

BOUNDEDNESS OF THE p -PRIMARY TORSION OF THE BRAUER GROUP OF AN ABELIAN VARIETY

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ABSTRACT. We prove that the p^∞ -torsion of the transcendental Brauer group of an abelian variety over a finitely generated field of characteristic $p > 0$ is bounded. This answers a (variant of a) question asked by Skorobogatov and Zarhin for abelian varieties. To do this, we prove a “flat Tate conjecture” for divisors. In the text, we also study other geometric Galois-invariant p^∞ -torsion classes of the Brauer group which are not in the transcendental Brauer group. These classes, in contrast with our main theorem, can be infinitely p -divisible. We explain how the existence of these p -divisible towers is naturally related to the failure of surjectivity of specialisation morphisms of Néron–Severi groups in characteristic p .

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1. INTRODUCTION

In this article we want to study problems related to the finiteness of the p -primary torsion of the Brauer group of abelian varieties in positive characteristic p . If k is a finite field and A is an abelian variety over k , it is well-known that the Brauer group of A , defined as $\mathrm{Br}(A) := H_{\acute{e}t}^2(A, \mathbb{G}_m)$, is a finite group, [Tat94, Prop. 4.3]. The main input for this result is the Tate conjecture for divisors, proved by Tate in [Tat66]. If k is replaced by a finitely generated field extension of \mathbb{F}_p one can not expect anymore $\mathrm{Br}(A)$ to be finite (see [SZ08, §1]). On the other hand, if $\mathrm{Br}(A_{k_s})^k$ is the *transcendental Brauer group* of A , namely the image of $\mathrm{Br}(A) \rightarrow \mathrm{Br}(A_{k_s})$ where k_s is a separable closure of k , the group $\mathrm{Br}(A_{k_s})^k[\frac{1}{p}]$ is finite by [CS21, Thm. 16.2.3]. In [SZ08, Ques. 1], Skorobogatov and Zarhin asked whether the p -primary torsion of $\mathrm{Br}(A_{k_s})^k$ is finite as well. This question has a negative answer already for abelian

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surfaces, as we show in Proposition 5.4. Nonetheless, we prove the following alternative finiteness result. Write \bar{k} for an algebraic closure of k_s .

Theorem 1.1 (Theorem 5.2). *Let A be an abelian variety over a finitely generated field k of characteristic $p > 0$. The transcendental Brauer group $\text{Br}(A_{k_s})^k$ is a direct sum of a finite group and a finite exponent p -group. In addition, if the Witt vector cohomology group $H^2(A_{\bar{k}}, W\mathcal{O}_{A_{\bar{k}}})$ is a finite $W(\bar{k})$ -module, then $\text{Br}(A_{k_s})^k$ is finite.*

The condition on $H^2(A_{\bar{k}}, W\mathcal{O}_{A_{\bar{k}}})$ is necessary to remove the “supersingular pathologies” as the one of our counterexample and it is satisfied, for example, when the p -rank of A is g or $g - 1$, where g is the dimension of A (see [Ill83, Cor. 6.3.16]). Note that if the formal Brauer group of $A_{\bar{k}}$, denoted by $\hat{\text{Br}}(A_{\bar{k}})$, is a formal Lie group, then by [AM77, Cor. II.4.4] the cohomology group $H^2(A_{\bar{k}}, W\mathcal{O}_{A_{\bar{k}}})$ is a finite $W(\bar{k})$ -module if and only if $\hat{\text{Br}}(A_{\bar{k}})$ has finite height. Note also that the formal Brauer group of abelian surfaces is always a formal Lie group by [ibid., Cor. II.2.12]. As a consequence of Theorem 1.1, we deduce that the subgroup of Galois-fixed points of $\text{Br}(A_{k_s})$, denoted by $\text{Br}(A_{k_s})^{\Gamma_k}$, has finite exponent (Corollary 5.3). This is a variant of [SZ08, Ques. 2] for abelian varieties.

In this article, we also study the Galois-fixed points of $\text{Br}(A_{\bar{k}})$. Ulmer in [Ulm14, §7.3.1] conjectured that $T_p(\text{Br}(A_{\bar{k}}))^{\Gamma_k} = 0$ where $T_p(\text{Br}(A_{\bar{k}}))$ is the p -adic Tate module of $\text{Br}(A_{\bar{k}})$. Even in this case, we provide a counterexample to this conjecture. We use the following result.

Proposition 1.2 (Proposition 6.6). *Let B be an abelian variety over a finitely generated field k of characteristic $p > 0$. Write A for $B \times_k B$ and $T_p(\text{Br}(A_{\bar{k}}))$ for the p -adic Tate module of $\text{Br}(A_{\bar{k}})$. There is a natural exact sequence*

$$0 \rightarrow \text{Hom}(B, B^\vee)_{\mathbb{Z}_p} \rightarrow \text{Hom}(B_{\bar{k}}[p^\infty], B_{\bar{k}}^\vee[p^\infty])^{\Gamma_k} \rightarrow T_p(\text{Br}(A_{\bar{k}}))^{\Gamma_k},$$

where $\text{Hom}(B, B^\vee)$ denotes the group of homomorphisms $B \rightarrow B^\vee$ as abelian varieties over k .

The proposition implies, for example, that when $\text{End}(B) = \mathbb{Z}$ the Γ_k -module $T_p(\text{Br}(A_{\bar{k}}))$ admits non-zero Galois-fixed points (Corollary 6.7). In this case, $\text{Br}(A_{\bar{k}})^{\Gamma_k}$ has infinite exponent since

$$T_p(\text{Br}(A_{\bar{k}}))^{\Gamma_k} = T_p(\text{Br}(A_{\bar{k}}))^{\Gamma_k}.$$

Note that if we replace $T_p(\text{Br}(A_{\bar{k}}))$ with the ℓ -adic Tate module $T_\ell(\text{Br}(A_{\bar{k}}))$, where ℓ is a prime different from p , then $T_\ell(\text{Br}(A_{k_s})) = T_\ell(\text{Br}(A_{\bar{k}}))$ has no non-trivial Galois-fixed points.

These “exceptional classes” in $T_p(\text{Br}(A_{\bar{k}}))^{\Gamma_k}$ are naturally related to specialisation morphisms of Néron–Severi groups. We recall the following theorem, which was proved in [And96, Thm. 5.2] in characteristic 0 (see also [MP12]) and in [Amb18] and [Chr18] in positive characteristic.

Theorem 1.3 (André, Ambrosi, Christensen). *Let K be an algebraically closed field which is not an algebraic extension of a finite field, X a finite type K -scheme, and $\mathcal{Y} \rightarrow X$ a smooth proper morphism. For every geometric point $\bar{\eta}$ of X there is an $x \in X(K)$ such that $\text{rk}_{\mathbb{Z}}(\text{NS}(\mathcal{Y}_{\bar{\eta}})) = \text{rk}_{\mathbb{Z}}(\text{NS}(\mathcal{Y}_x))^1$.*

As it is well-known, the theorem is false when $K = \bar{\mathbb{F}}_p$ (see [MP12, Rmk. 1.12]). What we prove is that, in the known counterexamples, the elements in $T_p(\text{Br}(A_{\bar{k}}))^{\Gamma_k}$ explain the failure of Theorem 1.3. More precisely, we prove the following result.

¹If R is a domain with fraction field K and M is an R -module, we write $\text{rk}_R(M)$ for the dimension of $M \otimes_R K$ as a K -vector space.

Theorem 1.4 (Theorem 6.2). *Let X be a connected normal scheme of finite type over \mathbb{F}_p with generic point $\eta = \text{Spec}(k)$ and let $f : \mathcal{A} \rightarrow X$ be an abelian scheme over X with constant Newton polygon². For every closed point $x = \text{Spec}(\kappa)$ of X we have*

$$\text{rk}_{\mathbb{Z}}(\text{NS}(\mathcal{A}_{\bar{x}})^{\Gamma_{\kappa}}) - \text{rk}_{\mathbb{Z}}(\text{NS}(\mathcal{A}_{\bar{\eta}})^{\Gamma_{\kappa}}) \geq \text{rk}_{\mathbb{Z}_p}(\text{T}_p(\text{Br}(\mathcal{A}_{\bar{\eta}}))^{\Gamma_{\kappa}}).$$

Note that in the inequality the left term is “motivic”, while the right term comes from some p -adic object which, as far as we know, has no ℓ -adic analogue. Note also that $\text{T}_p(\text{Br}(\mathcal{A}_{\bar{x}}))^{\Gamma_{\kappa}} = 0$ by Corollary 5.3 since κ is a perfect field.

To prove Theorem 1.1 we use a flat variant of the Tate conjecture. For every n , let $H_{\text{fppf}}^2(A_{\bar{k}}, \mu_{p^n})^k$ be the image of the extension of scalars morphism $H_{\text{fppf}}^2(A, \mu_{p^n}) \rightarrow H_{\text{fppf}}^2(A_{\bar{k}}, \mu_{p^n})$.

Theorem 1.5 (Theorem 5.1). *After possibly replacing k with a finite separable extension, the cycle class map*

$$c_1 : \text{NS}(A)_{\mathbb{Z}_p} \rightarrow \varprojlim_n H_{\text{fppf}}^2(A_{\bar{k}}, \mu_{p^n})^k$$

becomes an isomorphism³.

We obtain this result by using the crystalline Tate conjecture for abelian varieties, proved by de Jong in [deJ98, Thm. 2.6]. The main issue that we have to overcome is the lack of a good comparison between crystalline and fppf cohomology of $\mathbb{Z}_p(1)$ over imperfect fields. To avoid this problem, we exploit the fact that we are working with abelian varieties. In this special case, the comparison is constructed using the p -divisible group of A (and its dual).

The technical issue that we have to solve using the groups $H_{\text{fppf}}^2(A_{\bar{k}}, \mu_{p^n})^k$ is that it is not clear a priori whether $H_{\text{fppf}}^2(A, \mathbb{Z}_p(1)) \rightarrow \varprojlim_n H_{\text{fppf}}^2(A_{\bar{k}}, \mu_{p^n})^k$ is surjective. This is done (after inverting p) in Proposition 3.9, where we reduce to the case when A is the Jacobian of a curve. This idea was inspired by the proof of [CS13, Thm. 2.1].

