Logarithmic Akizuki–Nakano vanishing theorems on weakly pseudoconvex Kähler manifolds

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Abstract. In this note, we obtain a logarithmic vanishing theorem on certain weakly pseudoconvex Kähler manifolds. It is a generalization of Norimatsu's result on compact Kähler manifolds. As a direct corollary, we obtain relative vanishing theorems of certain direct image sheaves.

Contents

1. Introduction

One of central topics in complex and algebraic geometry is cohomological vanishing theorem. The famous Akizuki–Nakano vanishing theorem shows that if F is a positive line bundle over an n-dimensional compact Kähler manifod X, then

$$H^q(X, \Omega_X^p \otimes F) = 0 \quad \text{for any} \quad p+q \geq n+1.$$

The generalization of Akizuki–Nakano vanishing theorem on weakly pseudoconvex or weakly 1-complete Kähler manifolds are finished by Nakano [Naka73, Naka74], Kazama [Kaza73], Abdelkader [Abde80], Takegoshi [Take81], Ohsawa–Takegoshi [OhTa81] and so on. On the other hand, in [Nori78] Norimatsu obtained the logarithmic vanishing theorem on compact Kähler manifold. In [EsVi86], Esnault and Viehweg studied the logarithmic de Rham complexes and vanishing theorems on complex algebraic manifolds. They obtain the logarithmic type vanishing theorems for the pair (X, D), here X is projective manifold and D is a simple normal croosing divisor. Their methods are based on the Hodge theory and the degeneration of Hodge to de Rham spectral sequence. Recently, in [HLWY16], Huang–Liu–Wan–Yang obtain the corresponding results on compact Kähler manifold by the standard anlalytic technique like L^2-method.

In this paper, we try to generalize Norimatsu, Esnalut–Viehweg and Huang–Liu–Wan–Yang's results to open pseudoconvex Kähler manifolds. More specifically, we get

Theorem 1.1 (=Corollary 4.2)**.** *Let X be an n-dimensional holomorphically convex Kähler manifold and F is a positive line bundle on X. Let D be a simple normal crossing divisor on X. We have*

$$H^q(X, \Omega_X^p(\log D) \otimes F) = 0$$

for any $p + q \geq n + 1$.

The important case is that when $p = n$, one thus have $\Omega_X^n(\log D) = K_X \otimes \mathcal{O}_X(D)$. In such a case, we extend this result to weakly pseudoconvex Kähler manifolds, we arrive at

Theorem 1.2 (= Theorem 4.3 + Theorem 4.12)**.** *Let X be a n-dimensional weakly pseudoconvex Kähler manifold and F is a positive line bundle on X. Let D be a simple normal crossing divisor on X. We have*

$$H^q(X, K_X \otimes \mathcal{O}_X(D) \otimes F) = 0.$$

for any $q \geq 1$.

In Theorem 1.2, we do not need to twist the sheaf $K_X \otimes \mathcal{O}_X(D) \otimes F$ with multiplier ideal sheaf $\mathscr{I}(D)$ of divisor. The vanishing of $H^q(X, K_X \otimes \mathcal{O}_X(D) \otimes F \otimes \mathscr{I}(D))$ is the direct consequence of Nadel vanishing theorem. Our method is the combining of L^2 technique in [HLWY16] and Runge-type approximation method rooted in [Naka74, Kaza73, Take81, OhTa81]. For a weakly pseudoconvex Kähler manifold X with smooth plurisubharmonic exhaustion function Φ and a sequence of positive real numbers tends to infinity. On each sublevel subset $X_c := \{x \in X : \Phi(x) < c\}$ which is relative compact in X, we obtain the logarithmic vanishing theorem by the L^2-technique. All these sublevel subsets formed a Leray covering of X and therefore we can focus on the Čech cohomology. By the approximation we obtain the global vanishing theorem.

