

# A block triangular preconditioner for a class of three-by-three block saddle point problems

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**Abstract.** This paper deals with solving a class of three-by-three block saddle point problems. The systems are solved by preconditioning techniques. Based on an iterative method, we construct a block upper triangular preconditioner. The convergence of the presented method is studied in details. Finally, some numerical experiments are given to demonstrate the superiority of the proposed preconditioner over some existing ones.

*Keywords:* three-by-three saddle point, convergence, preconditioning, Krylov methods, GMRES.

*AMS Subject Classification:* 65F10, 65F50, 65F08.

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## 1 Introduction

We are concerned with the following three-by-three block system of linear equations

$$\mathcal{A}\mathbf{x} \equiv \begin{pmatrix} A & B^T & 0 \\ B & 0 & C^T \\ 0 & C & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} f \\ g \\ h \end{pmatrix}, \quad (1)$$

where  $A \in \mathbb{R}^{n \times n}$  is a symmetric positive definite (SPD),  $B \in \mathbb{R}^{m \times n}$  and  $C \in \mathbb{R}^{l \times m}$  have full row rank,  $f \in \mathbb{R}^n$ ,  $g \in \mathbb{R}^m$  and  $h \in \mathbb{R}^l$  are known, and  $\mathbf{x} = (x; y; z)$  is an unknown vector to be determined. We use  $(x; y; z)$  to denote the vector  $(x^T, y^T, z^T)^T$ . Throughout the paper, we assume that  $n \geq m$  and  $m \geq l$ . These hypothesis guarantee the nonsingularity of (1), see [28] for further details. So, the solution of (1) exists and is unique. In this case, the coefficient matrix of the system (1) is of order  $\mathbf{n}$ , in which  $\mathbf{n} = n + m + l$ .

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Evidently, one can solve the equivalent linear system instead of the original system:

$$\mathcal{B}\mathbf{x} \equiv \begin{pmatrix} A & B^T & 0 \\ -B & 0 & -C^T \\ 0 & C & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} f \\ -g \\ h \end{pmatrix} = \mathbf{b}. \quad (2)$$

Although  $\mathcal{B}$  loses symmetry, it retains some noteworthy properties:

1.  $\mathcal{B}$  is semipositive real, that is,  $v^T \mathcal{B}v > 0$ , for all  $v \in \mathbb{R}^n$ ;
2.  $\mathcal{B}$  is positive semistable which means that  $\Re(\lambda) \geq 0$  for all  $\lambda \in \sigma(\mathcal{B})$ , where  $\sigma(\mathcal{B})$  denotes the spectrum of  $\mathcal{B}$ .

These properties are so important for Krylov subspace methods like GMRES (see [5, 22]).

Systems of linear equations with the form (1) are called three-by-three saddle point problems, which appears in many engineering applications, such as the least squares problems [30], the Karush-Kuhn-Tucker (KKT) conditions of a type of quadratic programming [18], the discrete finite element methods for solving time-dependent Maxwell equation with discontinuous coefficient [2, 12, 14] and so on.

The stationary iterative methods usually combined with the acceleration techniques, because they may fail to converge or converge too slowly. The acceleration techniques, such as Chebyshev or Krylov subspace methods, while very successful, have some limitations. For instance, the use of Krylov acceleration require the computation of an orthonormal basis for the Krylov subspace, which may to have an adverse impact on the efficiency of these methods, like GMRES. There are some alternative acceleration techniques investigated by researchers, which we do not discuss here.

The coefficient matrix  $\mathcal{A}$  in Eq. (1) can be viewed as a standard block saddle point problem of the form

$$\mathcal{A} = \left( \begin{array}{cc|c} A & B^T & 0 \\ B & 0 & C^T \\ \hline 0 & C & 0 \end{array} \right), \quad (3)$$

or

$$\mathcal{A} = \left( \begin{array}{c|cc} A & B^T & 0 \\ \hline B & 0 & C^T \\ 0 & C & 0 \end{array} \right). \quad (4)$$