1.6. Outline of the article. In §3 we prove some general results on the cohomology of fppf sheaves. In particular, we prove Corollary 3.4, which is a first result on the relation between the Brauer group of a scheme over k_s and \bar{k} . In this section, we also prove in Proposition 3.8 the exactness of some fundamental sequences for the groups $H_{\text{fppf}}^2(X_{\bar{k}}, \mu_{p^n})^k$. In §4, we construct a morphism which relates $H_{\text{fppf}}^2(A, \mathbb{Z}_p(1))$ with $\text{Hom}(A[p^\infty], A^\vee[p^\infty])$ and we prove basic properties of this morphism as Proposition 4.5 and Proposition 4.6. In §5, we prove the flat variant of the Tate conjecture (Theorem 5.1) and the finiteness result for the transcendental Brauer group (Theorem 5.2). Finally, in §6, we look at the relation of our results with the theory of specialisation of Néron–Severi groups. In particular, we prove Theorem 6.2.

²With this we mean that for every algebraically closed field Ω and every $\bar{x} \in X(\Omega)$, the Newton polygons of the fibres $\mathcal{A}_{\bar{x}}$ are all equal. Note that in this case it is enough to check $\bar{\mathbb{F}}_p$ -points.

³For us, $\text{NS}(A)$ is the group of k -points of the group scheme $\pi_0(\text{Pic}_{A/k})$.

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2. NOTATION

If k is a field, we write \bar{k} for a fixed algebraic closure of k and k_s (resp. k_i) for the separable (resp. purely inseparable) closure of k in \bar{k} . We denote by Γ_k the absolute Galois group of k . If x is a k -point of a scheme, we denote by \bar{x} the induced \bar{k} -point. For an abelian group M , we write $T_p(M)$ for the p -adic Tate module of M , which is the projective limit $\varprojlim_n M[p^n]$, we write $V_p(M)$ for $T_p(M)[\frac{1}{p}]$, and we write M^\wedge for the p -adic completion of M . If M is endowed with a Γ_k -action, we denote by M^{Γ_k} the subgroup of fixed points. For a scheme X and an fppf sheaf \mathcal{F} , we denote by $H^\bullet(X, \mathcal{F})$ the fppf cohomology groups and when $X = \text{Spec}(k)$ we simply write $H^\bullet(k, \mathcal{F})$. If $f : X \rightarrow Y$ is a morphism of schemes, we denote by $R^\bullet f_* \mathcal{F}$ the fppf higher direct image functors over $(\mathbf{Sch}/Y)_{\text{fppf}}$. Finally, if X is a scheme over \mathbb{F}_p , we write X^{perf} for the projective limit $\varprojlim(\dots \xrightarrow{F} X \xrightarrow{F} X \xrightarrow{F} X)$ where F is the absolute Frobenius of X .

3. PRELIMINARY RESULTS

In this section we start by proving some results that we will use later on. We work over a field k of arbitrary characteristic and we consider a scheme X over k with structural morphism q .

Lemma 3.1. *Let \mathcal{F} be a sheaf over $(\mathbf{Sch}/k)_{\text{fppf}}$ such that $q_* \mathcal{F}_X = \mathcal{F}$ and suppose that X has a k -rational point. The group $H^0(k, R^1 q_* \mathcal{F}_X)$ is canonically isomorphic to $H^1(X, \mathcal{F}_X)/H^1(k, \mathcal{F})$. In addition, the natural morphism $H^2(X, \mathcal{F}_X) \rightarrow H^0(k, R^2 q_* \mathcal{F}_X)$ sits in an exact sequence*

$$0 \rightarrow K \rightarrow H^2(X, \mathcal{F}_X) \rightarrow H^0(k, R^2 q_* \mathcal{F}_X) \rightarrow H^2(k, R^1 q_* \mathcal{F}_X)$$

where K is an extension of $H^1(k, R^1 q_* \mathcal{F}_X)$ by $H^2(k, \mathcal{F})$.

Proof. We consider the Leray spectral sequence

$$E_2^{i,j} = H^i(k, R^j q_* \mathcal{F}_X) \Rightarrow H^{i+j}(X, \mathcal{F}_X).$$

The morphisms $E_2^{i,0} = H^i(k, q_* \mathcal{F}_X) = H^i(k, \mathcal{F}) \rightarrow H^i(X, \mathcal{F}_X)$ are injective since X admits a k -rational point. We deduce that $E_2^{1,1} = E_\infty^{1,1}$ and $E_2^{2,0} = E_\infty^{2,0}$. This implies that the kernel of $H^2(X, \mathcal{F}_X) \rightarrow E_\infty^{0,2}$ is an extension of $E_2^{1,1}$ by $E_2^{2,0}$, as we wanted. The obstruction for the map $H^2(X, \mathcal{F}_X) \rightarrow E_2^{0,2} = H^0(k, R^2 q_* \mathcal{F}_X)$ to be surjective lies in $E_2^{2,1} = H^2(k, R^1 q_* \mathcal{F}_X)$. This concludes the proof. \square

Definition 3.2. We say that a presheaf \mathcal{F} on $(\mathbf{Sch}^{\text{qcs}}/k)_{\text{fppf}}$ is *finitary* if for every inverse system $\{T^{(\ell)}\}_{\ell \in L}$ of quasi-compact quasi-separated k -schemes with affine transition maps, the natural morphism

$$\operatorname{colim}_{\ell \in L} \mathcal{F}(T^{(\ell)}) \rightarrow \mathcal{F}(\lim_{\ell \in L} T^{(\ell)})$$

is an isomorphism.

Lemma 3.3. *Let G be a commutative finite type group scheme over k . If X is quasi-compact quasi-separated, then $R^i q_* G_X$ is finitary for $i \geq 0$. In addition, the natural morphism $H^0(k, R^i q_* G_X) \rightarrow H^i(X_{\bar{k}}, G_{X_{\bar{k}}})$ is injective.*

Proof. Let $\mathcal{H}^i(q, G_X)$ be the higher presheaf pushforward of G_X on X with respect to q . We first want to prove that $\mathcal{H}^i(q, G_X)$ is finitary for $i \geq 0$. In other words, we want to prove that for every inverse system $\{T^{(\ell)}\}_{\ell \in L}$ of quasi-compact quasi-separated k -schemes, the natural morphism

$$\operatorname{colim}_{\ell \in L} H^i(X^{(\ell)}, G_{X^{(\ell)}}) \rightarrow H^i(X^{(\infty)}, G_{X^{(\infty)}})$$

is an isomorphism, where $X^{(\ell)} := X \times_k T^{(\ell)}$ and $X^{(\infty)} := \lim_{\ell \in L} X^{(\ell)}$. By [SP23, Tag 01H0],

$$H^i(X^{(\ell)}, G_{X^{(\ell)}}) = \operatorname{colim}_{U_{\bullet}^{(\ell)} \in \operatorname{HC}(X^{(\ell)})} \check{H}^i(U_{\bullet}^{(\ell)}, G_{U_{\bullet}^{(\ell)}})$$

for every $\ell \in L \amalg \{\infty\}$, where $\operatorname{HC}(X^{(\ell)})$ is the category of fppf hypercoverings of $X^{(\ell)}$. Since each $X^{(\ell)}$ is quasi-compact quasi-separated, by [ibid., Tag 021P], we can replace the category $\operatorname{HC}(X^{(\ell)})$ in the colimit with the subcategory $\operatorname{HC}(X^{(\ell)})^{\text{qcqs}}$, consisting of those hypercoverings such that $U_n^{(\ell)}$ is quasi-compact quasi-separated for every $n \geq 0$. By [ibid., Lem. 01ZM], for $U_{\bullet}^{(\infty)} \in \operatorname{HC}(X^{(\infty)})^{\text{qcqs}}$ and $n \geq 0$ there exists an $\ell \in L$ and $U_{\bullet}^{(\ell, n)} \in \operatorname{HC}(X^{(\ell)})^{\text{qcqs}}$ such that

$$\operatorname{tr}_n(U_{\bullet}^{(\ell, n)} \times_{T^{(\ell)}} T^{(\infty)}) \simeq \operatorname{tr}_n(U_{\bullet}^{(\infty)}),$$

where $\operatorname{tr}_n(-)$ denotes the n -th truncation of simplicial schemes and $T^{(\infty)} := \lim_{\ell \in L} T^{(\ell)}$. This implies that

$$H^i(X^{(\infty)}, G_{X^{(\infty)}}) = \operatorname{colim}_{\ell \in L} \operatorname{colim}_{U_{\bullet}^{(\ell)} \in \operatorname{HC}(X^{(\ell)})} \check{H}^i(U_{\bullet}^{(\ell)} \times_{T^{(\ell)}} T^{(\infty)}, G_{U_{\bullet}^{(\ell)} \times_{T^{(\ell)}} T^{(\infty)}}).$$

We are reduced to prove that for every $\ell \in L$ and $U_{\bullet}^{(\ell)} \in \operatorname{HC}(X^{(\ell)})^{\text{qcqs}}$ we have that

$$\check{H}^i(U_{\bullet}^{(\ell)} \times_{T^{(\ell)}} T^{(\infty)}, G_{U_{\bullet}^{(\ell)} \times_{T^{(\ell)}} T^{(\infty)}}) = \operatorname{colim}_{\ell' \leq \ell} \check{H}^i(U_{\bullet}^{(\ell')} \times_{T^{(\ell')}} T^{(\ell')}, G_{U_{\bullet}^{(\ell')} \times_{T^{(\ell')}} T^{(\ell')}}).$$

Since G is of finite type over k , this follows from [ibid., Lem. 01ZM] and the exactness of filtered colimits.

Knowing that $\mathcal{H}^i(q, G_X)$ is finitary, in order to prove that $R^i q_* G_X$ is finitary as well it is enough to prove that for every finitary presheaf \mathcal{F} on $(\mathbf{Sch}^{\text{qcqs}}/k)_{\text{fppf}}$, the ‘‘partial’’ sheafification \mathcal{F}^+ (defined as in [ibid., §00W1]) is finitary. Similarly to the previous paragraph, the proof of this fact follows from the observation that each finite quasi-compact quasi-separated fppf covering of $X^{(\infty)}$ descends to a covering of $X^{(\ell)}$ for some $\ell \in L$ and Čech cohomology commutes with filtered colimits (\check{H}^0 is enough in this case).

For the second part, we note that for every presheaf \mathcal{F} on $(\mathbf{Sch}/k)_{\text{fppf}}$ with sheafification \mathcal{F}^{\sharp} , the natural morphism

$$\mathcal{F}(\operatorname{Spec}(\bar{k})) \rightarrow \mathcal{F}^{\sharp}(\operatorname{Spec}(\bar{k}))$$

is an isomorphism because every fppf covering of $\mathrm{Spec}(\bar{k})$ admits a section. This implies that

$$H^0(\bar{k}, R^i q_* G_X) = H^i(X_{\bar{k}}, G_{X_{\bar{k}}}).$$

Thanks to the previous part, we deduce that the composition

$$H^0(k, R^i q_* G_X) \hookrightarrow \mathrm{colim}_{k'/k \text{ fin.}} H^0(k', R^i q_* G_X) \xrightarrow{\sim} H^0(\bar{k}, R^i q_* G_X) = H^i(X_{\bar{k}}, G_{X_{\bar{k}}})$$

is injective, where the colimit runs over all finite field extensions of k . This ends the proof. \square

With the previous results we can prove [Gro68, Prop. 5.6], which was stated by Grothendieck without a complete proof⁴.

