MIN-MAX MINIMAL HYPERSURFACES WITH HIGHER MULTIPLICITY

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Abstract. We exhibit the first set of examples of non-bumpy metrics on the $(n+1)$-sphere $(2 \leq n \leq 6)$ in which the varifold associated with the two-parameter min-max construction must be a multiplicity-two minimal n-sphere. This is proved by a new area-and-separation estimate for certain minimal hypersurfaces with Morse index two inspired by an early work of Colding-Minicozzi. We also construct non-bumpy projective spaces in which the first min-max hypersurfaces are one-sided, and non-bumpy balls in which the free boundary min-max hypersurfaces are improper.

1. Introduction

In the past decade, we have witnessed many important advancement in the development for minimal hypersurfaces, including the solution of Yau conjecture on minimal surfaces by Marques-Neves [20], and Song [30], and the establishment of a Morse theory for the area functional by Marques-Neves [19, 21]. One key challenge in these works is the a priori existence of integer multiplicity of the varifolds produced by the Almgren-Pitts min-max theory [1, 2, 22] (see also [4,6]). Now we have very well understanding of the multiplicity when the ambient manifold has a bumpy metric ([35]) thanks to the solution of the Multiplicity One Conjecture [38]; (see also [3]). For non-bumpy metrics, there are known trivial examples where some min-max varifolds have higher multiplicities, while some others have multiplicity one. For instance, on a thin and long flat torus, the min-max varifold can be two identical copies of the cross section, but one can move one copy parallelly away to obtain a multiplicity one varifold of the same mass. This made it tempting to conjecture that for any metric there always exists a min-max varifold of multiplicity one. However, in this paper, we will disprove this conjecture by constructing the first set of nontrivial and non-bumpy examples, where the varifold associated with the two-parameter min-max construction must have multiplicity two.

Theorem 1.1. *The $(n+1)$-sphere S^{n+1} of dimension $3 \leq (n+1) \leq 7$ admits metrics g with non-negative Ricci curvature so that the second volume spectrum $\omega_2(S^{n+1}, g)$ can only be achieved by a degenerate minimal n-sphere with multiplicity 2.*

Here $\{\omega_k\}$ is the *volume spectrum of M* defined by Gromov [9], Guth [11], and Marques-Neves [17] as a sequence of non-decreasing positive numbers

$$0 < \omega_1(M, g) \leq \omega_2(M, g) \leq \cdots \leq \omega_k(M, g) \to \infty,$$

depending only on M and g; see also [16][8][23][33] for further studies.

Our proof relies on a new area comparison argument. Roughly speaking, we show that if a connected multiplicity-one minimal hypersurface Σ is sufficiently close to a multiplicity-two degenerate stable hypersurface S_0^n, then the area of Σ is strictly greater than that of S_0^n. Similar area comparison arguments have played essential roles in the study of Morse index and

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multiplicity of min-max minimal hypersurfaces (see [14, 18, 36, 37]), but in a reserve way. In particular, the catenoid estimates by Ketover-Marques-Neves [14] imply that if S_0^n is unstable, then by pushing away the two copies of S_0^n and then adding a catenoid type neck therein, one can strictly decrease the area. More precisely, the area-decrease by pushing away the two copies of S_0^n dominates the area-increase by adding the catenoid neck, when S_0^n is unstable. This idea has been further extended by Haslhofer-Ketover [13] (see also [7]) to construct the second minimal 2-sphere in S^3 with a bumpy metric associated with the second width. In our situation, we will show the reverse phenomenon when S_0^n is degenerate stable. A crucial ingredient is to bound the area difference between Σ and $2S_0^n$ away from the neck region. To do so, we need a new distance-separation estimate between Σ and S_0^n in terms of the size of the neck region; see Theorem 4.6 and 4.9. This part is inspired by an early work of Colding-Minicozzi [5]; see the part on idea of proof for more details.

The Multiplicity One Theorem also implies that for bumpy metrics, the min-max minimal hypersurfaces can only be two-sided. However, our method can also be used to construct non-bumpy metrics on the projective spaces in which the first width must be achieved by one-sided hypersurfaces.

Corollary 1.2. *The $(n+1)$-projective space $\mathbb{RP}^{n+1}$ with $3 \leq (n+1) \leq 7$ admits metrics g so that the first volume spectrum $\omega_1(\mathbb{RP}^{n+1}, g)$ can only be achieved by a one-sided $\mathbb{RP}^n$ with multiplicity 2.*

In [15], Li-Zhou developed a free boundary min-max theory for compact manifolds with boundary $(M, \partial M)$. The minimal hypersurface Σ constructed therein may *not be properly embedded*, i.e. it may have interior touching with the boundary $\text{int}(\Sigma) \cap \partial M \neq \emptyset$. The touching phenomenon had caused major challenges in the application of this theory, e.g. [10,31,34]. It has been conjectured that the touching could happen even for Euclidean domains [15, Conjecture 1.5]. Here we exhibit examples where the touching phenomenon does happen.

Corollary 1.3. *B^{n+1} admits a metric with minimal boundary and non-negative Ricci curvature so that its first volume spectrum can only be achieved by its boundary with multiplicity one.*

Proof. Let (S^{n+1}, g) be as in Theorem 1.1. Then the round S_0^n divides the $(n+1)$-sphere into two connected components, denoted by M_+ and M_-; see Section 3 for explicit description. Note that (M_+, g) is an $(n+1)$-ball that has positive Ricci curvature away from its boundary. By [31], $\omega_1(M_+, g)$ is realized by a free boundary minimal hypersurface Σ with multiplicity one, whose index is bounded by one from above. Then by reflection along S_0^n, one can obtain a smoothly embedded minimal hypersurface $\widetilde{\Sigma} \subset (S^{n+1}, g)$ with index less than or equal to 2. Clearly, $\omega_1(M_+, g) \leq \text{Area}(S_0^n)$. It follows that

$$\text{Area}(\widetilde{\Sigma}) \leq 2\text{Area}(S_0^n).$$

Then by Theorem 1.1, $\widetilde{\Sigma}$ can only be the S_0^n with multiplicity two, which implies that Σ is the boundary of M_+. □

Idea of the proof. We construct a sequence of Riemannian $(n+1)$-spheres M_k (isometrically embedded in $\mathbb{R}^{n+2}$) that converges locally smoothly to $S_0^n \times \mathbb{R}$ (see Section 3 for details), where S_0^n is a round n-sphere in $\mathbb{R}^{n+1}$ and embedded in M_k as the unique degenerate stable minimal hypersurface for each k. Moreover, for each k, the level sets (denoted by $\{S_t\}$) of the distance function to S_0^n have area lower bound $\Omega_n(1-|t|^{n+1})$, where Ω_n is the volume of the unit n-sphere.

Suppose on the contrary that each $\omega_2(M_k)$ is realized by a minimal hypersurface Σ_k that is not S^n_0. Then by our construction, Σ_k has to be connected, unstable, of index less than or equal to 2 (by [19]) and multiplicity one ([38]). Then the compactness [27] gives that Σ_k converges locally smoothly to an embedded minimal hypersurface with integer multiplicity in $S^n_0\times\mathbb{R}$. By the lower bounds for $\omega_2(M)$, the limit of Σ_k is exactly S^n_0 with multiplicity 2. Moreover, by the classification of embedded minimal hypersurfaces with two ends ([26]), after suitable scaling, the blowup around each singular point is a standard catenoid. Let r_k denote the "radius" of link of the small catenoid in Σ_k, i.e. the distance of the small catenoid to the center point. First, the Hausdorff distance between Σ_k and S^n_0 is bounded from above by $10r_k|\log r_k|$ (to be discussed next). After replacing the annuli in neck regions of Σ_k by two minimizing n-disks, by using the one-sided minimizing property of $\{S_t\}$, the area of the new hypersurface is at least $2|S^n_0|-\mathcal{O}(r_k^{n+1}|\log r_k|^{n+1})$; see (3.4). On the other hand, the neck regions of Σ_k contribute at least $c(n)r_k^n$ amount of area more than that of the n-disks; see (3.2). Therefore, $|\Sigma_k|-2|S^n_0|\geq c(n)r_k^n-\mathcal{O}(r_k^{n+1}|\log r_k|^{n+1})>0$, contradicting $\omega_2(M_k)\leq 2\omega_1(M_1)\leq 2\mathrm{Area}(S^n_0)$.

To bound the Hausdorff distance $d_H(\Sigma_k,S^n_0)$, the key challenge arises from intermediate annuli regions $A(y_k,R_k,\epsilon;M_k)$, where y_k is a singular point, $R_k/r_k\nearrow\infty$, $R_k\searrow 0$. In fact, Σ_k is close in smooth topology to the catenoid of radius r_k inside $B(y_k,R_k;M_k)$ and to $2S^n_0$ outside $B(y_k,\epsilon;M_k)$ by smooth convergence. However, Σ_k may not be a 2-sheeted graph over S^n_0 inside $A(y_k,R_k,\epsilon;M_k)$, and we are forced to write one component Σ^2_k (of Σ_k) as a graph over the other Σ^1_k. First, $d_H(\Sigma^1_k,\Sigma^2_k)$ when restricted to $\partial B(y_k,R_k;M_k)$ is at most $3r_k\log\frac{R_k}{r_k}$; see (4.19). We need to prove that this bound keeps roughly at the order $r_k\log\frac{s}{r_k}$ on $\partial B(y_k,s;M_k)$ when $s\nearrow\epsilon$. Our proof relies on several new monotonicity formulas inspired by Colding-Minicozzi [5]. In particular, we study carefully the evolution of the averages of the height function w_k (between Σ^1_k,Σ^2_k): $s\mapsto \fint_{\Sigma^1_k\cap\partial B(y_k,s;M_k)}w_k$; see Proposition 4.4 and compare with [5, Lemma 2.1]. The main new challenge as compared with [5] is that w_k is not a graph function over a fixed hypersurface, and this causes many higher-order terms in our monotonicity formulas, particularly as the height function w_k only satisfies a highly nonlinear PD-inequality (4.6). Note that some extra care is needed when the two singular points are not too far from each other; see Proposition 4.5.

Outline. In Section 2, we will classify the limit cones of the singular set arising from the compactness of minimal hypersurfaces. In Section 3, we describe the concrete constructions and prove our main results by assuming the key Hausdorff distance upper bound estimates, which will be proved in Section 4. Finally in Appendix A, we derive a general inequality for minimal graphs over another minimal hypersurface. Then we list some basic results of catenoids in Appendix B.

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2. Blowing-up analysis

Let $\{M_k\}$ be a sequence of $(n+1)$-spheres embedded in $\mathbb{R}^{n+2}=\mathbb{R}^{n+1}\times\mathbb{R}$. Denote by S^n_0 the unit sphere in $\mathbb{R}^{n+1}$. Let Σ_k be a closed embedded connected minimal hypersurface in M_k. We use $B_r(p)$ and $B(p,r;M_k)$ to denote the geodesic ball in $\mathbb{R}^{n+1}$ and M_k, respectively. In this section, M_k and Σ_k always satisfy the following requirements.

(1) For each k, S_0^n is a stable minimal hypersurface in M_k with constant Jacobi fields;
(2) M_k locally smoothly converges to $S_0^n \times \mathbb{R}$ in $\mathbb{R}^{n+2}$ as $k \to \infty$;
(3) Σ_k converges to twice of S_0^n in the sense of varifolds;
(4) Σ_k has Morse index less than or equal to 2.

Example 1. Let M_k be the $(n+1)$-sphere given by

$$x_1^2 + x_2^2 + \cdots + x_{n+1}^2 + \frac{x_{n+2}^{2n}}{k^{2n}} = 1.$$

Suppose that M_k contains an embedded minimal hypersurface Σ_k with index less than or equal to 2 and

$$\mathrm{Area}(S_0^n) + \delta \leq \mathrm{Area}(\Sigma_k) \leq 3\mathrm{Area}(S_0^n) - \delta$$

for some $\delta > 0$. Then $\{\Sigma_k\}$ and $\{M_k\}$ satisfy our requirements in this section.

By compactness [27], Σ_k locally smoothly converges to S_0^n with multiplicity two away from a set $\mathcal{W}$ consisting of one or two points. Let $p \in \mathcal{W}$. Then we can take a positive constant $\epsilon < 10^{-1000}$ small enough such that the convergence for Σ_k is locally smooth in $B_{2\epsilon}(p) \setminus \{p\}$, and

$$\mathrm{Area}(B(p, 2\epsilon; M_k) \cap \Sigma_k) \leq \frac{5}{2} \cdot \frac{\Omega_{n-1}}{n} (2\epsilon)^n, \tag{2.1}$$

where Ω_m is the volume of unit m-spheres. Choose $p_k \in \Sigma_k$ so that

$$|A^{\Sigma_k}(p_k)| = \max_{\Sigma_k \cap B(p,\epsilon;M_k)} |A^{\Sigma_k}(x)|.$$

Clearly, $|A^{\Sigma_k}(p_k)| \to \infty$ as $k \to \infty$. In the following, we classify the limit cones of $\{\Sigma_k\}$ at p, which is either a hyperplane with multiplicity or a catenoid.

Proposition 2.1 (Classification of limit cones)**.** *Let $\{c_k\}$ be a sequence of positive numbers with $c_k \to \infty$.*

(1) If $\lim_{k\to\infty} |A^{\Sigma_k}(p_k)|/c_k = \ell > 0$, then $c_k(\Sigma_k - p_k)$ locally smoothly converges to a catenoid contained in $\mathbb{R}^{n+1}$;
(2) If $\lim_{k\to\infty} |A^{\Sigma_k}(p_k)|/c_k = 0$, then $c_k(\Sigma_k - p_k)$ locally smoothly converges to a hyperplane contained in $\mathbb{R}^{n+1}$;
(3) If $\lim_{k\to\infty} |A^{\Sigma_k}(p_k)|/c_k = \infty$, then $c_k(\Sigma_k - p_k)$ converges to a multiplicity-two hyperplane contained in $\mathbb{R}^{n+1}$. Moreover, the convergence is locally smooth away from at most two points including 0.