Since the attributes of the submatrix in (3) and (4) are different from the standard saddle point problems, many preconditioning strategies in the literature for standard two-by-two saddle point problems can not be directly applied for solving (1), for instance, shift-splitting preconditioners [3, 10, 11, 13, 24–27], block triangular preconditioners [4, 6, 7, 9, 16] and parameterized preconditioners [21]. In recent years, the iterative solution of the three-by-three saddle point problems has attracted substantial attention. Recently, Abdolmaleki et al. [1] proposed the following block diagonal preconditioner

$$\mathcal{P}_{D1} = \begin{pmatrix} A & 0 & 0 \\ 0 & \alpha I + \beta BB^T & 0 \\ 0 & 0 & \alpha I + \beta CC^T \end{pmatrix}, \quad (5)$$

where  $\alpha, \beta > 0$ . They also discussed properties of the corresponding iteration matrix  $\mathcal{P}_{D1}^{-1}\mathcal{B}$ . In [20], the following preconditioner was applied for accelerating the convergence rate of Krylov subspace method

$$\mathcal{P}_{D2} = \begin{pmatrix} A & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & CS^{-1}C^T \end{pmatrix}, \quad (6)$$

where  $S = BA^{-1}B^T$ . The preconditioner  $\mathcal{P}_{D1}$  received wide attention. Xie and Li [28] introduced the following preconditioners

$$\mathcal{P}_1 = \begin{pmatrix} A & 0 & 0 \\ B & -S & C^T \\ 0 & 0 & CS^{-1}C^T \end{pmatrix}, \quad \mathcal{P}_2 = \begin{pmatrix} A & 0 & 0 \\ B & -S & C^T \\ 0 & 0 & -CS^{-1}C^T \end{pmatrix}, \quad \mathcal{P}_3 = \begin{pmatrix} A & B^T & 0 \\ B & -S & 0 \\ 0 & 0 & -CS^{-1}C^T \end{pmatrix},$$

These three block preconditioners lead to the corresponding preconditioned matrices  $\mathcal{P}_1^{-1}\mathcal{A}$ ,  $\mathcal{P}_2^{-1}\mathcal{A}$  and  $\mathcal{P}_3^{-1}\mathcal{A}$ , that have only eigenvalues  $\{1\}$ ,  $\{\pm\frac{1}{2}, 1\}$  and  $\{\pm 1\}$ , respectively. Numerical results in [1] confirmed the robustness of the preconditioner  $\mathcal{P}_{D1}$ , for solving (2). In this work, a development of the block diagonal preconditioner  $\mathcal{P}_{D1}$  is employed. This new preconditioner is induced using a splitting of the coefficient matrix in (2). The corresponding splitting iteration method and its convergence properties are given.

The rest of paper is arranged as follows. Section 2 is devoted to introduce and convergence analysis of the proposed method. Furthermore, implementation issues of the corresponding preconditioner are briefly discussed. Numerical experiments are presented in Section 3. Finally, in Section 4 some concluding remarks are given.

Throughout the paper,  $I$  stands for the identity matrix of suitable order.  $x^H$  indicates the conjugate transpose of any arbitrary complex vectors  $x$ . For a given matrix  $A$  with real eigenvalues,  $\lambda_{\min}$  and  $\lambda_{\max}$  stand for the minimum and maximum eigenvalue of  $A$ , respectively. Moreover, the notations  $\sigma(A)$  and  $\rho(A)$  denote the set of all eigenvalues of  $A$  and the spectral radius of  $A$ , respectively. The minimum and maximum singular value of  $A$  are represented by  $\sigma_{\min}$  and  $\sigma_{\max}$ , respectively.

## 2 Preconditioner and convergence analysis

We first split the coefficient matrix in (2) as  $\mathcal{B} = \mathcal{P} - \mathcal{R}$ , where

$$\mathcal{P} = \begin{pmatrix} A & B^T & 0 \\ 0 & \alpha I + \beta BB^T & -C^T \\ 0 & 0 & \alpha I + \beta CC^T \end{pmatrix}, \quad \mathcal{R} = \begin{pmatrix} 0 & 0 & 0 \\ B & 0 & -C^T \\ 0 & -C & 0 \end{pmatrix},$$

in which  $\alpha$  and  $\beta$  are given positive constants. Evidently, the matrix  $\mathcal{P}$  is nonsingular. So, the iterative scheme associated with the splitting  $\mathcal{B} = \mathcal{P} - \mathcal{R}$ , can be constructed as

$$\mathbf{x}^{(k+1)} = \mathcal{G}_{\alpha,\beta}\mathbf{x}^{(k)} + f, \quad k = 0, 1, 2, \dots, \quad (7)$$

where  $\mathbf{x}^{(0)}$  is arbitrary and  $\mathcal{G}_{\alpha,\beta} = \mathcal{P}^{-1}\mathcal{R}$  is the iteration matrix and  $f = \mathcal{P}^{-1}\mathbf{b}$ .

In the sequel, we investigate the convergence properties of the proposed iterative method for solving the double saddle point problem (2). To do so, we need to recall a result about the evaluation of the roots of a quadratic equations as follows.