Corollary 3.4. *If k is separably closed and X is a proper k -scheme, then there is a natural exact sequence*

$$0 \rightarrow H^1(k, \mathrm{Pic}_{X/k}) \rightarrow \mathrm{Br}(X) \rightarrow \mathrm{Br}(X_{\bar{k}}).$$

In particular, if $\mathrm{Pic}_{X/k}$ is smooth then the natural morphism $\mathrm{Br}(X) \rightarrow \mathrm{Br}(X_{\bar{k}})$ is injective.

Proof. As in Lemma 3.1, we consider the Leray spectral sequence

$$E_2^{i,j} = H^i(k, R^j q_* \mathbb{G}_{m,X}) \Rightarrow H^{i+j}(X, \mathbb{G}_{m,X}).$$

Since X is proper over k , by [SP23, Tag 0BUG] we deduce that $A := H^0(X, \mathcal{O}_X)$ is a finite k -algebra. This implies that $q_* \mathbb{G}_m$ is represented by a smooth group scheme over k . Thanks to [Gro68, Thm. 11.7], we deduce that $E_2^{i,j} = 0$ for $i > 0$ and $j = 0$, so that $E_2^{1,1} = E_\infty^{1,1}$. The Leray spectral sequence produces then the exact sequence

$$0 \rightarrow H^1(k, \mathrm{Pic}_{X/k}) \rightarrow \mathrm{Br}(X) \rightarrow H^0(k, R^2 q_* \mathbb{G}_{m,X}).$$

To get the first part of the statement it is then enough to apply Lemma 3.3. For the second part, we note that when $\mathrm{Pic}_{X/k}$ is smooth, thanks to [ibid., Thm. 11.7], the group $H^1(k, \mathrm{Pic}_{X/k})$ vanishes. \square

Definition 3.5. For a scheme X over k and a prime p , we define $H^2(X, \mathbb{Z}_p(1))$ as the projective limit

$$\varprojlim_n H^2(X, \mu_{p^n}).$$

Remark 3.6. Note that we are defining $H^2(X, \mathbb{Z}_p(1))$ without taking into account higher inverse limits. Nonetheless, if k is algebraically closed of characteristic p and X is smooth and proper over k , then $R^1 \varprojlim_n H^1(X, \mu_{p^n}) = R^1 \varprojlim_n \mathrm{Pic}(X)[p^n] = 0$ since $\mathrm{Pic}(X)[p^\infty]$ is a direct sum of a p -divisible group and a finite group and $R^1 \varprojlim_n H^2(X, \mu_{p^n}) = 0$ by [Ill79, Chap. II, Prop. 5.9].

Construction 3.7. The Kummer exact sequences for X and $X_{\bar{k}}$ (for the fppf topology) induce the following commutative diagram with exact rows

$$(3.7.1) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Pic}(X)/p^n & \longrightarrow & H^2(X, \mu_{p^n}) & \longrightarrow & \mathrm{Br}(X)[p^n] \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{Pic}(X_{\bar{k}})/p^n & \longrightarrow & H^2(X_{\bar{k}}, \mu_{p^n}) & \longrightarrow & \mathrm{Br}(X_{\bar{k}})[p^n] \longrightarrow 0. \end{array}$$

⁴Note that the result is also proven in [CS21, Thm. 5.2.5.i], but their proof has a gap since the justification of the fact that $H^0(k, R^2 p_* \mathbb{G}_{m,X}) \rightarrow H^0(\bar{k}, R^2 p_* \mathbb{G}_{m,X})$ is injective is not correct.

We write

$$C_n(X) := (\mathrm{Pic}(X_{\bar{k}})/p^n)^k \rightarrow H^2(X_{\bar{k}}, \mu_{p^n})^k \rightarrow (\mathrm{Br}(X_{\bar{k}})[p^n])^k \rightarrow 0 \rightarrow \dots$$

for the complex obtained by taking images of the vertical arrows. Note that a priori $(\mathrm{Br}(X_{\bar{k}})[p^n])^k$ is smaller than $\mathrm{Br}(X_{\bar{k}})^k[p^n]$, where $\mathrm{Br}(X_{\bar{k}})^k := \mathrm{im}(\mathrm{Br}(X) \rightarrow \mathrm{Br}(X_{\bar{k}}))$.

Since both $R^1 \varprojlim_n \mathrm{Pic}(X)/p^n$ and $R^1 \varprojlim_n \mathrm{Pic}(X_{\bar{k}})/p^n$ vanish, we can also consider the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Pic}(X)^\wedge & \longrightarrow & H^2(X, \mathbb{Z}_p(1)) & \longrightarrow & \mathrm{T}_p(\mathrm{Br}(X)) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Pic}(X_{\bar{k}})^\wedge & \longrightarrow & H^2(X_{\bar{k}}, \mathbb{Z}_p(1)) & \longrightarrow & \mathrm{T}_p(\mathrm{Br}(X_{\bar{k}})) & \longrightarrow & 0, \end{array}$$

obtained by taking the projective limit of the diagrams (3.7.1) for various n . We denote by

$$\hat{C}(X) := (\mathrm{Pic}(X_{\bar{k}})^\wedge)^k \rightarrow H^2(X_{\bar{k}}, \mathbb{Z}_p(1))^k \rightarrow \mathrm{T}_p(\mathrm{Br}(X_{\bar{k}}))^k \rightarrow 0 \rightarrow \dots$$

the complex obtained by taking images of the vertical arrows.

Proposition 3.8. *If $\mathrm{char}(k) = p$ and A is an abelian variety over k such that the morphism $\mathrm{Pic}(A) \rightarrow \mathrm{NS}(A_{\bar{k}})$ is surjective, then the complexes $C_n(A)$ and $\hat{C}(A)$ are acyclic.*

Proof. If $K_{1,n}$ is the kernel of $H^2(A, \mu_{p^n}) \rightarrow H^2(A_{\bar{k}}, \mu_{p^n})$ and K_2 is the kernel of $\mathrm{Br}(A) \rightarrow \mathrm{Br}(A_{\bar{k}})$, in order to prove that $C_n(A)$ is acyclic we have to show that $K_{1,n} \rightarrow K_2[p^n]$ is surjective. Combining Lemma 3.1 and Lemma 3.3, we deduce the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Br}(k)[p^n] & \longrightarrow & K_{1,n} & \longrightarrow & H^1(k, \mathrm{Pic}_{A/k}[p^n]) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Br}(k) & \longrightarrow & K_2 & \longrightarrow & H^1(k, \mathrm{Pic}_{A/k}) & \longrightarrow & 0. \end{array}$$

The morphism of exact sequences factors through the complex

$$\mathrm{Br}(k)[p^n] \rightarrow K_2[p^n] \rightarrow H^1(k, \mathrm{Pic}_{A/k}[p^n]) \rightarrow 0 \rightarrow \dots$$

which is acyclic because $\mathrm{Br}(k)$ is p -divisible by [CS21, Thm. 1.3.7]. The image of

$$H^1(k, \mathrm{Pic}_{A/k}[p^n]) \rightarrow H^1(k, \mathrm{Pic}_{A/k}^\circ)$$

is $H^1(k, \mathrm{Pic}_{A/k}^\circ)[p^n]$, thus we are reduced to prove that

$$H^1(k, \mathrm{Pic}_{A/k}^\circ)[p^n] \rightarrow H^1(k, \mathrm{Pic}_{A/k}[p^n])$$

is surjective. Since $\mathrm{Pic}(A) \rightarrow \mathrm{NS}(A_{\bar{k}})$ is surjective, we know that $\pi_0(\mathrm{Pic}_{A/k})$ is a constant finitely generated torsion-free group over k such that $\mathrm{Pic}_{A/k}(k) \rightarrow \pi_0(\mathrm{Pic}_{A/k})(k)$ is surjective. Looking at the cohomology long exact sequence associated to

$$0 \rightarrow \mathrm{Pic}_{A/k}^\circ \rightarrow \mathrm{Pic}_{A/k} \rightarrow \pi_0(\mathrm{Pic}_{A/k}) \rightarrow 0,$$

we then deduce that $H^1(k, \mathrm{Pic}_{A/k}^\circ) \xrightarrow{\sim} H^1(k, \mathrm{Pic}_{A/k})$, which yields the desired result.

We now prove that $\hat{C}(A)$ is acyclic. The kernel of $H^2(A, \mathbb{Z}_p(1)) \rightarrow H^2(A_{\bar{k}}, \mathbb{Z}_p(1))$ is $\varprojlim_n K_{1,n}$ and the kernel of $T_p(\text{Br}(A)) \rightarrow T_p(\text{Br}(A_{\bar{k}}))$ is $T_p(K_2)$. Thus, again, we have to prove that $\varprojlim_n K_{1,n} \rightarrow T_p(K_2)$ is surjective. Combining the previous discussion and the fact that $\text{Br}(k)$ is p -divisible, we deduce that the two groups sit in the following diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & T_p(\text{Br}(k)) & \longrightarrow & \varprojlim_n K_{1,n} & \longrightarrow & \varprojlim_n H^1(k, \text{Pic}_{A/k}[p^n]) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & T_p(\text{Br}(k)) & \longrightarrow & T_p(K_2) & \longrightarrow & T_p(H^1(k, \text{Pic}_{A/k})) \longrightarrow 0. \end{array}$$

For every $n > 0$, the kernel of $H^1(k, \text{Pic}_{A/k}[p^n]) \rightarrow H^1(k, \text{Pic}_{A/k})[p^n]$ is $\text{Pic}(A)/p^n$ and the groups $(\text{Pic}(A)/p^n)_{n>0}$ form a Mittag-Leffler system. We deduce that the morphism

$$\varprojlim_n H^1(k, \text{Pic}_{A/k}[p^n]) \rightarrow T_p(H^1(k, \text{Pic}_{A/k}))$$

is surjective. This implies that $\hat{C}(A)$ is acyclic, as we wanted. \square

The proof of the following proposition was inspired by [CS13, Thm. 2.1].