One of motivation to study the cohomology on weakly pseudoconvex Kähler manifolds is that one can investigate the corresponding higher direct image sheaves. As a direct corollary, we acquire

Corollary 1.3. *Let $f : X \to S$ be a proper holomorphic morphism from a Kähler manifold X onto the reduced complex space S. Let D be an simple normal crossing divisor such that $f|_D$ is proper. If F is a positive holomorphic line bundle on X, then*

$$R^q f_*(\Omega_X^p(\log D) \otimes F) = 0 \quad \text{for any} \quad p + q \geq n + 1.$$

In corollary 1.3 above, if we replace the positive line bundle by the positive vector bundle in the sense of Nakano, the claim still be true. Also as Professor Ohsawa pointed in [Ohsa21], it may be interested to generalize the results in [LRW19] and [LWY19] to the weakly pseudoconvex situation.

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2. Preliminaries

In this section, we introduce some basic definitions and results in complex geometry. Unless otherwise mentioned, X denotes a complex manifold of dimension n. The basic reference is [Dem12b].

Definition 2.1 (Chern connection and curvature form of vector bundle)**.** Let (E, h) be a holomorphic vector bundle on X. Corresponding to this metric h, there exists the unique Chern connection $D = D_{(E,h)}$, which can be split in a unique way as a sum of a $(1,0)$ and a $(0,1)$ connection, i.e., $D = D'_{(E,h)} + D''_{(E,h)}$. Furthermore, the $(0,1)$ part of the Chern connection $D''_{(E,h)} = \overline{\partial}$. The curvature form is defined to be $\Theta_{E,h} := D^2_{(E,h)}$. On a coordinate patch $\Omega \subset X$ with complex coordinate $(z_1, \cdots, z_n)$, denote by $(e_1, \cdots, e_r)$ an orthonormal frame of vector bundle E with rank r. Set

$$\sqrt{-1}\Theta_{E,h} = \sqrt{-1} \sum_{1\leq j,k\leq n, 1\leq \lambda,\mu\leq r} c_{jk\lambda\mu} dz_j \wedge d\overline{z}_k \otimes e^*_\lambda \otimes e_\mu, \quad c_{jk\mu\lambda} = \overline{c}_{jk\lambda\mu}.$$

Corresponding to $\sqrt{-1}\Theta_{E,h}$, there is a Hermitian form $\theta_{E,h}$ on $TX \otimes E$ defined by

$$\theta_{E,h}(\phi, \phi) = \sum_{jk\lambda\mu} c_{jk\lambda\mu}(x)\phi_{j\lambda}\overline{\phi}_{k\mu}, \quad \phi \in T_xX \otimes E_x.$$

Definition 2.2 (Positive vector bundle)**.** A holomorphic vector bundle (E, h) is said to be

(1) positive in the sense of Nakano (resp. Nakano semipositive) if for every nonzero tensor $\phi \in TX \otimes E$, we have

$$\theta_{E,h}(\phi, \phi) > 0 \quad (\text{resp.} \geq 0).$$

(2) positive in the sense of Griffiths (resp. Griffiths semipositive) if for every nonzero decomposable tensor $\xi \otimes e \in TX \otimes E$, we have

$$\theta_{E,h}(\xi \otimes e, \xi \otimes e) > 0 \quad (\text{resp.} \geq 0).$$

It is clear that Nakano positivity implies Griffiths positivity and that both concepts coincide if $r = 1$. In the case of line bundle, E is merely said to be positive (resp. semipositive).

Definition 2.3 (Singular metric and curvature current on line bundle)**.** Let (F, h) be a holomorphic line bundle on complex manifold X endowed with possible singular Hermitian metric h. For any given trivilization $\theta : F|_\Omega \simeq \Omega \times \mathbb{C}$ by

$$\|\xi\|_h = |\theta(\xi)|e^{-\phi(x)}, \quad x \in \Omega, \xi \in F_x,$$

where $\phi \in L^1_{loc}(\Omega)$ is an arbitrary function, called the weight of the metric. The curvature $\sqrt{-1}\Theta_h(F)$ of h is defined by

$$\sqrt{-1}\Theta_h(F) = \sqrt{-1}2\partial\overline{\partial}\phi.$$

The Levi form $\sqrt{-1}\partial\overline{\partial}\phi$ is taken in the sense of distributions and thus the curvature is a $(1, 1)$-current but not always a smooth $(1, 1)$-form. It is globally defined on X and independent of the choice of trivializations. The curvature $\sqrt{-1}\Theta_h(F)$ of h is said to be positive (resp. semi-positive) if $\sqrt{-1}\Theta_h(F)>0$ (resp. ≥ 0) in the sense of current.