**Lemma 1.** [29] Consider the quadratic equation  $x^2 - bx + c = 0$ , where  $b$  and  $c$  are real numbers. Both roots of the equation are less than one in modulus if and only if  $|c| < 1$  and  $|b| < 1 + c$ .

**Theorem 1.** Suppose that  $A \in \mathbb{R}^{n \times n}$  is SPD,  $B \in \mathbb{R}^{m \times n}$  and  $C \in \mathbb{R}^{m \times l}$  are full row rank matrices. Then, the iterative method (7) converges to the unique solution of (2) for any initial guess, if

$$\frac{\sigma_{\max}^2(C)}{\alpha + \beta\sigma_{\min}^2(C^T)} + 2\frac{\sigma_{\max}^2(B^T)}{\lambda_{\min}(A)} < 4(\alpha + \beta\sigma_{\min}^2(B^T)). \quad (8)$$

*Proof.* Assume that  $(\lambda; \mathbf{x})$  is an eigenpair of the iteration matrix  $\mathcal{G}_{\alpha, \beta}$ , where  $\mathbf{x} := (x; y; z)$ . So, we have  $\mathcal{G}_{\alpha, \beta}\mathbf{x} = \lambda\mathbf{x}$  which is equivalent to say that

$$\begin{cases} \lambda(Ax + B^T y) = 0, & (9) \\ \lambda((\alpha I + \beta BB^T)y - C^T z) = Bx + (\alpha I + \beta BB^T)y, & (10) \\ \lambda(\alpha I + \beta CC^T)z = -Cy + (\alpha I + \beta CC^T)z. & (11) \end{cases}$$

If  $\lambda = 0$ , then there is nothing to prove. So, we assume that  $\lambda \neq 0$ . We claim that  $y \neq 0$ . If not, from (9) we have  $Ax = 0$ . Since  $A$  is a SPD matrix, we deduce that  $x = 0$ . Hence, from (10) and the assumption that  $C$  has full row rank we conclude that  $z = 0$ . Therefore,  $\mathbf{x} = 0$  and it is contrary to the assumption that  $\mathbf{x}$  is an eigenvector.

Furthermore, we assert that  $\lambda \neq 1$ . Otherwise, the Eqs. (10), (11) and (12) are reduced to

$$x = -A^{-1}B^T y, \quad (12)$$

$$C^T z = -Bx, \quad (13)$$

$$y^H C^T = 0, \quad (14)$$

respectively. Pre-multiplying Eq. (13) by  $y^H$  and substituting (14) into it, gives  $y^H Bx = 0$ . This along with (12) leads to  $y^H BA^{-1}B^T y = 0$ , equivalently,  $(B^T y)^H A^{-1}(B^T y) = 0$ . In view of the positive definiteness of  $A$ , we get  $B^T y = 0$ . Then, since  $B$  is of full row rank, we deduce that  $y = 0$ , which is impossible.

In the following, we assume that  $\lambda \neq 0, 1$  and  $y \neq 0$ . Without loss of generality, we assume that  $\|y\|_2 = 1$ . From (9) and (11), we obtain

$$x = -A^{-1}B^T y, \quad (15)$$

$$z = \frac{1}{1-\lambda}(\alpha I + \beta CC^T)^{-1}Cy. \quad (16)$$

Substituting the above relations into (10), yields

$$\lambda \left( (\alpha I + \beta BB^T)y + \frac{1}{\lambda-1}C^T(\alpha I + \beta CC^T)^{-1}Cy \right) = -BA^{-1}B^T y + (\alpha I + \beta BB^T)y.$$

By multiplying both sides of the preceding equality on the left by  $\lambda-1$  and  $y^H$  and with some algebra, we obtain the following quadratic equation

$$\lambda^2 - \frac{b}{a}\lambda + \frac{c}{a} = 0,$$

where

$$a = y^H(\alpha I + \beta BB^T)y, \quad c = y^H(\alpha I + \beta BB^T)y - y^H BA^{-1}B^T y,$$

$$b = 2y^H(\alpha I + \beta BB^T)y - y^H C^T(\alpha I + \beta CC^T)^{-1}Cy - y^H BA^{-1}B^T y.$$

According to Lemma 1, the following inequalities

$$\left| \frac{c}{a} \right| < 1, \quad \left| \frac{b}{a} \right| < 1 + \frac{c}{a}, \quad (17)$$

imply  $|\lambda| < 1$ . Clearly, whenever the inequality

$$y^H BA^{-1}B^T y < 2y^H(\alpha I + \beta BB^T)y, \quad (18)$$

holds, the first inequality of (17) is on. On the other hand, by easy manipulations we can observe that the second relation of (17) holds, if

$$P_{\alpha,\beta} := y^H C^T(\alpha I + \beta CC^T)^{-1}Cy + 2y^H BA^{-1}B^T y < 4y^H(\alpha I + \beta BB^T)y =: Q_{\alpha,\beta}. \quad (19)$$

