\operatorname{str}_{2} P_{(-1)}^{(+)} = \frac{1}{\operatorname{sdim} \mathfrak{g}} \operatorname{str}_{2} \mathbf{K} = \frac{\alpha \beta \gamma}{(\alpha - 2t)(\beta - 2t)(\gamma - 2t)} I. \tag{6.15}$$

Substituting (6.12) into (6.7) and using (6.13), (6.14), (6.15) results in

$$\sum_{p=0}^{\infty} \widehat{C}_{ad} z^{p} = \sum_{p=0}^{\infty} c_{p} z^{p} \cdot I = \frac{(\alpha - 2t)(\beta - 2t)(\gamma - 2t)}{\alpha \beta \gamma} \cdot I + z^{2} \frac{96t^{3} + 168t^{3}z + 6(14t^{3} + tt_{2} - t_{3})z^{2} + (13t + 3tt_{2} - 4t_{3})z^{3}}{6(2t + \alpha z)(2t + \beta z)(2t + \gamma z)(2 + z)(1 + z)} \cdot I,$$
(6.16)

where $t_2 = \alpha^2 + \beta^2 + \gamma^2$ and $t_3 = \alpha^3 + \beta^3 + \gamma^3$. Therefore, the generating function for the eigenvalues of the higher Casimir operators of the osp(M|N) and $s\ell(M|N)$ Lie superalgebras in the adjoint representation is

$$c(z) = \frac{(\alpha - 2t)(\beta - 2t)(\gamma - 2t)}{\alpha\beta\gamma} + z^2 \frac{96t^3 + 168t^3z + 6(14t^3 + tt_2 - t_3)z^2 + (13t + 3tt_2 - 4t_3)z^3}{6(2t + \alpha z)(2t + \beta z)(2t + \gamma z)(2 + z)(1 + z)}.$$
(6.17)

Formula (6.17) is in agreement with the results of [10], where this expression was found by using the formula for the values of the higher Casimir operators obtained in [1].

7 Conclusion

We have found explicit formulas for the projectors onto the invariant subspaces of the tensor product of two adjoint representations of the osp(M|N) for $M-N\neq 0,1,2$ and $s\ell(M|N)$ Lie superalgebras for $M-N\neq 0,\pm 1,\pm 2$. The construction was performed by finding the characteristic identities for the split Casimir operator of the corresponding algebras. In the case of the $s\ell(M|N)$ Lie superalgebras, an additional ad-invariant operator was defined by means of the so-called symmetric structure constants of $s\ell(M|N)$. It was also shown that the dimensions of the invariant subspaces and the values of the quadratic Casimir operator in those subspaces are in agreement with [7]–[9], where these quantities are written by means of the Vogel parameters in the context of the universal Lie algebra. Furthermore, the generating function of the eigenvalues of the higher Casimir operators in the adjoint representation was found and expressed in terms of the Vogel parameters. The last result is in accordance with [10].

Acknowledgements

The authors are thankful to S.O.Krivonos and R.L.Mkrtchyan for useful discussions. A.P.I. acknowledges the support of the Russian Science Foundation, grant No. 19-11-00131.

References

- [1] S. Okubo, Casimir invariants and vector operators in simple and classical Lie algebras. J. Math. Phys. 18 (1977), 2382-2394.
- [2] V. Chari, A. N. Pressley, A guide to quantum groups, Cambridge university press (1995).
- [3] Z. Ma, Yang-Baxter equation and quantum enveloping algebras, World Scientific, (1993) 91-123.
- [4] A. N. Sergeev, The tensor algebra of the identity representation as a module over the Lie superalgebras Gl(n,m) and Q(n), Math. USSR-Sb. 51(2), (1985) 419–427.
- [5] A. Berele, A. Regev, Hook young diagrams with applications to combinatorics and to representations of Lie superalgebras, Advances in Mathematics **62**(2), (1987) 118-175.
- [6] M. Ehrig, C. Stroppel, Schur-Weyl duality for the Brauer algebra and the ortho-symplectic Lie superalgebra, Mathematische Zeitschrift **284**, (2016) 595-613.
- [7] P. Vogel, The universal Lie algebra, Preprint (1999).
- [8] P. Deligne, La serie exceptionnelle des groupes de Lie, C. R. Acad. Sci. 322, (1996) 321-326.