Proposition 3.9. *If $\text{char}(k) = p$ and A is an abelian variety over k , we have $H^2(A_{\bar{k}}, \mathbb{Z}_p(1))^k[\frac{1}{p}] = (\varprojlim_n H^2(A_{\bar{k}}, \mu_{p^n})^k)[\frac{1}{p}]$ and $T_p(\text{Br}(A_{\bar{k}}))^k[\frac{1}{p}] = V_p(\text{Br}(A_{\bar{k}})^k)$.*

Proof. We first note that the four \mathbb{Q}_p -vector spaces are invariant under isogenies of A and finite separable extension of k . Indeed, for every isogeny $\varphi : B \rightarrow A$ there exists an isogeny $\psi : A \rightarrow B$ such that the composition $\varphi \circ \psi$ is the multiplication by some positive integer n . Since n is invertible in \mathbb{Q}_p , we deduce that φ^* is an isomorphism at the level of cohomology groups. Similarly, if k'/k is a finite separable extension, then the pullback morphisms with respect to $A_{k'} \rightarrow A$ admit as inverse the morphisms $\frac{1}{[k':k]} \text{Tr}_{A_{k'}/A}$.

Next, thanks to [Kat99, Thm. 11], we note that there exists a proper smooth connected curve C with a rational point and a morphism $C \rightarrow A$ such that $B := \text{Jac}(C)$ maps surjectively to A . By Poincaré's complete reducibility theorem, B is isogenous to a product $A \times_k A'$ with A' an abelian variety over k . Since $H^2(A_{\bar{k}}, \mathbb{Z}_p(1))^k$ (resp. $\text{Br}(A_{\bar{k}})^k$) is a direct summand of $H^2(A_{\bar{k}} \times_{\bar{k}} A'_{\bar{k}}, \mathbb{Z}_p(1))^k$ (resp. $\text{Br}(A_{\bar{k}} \times_{\bar{k}} A'_{\bar{k}})^k$) and the property we want to prove is invariant by isogenies, it is then enough to prove the result for B . In addition, since in the statement it is harmless to extend k to a finite separable extension, we may assume that $\text{Pic}(B) \rightarrow \text{NS}(B_{\bar{k}})$ is surjective, so that $H^1(k, \text{Pic}_{B/k}^\circ) = H^1(k, \text{Pic}_{B/k})$.

Let $K_{1,n}$ be the kernel of the morphism $H^2(B, \mu_{p^n}) \rightarrow H^2(B_{\bar{k}}, \mu_{p^n})$. By Lemma 3.1 and Lemma 3.3, the group $K_{1,n}$ is an extension of $H^1(k, \text{Pic}_{B/k}[p^n])$ by $\text{Br}(k)[p^n]$ and by the assumption $\text{Pic}_{C/k}[p^n] = \text{Pic}_{B/k}[p^n]$. We deduce that $K_{1,n} = \ker(H^2(C, \mu_{p^n}) \rightarrow H^2(C_{\bar{k}}, \mu_{p^n}))$. By [Gro68, Rmq. 2.5.b], the group $\text{Br}(C_{\bar{k}})$ vanishes, thus $H^2(C_{\bar{k}}, \mathbb{Z}_p(1)) = \mathbb{Z}_p$ and the morphism $H^2(C, \mathbb{Z}_p(1)) \rightarrow H^2(C_{\bar{k}}, \mathbb{Z}_p(1))$ is surjective because C has a rational point. This implies that $R^1 \varprojlim_n K_{1,n} \rightarrow R^1 \varprojlim_n H^2(C, \mu_{p^n})$ is injective. Since

$$R^1 \varprojlim_n K_{1,n} \rightarrow R^1 \varprojlim_n H^2(C, \mu_{p^n})$$

factors through $R^1 \varprojlim_n H^2(B, \mu_{p^n})$, we deduce that $R^1 \varprojlim_n K_{1,n} \rightarrow R^1 \varprojlim_n H^2(B, \mu_{p^n})$ is injective as well. Therefore, the morphism $H^2(B, \mathbb{Z}_p(1)) \rightarrow \varprojlim_n H^2(B_{\bar{k}}, \mu_{p^n})^k$ is surjective. Thanks to Proposition 3.8, for every n the morphism $H^2(B_{\bar{k}}, \mu_{p^n})^k \rightarrow (\mathrm{Br}(B_{\bar{k}})[p^n])^k$ is surjective with finite kernel, so that $\varprojlim_n H^2(B_{\bar{k}}, \mu_{p^n})^k \rightarrow \varprojlim_n (\mathrm{Br}(B_{\bar{k}})[p^n])^k$ is surjective as well. This implies that $T_p(\mathrm{Br}(B_{\bar{k}}))^k \rightarrow \varprojlim_n (\mathrm{Br}(B_{\bar{k}})[p^n])^k$ is surjective.

It remains to prove that for every n we have $(\mathrm{Br}(B_{\bar{k}})[p^n])^k = \mathrm{Br}(B_{\bar{k}})^k[p^n]$. Consider the natural morphism $K_3 \rightarrow \mathrm{Br}(C)$ where K_3 is the kernel of $\mathrm{Br}(B) \rightarrow \mathrm{Br}(B_{\bar{k}})$. Thanks to Lemma 3.1 and Lemma 3.3 and using the fact that $\mathrm{Br}(C_{\bar{k}}) = 0$, this morphism sits in the following commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Br}(k) & \longrightarrow & K_3 & \longrightarrow & H^1(k, \mathrm{Pic}_{B/k}) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{Br}(k) & \longrightarrow & \mathrm{Br}(C) & \longrightarrow & H^1(k, \mathrm{Pic}_{C/k}) \longrightarrow 0. \end{array}$$

Since $\mathrm{Pic}(B) \rightarrow \mathrm{NS}(B_{\bar{k}})$ is surjective and C is a curve, we have that

$$H^1(k, \mathrm{Pic}_{B/k}) = H^1(k, \mathrm{Pic}_{B/k}^\circ) \simeq H^1(k, \mathrm{Pic}_{C/k}^\circ) = H^1(k, \mathrm{Pic}_{C/k}).$$

We deduce that $K_3 \rightarrow \mathrm{Br}(C)$ is an isomorphism, thus $\mathrm{Br}(B) \rightarrow \mathrm{Br}(C) \xrightarrow{\sim} K_3$ provides a splitting of the exact sequence

$$0 \rightarrow K_3 \rightarrow \mathrm{Br}(B) \rightarrow \mathrm{Br}(B_{\bar{k}})^k \rightarrow 0.$$

This implies that $\mathrm{Br}(B)[p^n] \rightarrow \mathrm{Br}(B_{\bar{k}})^k[p^n]$ is surjective and this yields the desired result. \square

4. CONSTRUCTING A MORPHISM

Let A be an abelian variety over a field k . For a line bundle \mathcal{L} of A we write $\varphi_{\mathcal{L}} : A \rightarrow A^\vee$ for the morphism which sends $x \mapsto t_x^* \mathcal{L} \otimes \mathcal{L}^{-1}$, where t_x is the translation by x . In this section we want to complete the following solid square

$$\begin{array}{ccc} \mathrm{Pic}(A)^\wedge & \xrightarrow{c_1} & H^2(A, \mathbb{Z}_p(1)) \\ \downarrow \mathcal{L} \mapsto \varphi_{\mathcal{L}} & & \downarrow \\ \mathrm{Hom}(A, A^\vee)_{\mathbb{Z}_p} & \longrightarrow & \mathrm{Hom}(A[p^\infty], A^\vee[p^\infty]). \end{array}$$

If k is an algebraically closed field of characteristic 0 such a commutative diagram is constructed in [OSZ21, Lem. 2.6] using an analytic method. We propose instead an algebraic construction which works for any field.

4.1. Consider the morphism $h_1 : H^2(A, \mu_{p^n}) \rightarrow H^2(A \times_k A, \mu_{p^n})$ which sends a class α to $m^*(\alpha) - \pi_1^*(\alpha) - \pi_2^*(\alpha)$, where π_1 and π_2 are the two projections of $A \times_k A$. This morphism has the property that the first Chern class $c_1(\mathcal{L}) \in H^2(A, \mu_{p^n})$ of a line bundle \mathcal{L} is sent to $c_1(\Lambda(\mathcal{L}))$, the first Chern class of the associated Mumford bundle $\Lambda(\mathcal{L}) := m^* \mathcal{L} \otimes \pi_1^* \mathcal{L}^{-1} \otimes \pi_2^* \mathcal{L}^{-1}$. The Leray spectral sequence

$$(4.1.1) \quad E_2^{i,j} := H^i(A, R^j \pi_{2*} \mu_{p^n}) \Rightarrow H^{i+j}(A \times_k A, \mu_{p^n})$$

induces a filtration $0 \subseteq F^2 H^2(A \times_k A, \mu_{p^n}) \subseteq F^1 H^2(A \times_k A, \mu_{p^n}) \subseteq H^2(A \times_k A, \mu_{p^n})$.

Lemma 4.2. *The image of h_1 lies in $F^1 H^2(A \times_k A, \mu_{p^n})$.*

Proof. The spectral sequence (4.1.1) gives the exact sequence

$$0 \rightarrow F^1 H^2(A \times_k A, \mu_{p^n}) \rightarrow H^2(A \times_k A, \mu_{p^n}) \rightarrow E_\infty^{0,2} \rightarrow 0.$$

Therefore, it is enough to check that the composition

$$H^2(A, \mu_{p^n}) \xrightarrow{h_1} H^2(A \times_k A, \mu_{p^n}) \rightarrow H^0(A, R^2 \pi_{2*} \mu_{p^n})$$

is the 0-morphism. By [BO21, Cor. 1.4], there exists a commutative linear algebraic group⁵ G over k which represents $R^2 q_* \mu_{p^n}$ on the big fppf site $(\mathbf{Sch}/k)_{\text{fppf}}$. Since $R^2 \pi_{2*} \mu_{p^n}$ is the restriction of $R^2 q_* \mu_{p^n}$ from $(\mathbf{Sch}/k)_{\text{fppf}}$ to $(\mathbf{Sch}/A)_{\text{fppf}}$, this implies that $H^0(A, R^2 \pi_{2*} \mu_{p^n})$ can be computed as $\text{Mor}_{\mathbf{Sch}/k}(A, G)$. Thanks to the fact that G is affine, every morphism $A \rightarrow G$ contracts A to a point. We deduce that $\text{Mor}_{\mathbf{Sch}/k}(A, G) = \text{Mor}_{\mathbf{Sch}/k}(0_A, G) = H^0(k, R^2 q_* \mu_{p^n})$. By Lemma 3.3, the group $H^0(k, R^2 q_* \mu_{p^n})$ is naturally a subgroup of $H^2(A_{\bar{k}}, \mu_{p^n})$ and the induced morphism

$$H^2(A \times_k A, \mu_{p^n}) \rightarrow H^0(k, R^2 q_* \mu_{p^n}) \hookrightarrow H^2(A_{\bar{k}}, \mu_{p^n})$$

is given by the pullback via $i_1 : A = A \times_k 0_A \hookrightarrow A \times_k A$ followed by the extension of scalars to \bar{k} . By construction, we have that $i_1^* \circ h_1 = i_1^* \circ m^* - i_1^* \circ \pi_1^* = 0$. This concludes the proof. \square