Notice that, inequality (18) is ensured when (19) holds true. Hence, Eq. (18) is ignored. We first assume that  $w := Cy \neq 0$  (note that  $v = B^T y \neq 0$ ). According to Courant-Fisher inequality [22] we have

$$\begin{aligned} P_{\alpha,\beta} &= \frac{w^H(\alpha I + \beta CC^T)^{-1}w}{w^H w} \frac{y^H C^T C y}{y^H y} + 2 \frac{v^H A^{-1}v}{v^H v} \frac{y^H BB^T y}{y^H y} \\ &\leq \lambda_{\max}(\alpha I + \beta CC^T)^{-1} \lambda_{\max}(C^T C) + 2 \lambda_{\max}(A^{-1}) \lambda_{\max}(BB^T) \\ &= \frac{\sigma_{\max}^2(C)}{\alpha + \beta \sigma_{\min}^2(C^T)} + 2 \frac{\sigma_{\max}^2(B^T)}{\lambda_{\min}(A)}. \end{aligned} \quad (20)$$

It is necessary to mention that the upper bound for  $P_{\alpha,\beta}$  given above is valid even if  $w = 0$ . On the other hand, we have

$$Q_{\alpha,\beta} = y^H(\alpha I + \beta BB^T)y \geq \alpha + \beta \sigma_{\min}^2(B^T). \quad (21)$$

Now, from the Eqs. (20) and (21) we deduce that if the inequality (8) holds true, then the convergence of the proposed method is deduced.  $\square$

Since both of the matrices  $B$  and  $C$  are of full row rank, we deduce that  $\sigma_{\min}(B^T), \sigma_{\min}(C^T) > 0$ . Hence, it follows from Eq. (8) that for a large enough value of  $\alpha$  or  $\beta$  the method is convergent. However, for large values of  $\alpha$  and  $\beta$  the corresponding preconditioner may be inefficient. In the sequel we propose a method for choosing suitable.

Let

$$P = 2 \frac{\sigma_{\max}^2(B^T)}{\lambda_{\min}(A)}.$$

Based on Theorem 1, a sufficient condition for convergence of the proposed method is as follows

$$\frac{\sigma_{\max}^2(C)}{\alpha \left(1 + \frac{\beta}{\alpha} \sigma_{\min}^2(C^T)\right)} + P < 4\alpha \left(1 + \frac{\beta}{\alpha} \sigma_{\min}^2(B^T)\right). \quad (22)$$

Now, if  $\beta \geq \alpha$  and

$$\frac{\sigma_{\max}^2(C)}{\alpha(1 + \sigma_{\min}^2(C^T))} + P < 4\alpha(1 + \sigma_{\min}^2(B^T)), \quad (23)$$

then the inequality (22) holds true. By a little algebra, we can rewrite (23) as the following quadratic inequality

$$q(\alpha) := -4(1 + \sigma_{\min}^2(B^T))(1 + \sigma_{\min}^2(C^T))\alpha^2 + P(1 + \sigma_{\min}^2(C^T))\alpha + \sigma_{\max}^2(C) < 0. \quad (24)$$

Notice that the coefficient of  $\alpha^2$  in the polynomial  $q$  is negative and  $q(0) = \sigma_{\max}^2(C) > 0$ . Therefore, the polynomial  $q$  has two real roots, one negative and a positive. The positive one is given by

$$\tilde{\alpha} = \frac{P(1 + \sigma_{\min}^2(C^T)) + \sqrt{\Delta}}{8(1 + \sigma_{\min}^2(B^T))(1 + \sigma_{\min}^2(C^T))},$$

where  $\Delta = P^2(1 + \sigma_{\min}^2(C^T))^2 + 16(1 + \sigma_{\min}^2(B^T))(1 + \sigma_{\min}^2(C^T))\sigma_{\max}^2(C)$ .

According to the above results, we can claim that if

- (i)  $\beta \geq \alpha$ ,
- (ii)  $\alpha > \tilde{\alpha}$ ,

then  $q(\alpha) < 0$  and the proposed method is convergent for any initial choice of  $\mathbf{x}^{(0)}$ , i.e.,  $\rho(\mathcal{G}_{\alpha,\beta}) < 1$ .