- [9] J. M. Landsberg, L. Manivel, Triality, exceptional Lie algebras and Deligne dimension formulas, Adv. Math. 171, (2002) 59-85.
- [10] R. L. Mkrtchyan, A. N. Sergeev, A. P. Veselov, Casimir eigenvalues for universal Lie algebra, Journal of Mathematical Physics 53, (2012) 102-106.
- [11] J. M. Landsberg, L. Manivel, A universal dimension formula for complex simple Lie algebras, Adv. Math. 201, (2006) 379-407.
- [12] M.Y. Avetisyan, R.L. Mkrtchyan, X2 series of universal quantum dimensions, Jour. of Phys. A, 53.4 (2020) 045202; On (ad) n (X 2) k series of universal quantum dimensions, Jour. of Math. Phys. 61.10 (2020) 101701; arXiv:1909.02076 [math-ph].
- [13] A. Mironov, R. Mkrtchyan, A. Morozov, On universal knot polynomials, Journal of High Energy Physics **02**, (2016) 078; arXiv:1510.05884 [hep-th].
- [14] A. Mironov, A. Morozov, Universal Racah matrices and adjoint knot polynomials: Arborescent knots, Phys.Lett. B 755 (2016), 47; arXiv:1511.09077 [hep-th].
- [15] A. P. Isaev and S. O. Krivonos, Split Casimir operator for simple Lie algebras, solutions of Yang-Baxter equations and Vogel parameters, Journal of Math. Phys. 62, (2021) 083503, arXiv:2102.08258 [math-ph]; A. P. Isaev and S. O. Krivonos, Split Casimir Operator and Universal Formulation of the Simple Lie Algebras, Symmetry 13(6), (2021) 1046; arXiv:2106.04470 [math-ph].
- [16] A. P. Isaev and A. A. Provorov, Projectors on invariant subspaces of representations $ad^{\otimes 2}$ of Lie algebras so(N) and sp(2r) and Vogel parameterization, TMF, **206**(1) (2021), 3–22; arXiv:2012.00746 [math-ph].
- [17] V.G. Kac, Lie superalgebras, Advances in Mathematics 26(1), (1977) 8-96.
- [18] F.A. Berezin, *Introduction to algebra and analysis with anticommuting variables*, Moscow State University Press, Moscow (1983).
- [19] J. Fuksa, A. P. Isaev, D. Karakhanyan and R. Kirschner, Yangians and Yang-Baxter R-operators for orthosymplectic superalgebras, Nucl. Phys. B **917**, (2017) 44; arXiv:1612.04713 [math-ph].
- [20] A. P. Isaev, D. Karakhanyan, R. Kirschner, Yang-Baxter R-operators for osp superalgebras, Nucl. Phys. B 965 (2021) 115355; arXiv:2009.08143 [math-ph].
- [21] V.G. Kac, A sketch of Lie superalgebra theory, Comm. Math. Phys. 53 (1), (1977) 31-64.
- [22] F. A. Berezin, *Introduction to superanalysis*, Mathematical Physics and Applied Mathematics, Springer Netherlands, 9, (1987).
- [23] L.Frappat, A.Sciarrino, P.Sorba, Dictionary on Lie algebras and superalgebras, Academic Press (London), (2000) 410.
- [24] L.Frappat, A.Sciarrino, P.Sorba, Structure of basic Lie superalgebras and of their affine extensions, Communications in Mathematical Physics, 121, (1989) 457–500.
- [25] P. Cvitanović, Birdtracks, Lie's, and Exceptional Groups, Princeton; Oxford: Princeton University Press (2008); http://cns.physics.gatech.edu/grouptheory/chapters/draft.pdf
- [26] A.P. Isaev, V.A. Rubakov, Theory of groups and symmetries II. Representations of groups and Lie algebras, applications, World Scientific, (2020).
- [27] P.P. Kulish, E.K. Sklyanin, On solutions of the Yang-Baxter equation, J. Sov. Math. 19(5) (1982) 1596, Zap. Nauch. Semin. POMI 95 (1980) 129.
- [28] A.P.Isaev, Quantum groups and Yang–Baxter equations, preprint MPIM (Bonn), MPI 2004-132, (2004), http://webdoc.sub.gwdg.de/ebook/serien/e/mpi_mathematik/2004/132.pdf.
- [29] V. G. Kac, M. Wakimoto, Integrable highest weight modules over affine superalgebras and number theory, Progr. Math. 123 (1994), 415–456.
- [30] A.P. Isaev, V.A. Rubakov, Theory of groups and symmetries I. Finite groups, Lie groups and Lie algebras, World Scientific, (2018).