Proof. Note that $c_k(\Sigma_k - p_k)$ is always a minimal hypersurface in $c_k(M_k - p_k)$ with index ≤ 2. By the monotonicity formula and (2.1),

$$\lim_{k\to\infty} \mathrm{Area}(B_r(0) \cap c_k(\Sigma_k - p_k)) \leq \frac{5}{2} \cdot \frac{\Omega_{n-1}}{n} r^n. \tag{2.2}$$

Clearly, $c_k(M_k - p_k)$ converges locally smoothly to $\mathbb{R}^{n+1}$. Thus $c_k(\Sigma_k - p_k)$ converges to an embedded minimal hypersurface with integer multiplicity in $\mathbb{R}^{n+1}$ with index ≤ 2. By the work of Tysk [32], the limit minimal hypersurface, which has finite Morse index and polynomial volume growth (2.2), must have a unique tangent cone at infinity; therefore using the work of Schoen [26, Theorem 3] (see also [12, Theorem 1.3]), the limit minimal hypersurface is either a catenoid or a hyperplane.

If $\lim_{k\to\infty} |A^{\Sigma_k}(p_k)|/c_k = \ell < \infty$, then for any compact set $\Omega \subset \mathbb{R}^{n+2}$, $c_k(\Sigma_k - p_k) \cap \Omega$ has uniformly bounded second fundamental form, and this implies that the convergence is locally smooth. When $\ell > 0$, the limit hypersurface has at least one point that has non-zero curvature. Thus the limit is a catenoid. When $\ell = 0$, the limit hypersurface is flat and hence is a hyperplane.

If $\lim_{k\to\infty} |A^{\Sigma_k}(p_k)|/c_k = \infty$, then the convergence is non-smooth. By Allard's Regularity Theorem [28, Theorem 24.2], the multiplicity of the convergence is larger than or equal to two. Thus the limit is a stable minimal hypersurface in $\mathbb{R}^{n+1}$ with polynomial (degree n) area growth, which can only be a hyperplane by Schoen-Simon [24] and Schoen-Simon-Yau [25]. Since the index of Σ_k is bounded from above by 2, the convergence is locally smooth away from at most two points. Together with (2.2), we also have that the multiplicity of the convergence is exactly 2. Hence Proposition 2.1 is proved. □

Remark 2.2. Let $x_k \in \Sigma_k$ and $c_k \to \infty$. Then $c_k(\Sigma_k - x_k)$ converges to a multiplicity-one catenoid or hyperplane with multiplicity one or two in the sense of varifold. Suppose that the convergence is not locally smooth. Then the limit is a multiplicity-two hyperplane.

Denote by $r_{k,1} = \sqrt{n(n-1)}|A^{\Sigma_k}(p_k)|^{-1}$. Take $y_{k,1} \in \Sigma_k$ so that $r_{k,1}^{-1}(\Sigma_k - y_{k,1})$ converges to a standard catenoid $\mathcal{C}$ in $\mathbb{R}^{n+1}$; see Appendix B for several properties of the geometry of $\mathcal{C}$. In the remaining of this section, we are going to find a sequence of "bad balls", in which Σ_k is not flat enough even after rescaling.

So long as

$$\max_{x\in B(y_{k,1},\epsilon;M_k)\cap\Sigma_k} |A^{\Sigma_k}(x)||x - y_{k,1}| < 2\sqrt{n(n-1)},$$

we let $y_{k,2} := y_{k,1}$. Otherwise, take $y'_{k,2} \in \Sigma_k$ so that

$$\max_{x\in B(y_k,\epsilon;M_k)\cap\Sigma_k} |A^{\Sigma_k}(x)|x - y_k| = |A^{\Sigma_k}(y'_{k,2})||y'_{k,2} - y_{k,1}| \geq 2\sqrt{n(n-1)}.$$

Then by Proposition 2.1, $|A^{\Sigma_k}(y'_{k,2})|(\Sigma_k - y'_{k,2})$ converges locally smoothly to a catenoid in $\mathbb{R}^{n+1}$. Then we can take $y_{k,2}$ around $y'_{k,2}$ and $r_{k,2} > 0$ so that $r_{k,2}^{-1}(\Sigma_k - y_{k,2})$ converges locally smoothly to a standard catenoid. Let

$$r_k = \max\{r_{k,1}, r_{k,2}\}.$$

Denote by

$$b_k := |y_{k,1} - y_{k,2}|. \tag{2.3}$$

By the locally smooth convergence of Σ_k in $B_\epsilon(p) \setminus \{p\}$, we know that

$$b_k \to 0, \text{ as } k \to \infty.$$

Note that in a standard catenoid (see Appendix B), we have $|x| \cdot |A(x)| \leq \sqrt{n(n-1)}$. Therefore, we also have that

$$b_k/r_k \to \infty, \text{ if } y_{k,1} \neq y_{k,2}.$$

For simplicity, denote by $A(p, r, s; M_k) = B(p, s; M_k) \setminus B(p, r; M_k)$, $\mathcal{W}_k = \{y_{k,1}, y_{k,2}\}$ and

$$d_k(x) = \mathrm{dist}_{M_k}(x, \mathcal{W}_k). \tag{2.4}$$

We now claim that for all sufficiently large k,

$$\max_{x\in B(p,\epsilon;M_k)\cap\Sigma_k} |A^{\Sigma_k}(x)|d_k(x) < 2\sqrt{n(n-1)}. \tag{2.5}$$

Suppose not, then one can find $y_{k,3}$ and another small cantenoid. Observe that each small catenoid will contribute index 1. This contradicts that Σ_k has index less than or equal to 2.

From now on, let Σ_k^1 and Σ_k^2 be the two connected components of $\Sigma_k \setminus \cup_j B(y_{k,j}, 2r_{k,j}; M_k)$. Without loss of generality, outside $B(p, \epsilon; M_k)$, we choose the unit normal vector field of Σ_k^1 pointing towards Σ_k^2.

Proposition 2.3. *Given $\delta > 0$, there exists $K > 0$ such that for all $k \geq K$, the following statements hold true.*

(1) The second fundamental forms satisfy for $x \in \Sigma_k \setminus \cup_{j=1}^2 B(y_{k,j}, Kr_{k,j}; M_k)$,

$$d_k \cdot |A^{\Sigma_k}(x)| + d_k^2 \cdot |\nabla A^{\Sigma_k}(x)| < \delta. \tag{2.6}$$

Moreover, Σ_k^1 is a minimal graph over Σ_k^2. Denote by w_k the graph function.

(2) For every $x \in \Sigma_k^1 \setminus \cup_{j=1}^2 B(y_{k,j}, Kr_{k,j}; M_k)$,

$$d_k^2 \frac{|\nabla^2 w_k|}{w_k}(x) + d_k \frac{|\nabla w_k|}{w_k} + \frac{w_k}{d_k} < \delta. \tag{2.7}$$

(3) For any $s > Kr_k$ and $x, y \in \Sigma_k^1 \cap A(y_k, s, 4s; M_k) \setminus B(z_k, s/4; M_k)$,

$$w_k(x) \leq (1 + \delta) w_k(y).$$

Proof. Note that by Remark 2.2, the limit of $d_k^{-1}(x_k)(\Sigma_k - x_k)$ can only be a multiplicity-two hyperplane if $d_k(x_k)/r_{k,j} \to \infty$. Then the first item follows from standard blowup and contradiction arguments. Observe the third one follows directly from the second one. Hence it suffices to prove the second item.

Suppose on the contrary that there exists $\delta > 0$ and $\alpha_k \to \infty$ such that for a subsequence of $k \geq \alpha_k$, there exists $x_k \in \Sigma_k \setminus \cup_{j=1}^2 B(y_{k,j}, \alpha_k r_{k,j}; M_k)$ violating (2.7). So long as $\lim_{k\to\infty} d_k(x_k) \neq 0$, after renormalizations, the sequence, still denoted by $\{w_k\}$, will converge locally smoothly to a Jacobi field of S_0^n away from at most two points (see [29]), which is a constant function. This will give the desired contradiction.

It remains to consider the situation $d_k \to 0$ and $d_k/r_{k,j} \geq \alpha_k \to \infty$. By Remark 2.2, $d_k^{-1}(\Sigma_k - x_k)$ converges to a hyperplane with multiplicity two. Then the normalizations of $\{w_k\}$ will converge locally smoothly to a positive Jacobi field of the hyperplane, which is also a constant function. This will also give a contradiction. □

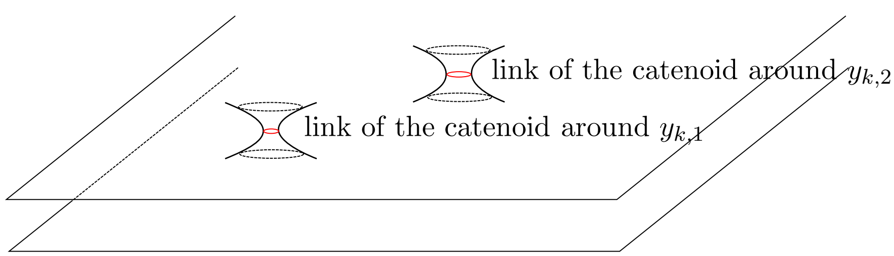

FIGURE I. Structure of Σ_k.

Remark 2.4. For $\{\Sigma_k\}$ satisfying the requirements at the beginning of this section, we summarize the properties of Σ_k (see Figure I) for large k:

(1) there exists $\{y_{k,j}\}$ containing one or two points such that for each j, Σ_k has a small catenoid with radius $r_{k,j}$ around $y_{k,j}$;

(2) Σ_k has exactly two connected components (denoted as Σ_k^1 and Σ_k^2) after removing such two (possibly one) small catenoids;
(3) Σ_k^2 is a graph over Σ_k^1 and the graph function satisfies Proposition 2.3.

3. Proof of the main theorem

For any $a \geq 1$ and $3 \leq (n+1) \leq 7$, let M_a be the embedded $(n+1)$-sphere in $\mathbb{R}^{n+2}$ given by

$$x_1^2 + x_2^2 \cdots + x_{n+1}^2 + \frac{x_{n+2}^{2n}}{a^{2n}} = 1.$$

For simplicity, let $S_t := \{x_{n+2} = t\} \cap M_a$, where we omit the a without ambiguity. These $(n+1)$-spheres satisfy the following properties:

(A) it has non-negative Ricci curvature;
(B) $S_0^n = S_0$ is the unique stable minimal hypersurface for each $a \geq 1$; it follows that each minimal hypersurface is two-sided;
(C) $M_a \to S_0^n \times \mathbb{R}$ locally smoothly as $a \to \infty$;
(D) $\{S_t\}_{t=-a}^a$ forms a foliation of M_a and $\{x_{n+2} = t\}$ is an embedded n-sphere with mean curvature vector pointing away from S_0 for $0 < |t| < a$. In particular, any two embedded minimal hypersurfaces in M_a intersect each other;
(E) the area of $S_t \subset M_a$ satisfies that for all t,

$$\text{Area}(S_t) = \Omega_n(1 - \frac{t^{2n}}{a^{2n}}) \geq \Omega_n(1 - |t|^{2n}). \tag{3.1}$$

(F) for each t with $0 < |t| < a$,

$$|t| \leq \text{dist}_{M_a}(S_0, S_t) \leq 2|t|.$$

Before stating our main results, we introduce the key height estimates, which will be proved in the next section (Theorem 4.6 and 4.9).

Theorem 3.1. *Let $\Sigma_k \subset M_k^{n+1}$ be a sequence of embedded minimal hypersurfaces with* index ≤ 2 *and*

$$\text{Area}(\Sigma_k) \leq 3\text{Area}(S_0^n) - \delta$$

for some $\delta > 0$. Then for all sufficiently large k,

$$\max_{\Sigma_k} \text{dist}_{M_k}(x, S_0^n) \leq 8r_k|\log r_k|.$$

Remark 3.2. Note that the estimates are sharp for dimension $n+1 = 3$ (Theorem 4.6). In higher dimensions $4 \leq n+1 \leq 7$, the height estimates are much better (Theorem 4.9).

In the following, we prove a rigidity theorem for minimal hypersurfaces with low index and area. This will be useful to identify the min-max solutions that realize the second width.

Theorem 3.3. *Suppose that Σ_k is a closed embedded minimal hypersurface in M_k with* index ≤ 2 *and*

$$\text{Area}(\Sigma_k) \leq 2\text{Area}(S_0^n).$$

Then for sufficiently large k, Σ_k is exactly S_0^n.

Proof. Suppose not, then Σ_k converges to a closed minimal hypersurface (with multiplicity) in $S^n_0\times\mathbb{R}$. By the monotonicity formula, the limit is $S^n_0\times\{0\}$ with multiplicity less than or equal to 2. We claim the multiplicity has to be two. So long as the the convergence has multiplicity one, the normalization of the difference between S^n_0 and Σ_k converges to a non-trivial Jacobi field on S^n_0 (see [29]). Note that such a Jacobi field is a constant. Thus Σ_k lies on one side of S^n_0, which contradicts (D) at the beginning of this section. Therefore, Σ_k converges to Σ with multiplicity two.