Definition 2.4 (Psh function and quasi-psh)**.** A function $u : \Omega \to [-\infty, \infty)$ defined on a open subset $\Omega \in \mathbb{C}^n$ is called plurisubharmonic (psh, for short) if

(1) u is upper semi-continuous;
(2) for every complex line $Q \subset \mathbb{C}^n$, $u|_{\Omega\cap Q}$ is subharmonic on $\Omega \cap Q$.

A quasi-plurisubharmonic (quasi-psh, for short) function is a function v which is locally equal to the sum of a psh function and of a smooth function.

Definition 2.5 (Multiplier ideal sheaves)**.** Let ϕ be a quasi-psh function on a complex manifold X, the multiplier ideal sheaf $\mathscr{I}(\phi) \subset \mathcal{O}_X$ is defined by

$$\Gamma(U, \mathscr{I}(\phi)) = \{f \in \mathcal{O}_X(U) : \quad |f|^2 e^{-2\phi} \in L^1_{loc}(U)\}$$

for every open set $U \subset X$. For a line bundle (F, h), if the local weight of metric h is ϕ, then we denote the multiplier ideal sheaf interchangeably by $\mathscr{I}(\phi)$ or $\mathscr{I}(h)$.

The basic properties of the sheaf of logarithmic differential forms and the logarithmic integrable connections on complex algebraic manifolds were developed by Deligne in [De70], Esnault and Viehweg in [EsVi86] studied the relations between logarithmic de Rham complexes and vanishing theorems on complex algebraic manifolds.

Definition 2.6 (Simple normal divisor and logarithmic forms)**.** Let X be a complex manifold and D be a simple normal crossing divisor on it, i.e., $D = \sum_i D_i$, where each D_i are distinct smooth hypersurfaces intersecting transversely in X. The sheaf of germs of differential p-forms on X with at most logarithmic poles along D, we denote it by $\Omega^p_X(\log D)$. Its space of sections on any open subset W of X are

$$\Gamma(W, \Omega^p_X(\log D)) := \{\alpha \in \Gamma(W, \Omega^p_X \otimes \mathcal{O}_X(D)) : d\alpha \in \Gamma(W, \Omega^{p+1}_X \otimes \mathcal{O}_X(D))\}.$$

Denote by $j : Y = X\backslash D \to X$ the natural inclusion and we can choose a local coordinate chart $(W; z_1, \cdots, z_n)$ of X such that the locus of D is given by $z_1 \cdots z_k = 0$

and $Y \cap W = W_r^* = (\Delta_r^*)^k \times (\Delta_r)^{n-k}$ where Δ_r (resp. Δ_r^*) is the (resp. punctured) open disk of radius r in the complex plane.

Definition 2.7 (Poincaré type metric)**.** Under the above setting, we say that the metric ω_Y on Y is of Poincaré type along D, if for each local coordinate chart $(W; z_1, \cdots, z_n)$ along D the restriction $\omega_Y|_{W_r^*}$ is equivalent to the usual Poincaré type metric ω_P defined by

$$\omega_P = \sqrt{-1}\sum_{j=1}^{k} \frac{dz_j \wedge d\overline{z}_j}{|z_j|^2 \cdot \log^2 |z_j|^2} + \sqrt{-1} \sum_{j=k+1}^{n} dz_j \wedge d\overline{z}_j.$$

We now turn to some basic definitions of pseudoconvex manifolds.

Definition 2.8 (Weakly pseudoconvex = Weakly 1-complete manifolds)**.** A function $\phi : X \to [-\infty, +\infty)$ on a manifold X is said to be exhaustive if all sublevel sets

$$X_c := \{x \in X : \phi(x) < c\} \quad c < \sup \phi,$$

are relatively compact. A complex manifold X is called weakly pseudoconvex if there exists a smooth plurisubharmonic exhaustion function $\phi : X \to \mathbb{R}$ with $\sup \phi = +\infty$. Similarly, a complex manifold X is said strongly pseudoconvex if the exhaustion function is smooth strictly plurisubharmonic.