Lemma 4.3. *Let G be a finite commutative group scheme killed by a positive integer n . There is a natural injective morphism $f_n : \text{Hom}(A[n], G) \rightarrow H^1(A, G)$ which admits a retraction g_n .*

Proof. Write P_n for the $A[n]$ -torsor over A given by the multiplication by n . The morphism f_n is then defined by $f_n(\sigma) := \sigma_* P_n$ for every $\sigma \in \text{Hom}(A[n], G)$. We want to define now g_n which sends a G -torsor P over A to an homomorphism $g_n(P) : A[n] \rightarrow G$. By Cartier duality, this is the same as defining a morphism $g_n(P)^\vee : G^\vee \rightarrow (A[n])^\vee = A^\vee[n]$. For a scheme T over k and a T -point of G^\vee corresponding to a morphism $\tau : G_T \rightarrow \mathbb{G}_{m,T}$ we define $g_n(P)^\vee(\tau)$ as $\tau_* P_T \in H^1(A_T, \mathbb{G}_{m,T})[n] = A^\vee[n](T)$. To prove that $g_n \circ f_n = \text{id}$ it is enough to note that for every $\sigma \in \text{Hom}(A[n], G)$, every scheme T over k , and every $\tau \in \text{Hom}(G_T, \mathbb{G}_{m,T})$ we have that $g_n(f_n(\sigma))^\vee(\tau) = \tau_*(\sigma_* P_n)_T = (\tau \circ \sigma_T)_* P_{n,T}$ is the line bundle over A_T associated to $\tau \circ \sigma_T \in (A[n])^\vee(T)$ under the identification $(A[n])^\vee = A^\vee[n]$. \square

4.4. Thanks to Lemma 4.2, we can define

$$\bar{h}_1 : H^2(A, \mu_{p^n}) \rightarrow H^1(A, A^\vee[p^n])$$

as the composition of h_1 and the natural morphism

$$F^1 H^2(A \times_k A, \mu_{p^n}) \rightarrow H^1(A, R^1 \pi_{2*} \mu_{p^n}) = H^1(A, A^\vee[p^n]).$$

In addition, by Lemma 4.3 applied to $G = A^\vee[p^n]$, we get a morphism

$$h_2 : H^1(A, A^\vee[p^n]) \rightarrow \text{Hom}(A[p^n], A^\vee[p^n]).$$

We write

$$h : H^2(A, \mu_{p^n}) \rightarrow \text{Hom}(A[p^n], A^\vee[p^n])$$

for the composition $h_2 \circ \bar{h}_1$ and we denote with the same letter the induced morphism $H^2(A, \mathbb{Z}_p(1)) \rightarrow \text{Hom}(A[p^\infty], A^\vee[p^\infty])$.

⁵Recall that a *linear algebraic group* over k is an affine group scheme of finite type over k .

Proposition 4.5. *The square*

$$(4.5.1) \quad \begin{array}{ccc} \mathrm{Pic}(A)/p^n & \xrightarrow{c_1} & H^2(A, \mu_{p^n}) \\ \downarrow & & \downarrow h \\ \mathrm{Hom}(A, A^\vee)/p^n & \longrightarrow & \mathrm{Hom}(A[p^n], A^\vee[p^n]) \end{array}$$

is commutative.

Proof. We have to show that for every line bundle $\mathcal{L} \in \mathrm{Pic}(A)$ we have

$$h(c_1(\mathcal{L})) = \varphi_{\mathcal{L}}|_{A[p^n]}.$$

Consider the Leray spectral sequence

$$(4.5.2) \quad E_2^{i,j} = H^i(A^\vee, R^j \pi_{2*} \mu_{p^n}) \Rightarrow H^{i+j}(A \times_k A^\vee, \mu_{p^n}).$$

The morphism $A \times_k A \xrightarrow{\mathrm{id}_A \times \varphi_{\mathcal{L}}} A \times_k A^\vee$ induces via pullback a morphism from (4.5.2) to (4.1.1). This produces the following commutative diagram

$$\begin{array}{ccccc} F^1 H^2(A \times_k A^\vee, \mu_{p^n}) & \longrightarrow & H^1(A^\vee, A^\vee[p^n]) & & \\ \downarrow (\mathrm{id}_A \times \varphi_{\mathcal{L}})^* & & \downarrow \varphi_{\mathcal{L}}^* & & \\ H^2(A, \mu_{p^n}) & \xrightarrow{h_1} & F^1 H^2(A \times_k A, \mu_{p^n}) & \longrightarrow & H^1(A, A^\vee[p^n]) \xrightarrow{h_2} \mathrm{Hom}(A[p^n], A^\vee[p^n]) \end{array}$$

where the composition of the lower horizontal arrows is h . If $\mathcal{P} \in \mathrm{Pic}(A \times_k A^\vee)$ is the Poincaré bundle of A , we have that $\Lambda(\mathcal{L}) = (\mathrm{id}_A \times \varphi_{\mathcal{L}})^* \mathcal{P}$. This implies that $h_1(c_1(\mathcal{L})) = c_1(\Lambda(\mathcal{L})) = (\mathrm{id}_A \times \varphi_{\mathcal{L}})^* c_1(\mathcal{P})$. In addition, by direct inspection, we note that $h_2(\varphi_{\mathcal{L}}^*([P])) = \varphi_{\mathcal{L}}|_{A[p^n]}$, where $[P] \in H^1(A^\vee, A^\vee[p^n])$ is the class of the torsor $A^\vee \xrightarrow{p^n} A^\vee$. It remains to prove that the morphism $F^1 H^2(A \times_k A^\vee, \mu_{p^n}) \rightarrow H^1(A^\vee, A^\vee[p^n])$ sends $c_1(\mathcal{P})$ to $[P]$. For this purpose, we introduce the Leray spectral sequence

$$(4.5.3) \quad E_2^{i,j} = H^i(A^\vee, R^j \pi_{2*} \mathbb{G}_m[1]) \Rightarrow H^{i+j}(A \times_k A^\vee, \mathbb{G}_m[1]).$$

The morphism $\delta : \mathbb{G}_m[1] \rightarrow \mu_{p^n}$ associated to the Kummer exact sequence induces a morphism from (4.5.3) to (4.5.2) which we denote with the same symbol. In turn, this produces the commutative diagram

$$\begin{array}{ccc} H^1(A \times_k A^\vee, \mathbb{G}_m) = F^1 H^1(A \times_k A^\vee, \mathbb{G}_m) & \longrightarrow & H^0(A^\vee, A^\vee). \\ \downarrow c_1 & & \downarrow \delta \\ F^1 H^2(A \times_k A^\vee, \mu_{p^n}) & \longrightarrow & H^1(A^\vee, A^\vee[p^n]) \end{array}$$

The upper horizontal arrow sends the line bundle \mathcal{P} to $\mathrm{id}_{A^\vee} \in H^0(A^\vee, A^\vee)$, while δ sends id_{A^\vee} to $[P]$. This yields the desired result. \square

Proposition 4.6. *The morphism $h : H^2(A_{\bar{k}}, \mathbb{Z}_p(1)) \rightarrow \mathrm{Hom}(A_{\bar{k}}[p^\infty], A_{\bar{k}}^\vee[p^\infty])$ is an injective morphism with image $\mathrm{Hom}^{\mathrm{sym}}(A_{\bar{k}}[p^\infty], A_{\bar{k}}^\vee[p^\infty])$, the group of homomorphisms which are fixed by the involution $\tau \mapsto \tau^\vee$.*

Proof. Suppose $\text{char}(k) = p$ and write W for the ring of Witt vectors of \bar{k} . The crystalline cohomology groups of an abelian variety are torsion free by [BBM82, Cor. 2.5.5]. Therefore, thanks to the Künneth formula, [Ber74, Thm. V.4.2.1], we have that $H_{\text{crys}}^*(A_{\bar{k}} \times_{\bar{k}} A_{\bar{k}}/W) = H_{\text{crys}}^*(A_{\bar{k}}/W) \otimes H_{\text{crys}}^*(A_{\bar{k}}/W)$ so that $m : A \times_k A \rightarrow A$ induces a morphism

$$m^* : H_{\text{crys}}^*(A_{\bar{k}}/W) \rightarrow H_{\text{crys}}^*(A_{\bar{k}}/W) \otimes H_{\text{crys}}^*(A_{\bar{k}}/W).$$

In degree 2 we get a morphism

$$m^* : H_{\text{crys}}^2(A_{\bar{k}}/W) \rightarrow H_{\text{crys}}^2(A_{\bar{k}}/W) \oplus H_{\text{crys}}^1(A_{\bar{k}}/W)^{\otimes 2} \oplus H_{\text{crys}}^2(A_{\bar{k}}/W)$$

which in turn induces a morphism

$$m^* - \pi_1^* - \pi_2^* : H_{\text{crys}}^2(A_{\bar{k}}/W) \rightarrow H_{\text{crys}}^1(A_{\bar{k}}/W)^{\otimes 2}.$$

Write $\sigma : \bigwedge^2 H_{\text{crys}}^1(A_{\bar{k}}/W) \xrightarrow{\sim} H_{\text{crys}}^2(A_{\bar{k}}/W)$ for the natural isomorphism induced by the cup product, as in [BBM82, Cor. 2.5.5]. For every $v \in H_{\text{crys}}^1(A_{\bar{k}}/W)$, the pullback $m^*(v)$ is equal to $\pi_1^*(v) + \pi_2^*(v)$. Therefore, the composition $(m^* - \pi_1^* - \pi_2^*) \circ \sigma : \bigwedge^2 H_{\text{crys}}^1(A_{\bar{k}}/W) \hookrightarrow H_{\text{crys}}^1(A_{\bar{k}}/W)^{\otimes 2}$ is equal to the natural embedding $v \wedge w \mapsto v \otimes w - w \otimes v$. By [ibid., Thm. 5.1.8], we have that the F -crystal $H_{\text{crys}}^1(A_{\bar{k}}/W)$ over \bar{k} is canonically isomorphic to $H_{\text{crys}}^1(A_{\bar{k}}^\vee/W)^\vee$ with F -structure defined as the dual of the F -structure of $H_{\text{crys}}^1(A_{\bar{k}}^\vee/W)$ multiplied by p . Thus, we have that

$$(H_{\text{crys}}^1(A_{\bar{k}}/W)^{\otimes 2})^{F=p} = \text{Hom}_{\mathbf{F}\text{-Crys}(\bar{k})}(H_{\text{crys}}^1(A_{\bar{k}}^\vee/W), H_{\text{crys}}^1(A_{\bar{k}}/W)),$$

where $\mathbf{F}\text{-Crys}(\bar{k})$ is the category of F -crystals over \bar{k} . By [Ill79, Rmq. II.3.11.2], the F -crystals $H_{\text{crys}}^1(A_{\bar{k}}/W)$ and $H_{\text{crys}}^1(A_{\bar{k}}^\vee/W)$ are the contravariant crystalline Dieudonné modules of the p -divisible groups $A_{\bar{k}}[p^\infty]$ and $A_{\bar{k}}^\vee[p^\infty]$, thus we get

$$(H_{\text{crys}}^1(A_{\bar{k}}/W)^{\otimes 2})^{F=p} = \text{Hom}(A_{\bar{k}}[p^\infty], A_{\bar{k}}^\vee[p^\infty]).$$