Based on the above results, the eigenvalues of  $\mathcal{G}_{\alpha,\beta}$  are contained in a circle centered at origin with radius 1. In addition, we obviously have

$$\mathcal{P}^{-1}\mathcal{B} = I - \mathcal{G}_{\alpha,\beta}.$$

So, the eigenvalues of  $\mathcal{P}^{-1}\mathcal{B}$  included in a circle centered  $(1, 0)$  with radius 1. Therefore,  $\mathcal{P}$  serves a preconditioner for a Krylov subspace methods such as GMRES.

We end this section by applying the preconditioner  $\mathcal{P}$  within the Krylov subspace methods to solve the system  $\mathcal{B}\mathbf{x} = \mathbf{b}$ . In each iteration, we need to compute vectors of the form  $v = \mathcal{P}^{-1}w$ , equivalently,  $w = \mathcal{P}v$ . Now, by taking  $v = (v_1; v_2; v_3)$  and  $w = (w_1; w_2; w_3)$  the following algorithm can be given:

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Algorithm 1: Computation of  $(v_1; v_2; v_3) = \mathcal{P}^{-1}(w_1; w_2; w_3)$ .

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1. Solve  $(\alpha I + \beta CC^T)v_3 = w_3$  for  $v_3$ ;
  2. Solve  $(\alpha I + \beta BB^T)v_2 = w_2 + C^T v_3$  for  $v_2$ ;
  3. Solve  $Av_1 = w_1 - B^T v_2$  for  $v_1$ .
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In each step of this algorithm a system of linear equations should be solved. Since the coefficient matrices of these systems are SPD, they can be solved exactly using the Cholesky factorization or inexactly using the conjugate gradient (CG) method. In practice, in Step 1 of algorithm it is recommended to choose the values of  $\alpha$  and  $\beta$  such that (See [6, 15])

$$\beta = \alpha \frac{1}{\|C\|_2^2}.$$

In the same way to choose the values of  $\alpha$  and  $\beta$  in Step 2 satisfying

$$\beta = \alpha \frac{1}{\|B\|_2^2}.$$

However, since  $\alpha$  and  $\beta$  are in common in Steps 1 and 2 we propose to use

$$\beta = \frac{\alpha}{2} \left( \frac{1}{\|C\|_2^2} + \frac{1}{\|B\|_2^2} \right), \quad (25)$$

for both of the steps. We will shortly see in the section of the numerical results that a small value of  $\alpha$  along with the value of  $\beta$  using (25) give usually suitable results.

### 3 Numerical results

In this section, we give some numerical experiments to illustrate the superiority of the proposed preconditioner  $\mathcal{P}$  over the recently suggested ones in the literature. At each iteration of the preconditioners  $\mathcal{P}_{D1}$ ,  $\mathcal{P}_{D2}$  and  $\mathcal{P}_1$ , three linear subsystems with SPD coefficient matrices should be solved. These subsystems are solved by the CG method.

In our numerical experiments, the iteration is started from a zero vector and terminated as soon as

$$Res = \frac{\|\mathbf{b} - \mathcal{A}\mathbf{x}^{(k)}\|_2}{\|\mathbf{b}\|_2} \leq 10^{-6},$$

where  $\mathbf{x}^{(k)}$  is the computed solution at iteration  $k$ . The maximum number of iterations is set to be 1000. We have used the right-hand side vector  $\mathbf{b}$  such that the exact solution is a vector of all ones. For the inner CG iterations, the iteration is terminated as soon as the residual norm is reduced by a factor of  $10^3$ . In addition, the maximum number of inner iterations is set to be 100. For all the test problems, we set  $S = B(\text{diag}(A))^{-1}B^T$ .

In the following, we will compare the preconditioners from aspects of the number of total iteration steps (denoted by “IT”), and elapsed CPU times in seconds (denoted by “CPU”). As well as, the accuracy of the methods are compared under

$$Err = \frac{\|\mathbf{x}^{(k)} - \mathbf{x}^*\|}{\|\mathbf{x}^*\|},$$

where  $\mathbf{x}^{(k)}$  and  $\mathbf{x}^*$  stand for the current iteration and the exact solution of (2), respectively. The symbols “†” and “‡” show that the method has not converged in 1000 seconds and *maxit*, respectively. Also, by “§” we mean that the coefficient matrix  $\mathcal{B}$  does not satisfy the assumptions:

- (i)  $A$  is a SPD matrix,
- (ii)  $B$  and  $C$  are full row rank matrices.

All the computations are implemented in MATLAB R2019a on a Laptop with intel (R) Core(TM) i5-8265U CPU @ 1.60 GHz 8.GB.