Proposition 2.9. [Naka70, Dem12b] *Every weakly pseudoconvex Kähler manifold carries a complete Kähler metric.*

Proof. We show the proof of this because we will use it later. Let ϕ be an exhaustive psh function on X. Set $\hat{\omega} = \omega + \sqrt{-1}\partial\overline{\partial}(\chi \circ \phi)$, where χ is a smooth convex increasing function. Then

$$\begin{aligned} \hat{\omega} &= \omega + \sqrt{-1}(\chi' \circ \phi)\partial\overline{\partial}\phi + \sqrt{-1}(\chi'' \circ \phi)\partial\phi \wedge \overline{\partial}\phi \\ &\geq \omega + \sqrt{-1}\partial(\rho \circ \phi) \wedge \overline{\partial}(\rho \circ \phi) \end{aligned}$$

where $\rho = \int_0^t \sqrt{\chi''(u)}du$. We thus have complete metric $\hat{\omega}$ as soon as $\lim_{t \to +\infty} \rho(t) = +\infty$, i.e.

$$\int_0^{+\infty} \sqrt{\chi''(u)}du = +\infty.$$

□

The next theorem is very important for this paper.

Theorem 2.10 ($\overline{\partial}$-equation on complete Kähler manifolds)**.** [Dem12a, Theorem 5.1] *Let X be a complete Kähler manifold with a Kähler metric ω which is not necessarily complete. Let (E, h) be a Hermitian vector bundle of rank r over X, and assume that the curvature operator $B := [i\Theta_{E,h}, \Lambda_\omega]$ is semi-positive definite everywhere on $\wedge^{p,q}T_X^* \otimes E$, for some $q \geq 1$. Then for any form $g \in L^2(X, \wedge^{p,q}T_X^* \otimes E)$ satisfying*

$\bar\partial g = 0$ and $\int_X \langle B^{-1}g, g\rangle dV_\omega < +\infty$, there exists $f \in L^2(X, \wedge^{p,q-1}T_X^* \otimes E)$ such that $\bar\partial f = g$ and

$$\int_X |f|^2 dV_\omega \le \int_X \langle B^{-1}g, g\rangle dV_\omega.$$

Definition 2.11 (Holomorphically convex manifold)**.** A complex manifold X is called holomorphically convex if for any compact set $K \subset X$, its holomorphic hull $\hat{K} = \{x \in X : |f(x)| \le \sup\limits_K |f| \text{ for all } f \in \mathcal{O}_X(X)\}$ is compact too.

Definition 2.12 (Stein manifold)**.** A complex manifold X is called Stein manifold if X is holomorphically convex and for any $x, y \in X, x \neq y$, there exists a $f \in \mathcal{O}_X(X)$ with $f(x) \neq f(y)$.

We know that every holomorphically convex manifold is weakly pseudoconvex but the converse to it does not hold. For a holomorphically convex manifold, we have the classical Remmert reduction which relates it to Stein space.

Remark 2.13 (Remmert reduction)**.** If X is a holomorphically convex manifold, then by Remmert reduction, there exist a normal Stein space S and a proper, surjective, holomorphic morphism $f : X \to S$ such that

(1) $f_*\mathcal{O}_X = \mathcal{O}_S$,
(2) f has connected fibers,
(3) The map $f^* : \mathcal{O}_S(S) \to \mathcal{O}_X(X)$ is an isomorphism,
(4) The pair (f, S) is unique up to biholomorphism.

Remark 2.14 (Sublevel set of weakly pseudoconvex manifolds)**.** Let (X, ω, Φ) be a weakly pseudoconvex Kähler manifold with smooth psh exhaustion function Φ. Without loss of generality, we may assume Φ is positive. For any positive real number c, the sublevel set $X_c = \{x \in X : \Phi(x) < c\}$ is relative compact in X and again pseudoconvex with respect to the exhaustion function $\Phi_c := \frac{1}{c-\Phi}$. Set $\omega_c := \omega|_{X_c}$, then (X_c, ω_c, Φ) is again a weakly pseudoconvex Kähler manifold and thus we have an exhaustion sequence of pseudoconvex sublevel set (X_c, ω_c, Φ_c).