On the other hand, by [ibid., Thm. II.5.14], there is a canonical isomorphism $H^2(A_{\bar{k}}, \mathbb{Z}_p(1)) = H_{\text{crys}}^2(A_{\bar{k}}/W)^{F=p}$. This concludes the case when $\text{char}(k) = p$. If p is invertible in k one can replace crystalline cohomology with p -adic étale cohomology. \square

5. MAIN RESULTS

We are now ready to prove our main result, which is a flat version of the Tate conjecture for divisors of abelian varieties.

Theorem 5.1. *If A is an abelian variety over a finitely generated field k of characteristic $p > 0$, then $T_p(\text{Br}(A_{\bar{k}})^k) = 0$. Moreover, after possibly replacing k with a finite separable extension, the cycle class map*

$$c_1 : \text{NS}(A)_{\mathbb{Z}_p} \rightarrow \varprojlim_n H^2(A_{\bar{k}}, \mu_{p^n})^k$$

becomes an isomorphism.

Proof. To prove the statement we may assume that $\text{Pic}(A) \rightarrow \text{NS}(A_{\bar{k}})$ is surjective by extending k . The \mathbb{Z}_p -module $\text{Hom}(A[p^\infty], A^\vee[p^\infty])$ embeds into $\text{Hom}(A_{\bar{k}}[p^\infty], A_{\bar{k}}^\vee[p^\infty])$, therefore the morphism

$$h : H^2(A, \mathbb{Z}_p(1)) \rightarrow \text{Hom}(A[p^\infty], A^\vee[p^\infty])$$

induces a morphism $\tilde{h} : H^2(A_{\bar{k}}, \mathbb{Z}_p(1))^k \rightarrow \text{Hom}(A[p^\infty], A^\vee[p^\infty])$. By Proposition 4.6, we know that \tilde{h} is injective and $\text{im}(\tilde{h})$ is contained in $\text{Hom}^{\text{sym}}(A[p^\infty], A^\vee[p^\infty])$. In addition, by Proposition 4.5, we have the following commutative square

$$\begin{array}{ccc} \text{NS}(A)_{\mathbb{Z}_p} & \xleftarrow{c_1} & H^2(A_{\bar{k}}, \mathbb{Z}_p(1))^k \\ \downarrow & & \downarrow \tilde{h} \\ \text{Hom}(A, A^\vee)_{\mathbb{Z}_p} & \longrightarrow & \text{Hom}(A[p^\infty], A^\vee[p^\infty]). \end{array}$$

The lower arrow is an isomorphism by [deJ98, Thm. 2.6], and since $\text{NS}(A) = \text{Hom}^{\text{sym}}(A, A^\vee)$, we deduce that $\text{im}(\tilde{h} \circ c_1) = \text{Hom}^{\text{sym}}(A[p^\infty], A^\vee[p^\infty])$. This implies that

$$c_1 : \text{NS}(A)_{\mathbb{Z}_p} \rightarrow H^2(A_{\bar{k}}, \mathbb{Z}_p(1))^k$$

is surjective, thus by Proposition 3.9 we get that

$$c_1 : \text{NS}(A)_{\mathbb{Q}_p} \rightarrow \varprojlim_n H^2(A_{\bar{k}}, \mu_{p^n})^k \left[\frac{1}{p} \right]$$

is surjective. Combining this with Proposition 3.8, we deduce that

$$c_1 : \text{NS}(A)_{\mathbb{Z}_p} \rightarrow \varprojlim_n H^2(A_{\bar{k}}, \mu_{p^n})^k$$

is surjective. For the result about the Brauer group, we just note that by the previous argument and Proposition 3.8, the \mathbb{Z}_p -module $\text{T}_p(\text{Br}(A_{\bar{k}}))^k$ vanishes. Therefore, thanks to Proposition 3.9, we deduce that $\text{T}_p(\text{Br}(A_{\bar{k}}))^k = \text{T}_p(\text{Br}(A_{\bar{k}}))^k \left[\frac{1}{p} \right] = 0$. \square

Theorem 5.2. *Let A be an abelian variety over a finitely generated field k of characteristic $p > 0$. The transcendental Brauer group $\text{Br}(A_{k_s})^k$ is a direct sum of a finite group and a finite exponent p -group. In addition, if the Witt vector cohomology group $H^2(A_{\bar{k}}, W\mathcal{O}_{A_{\bar{k}}})$ is a finite $W(\bar{k})$ -module, then $\text{Br}(A_{k_s})^k$ is finite.*

Proof. By [CS21, Thm. 16.2.3], the group $\text{Br}(A_{k_s})^k \left[\frac{1}{p} \right]$ is finite. Moreover, thanks to Corollary 3.4, the morphism $\text{Br}(A_{k_s}) \rightarrow \text{Br}(A_{\bar{k}})$ is injective, which implies that the transcendental Brauer group is the same as $\text{Br}(A_{\bar{k}})^k$. Write $\mathbf{Ab}_p^* \subseteq \mathbf{Ab}$ for the full subcategory of the category of abstract abelian groups with objects those (possibly infinite) p -groups isomorphic to $(\mathbb{Q}_p/\mathbb{Z}_p)^{\oplus a} \oplus M$ for some $a \geq 0$ and M a finite exponent p -group. Equivalently, \mathbf{Ab}_p^* is the subcategory of those p -groups M such that $M[p^{n+1}]/M[p^n]$ is finite for n big enough. Note that this subcategory is closed under the operation of taking subobjects, quotients, and finite direct sums. We first want to prove that $H := \varinjlim_n H^2(A_{\bar{k}}, \mu_{p^n}) \in \mathbf{Ab}_p^*$ and when $H^2(A_{\bar{k}}, W\mathcal{O}_{A_{\bar{k}}})$ is a finite $W(\bar{k})$ -module then, in addition, $H[p]$ is finite (so that $H[p^n]$ is also finite for every $n \geq 0$). Thanks to the Kummer exact sequence this implies the same result for $\text{Br}(A_{\bar{k}})^k[p^\infty]$.

Write $q : A_{\bar{k}} \rightarrow \text{Spec}(\bar{k})$ for the structural morphism. By [BO21, Cor. 1.4], for every n there exists a commutative linear algebraic group G_n representing $R^2q_*\mu_{p^n}$ ⁶. Write U_n for the unipotent radical of G_n and D_n for the reductive quotient G_n/U_n . Since G_n is commutative, there is a canonical Levi decomposition $G_n = U_n \times D_n$. In particular, we have that $H = U \times D$, where $U := \varinjlim_n U_n(\bar{k})$ and

⁶One could use alternatively perfect groups rather than algebraic groups, as in [Mil86, Lem. 1.8].

$D := \varinjlim_n D_n(\bar{k})$. For every $n > 0$, the group scheme D_n is finite, because it is a reductive group killed by p^n . In addition, by [BO21, Prop. 10.7], there is a canonical isomorphism of formal groups $\varinjlim_n \hat{G}_n = \varinjlim_n \hat{U}_n = \Phi_{\mathbb{A}}^2(A_{\bar{k}}, \mathbb{G}_m)$, where \hat{G}_n and \hat{U}_n are the formal completions at the identity of G_n and U_n and $\Phi_{\mathbb{A}}^2(-, -)$ is as in [ibid., §10.6].

Applying Rq_* to the exact sequence

$$(5.2.1) \quad 1 \rightarrow \mu_{p^n} \rightarrow \mu_{p^{n+1}} \xrightarrow{p^n} \mu_p \rightarrow 1$$

and using the fact that $\varinjlim_n R^1q_*\mu_{p^n} = \text{Pic}_{A_{\bar{k}}/\bar{k}}[p^\infty]$ is a p -divisible group, we get the exact sequence

$$1 \rightarrow G_n \rightarrow G_{n+1} \xrightarrow{p^n} G_1.$$

As a first consequence, we deduce that for every $n > 0$ the group scheme D_n is the same as $D_{n+1}[p^n]$, thus $D[p] = D_1(\bar{k})$ is finite. In particular, the abstract group D is in \mathbf{Ab}_p^* . To bound U , we note that by [Mil86, Prop. 3.1] the dimension of the chain of algebraic groups $G_1 \subseteq G_2 \subseteq \dots$ is eventually constant. Therefore, there exists $N > 0$ such that for every $n \geq N$, the morphism $(U_n)_{\text{red}} \rightarrow (U_{n+1})_{\text{red}}$ is an isomorphism. This shows that U is a finite exponent p -group.

If $H^2(A_{\bar{k}}, W\mathcal{O}_{A_{\bar{k}}})$ is a finite $W(\bar{k})$ -module, then the formal group $\Phi_{\mathbb{A}}^2(A_{\bar{k}}, \mathbb{G}_m)$ does not contain any copy of \hat{G}_a . Indeed, by [BO21, Cor. 12.5], the group $H^2(A_{\bar{k}}, W\mathcal{O}_{A_{\bar{k}}})$ is the Cartier module of $\Phi_{\mathbb{A}}^2(A_{\bar{k}}, \mathbb{G}_m)$ and, by the assumption, it can not contain $\bar{k}[[V]]$, the Cartier module of \hat{G}_a . Therefore, in this case, we have that each group $U_n(\bar{k})$ is trivial, so that $H[p] = D[p]$ is finite.