By [27], the convergence is smooth away from a set $\mathcal{W}$ containing at most two points since the index of Σ_k is less than or equal to 2. Then $\{\Sigma_k\}$ and $\{M_k\}$ satisfy the conditions in Section 2 (see Example 1). Let $p\in\mathcal{W}$. By the argument therein, we can find $y_k(\in M_k)\to p$ and $r_k>0$ such that $r_k^{-1}(\Sigma_k-y_k)$ locally smoothly converges to a standard catenoid $\mathcal{C}\subset\mathbb{R}^{n+1}$, i.e. it has the center at 0 and

$$\max_{x\in\mathcal{C}}|A(x)|=\sqrt{n(n-1)}.$$

By Remark 2.4, here we have three possibilities:

1. $\mathcal{W}=\{p,q\}$ consists of two different points. In this case, there exists $z_k\to q$ and $\widetilde{r}_k$ such that $\widetilde{r}_k^{-1}(\Sigma_k-z_k)$ locally smoothly converges to a standard catenoid. Without loss of generality, we assume that $r_k\geq\widetilde{r}_k$ for all large k.
2. $\mathcal{W}=\{p\}$ consists of only one point and b_k described in Proposition 2.3 is equal to 0. It follows that $\Sigma_k\setminus B(y_k,2r_k;M_k)$ has exactly two connected components. In this case, we let $\widetilde{r}_k=0$.
3. $\mathcal{W}=\{p\}$ consists of only one point and b_k described in Proposition 2.3 is not 0, i.e. there exists $z_k\to p$, $|z_k-y_k|/r_k\to\infty$ and $\widetilde{r}_k>0$ such that $\widetilde{r}_k^{-1}(\Sigma_k-z_k)$ locally smoothly converges to a standard catenoid. Without loss of generality, we assume that $r_k\geq\widetilde{r}_k$ for large k.

Above all, outside two small balls with radii r_k and $\widetilde{r}_k$, Σ_k has exactly two connected components; see Remark 2.4. Moreover, we can take $R_k\to 0$ with $R_k/r_k\to\infty$, $r_k^{-1}(\Sigma_k-y_k)\cap B_{R_k/r_k}(0)$ is arbitrarily smoothly close to a standard catenoid $\mathcal{C}\cap B_{R_r/r_k}(0)$; if z_k exists, we can also take $\widetilde{R}_k\to 0$ with $\widetilde{R}_k/\widetilde{r}_k\to\infty$ so that $\widetilde{r}_k^{-1}(\Sigma_k-z_k)\cap B_{R_k/r_k}(0)$ is arbitrarily close to a standard catenoid $\mathcal{C}\cap B_{R_k/r_k}(0)$.

Denote by γ^1_k and γ^2_k the two components of $\Sigma_k\cap\partial B(y_k,R_k;M_k)$. Then each γ^j_k ($j=1$ or 2) bounds an area minimizing n-disk $\mathcal{D}^j_k$ in M_k. By Proposition (B.3),

$$\operatorname{Area}(B(y_k,R_k;M_k)\cap\Sigma_k)-\sum_j\operatorname{Area}(\mathcal{D}^j_k)\geq\frac{\mathcal{A}_n}{2}r_k^n>0. \tag{3.2}$$

Here $\mathcal{A}_n$ ($n\geq 3$) is defined by (B.2) and $\mathcal{A}_2$ can be any fixed real numbers because of (B.3). Similarly, in Case (1) or (3), there exist $\widetilde{\gamma}^j_k\subset\Sigma_k$ ($j=1,2$) surrounding z_k such that each $\widetilde{\gamma}^j_k$ bounds an area minimizing n-disk $\widetilde{\mathcal{D}}^j_k$ in M_k with

$$\operatorname{Area}(B(z_k,\widetilde{R}_k;M_k)\cap\Sigma_k)-\sum_j\operatorname{Area}(\widetilde{\mathcal{D}}^j_k)\geq\frac{\mathcal{A}_n}{2}\widetilde{r}_k^n>0. \tag{3.3}$$

Then we cut off $\Sigma_k\cap B(y_k,R_k;M_k)$ and $\Sigma_k\cap B(z_k,\widetilde{R}_k;M_k)$, which are close to a standard catenoid after scaling. After that, we add the n-disks $\{\mathcal{D}^j_k\}$ and $\{\widetilde{\mathcal{D}}^j_k\}$ to the new hypersurface; see Figure II. This yields a closed hypersurface with two connected components Γ_1 and Γ_2 which are both homologous to S^n_0 in M_k.

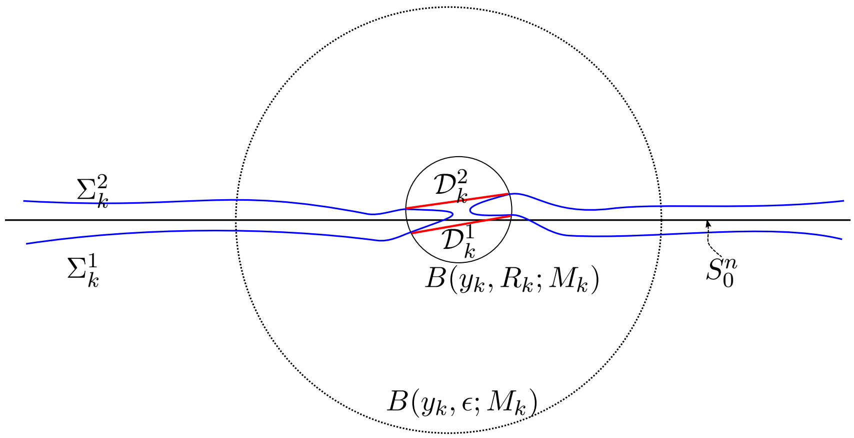

FIGURE II. Replacing the catenoids by two n-disks.

Note that Theorem 3.1 can be applied to obtain

$$\max_{x\in\Sigma_k} \mathrm{dist}_{M_k}(x, S_0^n) \le 8r_k|\log r_k|;$$

together with a standard minimal foliation argument (applied to $B(y_k, \epsilon; M_k)$), this implies that

$$\max_{x\in\Gamma_j} \mathrm{dist}_{M_k}(x, S_0^n) \le 10r_k|\log r_k|.$$

Let $\mathfrak{d}_k := 10r_k|\log r_k|$. Then by the one-sided minimizing property of $S_{\mathfrak{d}_k}$ and $S_{-\mathfrak{d}_k}$,

$$\mathrm{Area}(\Gamma_j) + \mathrm{Area}(S_0^n) > \mathrm{Area}(S_{\mathfrak{d}_k}) + \mathrm{Area}(S_{-\mathfrak{d}_k}).$$

By (3.1),

$$\mathrm{Area}(S_{\mathfrak{d}_k}) \ge \mathrm{Area}(S_0^n) - \Omega_n|\mathfrak{d}_k|^{2n} \quad \text{and} \quad \mathrm{Area}(S_{-\mathfrak{d}_k}) \ge \mathrm{Area}(S_0^n) - \Omega_n|\mathfrak{d}_k|^{2n};$$

these imply that

$$\mathrm{Area}(\Gamma_1) + \mathrm{Area}(\Gamma_2) > 2\mathrm{Area}(S_0^n) - 4\Omega_n\mathfrak{d}_k^{2n}. \tag{3.4}$$

By the construction of Γ_j,

$$\begin{aligned}
\mathrm{Area}(\Sigma_k) &= \sum_{j=1}^{2}\Big(\mathrm{Area}(\Gamma_j) - \mathrm{Area}(\mathcal{D}_k^j) - \mathrm{Area}(\widetilde{\mathcal{D}}_k^j)\Big) + \mathrm{Area}(B(y_k, R_k; M_k)\cap\Sigma_k) \\
&\quad + \mathrm{Area}(B(z_k, \widetilde{R}_k; M_k)\cap\Sigma_k) \\
&\ge \sum_{j=1}^{2}\Big(\mathrm{Area}(\Gamma_j) - \mathrm{Area}(\mathcal{D}_k^j)\Big) + \mathrm{Area}(B(y_k, R_k; M_k)\cap\Sigma_k) \\
&> 2\mathrm{Area}(S_0^n) - 10^{2n}r_k^{2n-1}\cdot(4\Omega_n r_k|\log r_k|^{2n}) + \frac{\mathcal{A}_n}{2}r_k^n \\
&> 2\mathrm{Area}(S_0^n) + \frac{\mathcal{A}_n}{4}r_k^n.
\end{aligned}$$

Here the first inequality is from (3.3); we used (3.2) and (3.4) in the second one; the last one follows from the fact that $r_k|\log r_k|^{2n}\to 0$ as $k\to\infty$. This gives a contradiction and we finish the proof of Theorem 3.3. □

Now we are going to prove our main results.

Theorem 3.4. *For sufficiently large k, the second width $\omega_2(M_k)$ can only be realized by S^n_0 with multiplicity two.*

Proof. By the work of Marques-Neves [19], together with the Frankel's property (D), $\omega_2(M_k)$ is realized by the area of a connected, closed, embedded, minimal hypersurface Σ_k with integer multiplicities and $\mathrm{index}(\Sigma_k)\leq 2$. Observe that

$$\lim_{k\to\infty}\omega_2(M_k)=\omega_2(S^n_0\times\mathbb{R})=2\mathrm{Area}(S^n_0). \tag{3.5}$$

We first prove that Σ_k is S^n_0. Suppose not, then Σ_k is unstable by Property (B); therefore using the direct corollary of the Multiplicity One Theorem in [38], Σ_k has multiplicity one, i.e. $\omega_2(M_k)=\mathrm{Area}(\Sigma_k)$. On the other hand, by the construction of optimal 1-sweepouts (c.f. [36]), $\omega_1(M_k)=\mathrm{Area}(S^n_0)$. Then it follows that (c.f. [20, Proof of Theorem 5.1]).

$$\omega_2(M_k)\leq 2\omega_1(M_k)=2\mathrm{Area}(S^n_0). \tag{3.6}$$

By Theorem 3.3, Σ_k has to be identical to S^n_0, but this contradicts (3.5). Thus we conclude that Σ_k has to be S^n_0. Then the multiplicity follows from (3.5).

Therefore, $\omega_2(M_k)$ can only be realized by S^n_0 with multiplicity two. □

Let M_k be the Riemannian $(n+1)$-sphere in Theorem 3.4. Denote by N_k and $\mathbb{RP}^n_0$ the quotient spaces $M_k/\{x\sim -x\}$ and $S^n_0/\{x\sim -x\}$, respectively. Then we have the following properties:

1. for each k, N_k satisfies the Frankel's property, i.e. any two embedded minimal hypersurfaces intersect;
2. $\mathbb{RP}^n_0$ is a one-sided embedded minimal hypersurface in N_k;
3. as $k\to\infty$, N_k locally smoothly converges to $S^n_0\times\mathbb{R}/\{x\sim -x\}$;
4. $\omega_1(S^n_0\times\mathbb{R}/\{x\sim -x\})=\mathrm{Area}(S^n_0)=2\mathrm{Area}(\mathbb{RP}^n_0)$.

Here the first three items follow from the Properties (A–D) of M_k at the beginning of this section. For the last one, since $S^n_0\times\mathbb{R}/\{x\sim -x\}$ has a minimal foliation, it follows that

$$\omega_1(S^n_0\times\mathbb{R}/\{x\sim -x\})\leq\mathrm{Area}(S^n_0)=2\mathrm{Area}(\mathbb{RP}^n_0).$$

On the other hand, as the limit contains $S^n_0\times(0,\infty)$, it follows that

$$\omega_1(S^n_0\times\mathbb{R}/\{x\sim -x\})\geq\omega_1(S^n_0\times\mathbb{R})=\mathrm{Area}(S^n_0).$$

Hence the last property is proved.

The following result directly implies Corollary 1.2.

Corollary 3.5. *For sufficiently large k, the first width $\omega_1(N_k)$ can only be realized by $\mathbb{RP}^n_0$ with multiplicity two.*

Proof. Notice that N_k also satisfies the Frankel's property. Then by Marques-Neves [19], $\omega_1(N_k)$ is realized by the area of a connected, closed, embedded, minimal hypersurface Γ_k with integer multiplicity m_k and $\mathrm{index}(\Gamma_k)\leq 1$. Observe that

$$\lim_{k\to\infty}\omega_1(N_k)=\omega_1(S^n_0\times\mathbb{R}/\{x\sim -x\})=2\mathrm{Area}(\mathbb{RP}^n_0). \tag{3.7}$$

It follows that $\mathrm{Area}(\Gamma_k)$ is uniformly bounded. Then by compactness [27], Γ_k converges to an embedded minimal hypersurface $\Gamma \subset S_0^n \times \mathbb{R}/\{x \sim -x\}$. Notice that such a space is foliated by minimal hypersurfaces. Thus Γ is $S_0^n \times \{t\}$ or $\mathbb{RP}_0^n$. Recall that Γ_k intersects $\mathbb{RP}_0^n$. Hence we conclude that the limit is $\mathbb{RP}_0^n$. Together with (3.7), we then have that $m_k \leq 2$.

Case I: $m_k = 1$.

Then Γ_k locally smoothly converges to $2\mathbb{RP}_0^n$ away from at most one point. Then by the same argument as in Theorem 3.3,

$$\mathrm{Area}(\Gamma_k) > 2\mathrm{Area}(\mathbb{RP}_0^n),$$

which contradicts $\omega_1(N_k) \leq 2\mathrm{Area}(\mathbb{RP}_0^n)$.

Case II: $m_k = 2$.