3. Vanishing theorem on each sublevel set

In this section, let (X, ω, Φ) be a weakly pseudoconvex Kähler manifold with smooth psh function Φ with $\sup \Phi = +\infty$. Let (F, h^F) be a holomorphic Hermitian line bundle on X. For a fixed positive real number c, we have a sublevel set (X_c, ω_c, Φ_c). We will focus on the sublevel set X_c because it is relative compact.

Definition 3.1 (Incomplete Poincaré type Kähler metric on X_c)**.** Let $D = \sum_i D_i$ be a simple normal crossing divisor of X, and σ_i be the defining section of D_i. Fix any smooth Hermitian metrics $\|\cdot\|_i$ on $\mathcal{O}(D_i)$ such that $\|\sigma_i\|_i < \frac{1}{2}$ on X_c for each i. Similar to [Zucker79], we set $\omega_{c,p} := (k_c\omega_c - \frac{1}{2}\sum \partial\overline{\partial} \log\log^2 \|\sigma_i\|_i^2)$ for large positive integer k_c which depends on X_c. In special coordinates W where D_i is defined by $z_i = 0$, and $\|\sigma_i\|_i^2 = |z_i|^2 e^u$ for some function u that is smooth on W. Then

$$-\frac{1}{2}\partial\overline{\partial}\log\log^2\|\sigma_i\|_i^2 = \frac{1}{(\log|z|^2+u)^2}(\frac{dz}{z}+\partial u)\wedge(\frac{d\overline{z}}{z}+\overline{\partial}u) - \frac{1}{\log|z|^2+u}\partial\overline{\partial}u.$$

It is clear that $\omega_{c,p}$ is positive on X_c and of Poincaré type along D provided k_c is sufficiently large. But it is obvious that $\omega_{c,p}$ is not complete along the boundary of X_c.

Now we will follow Huang–Liu–Wan–Yang's approach in [HLWY16] to acquire the L^2 resolution.

Definition 3.2 (L^2 fine sheaf)**.** Let (X_c, ω_c) be the fixed sublevel set, we denote the restriction of line bundle (F, h^F) on $Y_c := X_c \backslash D$ by $(F, h^F_{Y_c})$. The sheaf $\Xi^{p,q}_{(2)}(X_c, F, \omega_{c,p}, h^F_{Y_c})$ over X_c is defined as follows. On any open subset U of X_c, the section space $\Gamma(U, \Xi^{p,q}_{(2)}(X_c, F, \omega_{c,p}, h^F_{Y_c}))$ over U consists of F-valued (p,q)-forms u with measurable coefficients such that the L^2 norms of both u and $\overline{\partial}u$ are integrable on any compact subset K of U. Here the integrability means that both $|u|^2_{\omega_{c,p}\otimes h^F_{Y_c}}$ and $|\overline{\partial}u|^2_{\omega_{c,p}\otimes h^F_{Y_c}}$ are integrable on $K \backslash D$. Recall the sheaf $\mathcal{F}$ is called a fine sheaf if for any locally finite open covering $\{U_i\}$, there is a family of homomorphisms $\{f_i\}$, $f_i : \mathcal{F} \to \mathcal{F}$, such that

(1) supp $f_i \subset U_i$,
(2) $\sum_i f_i = 1$, i.e., $\sum_i f_i(s) = s$ for any section s.

If the metric $\omega_{c,p}$ is of Poincaré type as in Definition 3.1, then it is complete along the divisor D and is of finite volume, see for example [Zucker79, Proposition 3.4]. As a consequence, the sheaf $\Xi^{p,q}_{(2)}(X_c, F, \omega_{c,p}, h^F_{Y_c})$ would be a fine sheaf.