We can finally prove that $\text{Br}(A_{\bar{k}})^k[p^\infty]$ has finite exponent. Suppose by contradiction that this is not the case. Since $\text{Br}(A_{\bar{k}})^k[p^\infty] \in \mathbf{Ab}_p^*$, we deduce that it contains a copy of $\mathbb{Q}_p/\mathbb{Z}_p$. On the other hand, by Theorem 5.1, the group $T_p(\text{Br}(A_{\bar{k}})^k)$ vanishes, which leads to a contradiction. \square

Corollary 5.3. *The group $\text{Br}(A_{k_s})^{\Gamma_k}$ has finite exponent.*

Proof. This follows from Theorem 5.2 thanks to [CS21, Thm. 5.4.12]. \square

We end this section with some examples of abelian varieties over finitely generated fields with infinite transcendental Brauer group. Let E be a supersingular elliptic curve over an infinite finitely generated field k and let A be the product $E \times_k E$.

Proposition 5.4. *After possibly extending k to a finite separable extension, the transcendental Brauer group $\text{Br}(A_{k_s})^k$ becomes infinite.*

Proof. Even in this case we use that, thanks to Corollary 3.4, the transcendental Brauer group is the same as $\text{Br}(A_{\bar{k}})^k$. Moreover, after extending the scalars we may assume that the morphism $\text{Pic}(A) \rightarrow \text{NS}(A_{\bar{k}})$ is surjective. Combining Proposition 3.8 and the fact that $\text{NS}(A_{\bar{k}})/p$ is finite we deduce that it is enough to show that $H^2(A_{\bar{k}}, \mu_p)^k$ is infinite. We look at the Leray spectral sequence with respect to the second projection $\pi_2 : A = E \times_k E \rightarrow E$ (both over k and over \bar{k}). In the second page, we have that the boundary morphism $H^1(E, R^1\pi_{2*}\mu_p) \rightarrow H^3(E, \pi_{2*}\mu_p)$ vanishes because $H^3(E, \pi_{2*}\mu_p) \rightarrow H^3(A, \mu_p)$ admits a retraction induced by the zero section of π_2 . Since $H^0(E_{\bar{k}}, R^2\pi_{2*}\mu_p) = H^2(E_{\bar{k}}, \mu_p) = \mathbb{Z}/p$, it is then enough to show that the image of

$$H^1(E, E[p]) = H^1(E, R^1\pi_{2*}\mu_p) \rightarrow H^1(E_{\bar{k}}, R^1\pi_{2*}\mu_p) = H^1(E_{\bar{k}}, E_{\bar{k}}[p])$$

is infinite. By Lemma 4.3, we have that $\text{End}(E[p])$ (resp. $\text{End}(E_{\bar{k}}[p])$) admits a natural embedding in $H^1(E, E[p])$ (resp. $H^1(E_{\bar{k}}, E_{\bar{k}}[p])$). Since

$$k = \text{End}(\alpha_p) \subseteq \text{End}(E[p]) \subseteq \text{End}(E_{\bar{k}}[p])$$

by the assumption that E is supersingular, we deduce the desired result. \square

6. SPECIALISATION OF NÉRON–SEVERI GROUPS

6.1. We want to start this section with an explicative example. Let $\mathcal{E} \rightarrow X$ be a non-isotrivial family of ordinary elliptic curves, where X is a connected normal scheme of finite type over \mathbb{F}_p . Let \mathcal{A} be the fibred product $\mathcal{E} \times_X \mathcal{E}$. We denote by E and A the generic fibres over the generic point $\text{Spec}(k) \hookrightarrow X$. The Kummer exact sequence induces the exact sequence

$$0 \rightarrow \text{NS}(A_{\bar{k}})_{\mathbb{Z}_p} \rightarrow H^2(A_{\bar{k}}, \mathbb{Z}_p(1)) \rightarrow \text{T}_p(\text{Br}(A_{\bar{k}})) \rightarrow 0.$$

The group $\text{NS}(A_{\bar{k}})_{\mathbb{Z}_p}$ is of rank $2 + \text{rk}_{\mathbb{Z}}(\text{End}(E_{\bar{k}})) = 3$, while, by Proposition 4.6, the rank of $H^2(A_{\bar{k}}, \mathbb{Z}_p(1))$ is $2 + \text{rk}_{\mathbb{Z}_p}(\text{End}(E_{\bar{k}}[p^\infty])) = 4$. This shows that $\text{T}_p(\text{Br}(A_{\bar{k}}))$ is of rank 1. The endomorphisms of $E_{\bar{k}}[p^\infty]$ are all defined over k_i , which implies that the action of Γ_k on $H^2(A_{\bar{k}}, \mathbb{Z}_p(1))$ is trivial. In particular, the morphism $\text{NS}(A)_{\mathbb{Z}_p} \rightarrow H^2(A_{\bar{k}}, \mathbb{Z}_p(1))^{\Gamma_k}$ is not surjective and the cokernel $\text{T}_p(\text{Br}(A_{\bar{k}}))^{\Gamma_k}$ is isomorphic to \mathbb{Z}_p .

In this case, the Galois action on $H^2(A_{\bar{k}}, \mathbb{Z}_p(1))$ is not enough to detect what classes are \mathbb{Z}_p -linear combinations of algebraic cycles. There is an additional obstruction to descend cohomology classes through the purely inseparable extension k_i/k . This extra purely inseparable obstruction gives an explanation of the failure of surjectivity of specialisation morphisms of Néron–Severi groups. In the example, if $\text{Spec}(\kappa) \hookrightarrow X$ is a closed point, we have that $\text{NS}(\mathcal{A}_\kappa) = \text{NS}(\mathcal{A}_{\bar{\kappa}})$ is of rank 4 because $\text{End}(\mathcal{E}_\kappa) = \text{End}(\mathcal{E}_{\bar{\kappa}})$ is of rank 2 (there is an extra Frobenius endomorphism). Thus, the specialisation map $\text{NS}(A) \hookrightarrow \text{NS}(\mathcal{A}_\kappa)$ is never surjective even if the rank of $H^2(A_{\bar{\kappa}}, \mathbb{Z}_p(1))$ is 4 as the generic geometric fibre and the Galois action is trivial in both cases. One can interpret this failure by saying that the extra obstruction on $H^2(A_{\bar{k}}, \mathbb{Z}_p(1))$ coming from the purely inseparable extension k_i/k is trivial on $H^2(A_{\bar{\kappa}}, \mathbb{Z}_p(1))$ since κ is perfect. In general, we prove the following theorem.

Theorem 6.2. *Let X be a connected normal scheme of finite type over \mathbb{F}_p with generic point $\eta = \text{Spec}(k)$ and let $f : \mathcal{A} \rightarrow X$ be an abelian scheme over X with constant Newton polygon. For every closed point $x = \text{Spec}(\kappa)$ of X we have*

$$\text{rk}_{\mathbb{Z}}(\text{NS}(\mathcal{A}_{\bar{x}})^{\Gamma_\kappa}) - \text{rk}_{\mathbb{Z}}(\text{NS}(\mathcal{A}_{\bar{\eta}})^{\Gamma_k}) \geq \text{rk}_{\mathbb{Z}_p}(\text{T}_p(\text{Br}(\mathcal{A}_{\bar{\eta}}))^{\Gamma_k}).$$

Remark 6.3. Note that after replacing X with a finite étale cover the action of Γ_k on $\text{NS}(\mathcal{A}_{\bar{\eta}})$ is trivial. Thus we also get an inequality before taking Galois-fixed points.

To prove Theorem 6.2 we first need the following result.

Proposition 6.4. *Under the assumptions of Theorem 6.2, the functor \mathcal{F} which sends $T \in (X^{\text{perf}})_{\text{proét}}$ ⁷ to $\text{Hom}^{\text{sym}}(\mathcal{A}_T[p^\infty], \mathcal{A}_T^\vee[p^\infty])[\frac{1}{p}]$ is a semi-simple finite rank \mathbb{Q}_p -local system such that for every $\bar{x} \in X(\bar{\mathbb{F}}_p)$ we have*

$$\mathcal{F}_{\bar{x}} = \text{Hom}^{\text{sym}}(\mathcal{A}_{\bar{x}}[p^\infty], \mathcal{A}_{\bar{x}}^\vee[p^\infty])[\frac{1}{p}].$$

⁷We denote by $(-)_{\text{proét}}$ the pro-étale site of a scheme, as defined in [BS15].

Proof. Let $\tilde{\mathcal{F}}$ be the functor which sends $T \in (X^{\text{perf}})_{\text{proét}}$ to $\text{Hom}(\mathcal{A}_T[p^\infty], \mathcal{A}_T^\vee[p^\infty])[\frac{1}{p}]$. We first note that to prove the result we can replace \mathcal{F} with $\tilde{\mathcal{F}}$ since \mathcal{F} is the kernel of the \mathbb{Q}_p -linear endomorphism $\alpha - \text{id}_{\tilde{\mathcal{F}}} : \tilde{\mathcal{F}} \rightarrow \tilde{\mathcal{F}}$ where α sends $\tau \in \text{Hom}(\mathcal{A}_T[p^\infty], \mathcal{A}_T^\vee[p^\infty])[\frac{1}{p}]$ to τ^\vee . Write $\mathbf{F}\text{-Crys}(X)$ for the category of F -crystals over the absolute crystalline site of X and let $\mathcal{M}_1, \mathcal{M}_2 \in \mathbf{F}\text{-Crys}(X)$ be the contravariant crystalline Dieudonné modules of $\mathcal{A}[p^\infty]$ and $\mathcal{A}^\vee[p^\infty]$ over X constructed in [BBM82, Déf. 3.3.6]. By [ibid., Thm. 5.1.8], we have that $\mathcal{M}_1 = \mathcal{M}_2^\vee(-1)$ where $\mathcal{M}_2^\vee(-1)$ is the F -crystal $\mathcal{H}om(\mathcal{M}_2, \mathcal{O}_{X, \text{cris}})$ endowed with the dual of the F -structure of \mathcal{M}_2 multiplied by p . Thus $\mathcal{M}_1^{\otimes 2}$ is equal to $\mathcal{H}om(\mathcal{M}_2, \mathcal{M}_1)$ endowed with the natural F -structure multiplied by p . By [Lau13, Thm. D], for every perfect scheme $T \rightarrow X$ we have canonical isomorphisms

$$\Gamma(T, \mathcal{M}_1^{\otimes 2})^{F=p} = \text{Hom}_{\mathbf{F}\text{-Crys}(T)}(\mathcal{M}_{2,T}, \mathcal{M}_{1,T}) = \text{Hom}(\mathcal{A}_T[p^\infty], \mathcal{A}_T^\vee[p^\infty])$$

where $\mathcal{M}_{1,T}$ and $\mathcal{M}_{2,T}$ are the inverse images of \mathcal{M}_1 and \mathcal{M}_2 to T . These isomorphisms are equivariant with respect to the action of the abstract group $\text{Aut}(T/X)$.