Then Γ_k converges to $\mathbb{RP}_0^n$ with multiplicity one. By Allard's regularity, the convergence is smooth. Let $\widetilde{\Gamma}_k \in M_k$ be the double cover of Γ_k. Then $\widetilde{\Gamma}_k$ smoothly converges to S_0^n. This implies that $\widetilde{\Gamma}_k$ is identical to S_0^n. Hence Γ_k is identical to $\mathbb{RP}_0^n$. This completes the proof of Corollary 3.5. □

4. Upper bounds for the Hausdorff distance

We use the notation in Section 2. Recall that $\Sigma_k \subset M_k^{n+1}$ is a closed embedded minimal hypersurface such that the Morse index is bounded above by two and the area is uniformly bounded independent of $k \in \mathbb{N}$. As $k \to \infty$, M_k converges locally smoothly to the product space $S_0^n \times \mathbb{R}$, and Σ_k converges to a minimal n-sphere in the limit space of M_k, namely $S_0^n \times \mathbb{R}$. The convergence of Σ_k is locally smooth away from at most two points according to the Morse index and area bounds. At a point near which the convergence is not smooth, the limit cones are either planes or catenoids by Proposition 2.1. For simplicity, we use A for A^{Σ_k} and sometimes omit the subscription k when there is no ambiguity. Denote by ∇ and $\widetilde{\nabla}$ the Levi-Civita connections of Σ_k and M_k, respectively.

In this section, we prove the key height estimates, which says that the Hausdorff distance between Σ_k and S_0^n is bounded by a quantity associated with the catenoids arising from blowups. This result is essentially used in the previous section (see Theorem 3.1).

Recall that by Remark 2.4, Σ_k has two connected components (denoted by Σ_k^1 and Σ_k^2) by removing one or two small catenoids. Denote by y_k, z_k the centers and $r_k, \widetilde{r}_k$ the radii of links of such catenoids (we used $y_{k,j}$ and $r_{k,j}$ in Section 2 for general cases). Without loss of generality, we assume that $r_k \geq \widetilde{r}_k$.

Then by Proposition 2.3, Σ_k^2 is a minimal graph over Σ_k^1. Let $\mathbf{n}$ be the unit normal vector field of Σ_k^1 , and let ρ and $\widetilde{\rho}$ be the distance functions to y_k and z_k in M_k. Denote by

$$\boldsymbol{\eta} = \nabla\rho/|\nabla\rho|, \quad \widetilde{\boldsymbol{\eta}} = \nabla\widetilde{\rho}/|\nabla\widetilde{\rho}|; \quad \phi = |\nabla\rho|, \quad \widetilde{\phi} = |\nabla\widetilde{\rho}|. \tag{4.1}$$

Recall that b_k, defined in (2.3), is the distance between the two blowup points y_k, z_k. For any

$$\epsilon \geq s \geq 2b_k > 0 \quad \text{or} \quad b_k/2 \geq s \geq 4r_k, \tag{4.2}$$

we set

$$\gamma_s = \Sigma_k^1 \cap \partial B(y_k, s; M_k). \tag{4.3}$$

In the remaining part of this section, we assume that $R_k, \widetilde{R}_k \to 0$ are two sequence of real numbers satisfying $R_k/r_k \to \infty$ and $\widetilde{R}_k/\widetilde{r}_k \to \infty$. Moreover, we can also assume $r_k^{-1}(\Sigma_k \cap B(y_k, R_k; M_k) - y_k)$ (resp. $\widetilde{r}_k^{-1}(\Sigma_k \cap B(z_k, \widetilde{R}_k; M_k) - y_k))$ is arbitrarily close to the catenoid

$\mathcal{C}\cap B_{R_k/r_k}(0)$ (resp. $\mathcal{C}\cap B_{\widetilde{R}_k/\widetilde{r}_k}(0)$) in the smooth topology. By Proposition 2.1, for sufficiently large k, $s^{-1}(\gamma_s - y_k)$ is very close to the unit $(n-1)$-sphere in the smooth topology. In particular, we have

$$(1-\epsilon^2)\Omega_{n-1}s^{n-1} < |\gamma_s| < (1+\epsilon^2)\Omega_{n-1}s^{n-1}, \tag{4.4}$$

where Ω_m is the volume of unit m-spheres. Recall that w_k is the graph function of Σ_k^2 over Σ_k^1.

Lemma 4.1. *Given any $\delta > 0$, then for all sufficiently large k, we have*

$$\Delta w_k \le w_k + C(n)\delta\cdot\frac{w_k^3}{d_k^4(x)} \tag{4.5}$$

for $x\in\Sigma_k^1\setminus\big[B(y_k,R_k;M_k)\cup B(z_k,\widetilde{R}_k;M_k)\big]$. In particular,

(1) for all large k and $x\in\Sigma_k^1\cap A(y_k,R_k,\epsilon;M_k)\setminus B(z_k,b_k/2;M_k)$,

$$\Delta w_k \le w_k + \frac{1}{8}\cdot\frac{w_k^3}{|\rho(x)|^4}; \tag{4.6}$$

(2) for all large k and $x\in\Sigma_k^1\cap A(z_k,R_k,b_k/2;M_k)$,

$$\Delta w_k \le w_k + \frac{1}{8}\cdot\frac{w_k^3}{|\widetilde{\rho}(x)|^4}.$$

Proof. By Proposition 2.3, $|A^{\Sigma_k}|$ has an upper bound over $\Sigma_k^1\cap A(y_k,s,2s;M_k)\setminus B(z_k,b_k/2;M_k)$, so the level sets of the distance to Σ_k^1 will form a desired foliation as in Appendix A. By the Cauchy-Schwartz inequality, we have

$$8|A||\nabla w_k|^2 \le \frac{1}{2}|A|^2 w_k + \frac{32|\nabla w_k|^4}{w_k};\quad |\nabla^2 w_k||A|w_k \le \delta|A|^2 w_k + \frac{1}{4\delta}|\nabla^2 w_k|^2 w_k. \tag{4.7}$$

By plugging (2.6), (2.7) and (4.7) into (A.3) in Appendix A, we can show that

$$\Delta w_k + |A|^2 w_k \le 3\delta w_k + |A|^2 w_k + C(n)\delta\cdot\Big(\frac{w_k^3}{d_k^4(x)} + w_k\Big).$$

Here in (A.3), we deal with the first and third terms by (4.7); for the second term in (A.3), we used $|A|w_k \le d_k|A|\cdot\frac{w_k}{d_k} < \delta^2$; in the fourth term, we used $w_k + |\nabla w_k| + |\nabla w_k|^3 < 3\delta$ by (2.7); the others can be bounded similarly; for instance, we have

$$|\nabla w_k|^2|\nabla^2 w_k| < \Big(\frac{\delta w_k}{d_k}\Big)^2\cdot\frac{w_k}{d_k^2} < \delta\cdot\frac{w_k^3}{d_k^4}.$$

Hence the (4.5) is proved.

Then for $x\in\Sigma_k^1\cap A(y_k,R_k,\epsilon;M_k)\setminus B(z_k,b_k/2;M_k)$, it follows that

$$d_k(x) \ge \frac{1}{4}\rho(x).$$

Plugging this into (4.6), we then have

$$\Delta w_k \le w_k + \frac{1}{8}\cdot\frac{w_k^3}{|\rho(x)|^4}. \tag{4.8}$$

The last case can be proved similarly. This completes the proof of Lemma 4.1. ∎

Inspired by Colding-Minicozzi [5, (2.1)], we define

$$\mathcal{I}_k(s) = \frac{1}{\Omega_{n-1}s^{n-1}}\int_{\gamma_s} w_k|\nabla\rho|\, d\mathcal{H}^{n-1}(x), \tag{4.9}$$

$$\tau_k(s) = \frac{1}{\Omega_{n-1}}\int_{\gamma_s}\langle\nabla w_k, \boldsymbol{\eta}\rangle \quad \text{and} \quad F_k(s) = \frac{n}{\Omega_{n-1}s^n}\int_{\gamma_s}(\phi^{-1}-\phi), \tag{4.10}$$

where $\phi = |\nabla\rho|$ is defined in (4.1). Compared with [5], we introduce the weight $\phi = |\nabla\rho|$ in $\mathcal{I}_k$ as Σ_k^1 is not flat. Note that by Proposition B.1,

$$\lim_{k\to\infty}\frac{\tau_k(R_k)}{r_k^{n-1}} = 2. \tag{4.11}$$

4.1. **Inequalities for the first derivatives.** In this subsection, we take the derivative for the averages of the height functions between two sheets. The coefficients of leading terms are the functions τ_k defined above.

Proposition 4.2. *Using above notations, we then have*

$$\begin{aligned}\mathcal{I}_k'(s) - \frac{\tau_k(s)}{s^{n-1}} &= \frac{1}{\Omega_{n-1}s^{n-1}}\int_{\gamma_s} w_k\Big[\phi^{-1}\mathrm{div}_{\Sigma_k}\widetilde{\nabla}\rho + \frac{1-n}{s}\phi\Big]\, d\mathcal{H}^{n-1}\\ &\le \frac{n}{\Omega_{n-1}s^n}\int_{\gamma_s}(\phi^{-1}-\phi)w_k + C_0 s\mathcal{I}_k(s),\end{aligned}$$

where C_0 is a constant depending only on n and can be changed from line to line.

Proof. Recall that $\phi = |\nabla\rho|$. A direct computation gives that

$$\langle\nabla\phi,\boldsymbol{\eta}\rangle + \phi\,\mathrm{div}_{\gamma_s}\boldsymbol{\eta} = \langle\nabla\phi,\boldsymbol{\eta}\rangle + \phi\,\mathrm{div}_{\Sigma_k}\boldsymbol{\eta} = \mathrm{div}_{\Sigma_k}\phi\boldsymbol{\eta} = \mathrm{div}_{\Sigma_k}\nabla\rho = \mathrm{div}_{\Sigma_k}\widetilde{\nabla}\rho. \tag{4.12}$$

Here we used that $\langle\nabla_{\boldsymbol{\eta}}\boldsymbol{\eta},\boldsymbol{\eta}\rangle = 0$ in the first equality; and the last one follows from the minimality of Σ_k. Then by noting that $\frac{\partial}{\partial s} = \frac{\nabla\rho}{|\nabla\rho|^2} = \frac{\boldsymbol{\eta}}{\phi}$, we have

$$\begin{aligned}\mathcal{I}_k'(s) &= \frac{1-n}{\Omega_{n-1}s^n}\int_{\gamma_s} w_k\phi\, d\mathcal{H}^{n-1} + \frac{1}{\Omega_{n-1}s^{n-1}}\int_{\gamma_s}\langle\nabla(w_k\phi),\frac{\boldsymbol{\eta}}{\phi}\rangle + w_k\phi\,\mathrm{div}_{\gamma_s}\big(\frac{\boldsymbol{\eta}}{\phi}\big)\, d\mathcal{H}^{n-1}\\ &= \frac{1-n}{\Omega_{n-1}s^n}\int_{\gamma_s} w_k\phi\, d\mathcal{H}^{n-1} + \frac{1}{\Omega_{n-1}s^{n-1}}\int_{\gamma_s}\langle\nabla w_k,\boldsymbol{\eta}\rangle + w_k\big[\langle\nabla\phi,\frac{\boldsymbol{\eta}}{\phi}\rangle + \mathrm{div}_{\gamma_s}\boldsymbol{\eta}\big]\, d\mathcal{H}^{n-1}\\ &= \frac{1-n}{\Omega_{n-1}s^n}\int_{\gamma_s} w_k\phi\, d\mathcal{H}^{n-1} + \frac{1}{\Omega_{n-1}s^{n-1}}\int_{\gamma_s}\langle\nabla w_k,\boldsymbol{\eta}\rangle + w_k\phi^{-1}\mathrm{div}_{\Sigma_k}\widetilde{\nabla}\rho\, d\mathcal{H}^{n-1},\end{aligned}$$

where the last equality is from (4.12). Then we are going to prove the inequality. Recall that when restricting to $T(\partial B(p,s;M_k))$, we have

$$\widetilde{\nabla}^2\rho = \frac{1}{\rho}\delta_{ij} + \mathcal{O}(\rho). \tag{4.13}$$

Therefore, we have that for $x \in \gamma_s$,

$$\begin{aligned}\mathrm{div}_{\Sigma_k}\widetilde{\nabla}\rho &= \mathrm{div}_{M_k}\widetilde{\nabla}\rho - \langle\widetilde{\nabla}_{\mathbf{n}}\widetilde{\nabla}\rho,\mathbf{n}\rangle\\ &= \frac{n}{s} + \mathcal{O}(s) - \langle\widetilde{\nabla}_E\widetilde{\nabla}\rho, E\rangle\langle\mathbf{n}, E\rangle^2\\ &\le \frac{1}{s}(n-\phi^2) + C_0 s.\end{aligned} \tag{4.14}$$

In the above calculation, we used E for the following expression,

$$E := \frac{\mathbf{n} - \langle \mathbf{n}, \widetilde{\nabla}\rho\rangle \widetilde{\nabla}\rho}{\sqrt{1-\langle \mathbf{n}, \widetilde{\nabla}\rho\rangle^2}};$$

this is the unit projection of $\mathbf{n}$ to $T(\partial B(p,s;M_k))$. This finishes the proof of Proposition 4.2. □

In the next subsection, we will use the inequality in Proposition 4.2 to bound $\mathcal{I}_k$, which requires the following results for $F_k(t)$.