**Example 1.** Consider the saddle point problem (2) with (see [20, 28])

$$A = \begin{pmatrix} I \otimes T + T \otimes I & 0 \\ 0 & I \otimes T + T \otimes I \end{pmatrix} \in \mathbb{R}^{2p^2 \times 2p^2},$$

$B = (I \otimes F \quad F \otimes I) \in \mathbb{R}^{p^2 \times 2p^2}$  and  $C = E \otimes F \in \mathbb{R}^{p^2 \times p^2}$  where

$$T = \frac{1}{h^2} \cdot \text{tridiag}(-1, 2, -1) \in \mathbb{R}^{p \times p}, \quad F = \frac{1}{h} \cdot \text{tridiag}(0, 1, -1) \in \mathbb{R}^{p \times p},$$

and  $E = \text{diag}(1, p+1, 2p+1, \dots, p^2-p+1)$  in which  $\otimes$  denotes the Kronecker product and  $h = 1/(p+1)$  stands for the discretization meshsize. For the preconditioner  $\mathcal{P}$ , we set  $\alpha = 5 \times 10^{-2}$  and compute  $\beta$  using (25). These values are listed in Table 1.

We observe from Table 1 that  $\beta \geq \alpha$ , which is in agreement with what we claimed in Section 2.

We consider two choices for parameters  $\alpha$  and  $\beta$  in the preconditioner  $\mathcal{P}_{D1}$  as the following cases:

Case (i):  $\alpha = 10^{-3}$  and  $\beta = 1$ , as considered in [1];

Case (ii): According to the Table 1.

Numerical results of the flexible GMRES (FGMRES) method [22, 23] in conjunction with the preconditioners for solving the double saddle point problem (2) are presented in Table 2. These results

Table 1: The values of  $\beta$  involved in the preconditioner  $\mathcal{P}$  for Example 1 with  $\alpha = 10^{-2}$ .

$p$	16	32	64	128	256
$\beta$	0.94	1.83	3.60	7.14	14.22

Table 2: Numerical results for Example 1.

Precon.	$p$	16	32	64	128	256
$I$	IT	425	949	‡	‡	‡
	CPU	0.72	8.97	55.10	161.23	637.21
	Res	8.6e-07	9.9e-07	2.7e-03	6.7e-03	4.9e-02
	Err	2.6e-06	2.4e-5	1.8e-01	5.5e-01	7.8e-01
$\mathcal{P}$	IT	33	42	53	75	141
	CPU	0.03	0.11	0.75	4.05	70.81
	Res	8.7e-07	5.9e-07	6.2e-07	8.9e-07	8.8e-07
	Err	1.6e-06	1.6e-06	6.6e-07	2.2e-05	1.9e-05
$\mathcal{P}_{D1(\text{case(i)})}$	IT	109	80	65	89	191
	CPU	0.17	0.33	1.89	6.91	113.10
	Res	7.0e-07	8.4e-07	7.8e-07	8.6e-07	9.7e-07
	Err	3.0e-07	6.0e-07	1.2e-06	2.3e-05	4.3e-05
$\mathcal{P}_{D1(\text{case(ii)})}$	IT	49	53	69	103	181
	CPU	0.08	0.14	1.10	6.50	102.66
	Res	3.9e-07	6.6e-07	2.8e-07	9.6e-07	9.6e-07
	Err	2.1e-07	1.8e-07	8.2e-06	2.5e-05	4.0e-05
$\mathcal{P}_1$	IT	114	466	‡	-	-
	CPU	0.75	24.66	421.56	†	†
	Res	8.10e-07	2.7e-06	4.6e-02	-	-
	Err	1.9e-06	2.1e-06	4.2e-01	-	-
$\mathcal{P}_{D2}$	IT	170	792	‡	-	-
	CPU	1.20	48.91	406.2	†	†
	Res	9.2e-07	1.5e-04	5.7e-02	-	-
	Err	1.0e-6	1.8e-05	1.4e-01	-	-

clearly show that the preconditioner  $\mathcal{P}$  is quite effective. In this problem, we find that the overall computation times and the iteration numbers for the preconditioner  $\mathcal{P}$  is less than the other examined preconditioners.