By [Kat79, Thm. 2.5.1], the slope filtration of the F -crystal $\mathcal{M}_{1,X^{\text{perf}}}^{\otimes 2}$ (which exists since $\mathcal{A} \rightarrow X$ has constant Newton polygon) splits uniquely up to isogeny. We denote by $\mathcal{N}_{X^{\text{perf}}}^{[1]}$ the slope 1 subobject of $\mathcal{M}_{1,X^{\text{perf}}}^{\otimes 2}$, defined up to isogeny. Note that for every $T \rightarrow X^{\text{perf}}$ we have that

$$\Gamma(T, \mathcal{M}_{1,T}^{\otimes 2})^{F=p} = \Gamma(T, \mathcal{N}_T^{[1]})^{F=p} = \Gamma(T, \mathcal{N}_T^{[1]}(1))^{F=1}.$$

By construction, the F -crystal $\mathcal{N}_{X^{\text{perf}}}^{[1]}(1)$ is unit-root. Therefore, by [Kat73, Prop. 4.1.1], we deduce that $\tilde{\mathcal{F}}$ is a \mathbb{Q}_p -local system. In addition, by [ibid., Lem. 4.3.15], for every $S = \text{Spec}(R) \rightarrow X$ with R strictly henselian perfect ring we have

$$\text{Hom}(\mathcal{A}_S[p^\infty], \mathcal{A}_S^\vee[p^\infty])[\frac{1}{p}] = \text{Hom}(\mathcal{A}_s[p^\infty], \mathcal{A}_s^\vee[p^\infty])[\frac{1}{p}]$$

where s is the closed point of S . This implies that for every $\bar{x} \in X(\bar{\mathbb{F}}_p)$ we have

$$\mathcal{F}_{\bar{x}} = \text{Hom}^{\text{sym}}(\mathcal{A}_{\bar{x}}[p^\infty], \mathcal{A}_{\bar{x}}^\vee[p^\infty])[\frac{1}{p}].$$

For the semi-simplicity, since X is normal, we can shrink X and assume it smooth. Write \mathcal{N} for the F -isocrystal $(R^1 f_{\text{crys}*} \mathcal{O}_{\mathcal{A}, \text{crys}})^{\otimes 2}$ and $\mathcal{N}^{[1]}$ for the quotient $\mathcal{N}^{\leq 1} / \mathcal{N}^{< 1}$, where $\mathcal{N}^{\leq 1}$ (resp. $\mathcal{N}^{< 1}$) is the subobject of \mathcal{N} of slopes ≤ 1 (resp. < 1). Note that by [BBM82, Thm. 2.5.6.(ii)], the pullback of $\mathcal{N}^{[1]}$ to X^{perf} is isomorphic as an F -isocrystal with $\mathcal{N}_{X^{\text{perf}}}^{[1]}$ over X^{perf} defined above. Thanks to [D'Ad20, Thm. 1.1.2], we have that $\mathcal{N}^{[1]}$ is semi-simple as an F -isocrystal.

By [Cre87, Thm. 2.1], there is an equivalence between unit-root F -isocrystals over X and finite rank \mathbb{Q}_p -local systems. By construction, Crew's and Katz's correspondences are compatible, in the sense that they agree after pulling back the objects through $X^{\text{perf}} \rightarrow X$. Since the étale fundamental groups of X and X^{perf} are canonically isomorphic, we deduce that $\tilde{\mathcal{F}}$ is semi-simple as well. This yields the desired result. \square

6.5. *Proof of Theorem 6.2.* We look at the exact sequence

$$0 \rightarrow \mathrm{NS}(A_{\bar{k}})_{\mathbb{Q}_p} \rightarrow H^2(A_{\bar{k}}, \mathbb{Q}_p(1)) \rightarrow \mathrm{T}_p(\mathrm{Br}(A_{\bar{k}}))\left[\frac{1}{p}\right] \rightarrow 0.$$

Thanks to Proposition 6.4, the operation of taking Galois-fixed points is exact. We get the exact sequence

$$0 \rightarrow \mathrm{NS}(A_{\bar{k}})_{\mathbb{Q}_p}^{\Gamma_k} \rightarrow H^2(A_{\bar{k}}, \mathbb{Q}_p(1))^{\Gamma_k} \rightarrow \mathrm{T}_p(\mathrm{Br}(A_{\bar{k}}))\left[\frac{1}{p}\right]^{\Gamma_k} \rightarrow 0.$$

Looking at the ranks we deduce the following equality

$$(6.5.1) \quad \mathrm{rk}_{\mathbb{Z}_p}(H^2(A_{\bar{k}}, \mathbb{Z}_p(1))^{\Gamma_k}) = \mathrm{rk}_{\mathbb{Z}}(\mathrm{NS}(A_{\bar{k}})^{\Gamma_k}) + \mathrm{rk}_{\mathbb{Z}_p}(\mathrm{T}_p(\mathrm{Br}(A_{\bar{k}}))^{\Gamma_k}).$$

By Proposition 6.4, the action of Γ_k on $H^2(A_{\bar{k}}, \mathbb{Q}_p(1)) = \mathrm{Hom}^{\mathrm{sym}}(A_{\bar{k}}[p^\infty], A_{\bar{k}}^\vee[p^\infty])\left[\frac{1}{p}\right]$ factors through the étale fundamental group of X associated to $\bar{\eta}$, denoted by $\pi_1^{\mathrm{ét}}(X, \bar{\eta})$. In addition, if κ is the residue field of x , the inclusion $x \hookrightarrow X$ induces then an action of Γ_κ on $H^2(A_{\bar{k}}, \mathbb{Q}_p(1))$ which corresponds, up to conjugation, to the action of Γ_κ on $H^2(A_{\bar{x}}, \mathbb{Q}_p(1))$. Therefore, by the Tate conjecture over finite fields (or Corollary 5.3), we get $\mathrm{NS}(A_{\bar{x}})_{\mathbb{Q}_p}^{\Gamma_\kappa} = H^2(A_{\bar{k}}, \mathbb{Q}_p(1))^{\Gamma_\kappa}$. Since $H^2(A_{\bar{k}}, \mathbb{Q}_p(1))^{\Gamma_k}$ is a subspace of $H^2(A_{\bar{k}}, \mathbb{Q}_p(1))^{\Gamma_\kappa}$ we deduce that

$$(6.5.2) \quad \mathrm{rk}_{\mathbb{Z}}(\mathrm{NS}(A_{\bar{x}})^{\Gamma_\kappa}) = \mathrm{rk}_{\mathbb{Z}_p}(H^2(A_{\bar{k}}, \mathbb{Z}_p(1))^{\Gamma_\kappa}) \geq \mathrm{rk}_{\mathbb{Z}_p}(H^2(A_{\bar{k}}, \mathbb{Z}_p(1))^{\Gamma_k}).$$

Combining (6.5.1) and (6.5.2) we get the desired result. \square

We want to conclude this section with other examples of abelian varieties such that $\mathrm{T}_p(\mathrm{Br}(A_{\bar{k}}))^{\Gamma_k} \neq 0$. These are variants of the abelian surface of §6.1 and they all provide counterexamples to the conjecture in [Ulm14, §7.3.1] when $\ell = p$.

Proposition 6.6. *Let A be an abelian variety which splits as a product $B \times_k B$ with B an abelian variety over k . There is a natural exact sequence*

$$0 \rightarrow \mathrm{Hom}(B, B^\vee)_{\mathbb{Z}_p} \rightarrow \mathrm{Hom}(B_{\bar{k}}[p^\infty], B_{\bar{k}}^\vee[p^\infty])^{\Gamma_k} \rightarrow \mathrm{T}_p(\mathrm{Br}(A_{\bar{k}}))^{\Gamma_k}.$$

Proof. We consider the exact sequence

$$0 \rightarrow \mathrm{NS}(A_{\bar{k}})_{\mathbb{Z}_p}^{\Gamma_k} \rightarrow H^2(A_{\bar{k}}, \mathbb{Z}_p(1))^{\Gamma_k} \rightarrow \mathrm{T}_p(\mathrm{Br}(A_{\bar{k}}))^{\Gamma_k}.$$

Arguing as in the proof of Proposition 4.6, the \mathbb{Z}_p -module $\mathrm{Hom}(B_{\bar{k}}[p^\infty], B_{\bar{k}}^\vee[p^\infty])^{\Gamma_k}$ is naturally a direct summand of $H^2(A_{\bar{k}}, \mathbb{Z}_p(1))^{\Gamma_k}$. Its preimage in $\mathrm{NS}(A_{\bar{k}})_{\mathbb{Z}_p}^{\Gamma_k}$ corresponds to the \mathbb{Z}_p -module

$$\mathrm{Hom}(B_{\bar{k}}, B_{\bar{k}}^\vee)_{\mathbb{Z}_p}^{\Gamma_k} = \mathrm{Hom}(B, B^\vee)_{\mathbb{Z}_p}.$$

This concludes the proof. \square

Corollary 6.7. *If $\mathrm{End}(B) = \mathbb{Z}$, then $\mathrm{T}_p(\mathrm{Br}(A_{\bar{k}}))^{\Gamma_k} \neq 0$.*

Proof. By the assumption, $\mathrm{Hom}(B, B^\vee)_{\mathbb{Z}_p}$ is a \mathbb{Z}_p -module of rank 1. Therefore, by Proposition 6.6, it is enough to prove that the rank of $\mathrm{Hom}(B_{\bar{k}}[p^\infty], B_{\bar{k}}^\vee[p^\infty])^{\Gamma_k}$ is greater than 1. Since $\mathrm{End}(B) = \mathbb{Z}$, the abelian variety B is not supersingular, so that the p -divisible group $B[p^\infty]$ admits at least two slopes. By the Dieudonné–Manin classification, this implies that $B_{k_i}[p^\infty]$ is isogenous to a direct sum $\mathcal{G}_1 \oplus \mathcal{G}_2$ of non-zero p -divisible groups over k_i . Since $\mathrm{End}(\mathcal{G}_1)\left[\frac{1}{p}\right] \oplus \mathrm{End}(\mathcal{G}_2)\left[\frac{1}{p}\right]$ embeds into $\mathrm{End}(B_{k_i}[p^\infty])\left[\frac{1}{p}\right] \simeq \mathrm{Hom}(B_{\bar{k}}[p^\infty], B_{\bar{k}}^\vee[p^\infty])\left[\frac{1}{p}\right]^{\Gamma_k}$, we deduce that $\mathrm{rk}_{\mathbb{Z}_p}(\mathrm{Hom}(B_{\bar{k}}[p^\infty], B_{\bar{k}}^\vee[p^\infty])^{\Gamma_k}) > 1$, as we wanted. \square

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