Proposition 4.3. *Suppose that* $[R,s]\subset[R_k,b_k/2]$ *or* $[2b_k,\epsilon]$. *Then*

$$\frac{d}{ds}\Big[s^{-n}\Big(|\Sigma_k^1\cap A(y_k,R,s;M_k)|+\frac{R}{n}\int_{\gamma_R}\phi\Big)\Big]=s^{-n}\int_{\gamma_s}(\frac{1}{\phi}-\phi)+\frac{1}{s^{n+1}}\int_{A(y_k,R,s;M_k)\cap\Sigma_k^1}\mathcal{O}(\rho^2).$$

For all sufficiently large k,

$$\int_{R_k}^{b_k/2}F_k(t)dt+\int_{2b_k}^{\epsilon}F_k(t)dt<\epsilon.$$

Proof. By the divergence theorem,

$$\int_{A(y_k,R,s;M_k)\cap\Sigma_k^1}\Delta\rho^2=\int_{\gamma_s}2s\phi-\int_{\gamma_R}2R\phi. \tag{4.15}$$

Since $\Delta\rho^2=2n+\mathcal{O}(\rho^2)$, we have

$$|\Sigma_k^1\cap A(y_k,R,s;M_k)|+\frac{R}{n}\int_{\gamma_R}\phi=\frac{s}{n}\int_{\gamma_s}\phi+\int_{A(y_k,R,s;M_k)\cap\Sigma_k^1}\mathcal{O}(\rho^2).$$

Finally note that

$$\frac{d}{ds}\Big|\Sigma_k^1\cap A(y_k,R,s;M_k)\Big|=\int_{\gamma_s}\phi^{-1}.$$

The first identity follows by plugging all the above identities. Then we have

$$\begin{aligned}\int_{R_k}^{b_k/2}F_k(t)\,dt\le{}&\frac{n}{\Omega_{n-1}(b_k/2)^n}\Big(|\Sigma_k^1\cap A(y_k,R_k,b_k/2;M_k)|+\frac{R_k}{n}\int_{\gamma_{R_k}}\phi\Big)\\&-\frac{1}{\Omega_{n-1}R_k^{n-1}}\int_{\gamma_{R_k}}\phi+C_0b_k^2.\end{aligned}$$

Recall that by (4.4), for all sufficiently large k, we have

$$(1-\frac{\epsilon}{10})\Omega_{n-1}s^{n-1}\le|\gamma_s|\le(1+\frac{\epsilon}{10})\Omega_{n-1}s^{n-1};\quad 1\le\phi^{-1}\le1+\frac{\epsilon}{10}.$$

Then by the co-area formula, we have

$$\begin{aligned}&\frac{n}{\Omega_{n-1}(b_k/2)^n}\Big(|\Sigma_k^1\cap A(y_k,R_k,b_k/2;M_k)|+\frac{R_k}{n}\int_{\gamma_{R_k}}\phi\Big)\\&\le\frac{n}{\Omega_{n-1}(b_k/2)^n}\cdot\Omega_{n-1}\Big[\int_{R_k}^{b_k/2}(1+\frac{\epsilon}{10})s^{n-1}\,ds+\frac{R_k}{n}(1+\frac{\epsilon}{10})R_k^{n-1}\Big]=1+\frac{\epsilon}{10}.\end{aligned}$$

On the other hand,

$$\frac{1}{\Omega_{n-1}R_k^{n-1}}\int_{\gamma_{R_k}}\phi\ge1-\frac{\epsilon}{5}.$$

Combining them together, we then have

$$\int_{R_k}^{b_k/2} F_k(t)\,dt \leq \frac{3\epsilon}{10} + C_0\epsilon^2 < \frac{\epsilon}{2}.$$

Similarly,

$$\int_{2b_k}^{\epsilon} F_k(t)\,dt < \frac{\epsilon}{2}.$$

Hence Proposition 4.3 is proved. ☐

4.2. **Hausdorff distance upper bounds in dimension three.** In this part, we focus on the case $n = 2$ and prove the upper bounds of the Hausdorff distance between Σ_k and S^2_0. Recall that Σ^1_k and Σ^2_k are jointed by one or two small catenoids with radii r_k and $\widetilde{r}_k$; see Remark 2.4. Without loss of generality, we assume that $r_k \geq \widetilde{r}_k$. Recall that Σ^2_k is a graph over Σ^1_k with graph function w_k.

To prove the desired bound, we will derive several monotonicity formulas associated with the average of w_k. Then the smooth convergence on $\partial B(p, \epsilon; M_k)$ will give the desired upper bound.

Now let

$$\widetilde{\mathcal{I}}_k(s) = \mathcal{I}_k(s) - 3r_k \log\frac{s}{r_k} - 2sr_k\log\frac{s}{r_k} - 10\Big(\int_{[R_k,s]\setminus[b_k/2,2b_k]} F_k(t)\,dt\Big)\cdot r_k\log\frac{s}{r_k}; \tag{4.16}$$

$$\widetilde{\tau}_k(s) = \tau_k(s) + \frac{r_k^2}{2s} - sr_k\log\frac{s}{r_k}. \tag{4.17}$$

Then by Proposition 4.3,

$$\mathcal{I}_k(s) < \widetilde{\mathcal{I}}_k(s) + 4r_k\log\frac{s}{r_k}. \tag{4.18}$$

Recall that $R_k/r_k \to +\infty$ and $r_k^{-1}(\Sigma_k \cap B(y_k, R_k; M_k) - y_k)$ is arbitrarily close to the catenoid $\mathcal{C} \cap B_{R_k/r_k}(0)$ in the smooth topology. Then by Item (1)in Appendix B,

$$\mathcal{I}_k(R_k) \leq (1 + \frac{1}{100})\cdot 2r_k\int_1^{R_k/r_k}\frac{ds}{\sqrt{s^2-1}} < (1+\frac{1}{100})\cdot 2r_k\log\frac{2R_k}{r_k} < 3r_k\log\frac{R_k}{r_k}. \tag{4.19}$$

Then by (4.11), (4.16) and (4.19),

$$\widetilde{\mathcal{I}}_k(R_k) < 0, \quad \tau_k(R_k) < (2 + \frac{1}{10})r_k. \tag{4.20}$$

Now we are ready to show that $\widetilde{\mathcal{I}}_k(s)$ is decreasing.

Proposition 4.4. *$\widetilde{\mathcal{I}}_k(s)$ and $\widetilde{\tau}_k(s)$ are decreasing on $[R_k, b_k/2]$ (resp. $[R_k, \epsilon]$) if $b_k \neq 0$ (resp. $b_k = 0$). It follows that*

$$\mathcal{I}_k(s) < 4r_k\log\frac{s}{r_k}, \quad \tau_k(s) < 3r_k - \frac{r_k^2}{2s} + sr_k\log\frac{s}{r_k} \quad \textit{for} \ \ s \in [R_k, b_k/2].$$

Proof. We first assume that $b_k \neq 0$. To prove that $\widetilde{\mathcal{I}}_k$ and $\widetilde{\tau}_k$ are decreasing on $[R_k, b_k/2]$, let

$$s_1 := \sup\{s \in (R_k, b_k/2) : \widetilde{\tau}_k'(t) \leq 0,\ \widetilde{\mathcal{I}}_k'(t) \leq 0 \ \text{ for all } \ t \in [R_k, s)\}.$$

We claim that $s_1 = b_k/2$. Suppose on the contrary that $s_1 < b_k/2$. Then $\widetilde{\mathcal{I}}_k$ and $\widetilde{\tau}_k$ are decreasing on $[R_k, s_1]$, which together with (4.20) implies that

$$\widetilde{\mathcal{I}}_k(s_1) \leq \widetilde{\mathcal{I}}_k(R_k) < 0; \quad \widetilde{\tau}_k(s_1) \leq \widetilde{\tau}_k(R_k) < \frac{5r_k}{2}.$$

By (4.17) and (4.18), it follows that

$$\mathcal{I}_k(s_1) < \widetilde{\mathcal{I}}_k(s_1) + 4r_k \log\frac{s_1}{r_k} < 4r_k \log\frac{s_1}{r_k};$$
$$\tau_k(s_1) = \widetilde{\tau}_k(s_1) - \frac{r_k^2}{2s_1} + s_1 r_k \log\frac{s_1}{r_k} < 3r_k - \frac{r_k^2}{2s_1} + s_1 r_k \log\frac{s_1}{r_k}.$$

Then by Proposition 4.2,

$$\begin{aligned}\mathcal{I}_k'(s_1) &< \frac{\tau_k(s_1)}{s_1} + 5r_k \log\frac{s_1}{r_k}\cdot F_k(s_1) + C_0 s_1 r_k \log\frac{s_1}{r_k}\\ &< \frac{3r_k}{s_1} + 2r_k\log\frac{s_1}{r_k} + 5r_k\log\frac{s_1}{r_k}\cdot F_k(s_1).\end{aligned}$$

Here in the first inequality, we used that by Proposition 2.3 (3) and (4.4), for $x \in \gamma_{s_1}$,

$$w_k(x) \leq (1 + 1/100)^2 \mathcal{I}_k(s_1) < 5r_k \log\frac{s_1}{r_k}. \tag{4.21}$$

Then by (4.16), it follows that

$$\widetilde{\mathcal{I}}_k'(s_1) < \mathcal{I}_k'(s_1) - \frac{3r_k}{s_1} - 2r_k\log\frac{s_1}{r_k} - 10F_k(s_1)\cdot r_k\log\frac{s_1}{r_k} < 0. \tag{4.22}$$

On the other hand, by the divergence theorem, the co-area formula, (4.6) and (4.21),

$$\begin{aligned}\tau_k'(s_1) &= \frac{1}{2\pi}\int_{\gamma_{s_1}} \phi^{-1}\Delta w_k \leq \frac{1}{2\pi}\int_{\gamma_{s_1}} \phi^{-1}\Big(w_k + \frac{w_k^3}{8s_1^4}\Big)\\ &< 2s_1\cdot\Big(5r_k\log\frac{s_1}{r_k} + \frac{1}{8s_1^4}5^3 r_k^3\big(\log\frac{s_1}{r_k}\big)^3\Big) < r_k\log\frac{s_1}{r_k} + \frac{r_k^2}{2s_1^2}.\end{aligned} \tag{4.23}$$

Here the first inequality follows from (4.6); the second one follows from (4.21), (4.4) and $\phi^{-1}|_{\gamma_{s_1}} \leq 1+\epsilon$; in the last one, we used that as $k \to \infty$,

$$s_1 \to 0; \quad \frac{s_1}{r_k} \to \infty; \quad \frac{r_k}{s_1}(\log\frac{s_1}{r_k})^3 \to 0.$$

Together with (4.17), it follows that

$$\widetilde{\tau}_k'(s_1) < \tau_k'(s_1) - \frac{r_k^2}{2s_1^2} - r_k\log\frac{s_1}{r_k} < 0.$$

Combining with (4.22), this contradicts the choice of s_1.

If $b_k = 0$, then the same argument above gives that $\widetilde{\mathcal{I}}_k$ and $\widetilde{\tau}_k$ are decreasing on $[R_k, \epsilon]$. This finishes the proof of Proposition 4.4. □

Note that $\mathcal{I}_k(s)$ is not well-defined in a small neighborhood of b_k. Here we use Proposition 2.3 and the divergence theorem to jump over such a small interval.

Proposition 4.5. *Suppose that* $b_k \neq 0$. *Then* $\widetilde{\mathcal{I}}_k - 3r_k \log \frac{s}{r_k}$ *and* $\widetilde{\tau}_k$ *are decreasing on* $[2b_k, \epsilon]$. *It follows that*

$$\mathcal{I}_k(s) < 7r_k \log \frac{s}{r_k}, \qquad \tau_k(s) < 6r_k + sr_k \log \frac{s}{r_k} \quad \text{for} \quad s \in [2b_k, \epsilon].$$

In particular, we conclude that for sufficiently large k,

$$\max_{x \in \Sigma_k^1 \cap \partial B(y_k, \epsilon; M_k)} w_k < \frac{15}{2} r_k \log \frac{\epsilon}{r_k}.$$

Proof. By applying Proposition 2.3 (3), and then Proposition 4.4, we have

$$\max\{w_k; x \in \Sigma_k^1 \cap A(y_k, b_k/2, 2b_k; M_k) \setminus B(z_k, b_k/4; M_k)\} \leq (1+\delta)\mathcal{I}_k(\frac{b_k}{2}) < \frac{9}{2} r_k \log \frac{b_k}{2r_k}. \tag{4.24}$$

Then by (4.4),

$$\mathcal{I}_k(2b_k) < 5r_k \log \frac{b_k}{2r_k}. \tag{4.25}$$

Moreover, by the divergence theorem,

$$\begin{aligned}
\tau_k(2b_k) - \tau_k(b_k/2) - \tau_k(b_k/4; z_k) &= \frac{1}{2\pi} \int_{\Sigma_k^1 \cap A(y_k, b_k/2, 2b_k; M_k) \setminus B(z_k, b_k/4; M_k)} \Delta w_k \, dx \\
&\leq \frac{1}{2\pi} \int_{\Sigma_k^1 \cap A(y_k, b_k/2, 2b_k; M_k) \setminus B(z_k, b_k/4; M_k)} w_k + \frac{w_k^3}{8\rho^4} \, dx \\
&< 10 b_k^2 r_k \log \frac{b_k}{r_k} + \frac{5^3 r_k^3}{b_k^2} \Big(\log \frac{b_k}{r_k} \Big)^3 \\
&< \frac{1}{4} b_k r_k \log \frac{b_k}{2r_k} + \frac{r_k^2}{2b_k}.
\end{aligned}$$