Theorem 3.3 (An L^2-type Dolbeault isomorphism)**.** [HLWY16, Theorem 3.1] *Let (X, ω) be a weakly pseudoconvex Kähler manifold of dimension n and $D = \sum_{i=1}^s D_i$ be a simple normal crossing divosor in X. For a fixed real number c, let X_c be the sublevel set. Let $\omega_{c,p}$ be a smooth Kähler metric on Y_c which is of Poincaré type along D as in Definition 3.1. For a line bundle (F, h^F), there exists a smooth Hermitian metric $h^F_{Y_c,\alpha_c}$ on $F|_{Y_c}$ such that the sheaf $\Omega^p(\log D) \otimes F$ over X_c enjoys a fine resolution given by the L^2 Dolbeault complex $(\Xi^{p,*}_{(2)}(X_c, F, \omega_{c,p}, h^F_{Y_c,\alpha_c}), \overline{\partial})$, here α_c is a large positive constant depends on X_c. This is to say, we have an exact sequence of sheaves over X_c*

$$0 \to \Omega^p(\log D) \otimes F \to \Xi^{p,*}_{(2)}(X_c, F, \omega_{c,p}, h^F_{Y_c,\alpha_c}) \tag{3.1}$$

such that $\Xi^{p,q}_{(2)}(X_c, F, \omega_{c,p}, h^F_{Y_c,\alpha_c})$ is a fine sheaf for each $0 \le p, q \le n$. In particular, by Dolbeault isomorphism

$$H^q(X_c, \Omega^p(\log D) \otimes F) \simeq H^{p,q}_{(2)}(Y_c, F, \omega_{c,p}, h^F_{Y_c,\alpha_c}). \tag{3.2}$$

Proof. Firstly, for any fixed constants $\tau_i \in (0,1]$, we construct a smooth Hermitian metric on $F|_{Y_c}$.

$$h^F_{Y_c,\alpha_c} := \prod_{i=1}^{s} \|\sigma_i\|_i^{2\tau_i} (\log^2 \|\sigma_i\|_i^2)^{\frac{\alpha_c}{2}} h^F,$$

where α_c is a large positive constant to be determined later. Based on the Definition 3.2, it is sufficient to check the exactness of complex. The proof is almost identical to the proof in [HLWY16]. □

Even though $\omega_{c,p}$ is not complete, but Y_c admit a complete Kähler metric. Indeed, let $\tilde{\omega} = \hat{\omega} + \omega_{c,p}$, here $\hat{\omega}$ is complete along the boundary of X_c like in the Definition 2.9. We know that $\tilde{\omega}$ is complete on Y_c. Hence we can still solve the certain $\overline{\partial}$-equation on Y_c thanks to Theorem 2.10. Now we slight modify Huang–Liu–Wan–Yang's approach in [HLWY16] to get the local vanishing.

Theorem 3.4. *Let (F, h^F) be a positive holomorphic line bundle on an n-dimensional weakly pseudoconvex Kähler manifold X. For each real number c and on the corresponding sublevel set X_c, we have the vanishing of cohomology groups,*

$$H^q(X_c, \Omega^p(\log D) \otimes F) = 0 \quad \text{for any} \quad p + q \geq n + 1.$$

Proof. Let ω_c be a fixed Kähler metric on X_c. Set $\{\lambda^j_{\omega_c}(h^F)\}_{j=1}^n$ be the increasing sequence of eigenvalues of $\sqrt{-1}\Theta(F, h^F)$ with respect to ω_c. Since X_c is relative compact in X, there exists a positive constant c_0 such that $\lambda^1_{\omega_c}(h^F) \geq c_0$ everywhere over X_c. We construct a new metric on $F|_{Y_c}$ as following,

$$h_{\alpha,\epsilon,\tau} := \prod_{i=1}^{s} \|\sigma_i\|_i^{2\tau_i} (\log^2(\epsilon\|\sigma_i\|_i^2))^{\frac{\alpha}{2}} h^F.$$

Here the constant $\alpha > 0$ is chosen to be large enough to meet the condition in Theorem 3.3 and the constants $\tau_i, \epsilon \in (0,1]$ are to be determined later and of course they are all depend on X_c. On Y_c, a straightforward computation gives rise to

$$\begin{aligned}\sqrt{-1}\Theta(F, h_{\alpha,\epsilon,\tau}) =& \sqrt{-1}\Theta(F, h^F) + \sum_i^s \tau_i c_1(D_i) \\ &+ \sum_i^s \frac{\alpha c_1(D_i)}{\log(\epsilon\|\sigma_i\|_i^2)} + \sqrt{-1}\sum_i^s \frac{\alpha\partial \log\|\sigma_i\|_i^2 \wedge \overline{\partial}\log\|\sigma_i\|_i^2}{(\log(\epsilon\|\sigma_i\|_i^2))^2}.\end{aligned} \tag{3.3}$$