**Example 2.** We consider the three-by-three block saddle point problem (1) for which ( see [20, 28])

$$A = \text{blkdiag}(2W^T W + D_1, D_2, D_3) \in \mathbb{R}^{n \times n},$$

is a block-diagonal matrix,

$$B = [E, -I_{2\bar{p}}, I_{2\bar{p}}] \in \mathbb{R}^{m \times n} \quad \text{and} \quad C = E^T \in \mathbb{R}^{l \times m},$$



are both full row-rank matrices where  $\tilde{p} = p^2$ ,  $\hat{p} = p(p+1)$ ;  $W = (w_{i,j}) \in \mathbb{R}^{\hat{p} \times \hat{p}}$  with  $w_{i,j} = e^{-2((i/3)^2 + (j/3)^2)}$ ;  $D_1 = I_{\tilde{p}}$  is an identity matrix;  $D_i = \text{diag}(d_j^{(i)}) \in \mathbb{R}^{2\tilde{p} \times 2\tilde{p}}$ ,  $i = 2, 3$ , are diagonal matrices, with

$$d_j^{(2)} = \begin{cases} 1, & \text{for } 1 \leq j \leq \tilde{p}, \\ 10^{-5}(j - \tilde{p})^2, & \text{for } \tilde{p} + 1 \leq j \leq 2\tilde{p}, \end{cases}$$

$$d_j^{(3)} = 10^{-5}(j + \tilde{p})^2 \text{ for } 1 \leq j \leq 2\tilde{p},$$

and

$$E = \begin{pmatrix} \hat{E} \otimes I_{\tilde{p}} \\ I_p \otimes \hat{E} \end{pmatrix}, \quad \hat{E} = \begin{pmatrix} 2 & -1 & & & \\ & 2 & -1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & \ddots \\ & & & & 2 & -1 \end{pmatrix} \in \mathbb{R}^{p \times (p+1)}.$$

According to the above definitions, we have  $n = \hat{p} + 4\tilde{p}$ ,  $m = 2\tilde{p}$  and  $l = \hat{p}$ .

The parameters  $\alpha$  and  $\beta$  involved in the preconditioner  $\mathcal{P}_{D1}$  are chosen as  $\alpha = 10^{-1}$  and  $\beta = 1$  (See [1]). Also, in the preconditioner  $\mathcal{P}$ , we set  $\alpha = 5 \times 10^{-1}$  and  $\beta$  is computed similar to Example 1, that are listed in Table 3. In Table 4, we give numerical results for the FGMRES method incorporated with the preconditioners  $\mathcal{P}$ ,  $\mathcal{P}_{D1}$ ,  $\mathcal{P}_1$  and  $\mathcal{P}_{D2}$ . Hence, we have also reported the results of FGMRES without preconditioning. As observed, the preconditioner  $\mathcal{P}$  substantially accelerate the convergence rate of FGMRES. It should be mentioned that when  $p$  is large, only  $\mathcal{P}$  and  $\mathcal{P}_{D1}$  are feasible in practice.

**Example 3.** Consider the quadratic program [18, 19]:

$$\begin{aligned} \min_{x \in \mathbb{R}^n, y \in \mathbb{R}^l} \quad & \frac{1}{2}x^T A x + r^T x + q^T y \\ \text{s.t. :} \quad & Bx + C^T y = b, \end{aligned} \tag{26}$$

where the vector  $\lambda \in \mathbb{R}^m$  is the Lagrange multiplier. To solve the above problem we define the Lagrange function

$$L(x, y, \lambda) = \frac{1}{2}x^T A x + r^T x + q^T y + \lambda^T (Bx + C^T y - b),$$

where the vector  $\lambda \in \mathbb{R}^m$  is the Lagrange multiplier. Then the Karush-Kuhn-Tucker necessary conditions of (26) are as follows (see [8])

$$\nabla_x L(x, y, \lambda) = 0, \quad \nabla_y L(x, y, \lambda) = 0 \quad \text{and} \quad \nabla_\lambda L(x, y, \lambda) = 0.$$

These equations lead to a system of linear equations of the form (1). In this example, the matrices  $A, B$  and  $C$  have been chosen from the CUTER collection [17]. We note that for the test matrix MOSARQP1, the matrix  $C$  is not full row rank. So, the matrix  $CS^{-1}C^T$  is symmetric positive semidefinite. This means that the preconditioners  $\mathcal{P}_1$  and  $\mathcal{P}_{D2}$  are singular. Consequently,  $\mathcal{P}_1$  and  $\mathcal{P}_{D2}$  can not be applied as a preconditioner. Similarly, for the test matrices AUG2D and AUG2DC,  $A$  is symmetric positive semidefinite. Accordingly, the matrix  $S$  and as well as  $\mathcal{P}_1$  and  $\mathcal{P}_{D1}$  can not be formed.

In this example, for the preconditioners  $\mathcal{P}$  and  $\mathcal{P}_{D1}$  we set  $\alpha = 5 \times 10^{-1}$ , and  $\beta$  computed according to the formula (25), that are reported in Table 5. The result for FGMRES and application of the preconditioners are shown in Table 6. As seen in Table 6, the iteration steps and computational time for  $\mathcal{P}$  are less than the other ones.