Here we used the notation

$$\tau_k(s; z_k) = \frac{1}{2\pi} \int_{\Sigma_k^1 \cap \partial B(z_k, s; M_k)} \langle \nabla w_k, \boldsymbol{\eta}_z \rangle,$$

and $\boldsymbol{\eta}_z$ is the unit normal to $\partial B(z_k, s; M_k) \cap \Sigma_k^1$. The first inequality is from (4.6); we used (4.24) in the second one, and the area upper bound of $\Sigma_k^1 \cap A(y_k, b_k/2, 2b_k; M_k) \setminus B(z_k, b_k/4; M_k)$ is from the fact that $b_k^{-1}(\Sigma_k^1 - y_k)$ is very close to a hyperplane. Recall that Σ_k is very close to a catenoid with radius $\widetilde{r}_k \leq r_k$ around z_k. Then by the same argument as in Proposition 4.4, we also have

$$\tau_k(b_k/4; z_k) < 3\widetilde{r}_k + \frac{1}{4} b_k \widetilde{r}_k \log \frac{b_k}{4\widetilde{r}_k} \leq 3r_k + \frac{1}{4} b_k r_k \log \frac{b_k}{4r_k}.$$

Here the inequality is from $\widetilde{r}_k \leq r_k$, and the monotonicity of $r \mapsto r \log \frac{b_k}{4r}$ when $b_k/4r_k > e^{-1}$. It follows that

$$\begin{aligned}
\tau_k(2b_k) &< \tau_k(b_k/2) + \tau_k(b_k/4; z_k) + \frac{1}{4} b_k r_k \log \frac{b_k}{2r_k} + \frac{r_k^2}{2b_k} \\
&< 3r_k - \frac{r_k^2}{b_k} + \frac{1}{2} b_k r_k \log \frac{b_k}{2r_k} + 3r_k + \frac{1}{4} b_k r_k \log \frac{b_k}{4r_k} + \frac{1}{4} b_k r_k \log \frac{b_k}{2r_k} + \frac{r_k^2}{2b_k} \\
&< 6r_k - \frac{r_k^2}{4b_k} + b_k r_k \log \frac{b_k}{r_k}.
\end{aligned} \tag{4.26}$$

In the next, we are going to prove that $\widetilde{\mathcal{I}}_k(s) - 3r_k\log\frac{s}{r_k}$ and $\widetilde{\tau}_k$ are decreasing. Let

$$s_2 := \sup\{s\in[2b_k,\epsilon] : \widetilde{\tau}_k'(t)\leq 0,\ \widetilde{\mathcal{I}}_k'(t)\leq \frac{3r_k}{t}\ \text{ for all }\ t\in[2b_k,s]\}.$$

It suffices to prove that $s_2=\epsilon$. Suppose on the contrary that $s_2<\epsilon$. Then $\widetilde{\mathcal{I}}_k - 3r_k\log\frac{s}{r_k}$ and $\widetilde{\tau}_k$ are decreasing on $[2b_k,s_2]$, and this implies that

$$\widetilde{\mathcal{I}}_k(s_2) - 3r_k\log\frac{s_2}{r_k} \leq \widetilde{\mathcal{I}}_k(2b_k) - 3r_k\log\frac{2b_k}{r_k} < \mathcal{I}(2b_k) - 6r_k\log\frac{2b_k}{r_k} < 0, \text{ by (4.25)};$$

$$\widetilde{\tau}_k(s_2)\leq\widetilde{\tau}_k(2b_k) = \tau_k(2b_k) + \frac{r_k^2}{4b_k} - 2b_kr_k\log\frac{2b_k}{r_k} < 6r_k, \text{ by (4.26)}.$$

Together with (4.18) and (4.17), we then have

$$\mathcal{I}_k(s_2) < \widetilde{\mathcal{I}}_k(s_2) + 4r_k\log\frac{s_2}{r_k} < 7r_k\log\frac{s_2}{r_k}; \tag{4.27}$$

$$\tau_k(s_2) = \widetilde{\tau}_k(s_2) - \frac{r_k^2}{2s_2} + s_2r_k\log\frac{s_2}{r_k} < 6r_k - \frac{r_k^2}{2s_2} + s_2r_k\log\frac{s_2}{r_k}.$$

Recall that by Proposition 4.2,

$$\begin{aligned}\mathcal{I}_k'(s_2) &\leq \frac{\tau_k(s_2)}{s_2} + 9r_k\log\frac{s_2}{r_k}\cdot F_k(s_2) + C_0s_2r_k\log\frac{s_2}{r_k},\\ &< \frac{6r_k}{s_2} + 2r_k\log\frac{s_2}{r_k} + 9r_k\log\frac{s_2}{r_k}\cdot F_k(s_2),\end{aligned}$$

where in the first inequality, we used that for all $x\in\gamma_{s_2}$,

$$w_k(x)\leq(1+1/10)\mathcal{I}_k(s_2) < 9r_k\log\frac{s_2}{r_k}. \tag{4.28}$$

By (4.16), it follows that

$$\widetilde{\mathcal{I}}_k'(s_2) < \mathcal{I}_k'(s_2) - \frac{3r_k}{s_2} - 2r_k\log\frac{s_2}{r_k} - 10F_k(s_2)\cdot r_k\log\frac{s_2}{r_k} < \frac{3r_k}{s_2}. \tag{4.29}$$

On the other hand, by the divergence theorem and the co-area formula,

$$\begin{aligned}\tau_k'(s_2) &= \frac{1}{2\pi}\int_{\gamma_{s_2}}\phi^{-1}\Delta w_k \leq \frac{1}{2\pi}\int_{\gamma_{s_2}}\phi^{-1}\Big(w_k + \frac{w_k^3}{8s_2^4}\Big)\,dx\\ &\leq 4s_2\cdot 9r_k\log\frac{s_2}{r_k} + 9^3\frac{r_k^3}{s_2^3}(\log\frac{s_2}{r_k})^3 < r_k\log\frac{s_2}{r_k} + \frac{r_k^2}{2s_2^2}.\end{aligned}$$

Here the first inequality is from (4.6); the second one follows from (4.27), (4.28), (4.4) and $\phi^{-1}|_{\gamma_{s_1}}\leq 1+\epsilon$; in the last one, we used that as $k\to\infty$,

$$s_2\to 0;\quad \frac{s_2}{r_k}\to\infty;\quad \frac{r_k}{s_2}(\log\frac{s_2}{r_k})^3\to 0.$$

Then (4.17) can be applied to get

$$\widetilde{\tau}_k'(s_2) < \tau_k'(s_2) - \frac{r_k^2}{2s_2^2} - r_k\log\frac{s_2}{r_k} < 0.$$

Combining with (4.29), it contradicts the choice of s_2.

Then the bound of w_k on $\Sigma_k^1\cap\partial B(y_k,\epsilon;M_k)$ follows from Proposition 2.3(3). Hence Proposition 4.5 is proved. $\square$

Recall that Σ_k smoothly converges to $2S_0^2$ outside at most two balls. In particular, Σ_k^1 and Σ_k^2 are graphs over S_0^2 outside such two balls; see Remark 2.4. Observe that the normalization of graph functions of Σ_k^1 and Σ_k^2 can both give bounded Jacobi fields, which would be smooth across the singularities. Then w_k are equivalent to the difference of the two graph functions. By the maximum principle, such two graph functions should have opposite signs. Then the upper bound for w_k implies the Hausdorff distance between Σ_k and S_0^2.

Theorem 4.6. *For all sufficiently large k,*

$$\max_{\Sigma_k} \operatorname{dist}_{M_k}(x, S_0^2) \leq 8 r_k |\log r_k|.$$

Proof. Recall that Σ_k locally smoothly converges locally smoothly to $2 \cdot S_0^2$ away from $\mathcal{W}$ which contains at most two points. Then for any compact set $\Omega \subset S_0^2 \setminus \mathcal{W}$, Σ_k can be decomposed into two minimal graph functions in the neighborhood of Ω for sufficiently large k. Denote by u_k^1 and u_k^2 the graph functions. Note that u_k^1 and u_k^2 are defined on any compact domain in $S_0^2 \setminus \mathcal{W}$ by taking large k. Denote by

$$\lambda_k(x, s) = \max_{\partial B_s(x) \cap S_0^2} \{-u_k^1, u_k^2\} \quad \text{and} \quad \Lambda_{k,s} = \max_{x \in \mathcal{W}} \lambda_k(x, s).$$

Then by a standard argument (see [29]), $u_k^1/\Lambda_{k,\epsilon}$ and $u_k^2/\Lambda_{k,\epsilon}$ locally smoothly converges to c_1 and c_2, which are Jacobi fields of $S_0^2 \setminus \mathcal{W}$. By the minimal foliation argument, c_1 and c_2 are bounded, which implies that they can be extended smoothly across $\mathcal{W}$. Therefore, c_1 and c_2 are constants. Moreover, by the definition of $\Lambda_{k,\epsilon}$, $c_1 = -1$ or $c_2 = 1$. Without loss of generality, we assume that $c_2 = 1$. Note that Σ_k does not lie in one side of S_0^2, which implies that $c_1 \leq 0$. Then by the smooth convergence of Σ_k in $B_{2\epsilon}(\mathcal{W}) \setminus B_{\epsilon/2}(\mathcal{W})$, for sufficiently large k,

$$\lim_{k \to \infty} \max_{\Sigma_k^1 \cap B_\epsilon(\mathcal{W})} \frac{w_k}{\Lambda_{k,\epsilon}} = \lim_{k \to \infty} \max_{\partial B_\epsilon(\mathcal{W}) \cap \Sigma} \frac{u_k^2 - u_k^1}{\Lambda_{k,\epsilon}} = 1 - c_1 \geq 1.$$

By Propositions 4.4 and 4.5, for sufficiently large k,

$$\max_{x \in \Sigma_k^1 \cap \partial B_\epsilon(\mathcal{W})} w_k < (\frac{15}{2} + \frac{1}{100}) r_k \log \frac{\epsilon}{r_k}.$$

Thus we conclude that for all large k,

$$\Lambda_{k,\epsilon} < (\frac{15}{2} + \frac{1}{50}) r_k |\log r_k|.$$

By the minimal foliation argument inside $B_\epsilon(p)$ and Harnack inequalities outside such a ball,

$$\max_{x \in \Sigma_k} \operatorname{dist}_M(x, S_0^2) < 8 r_k |\log r_k|.$$

This completes the proof of Lemma 4.6. □

4.3. **Upper bounds for Hausdorff distance in high dimensions.** In this part, we restrict $3 \leq n \leq 6$ and prove the upper bounds of the Hausdorff distance between Σ_k and S_0^n. Recall that Σ_k^1 and Σ_k^2 are jointed by one or two small catenoids with radii r_k and $\widetilde{r}_k$; see Remark 2.4. Without loss of generality, we assume that $r_k \geq \widetilde{r}_k$. Recall that Σ_k^2 is a graph over Σ_k^1 with graph function $w_k > 0$.

To prove the desired bound, we start at one catenoid and derive new monotonicity formulas. Then the smooth convergence on $\partial B(p,\epsilon;M_k)$ will give the desired upper bound. Now let

$$\widetilde{\mathcal{I}}_k(s)=\mathcal{I}_k(s)+4s^{-\frac{1}{2}}r_k^{\frac{3}{2}}-sr_k-10r_k\int_{[R_k,s]\setminus[b_k/2,2b_k]}F_k(t)\,dt; \tag{4.30}$$

$$\widetilde{\tau}_k(s)=\tau_k(s)-9s^nr_k-r_k^{\frac{3}{2}}s^{n-\frac{5}{2}}. \tag{4.31}$$

Then by (4.3),

$$\mathcal{I}_k(s)\le\widetilde{\mathcal{I}}_k(s)+\frac{r_k}{20}, \tag{4.32}$$

Recall that $R_k/r_k\to 0$ and $r_k^{-1}(\Sigma_k\cap B(y_k,R_k;M_k)-y_k)$ is arbitrarily close to the catenoid $\mathcal{C}\cap B_{R_k/r_k}(0)$ in the smooth topology. Then by Item (1) in Appendix B,

$$\mathcal{I}_k(R_k)\le(1+\frac{1}{100})\cdot 2r_k\int_1^\infty\frac{ds}{\sqrt{s^{2(n-1)}-1}}<\frac{27}{10}r_k. \tag{4.33}$$

And by (4.11), (4.30) and (4.33),

$$\widetilde{\mathcal{I}}_k(R_k)<\frac{14}{5}r_k,\quad \tau_k(R_k)<(2+\frac{1}{10})r_k^{n-1}. \tag{4.34}$$

Now we are ready to show that $\widetilde{\mathcal{I}}_k(s)$ is decreasing in two disjoint intervals.