Set $\delta = \frac{c_0}{8n-1}$, we can choose τ_i and ϵ small enough such that

$$-\frac{\delta}{2}\omega_c \leq \sum_i^s \tau_i c_1(D_i) \leq \frac{\delta}{2}\omega_c, \qquad -\frac{\delta}{2}\omega_c \leq \sum_i^s \frac{\alpha c_1(D_i)}{\log(\epsilon\|\sigma_i\|_i^2)} \leq \frac{\delta}{2}\omega_c, \tag{3.4}$$

hold on Y_c. Note that the constants τ_i and ϵ are thus fixed. We set

$$\omega_{Y_c} := \sqrt{-1}\Theta(F, h_{\alpha,\epsilon,\tau}) + 2\delta\omega_c.$$

It follows from formula (3.3) that ω_{Y_c} is of Poincaré type Kähler form on Y_c. And it is apparent form formula (3.4) that we have

$$\sqrt{-1}\Theta(F, h_{\alpha,\epsilon,\tau}) \geq \sqrt{-1}\Theta(F, h^F) - \delta\omega_c \tag{3.5}$$

on Y_c. Since $\sqrt{-1}\Theta(F, h^F)$ is a positive $(1,1)$-form, we know on Y_c

$$\omega_{Y_c} = \sqrt{-1}\Theta(F, h_{\alpha,\epsilon,\tau}) + 2\delta\omega_c \geq \delta\omega_c,$$

which means ω_{Y_c} is a positive $(1,1)$-form and thus a metric. On a local chart of Y_c, we may assume that $\omega_c = \sqrt{-1}\sum_{i=1}^n \eta_i \wedge \overline{\eta}_i$ and

$$\begin{aligned}\sqrt{-1}\Theta(F, h_{\alpha,\epsilon,\tau}) &= \sqrt{-1}\sum_{i=1}^n \lambda^i_{\omega_c}(h_{\alpha,\epsilon,\tau})\eta_i \wedge \overline{\eta}_i \\ &= \sqrt{-1}\sum_{i=1}^n \frac{\lambda^i_{\omega_c}(h_{\alpha,\epsilon,\tau})}{\lambda^i_{\omega_c}(h_{\alpha,\epsilon,\tau}) + 2\delta}\eta'_i \wedge \overline{\eta}'_i\end{aligned}$$

where

$$\eta'_i = \eta_i\sqrt{\lambda^i_{\omega_c}(h_{\alpha,\epsilon,\tau}) + 2\delta}.$$

Note that $\omega_{Y_c} = \sqrt{-1}\sum_{i=1}^n \eta'_i \wedge \overline{\eta}'_i$, and so the eigenvalues of $\sqrt{-1}\Theta(F, h_{\alpha,\epsilon,\tau})$ with respect to ω_{Y_c} are

$$\gamma_i := \frac{\lambda^i_{\omega_c}(h_{\alpha,\epsilon,\tau})}{\lambda^i_{\omega_c}(h_{\alpha,\epsilon,\tau}) + 2\delta} < 1.$$

On the other hand, due to formula (3.5) one has

$$\lambda^i_{\omega_c}(h_{\alpha,\epsilon,\tau}) \geq c_0 - \delta.$$

It implies

$$\gamma_i = \frac{\lambda^i_{\omega_c}(h_{\alpha,\epsilon,\tau})}{\lambda^i_{\omega_c}(h_{\alpha,\epsilon,\tau}) + 2\delta} \geq \frac{c_0 - \delta}{c_0 + \delta} = 1 - \frac{1}{4n}.$$