Table 3: The values of  $\beta$  involved in the preconditioner  $\mathcal{P}$  for Example 2 with  $\alpha = 5 \times 10^{-1}$ .

$p$	16	32	64	128	256	512
$\beta$	0.36	0.35	0.35	0.35	0.35	0.34

Table 4: Numerical results for Example 2.

Precon.	$p$	16	32	64	128	256	512
$I$	IT	186	190	187	180	-	-
	CPU	0.25	1.04	3.01	27.87	†	†
	Res	1.0e-06	9.9e-07	1.0e-06	9.8e-07	-	-
	Err	1.3e-06	1.4e-5	1.4e-05	1.4e-05	-	-
$\mathcal{P}$	IT	53	55	56	54	52	50
	CPU	0.06	0.15	0.55	2.89	14.50	57.30
	Res	8.2e-07	9.3e-07	9.9e-07	9.9e-07	8.9e-07	8.4e-07
	Err	1.2e-05	1.5e-05	1.5e-05	1.6e-05	1.6e-05	1.2e-06
$\mathcal{P}_{D1}$	IT	70	69	68	65	63	60
	CPU	0.11	0.28	0.93	5.46	24.08	99.58
	Res	1.0e-06	9.5e-07	8.8e-07	9.3e-07	8.5e-07	9.5e-07
	Err	5.6e-06	5.7e-06	5.0e-6	5.5e-06	4.9e-06	5.2e-06
$\mathcal{P}_1$	IT	10	10	10	9	-	-
	CPU	0.04	0.14	1.96	41.00	†	†
	Res	2.8e-07	3.1e-07	2.4e-07	8.8e-07	-	-
	Err	1.1e-06	2.1e-06	8.5e-07	1.7e-06	-	-
$\mathcal{P}_{D2}$	IT	19	19	19	18	-	-
	CPU	0.05	0.24	3.51	63.50	†	†
	Res	2.7e-07	2.0e-07	4.4e-07	8.7e-07	-	-
	Err	7.7e-07	4.7e-07	1.6e-06	3.7e-06	-	-

## 4 Conclusion

We have proposed a new iteration method for solving a class of three-by-three saddle point problems. The convergence theory of the method have been studied. The exploited preconditioner from the presented method, has been applied for accelerating the convergence rate of Krylov subspaces method, especially for GMRES method. The remarkable point was that introduced preconditioner is easy to implement. Numerical results indicate that the presented preconditioner is effective.

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Table 5: The values of  $\beta$  involved in the preconditioner  $\mathcal{P}$  for Example 3 with  $\alpha = 5 \times 10^{-1}$ .

Matrix	MOSARQP1	AUG2DC	AUG2D	YAO	LISWET12	HUESMOD
$\beta$	0.66	0.82	0.53	0.60	0.60	0.49

Table 6: Numerical results for Example 3.

Precon.	Matrix	MOSARQP1	AUG2DC	AUG2D	YAO	LISWET12	HUESMOD
	<b>n</b>	5700	50400	50400	6004	30004	20002
	<i>nnz</i>	14434	140600	140200	18006	90006	70000
$I$	IT	110	69	69	61	56	9
	CPU	0.24	0.65	0.63	0.08	0.28	0.01
	Err	9.9e-07	8.4e-07	8.5e-07	9.0e-07	8.3e-07	6.2e-10
	Res	3.9e-06	3.8e-06	3.8e-06	4.1e-06	4.1e-06	6.3e-10
$\mathcal{P}$	IT	31	32	46	35	33	7
	CPU	0.04	0.26	0.54	0.06	0.19	0.02
	Err	9.8e-07	8.5e-07	9.4e-07	8.8e-07	8.7e-07	5.1e-09
	Res	5.2e-06	3.2e-06	4.2e-06	3.8e-06	3.7e-06	5.2e-09
$\mathcal{P}_{D1}$	IT	60	55	58	49	46	11
	CPU	0.11	0.61	0.61	0.09	0.32	0.04
	Err	8.4e-07	8.6e-7	9.0e-07	9.1e-07	8.0e-07	8.1e-08
	Res	4.1e-06	3.6e-06	3.9e-06	4.5e-06	4.0e-06	1.4e-07
$\mathcal{P}_1$	IT	§	§	§	†	†	9
	CPU	-	-	-	-	-	10.65
	Err	-	-	-	-	-	3.0e-06
	Res	-	-	-	-	-	3.0e-06
$\mathcal{P}_{D2}$	IT	§	§	§	†	†	10
	CPU	-	-	-	-	-	10.48
	Err	-	-	-	-	-	1.8e-06
	Res	-	-	-	-	-	1.7e-06

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