Proposition 4.7. *$\widetilde{\mathcal{I}}_k(s)$ and $\widetilde{\tau}_k(s)$ are decreasing on $[R_k,b_k/2)$ (resp. $[R_k,\epsilon]$) if $b_k\neq 0$ (resp. $b_k=0$). It follows that*

$$\mathcal{I}_k(s)<3r_k,\quad \tau_k(s)<3r_k^{n-1}+9s^nr_k+r_k^{\frac{3}{2}}s^{n-\frac{5}{2}}\quad \textit{for}\quad s\in[R_k,b_k/2].$$

Proof of Proposition 4.7. Suppose $b_k\neq 0$. To prove that $\widetilde{\mathcal{I}}_k$ and $\widetilde{\tau}_k$ are decreasing on $[R_k,b_k/2]$, we let

$$s_1:=\sup\{s\in(R_k,b_k/2):\widetilde{\tau}_k'(t)\le 0,\ \widetilde{\mathcal{I}}_k'(t)\le 0\ \text{ for all }\ t\in[R_k,s)\}.$$

We claim that $s_1=b_k/2$. Suppose on the contrary that $s_1<b_k/2$. Then it follows that $\widetilde{\mathcal{I}}_k(s)$ and $\widetilde{\tau}_k(s)$ are decreasing on $[R_k,s_1]$. Hence by (4.34),

$$\widetilde{\mathcal{I}}_k(s_1)\le\widetilde{\mathcal{I}}_k(R_k)<\frac{14r_k}{5};\quad \widetilde{\tau}_k(s_1)\le\widetilde{\tau}_k(R_k)<\frac{5r_k^{n-1}}{2}.$$

Together with (4.32) and (4.31), we have that

$$\mathcal{I}_k(s_1)\le\widetilde{\mathcal{I}}_k(s_1)+\frac{r_k}{20}<\frac{29r_k}{10}; \tag{4.35}$$
$$\tau_k(s_1)=\widetilde{\tau}_k(s_1)+9s_1^nr_k+s_1^{n-\frac{5}{2}}r_k^{\frac{3}{2}}<\frac{5}{2}r_k^{n-1}+9s_1^nr_k+s_1^{n-\frac{5}{2}}r_k^{\frac{3}{2}}.$$

Note that the first inequality together with (4.4) and Proposition 2.3 (3) implies that for $x\in\gamma_{s_1}$,

$$w_k(x)\le(1+1/100)\mathcal{I}_k(s_1)<3r_k. \tag{4.36}$$

Then by Proposition 4.2,

$$\begin{aligned}\mathcal{I}_k'(s_1) &\le \frac{\tau_k(s_1)}{s_1^{n-1}} + \frac{n}{\Omega_{n-1}s_1^n}\int_{\gamma_{s_1}}(\phi^{-1}-\phi)w_k + C_0 s_1\mathcal{I}_k(s_1)\\ &\le \frac{5}{2}\Big(\frac{r_k}{s_1}\Big)^{n-1} + 9s_1r_k + s_1^{-\frac{3}{2}}r_k^{\frac{3}{2}} + 3r_kF_k(s_1) + C_0s_1\cdot 3r_k\\ &< 2\Big(\frac{r_k}{s_1}\Big)^{\frac{3}{2}} + r_k + 3r_kF_k(s_1),\end{aligned}$$

where the second inequality is from (4.36); the last one follows from $n-1\ge 2$, $s_1\to 0$ and $r_k/s_k\to 0$ as $k\to\infty$. Together with (4.30), we then have

$$\widetilde{\mathcal{I}}_k'(s_1) = \mathcal{I}_k'(s_1) - 2\Big(\frac{r_k}{s_1}\Big)^{\frac{3}{2}} - r_k - 10r_kF_k(s_1) < 0. \tag{4.37}$$

On the other hand, by the divergence theorem and the co-area formula,

$$\begin{aligned}\tau_k'(s_1) = \frac{1}{\Omega_{n-1}}\int_{\gamma_{s_1}}\phi^{-1}\Delta w_k &\le \frac{1}{\Omega_{n-1}}\int_{\gamma_{s_1}}\phi^{-1}\Big(w_k + \frac{w_k^3}{8s_1^4}\Big)\\ &< (1+\frac{1}{100})^2\cdot\frac{29}{10}s_1^{n-1}r_k + \frac{3^3}{8}\cdot 4s_1^{n-5}r_k^3\\ &< 3s_1^{n-1}r_k + r_k^2s_1^{n-4}.\end{aligned}$$

Here the first inequality follows from (4.6); in the second one, we used (4.4), (4.35), (4.36) and $\phi^{-1}\Big|_{\gamma_{s_1}} < 1+\frac{1}{100}$; the last one is from $r_k/s_1\to 0$. Together with (4.31),

$$\widetilde{\tau}_k'(s_1) = \tau_k'(s_1) - 9ns_1^{n-1}r_k - (n-\frac{5}{2})r_k^{\frac{3}{2}}s_1^{n-\frac{7}{2}} < 0.$$

Combining with (4.37), this contradicts the choice of s_1.

If $b_k = 0$, then the same argument above gives that $\widetilde{\mathcal{I}}_k$ and $\widetilde{\tau}_k$ are decreasing on $[R_k,\epsilon]$. Hence Proposition 4.7 is proved. ☐

Note that $\mathcal{I}(s)$ is not well-defined in a small neighborhood of b_k. Here we use Proposition 2.3 and the divergence theorem to jump over such a small interval.

Proposition 4.8. *Suppose that $b_k\neq 0$. Then $\widetilde{\mathcal{I}}_k$ and $\widetilde{\tau}_k$ are decreasing on $[2b_k,\epsilon]$. It follows that*

$$\mathcal{I}_k(s) < 4r_k,\qquad \tau_k(s) < 6r_k^{n-1} + 9s^nr_k + r_k^{\frac{3}{2}}s^{n-\frac{5}{2}}\quad \text{for}\ \ s\in[2b_k,\epsilon].$$

In particular, we conclude that for sufficiently large k,

$$\max_{x\in\Sigma_k^1\cap\partial B(y_k,\epsilon;M_k)} w_k < \frac{9}{2}r_k.$$

Proof. By applying Proposition 2.3 (3), and then Proposition 4.7, together with (4.4), we have

$$\max\{w_k; x\in\Sigma_k^1\cap A(y_k,b_k/2,2b_k;M_k)\setminus B(z_k,b_k/4;M_k)\} \le (1+\delta)\mathcal{I}_k\big(\frac{b_k}{2}\big) \le \frac{10}{3}r_k, \tag{4.38}$$

as well as

$$\mathcal{I}_k(2b_k) < \frac{11r_k}{3}. \tag{4.39}$$

Moreover, by the divergence theorem and (4.6),

$$\begin{aligned}
&\tau_k(2b_k) - \tau_k(b_k/2) - \tau_k(b_k/4; z_k) = \frac{1}{\Omega_{n-1}} \int_{\Sigma_k^1 \cap A(y_k, b_k/2, 2b_k; M_k) \setminus B(z_k, b_k/4; M_k)} \Delta w_k \, dx \\
&\leq \frac{1}{\Omega_{n-1}} \int_{\Sigma_k^1 \cap A(y_k, b_k/2, 2b_k; M_k) \setminus B(z_k, b_k/4; M_k)} \Big(w_k + \frac{w_k^3}{8\rho^4} \Big) \, dx \\
&< \frac{10}{3} r_k \cdot (1 + \frac{1}{100})(2b_k)^n + (1 + \frac{1}{100})(\frac{10}{3})^3 \cdot \frac{1}{8} r_k^3 (2b_k)^{n-4} < 4r_k(2b_k)^n + r_k^2(2b_k)^{n-3}.
\end{aligned}$$

Here

$$\tau_k(s; z_k) = \frac{1}{\Omega_{n-1}} \int_{\Sigma_k^1 \cap \partial B(z_k, s, M_k)} \langle \nabla w_k, \boldsymbol{\eta}_z \rangle,$$

and $\boldsymbol{\eta}_z$ is the unit normal to $\partial B(z_k, s; M_k) \cap \Sigma_k^1$. The first inequality is from (4.6); we used (4.38) in the second one, and the area upper bound is from the fact that $b_k^{-1}(\Sigma_k^1 - y_k)$ is very close to a hyperplane. Recall that Σ_k is very close to a catenoid with radius $\widetilde{r}_k \leq r_k$ around z_k. Then by the same argument as in Proposition 4.7, we also have

$$\tau_k(b_k/4; z_k) < 3\widetilde{r}_k^{n-1} + 9\Big(\frac{b_k}{4}\Big)^n \widetilde{r}_k + \Big(\frac{b_k}{4}\Big)^{n-\frac{5}{2}} \widetilde{r}_k^{\frac{3}{2}} \leq 3r_k^{n-1} + 9\Big(\frac{b_k}{4}\Big)^n r_k + \Big(\frac{b_k}{4}\Big)^{n-\frac{5}{2}} r_k^{\frac{3}{2}}.$$

It follows that

$$\begin{aligned}
\tau_k(2b_k) &< \tau_k(b_k/2) + \tau_k(b_k/4; z_k) + 4r_k(2b_k)^n + r_k^2(2b_k)^{n-3} \\
&< 6r_k^{n-1} + 9\Big(\frac{b_k}{2}\Big)^n r_k + \Big(\frac{b_k}{2}\Big)^{n-\frac{5}{2}} r_k^{\frac{3}{2}} + 9\Big(\frac{b_k}{4}\Big)^n r_k + \Big(\frac{b_k}{4}\Big)^{n-\frac{5}{2}} r_k^{\frac{3}{2}} + 4r_k(2b_k)^n + r_k^2(2b_k)^{n-3} \\
&< 6r_k^{n-1} + 9(2b_k)^n r_k + (2b_k)^{n-\frac{5}{2}} r_k^{\frac{3}{2}},
\end{aligned}$$

which implies that

$$\widetilde{\tau}_k(2b_k) = \tau_k(2b_k) - 9(2b_k)^n r_k - r_k^{\frac{3}{2}}(2b_k)^{n-\frac{5}{2}} < 6r_k^{n-1}.$$

Then one can prove that $\widetilde{\mathcal{I}}_k(s)$ and $\widetilde{\tau}_k(s)$ are decreasing on $[2b_k, \epsilon]$ by the same argument as in Proposition 4.7.

Applying Proposition 2.3(3) and (4.4) again,

$$\max_{x \in \Sigma_k^1 \cap \partial B(y_k, \epsilon; M_k)} w_k \leq (1 + 1/10)\mathcal{I}_k(\epsilon) \leq \frac{9}{2} r_k.$$

This completes the proof of Proposition 4.7. □

Then using the same argument as Theorem 4.6, one can prove the Hausdorff distance upper bounds between Σ_k and S_0^n. The only modification is to replace Propositions 4.4 and 4.5 by Propositions 4.7 and 4.8.

Theorem 4.9. *For all sufficiently large k,*

$$\max_{\Sigma_k} \operatorname{dist}_{M_k}(x, S_0^n) < 5r_k.$$

Appendix A. Minimal graphs

Let $\mathcal{N}\subset M$ be a two-sided minimal hypersurface in (M^{n+1},g) with a chosen unit normal vector field $\mathbf{n}$. Let d be the oriented distance function to $\mathcal{N}$, and $\mathcal{N}_s$ be the level set of d. Then for some small $\mathfrak{d}>0$, $\{\mathcal{N}_t\}_{t\in(-\mathfrak{d},\mathfrak{d})}$ forms a foliation of a neighborhood of $\mathcal{N}$. Denote by $\widetilde{\nabla}$ and ∇ the Levi-Civita connections of M and $\mathcal{N}_s$ respectively. Then $\widetilde{\nabla}d$ is the unit normal vector field on $\mathcal{N}_s$. In this section, we always assume that M and $\mathcal{N}$ satisfy the following conditions:

$$|A|_{\mathcal{N}}<\epsilon/\mathfrak{d},\quad |R|\leq 1,\quad |\widetilde{\nabla}R|<C(n). \tag{A.1}$$

Here R is the Riemannian curvature tensor of M; ϵ is a small constant depending only on n, e.g. $\epsilon=10^{-1000n}$; $C(n)>1$ is a constant that can be changed from line to line.

Let Σ be a minimal graph over $\mathcal{N}$ with

$$\max_{x\in\Sigma}\operatorname{dist}_M(x,\mathcal{N})<\mathfrak{d}.$$

Denote by u the graph function. Then such a function can be extended to a neighborhood of $\mathcal{N}$ by taking

$$u(p,s)=u(p)-s.$$

When restricted to $\mathcal{N}_s$, u is the graph function of Σ over $\mathcal{N}_s$. Moreover, $u=0$ when restricted to Σ, and hence $\widetilde{\nabla}u|_\Sigma$ is the normal vector field on Σ and is non-zero everywhere.

For $p\in\Sigma\cap\mathcal{N}_s$, let $\{e_i\}$ be an orthonormal basis of $T_p\mathcal{N}_s$. Let A and H denote respectively the second fundamental form and mean curvature of $\mathcal{N}_s$ with respect to $\widetilde{\nabla}d$. A direct computation gives that

$$\frac{\partial}{\partial s}H=-\operatorname{Ric}(\widetilde{\nabla}d,\widetilde{\nabla}d)-|A|^2;$$

$$\widetilde{\nabla}_{\widetilde{\nabla}d}\widetilde{\nabla}u=\widetilde{\nabla}_{\widetilde{\nabla}d}\nabla u=-A(\nabla u);\quad \frac{\partial}{\partial s}|\widetilde{\nabla}u|^2=-2A(\nabla u,\nabla u);$$

$$\operatorname{div}_M A(\nabla u)=\operatorname{div}_{\mathcal{N}_s}A(\nabla u)=\langle\nabla^2u,A\rangle+\operatorname{Ric}(\nabla u,\widetilde{\nabla}d)+\langle\nabla u,\nabla H\rangle;$$

$$\frac{\partial}{\partial s}\operatorname{div}_M\widetilde{\nabla}u=-\operatorname{Ric}(\widetilde{\nabla}d,\widetilde{\nabla}u+\nabla u)+|A|^2-2\langle\nabla^2u,A\rangle-\langle\nabla u,\nabla H\rangle;$$

$$\frac{\partial}{\partial s}|\nabla H|^2=-2\langle\nabla(\operatorname{Ric}(\widetilde{\nabla}d,\widetilde{\nabla}d)+|A|^2),\nabla H\rangle-2A(\nabla H,\nabla H);$$

$$\frac{\partial}{\partial s}|A|^2=2R(\widetilde{\nabla}d,e_i,\widetilde{\nabla}d,e_j)A(e_i,e_j)-2A(e_i,e_j)A(e_j,e_k)A(e_k,e_i);$$

$$\frac{\partial}{\partial s}|\nabla A|^2=2R_{sjsk,i}A_{jk,i}+4R_{silk}A_{jl}A_{ki,j}+4R_{silj}A_{kl}A_{jk,i}-2A_{jk,i}A_{jk,l}A_{li}-4A_{ik,j}A_{lk,j}A_{il};$$

$$\frac{\partial}{\partial s}|\nabla^2u|^2=-2A_{kj,i}u_ku_{ij}+2R_{sikj}u_ku_{ij}-4u_{ij}u_{kj}A_{ik}.$$

We pause to state a standard differential inequality, whose proof is left to readers.

