## A REMARK ON EHRESMANN'S FIBRATION THEOREM

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If  $f\colon Z\to Y$  is a smooth proper morphism of smooth varieties, and  $\mathcal L$  a local system on Z, then the sheaves  $R^qf_*\mathcal L$  are local systems on Y. This is typically seen as a consequence of Ehresmann's Theorem - f is a topological fiber bundle over each component of Y ([Vo, Theorem 9.3] is a convenient reference). This note records that the cohomological consequence holds without the smooth assumption on Y or Z.

**Conventions.** A 'sheaf' means a 'sheaf of vector spaces over some fixed field', and 'variety' = 'separated reduced scheme of finite type over Spec(C)'. Sheaves on varieties are with respect to the complex analytic site. A proper map of topological spaces is a separated and universally closed map.

J-L. Verdier asserts the following without the locally connected hypothesis [Ve, Lemme 2.2.2]. I was unable to understand his proof without this assumption.

**1. Lemma.** Let  $p: X \to Y$  be a proper surjective map of topological spaces. Assume X is locally connected. Let  $\mathcal{F}$  be a sheaf on Y with finite dimensional stalks. If  $p^*\mathcal{F}$  is a local system, then so is  $\mathcal{F}$ .

*Proof.* Let  $y \in Y$ . The stalk  $\mathcal{F}_y$  is finite dimensional, so there exist sections  $s_1, \ldots, s_n$ , of  $\mathcal{F}$  over some open neighborhood of y, which restrict to a basis of  $\mathcal{F}_y$ . Since our problem is local, we may assume this neighborhood is all of Y. Let  $\mathcal{G}$  be the constant sheaf on Y with stalk span $\{s_1, \ldots, s_n\}$ . Then the evident map  $u \colon \mathcal{G} \to \mathcal{F}$  induces an isomorphism  $\mathcal{G}_y \stackrel{\sim}{\to} \mathcal{F}_y$ . Consequently,  $p^*u$  induces isomorphisms:

$$(p^*\mathcal{G})_x \xrightarrow{\sim} (p^*\mathcal{F})_x$$
 for all  $x \in p^{-1}(y)$ .

For a locally connected space, the set of points at which a morphism of local systems induces an isomorphism on stalks defines an open set. Hence, the set  $V \subset X$  of points at which  $p^*u$  induces isomorphisms on stalks is open. As p is proper, U = Y - f(X - V) is an open neighborhood of y. As p is surjective, u yields an isomorphism  $\mathcal{G}|_{U} \xrightarrow{\sim} \mathcal{F}|_{U}$ .

**2. Proposition.** Let  $f: Z \to Y$  be a smooth and proper morphism of varieties. Let  $\mathcal{L}$  be a local system on Z with finite dimensional stalks. Then the sheaves  $R^q f_* \mathcal{L}$  are local systems.

*Proof.* Resolution of singularities (the version in [BP] suffices), the Lemma and proper base change reduce us to the situation where Z and Y are smooth. Here the usual form of Ehresmann's Theorem applies.

## REFERENCES

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<sup>[</sup>BP] F. BOGOMOLOV, T. PANTEV, Weak Hironaka Theorem, arXiv:alg-geom/9603019v2.

<sup>[</sup>Ve] J-L. Verdier, Classe d'Homologie associée un Cycle, Asterisque 36-37, p. 101-151 (1976).

<sup>[</sup>Vo] C. Voisin, Hodge Theory and Complex Algebraic Geometry I, Cambridge Studies in Math. 76 (2002).