Lemma A.1. *Let $f:[0,\mathfrak{d}]\to\mathbb{R}$ be a non-negative differentiable function. Suppose that*

$$f'(t)\leq a+bf(t)$$

for real numbers a and b. Then for each $t\in[0,\mathfrak{d}]$,

$$f(t)\leq e^{bt}f(0)+\frac{a}{b}(e^{bt}-1).$$

Now Lemma A.1 can be applied to bound those terms on $\mathcal{N}_s$ by their restriction on $\mathcal{N}$.

Lemma A.2.

$$|\nabla u(x,t)| \le 2|\nabla u(x)|; \qquad |A(x,t)| \le 2|A(x)| + 2t;$$
$$\Big||A(x,t)|^2 - |A(x)|^2\Big| \le C(n)\Big(|A(x)|t + t + |A(x)|^3 t\Big);$$
$$|\nabla A(x,t)| \le C(n)\Big(|\nabla A(x)| + t(|A(x)| + 1)\Big);$$
$$|\nabla H(x,t)| \le C(n)\Big(t + t(|A(x)| + t)|\nabla A(x)| + |A(x)|^2 t^2\Big);$$
$$|\nabla^2 u(x,t)| \le C(n)\Big(|\nabla^2 u(x)| + t(1 + |\nabla A(x)|)|\nabla u(x)|\Big).$$

Proof. Note that $\frac{\partial}{\partial s}|A|^2 \le 2|A| + 2|A|^3$. Then by classical OD-inequalities, we have

$$\arctan|A(x,t)| \le \arctan|A(x)| + t.$$

Since $|tA(x)| \le \delta|A(x)| < \epsilon$,

$$|A|(x,t) \le 2|A(x)| + 2t. \tag{A.2}$$

The others can be derived similarly as follows: note that

$$\frac{\partial}{\partial s}|\nabla A|^2 \le C(n)\Big(|\nabla A(x,t)| + |A(x,t)||\nabla A(x,t)| + |A(x,t)||\nabla A(x,t)|^2\Big).$$

Fix $x \in \mathcal{N}$. Now let $f(t) = \sqrt{1 + |\nabla A(x,t)|^2}$. It follows that

$$f'(t) \le C(n)\Big(1 + |A(x,t)| + |A(x,t)|f\Big) \le C(n)\Big(1 + |A(x)| + (1 + |A(x)|)f(t)\Big)$$

Here the second inequality comes from (A.2). Then Lemma (A.1) can be applied to obtain

$$|\nabla A(x,t)| + 1 \le e^{C(n)(1+|A(x)|)t}(1 + |\nabla A(x)|) + e^{C(n)(1+|A(x)|)t} - 1,$$

which yields

$$|\nabla A(x,t)| \le 2|\nabla A(x)| + C(n)(1 + |A(x)|)t.$$

Here we used the condition that $C(n)|A(x)|t \le C(n)|A(x)|\mathfrak{d} \ll 1$ by (A.1). □

Now we are ready to derive our inequality for u. Indeed, by the minimality of Σ,

$$\begin{aligned} 0 = \operatorname{div}_\Sigma \widetilde{\nabla} u &= \operatorname{div}_M \widetilde{\nabla} u - \widetilde{\nabla}^2 u\Big(\frac{\widetilde{\nabla} u}{|\widetilde{\nabla} u|}, \frac{\widetilde{\nabla} u}{|\widetilde{\nabla} u|}\Big) \\ &= \operatorname{div}_M \widetilde{\nabla} u - (A + \nabla^2 u)\Big(\frac{\nabla u}{|\widetilde{\nabla} u|}, \frac{\nabla u}{|\widetilde{\nabla} u|}\Big). \end{aligned}$$

It follows that

$$\begin{aligned} \frac{(A + \nabla^2 u)(\nabla u, \nabla u)}{1 + |\nabla u|^2} &= \operatorname{div}_{\mathcal{N}} \nabla u + \int_0^s \frac{\partial}{\partial t} \operatorname{div}_M \widetilde{\nabla} u \\ &= \Delta_{\mathcal{N}} u + \int_0^s -\operatorname{Ric}(\widetilde{\nabla} d, \widetilde{\nabla} u + \nabla u) + |A|^2 - 2\langle \nabla^2 u, A\rangle - \langle \nabla u, \nabla H\rangle \, dt. \end{aligned}$$

Since $\widetilde{\nabla} u = \nabla u - \widetilde{\nabla} d$, then

$$\operatorname{Ric}(\widetilde{\nabla} d, \widetilde{\nabla} u + \nabla u) = -\operatorname{Ric}(\widetilde{\nabla} d, \widetilde{\nabla} d) + 2\operatorname{Ric}(\widetilde{\nabla} d, \nabla u).$$

It follows that

$$
\begin{aligned}
&(\Delta u+|A|^2u)(x,0)+\int_0^s \mathrm{Ric}(\widetilde{\nabla}d,\widetilde{\nabla}d)\,dt\\
&\quad\le (|A|+|\nabla^2u|)|\nabla u|^2(x,s)+\int_0^s 2n|\nabla u|+\Big||A(x)|^2-|A(x,t)|^2\Big|+2|\nabla^2u||A|+|\nabla u||\nabla H|\,dt.
\end{aligned}
$$

Combining all of them, we conclude that

$$
\begin{aligned}
&\Delta_{\mathcal{N}}u+|A|^2_{\mathcal{N}}u+\int_0^{u(x)} \mathrm{Ric}(\widetilde{\nabla}d,\widetilde{\nabla}d)\,dt\\
&\quad\le 8|A|_{\mathcal{N}}|\nabla u|^2+|A|^3_{\mathcal{N}}u^2+|\nabla^2u||A|_{\mathcal{N}}u+C(n)\Big(u+|\nabla u|+|\nabla u|^3\Big)u+\\
&\quad+C(n)\Big\{|\nabla u(x)|^2|\nabla^2u|+|\nabla u|^3|\nabla A|u+|A|u^2(1+|\nabla u|)+|\nabla^2u|u^2+\\
&\quad+|\nabla A||\nabla u|u^2(|A|+u)+|\nabla u|u^2(1+|A||\nabla u|+|\nabla A|u+|A|^2u)\Big\}_{\mathcal{N}}.
\end{aligned}
\tag{A.3}
$$

Appendix B. Catenoids

In this section, we collect some basic results for the catenoids. In $\mathbb{R}^{n+1}$, given $r>0$, there is an associated catenoid given by

$$
|x_{n+1}|=\int_r^t \frac{ds}{\sqrt{(s/r)^{2(n-1)}-1}},\quad t=\sqrt{x_1^2+x_2^2+\cdots x_n^2}.
$$

Here r is called the *radius*. When $r=1$, this catenoid is said to be *standard*. Let t,h,R be the solutions of

$$
t^2+h^2=R^2\quad\text{and}\quad |h|=\int_1^t\frac{ds}{\sqrt{s^{2(n-1)}-1}}.
\tag{B.1}
$$

For a standard catenoid $\mathcal{C}$, we have that

(1) for $n=2$, $\log t<|h|<\log(2t)$ and for $n\ge 3$, $|h|<1.31103$;
(2) $|A(x)|\le\sqrt{n(n-1)}$ and $|x||A|\to 0$ as $|x|\to\infty$;
(3) for each connected component γ of $\partial B_R(0)\cap\mathcal{C}$, it bounds an n-dimensional ball D with area $\frac{\Omega_{n-1}}{n}t^n$, where Ω_{n-1} is the area of unit $(n-1)$-sphere;
(4) for $4\le(n+1)\le 7$,

$$
\mathcal{A}_n:=\lim_{R\to\infty}\mathrm{Area}(\mathcal{C}\cap B_R(0))-2\cdot\frac{\Omega_{n-1}}{n}t^n>0;
\tag{B.2}
$$

(5) for $n=2$,

$$
\mathrm{Area}(\mathcal{C}\cap B_R(0))-2\mathrm{Area}(D)=2\pi(|h|+t\sqrt{t^2-1})-2\pi t^2>2\pi(\log R-1).
\tag{B.3}
$$

Note that outside $B_2(0)$, $\mathcal{C}$ has two connected components Σ^1 and Σ^2. Here we assume $\Sigma^1\subset\{x_{n+1}<0\}$. Then Σ^2 is a graph over Σ^1. Let w denote the graph function. Denote by $\boldsymbol{\eta}$ the unit outward normal to $\partial B_R(0)\cap\Sigma^1$ of $B_R(0)\cap\Sigma^1$.

Proposition B.1.

$$
\lim_{R\to\infty}\frac{1}{\Omega_{n-1}}\int_{\Sigma_k^1\cap\partial B_R(0)}\langle\nabla w,\boldsymbol{\eta}\rangle=2.
$$

Proof of Proposition B.1. By computation,

$$\partial_h = (-\sqrt{t^{2n-2}-1}Y, 1),$$

where $Y = (y_1, y_2, \cdots, y_n)$ with $|Y| = 1$. Then the normal line at (tY, h) intersects $\mathcal{C}$ at another point

$$P = (t_h Y, z_h),$$

where $z_h > 0$ is given by

$$z_h = h + (t_h - t)\sqrt{t^{2n-2}-1}. \tag{B.4}$$

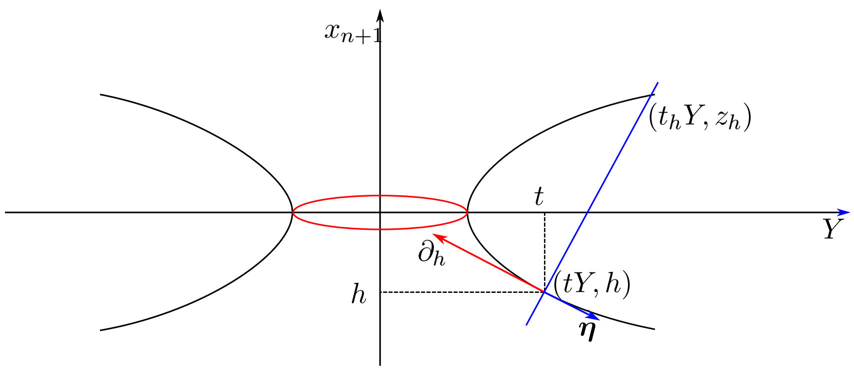

FIGURE III. The normal line of a catenoid.

Observe that $z_h > 0$ and $h < 0$. Then by (B.4), $t_h > t$. By (B.1), $z_h < 2\log t_h < t_h$ and $|h| < 2\log t < t$ for large t. Plugging them into (B.4), it follows that

$$t_h + t > z_h - h = (t_h - h)\sqrt{t^{2n-2}-1} \geq 3(t_h - t),$$

which implies that $t_h < 2t$. Plugging it back into (B.4) again,

$$2\log(2t) + 2\log t > z_h - h = (t_h - t)\sqrt{t^{2n-2}-1} > (t_h - t)(t-1).$$

It follows that $t < t_h < t + 1$ for large t. Observe that by (B.1),

$$\partial_h t_h = \partial_h z_h \cdot \sqrt{t_h^{2n-2}-1}; \quad \partial_h t = -\sqrt{t^{2n-2}-1}.$$

Differentiating both sides in (B.4), we then have

$$\partial_h t_h = 1 + (\partial_h t_h + \partial_h t)\sqrt{t^{2n-2}-1} + (t_h - t)(n-1)\frac{t^{2n-3}\cdot \partial_h t}{\sqrt{t^{2n-2}-1}}.$$

Combining them together, it follows that

$$\partial_h z_h = \frac{t^{2n-2} - (t_h - t)(n-1)t^{2n-3}}{1 - \sqrt{(t_h^{2n-2}-1)(t^{2n-2}-1)}} \to -1 \quad \text{as} \quad h \to -\infty.$$

Note that

$$w^2 = (t_h - t)^2 + (z_h - h)^2.$$

Then by computation,

$$\begin{aligned}\partial_h w^2 &= 2(t_h - t)(\sqrt{t_h^{2n-2} - 1}\partial_h z_h + \sqrt{t^{2n-2} - 1}) + 2(z_h - h)(\partial_h z_h - 1)\\ &= 2 \cdot \Big(\frac{\sqrt{t_h^{2n-2} - 1}}{\sqrt{t^{2n-2} - 1}} + 1 \Big) \cdot (z_h - h)\partial_h z_h,\end{aligned}$$

which implies that

$$\lim_{h\to-\infty} \partial_h w = \lim_{h\to-\infty} -\frac{2(z_h - h)}{w} = \lim_{h\to-\infty} -2 \cdot \frac{(t_h - t)\sqrt{t^{2n-2} - 1}}{t^{n-1}(t_h - t)} = -2.$$

On the other hand,

$$\frac{1}{\Omega_{n-1}} \int_{\Sigma\cap\{x_{n+1}=h\}} \langle \nabla w, \boldsymbol{\eta} \rangle = \frac{1}{\Omega_{n-1}} \int_{\Sigma\cap\{z=h\}} \frac{-\partial_h w}{t^{n-1}} = -\partial_h w.$$

This finishes the proof of Proposition B.1. □

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