For any section $u \in \Gamma(Y_c, \wedge^{p,q}T^*Y_c \otimes F)$, we get

$$\begin{aligned}\big\langle[\sqrt{-1}\Theta(F, h_{\alpha,\epsilon,\tau}), \Lambda_{\omega_{Y_c}}]u, u\big\rangle &\geq \big(\sum_{i=1}^q \gamma_i - \sum_{j=p+1}^n \gamma_j\big)|u|^2 \\ &\geq \Big(q(1 - \frac{1}{4n}) - (n-p)\Big)|u|^2 \\ &\geq \frac{1}{2}|u|^2.\end{aligned} \tag{3.6}$$

The last inequality holds because of $p + q \geq n + 1$. We know ω_{Y_c} is of Poincaré type metric along D on X_c, and α is large enough, by Thorem 3.3 we have

$$H^q(X_c, \Omega^p(\log D) \otimes F) \simeq H^{p,q}_{(2)}(Y_c, F, \omega_{Y_c}, h_{\alpha,\epsilon,\tau}).$$

The inequality (3.6) and Theorem 2.10 show that the vanishing of $H^{p,q}_{(2)}(Y_c, F, \omega_{Y_c}, h_{\alpha,\epsilon,\tau})$. Hence we get the desired vanishing theorem. □

We introduce one important result of Le Potier which enables one to carry vanishing theorems for line bundles over to vector bundles.

Theorem 3.5. [ShSo85, Theorem 5.16] *Let $\pi : E \to X$ be a holomorphic vector bundle on a complex manifold X and let $\mathcal{F}$ be a coherent analytic sheaf on X. Then for all $p, q \geq 0$,*

$$H^q(X, \mathcal{F} \otimes \Omega_X^p \otimes E) \simeq H^q(\mathbb{P}(E^*), \pi^*\mathcal{F} \otimes \Omega^p_{\mathbb{P}(E^*)} \otimes \mathcal{O}_{\mathbb{P}(E^*)}(1)).$$

If X is a weakly pseudoconvex Kähler manifold, E and its dual E^* are vector bundle over it. We know the dual projectivized $\mathbb{P}(E^*)$ is weakly pseudoconvex manifold but not necessary a Kähler one. But if we restrict the vector bundle E^* on a sublevel set X_c and denote it by $E_c^* := E^*|_{X_c}$. Then we know $\mathbb{P}(E_c^*)$ is weakly pseudoconvex Kähler manifold. Indeed, if $\pi_1 : \mathbb{P}(E_c^*) \to X_c$ is the natural projection, there is a tautological hyperplane sunbundle S of π_1^*E over $\mathbb{P}(E_c^*)$ such that $S_{[\xi]} = \xi^{-1}(0) \subset E_x$ for all $\xi \in E_x^* - \{0\}$. The quotient line bundle π_1^*E/S is called the tautological line bundle associated to E and denoted by $\mathcal{O}_{\mathbb{P}(E^*)}(1)$. Therefore there is an exact sequence

$$0 \to S \to \pi_1^*E \to \mathcal{O}_{\mathbb{P}(E^*)}(1) \to 0$$

of vector bundles over $\mathbb{P}(E_c^*)$. Suppose that E is equipped with a Hermitian metric, then the above morphism endows $\mathcal{O}_{\mathbb{P}(E^*)}(1)$ with a quetient metric. The Chern form ω_{E^*} associated to this metric is not necessarily positive on $\mathbb{P}(E^*)$, but its restriction to each fibre $\mathbb{P}(E_x^*)$ is positive. Suppose now that X is Kähler with Kähler form ω. Let ω_c be the restriction on X_c. Since X_c is relatively compact, it is easy to see that for $\lambda \gg 0$, the real closed form of type $(1,1)$

$$\omega = \omega_{E^*} + \lambda \pi_1^* \omega_c$$

is positive on $\mathbb{P}(E_c^*)$. Thus we have this claim. Moreover, if the vector bundle E be positive in the sense of Griffiths, then the line bundle $\mathcal{O}_{\mathbb{P}(E^*)}(1)$ is a positive line bundle, the reader can find the curvature formula in [Dem12b, Chapter V Formula 15.15].

Corollary 3.6. *Let X be a weakly pseudoconvex Kähler manifold of dimension n and D be a simple normal crossing divisor. Suppose that $\pi : E \to X$ is a Nakano positive vector bundle of rank r. Then for each sublevel set X_c*

$$H^q(X_c, \Omega^p(\log D) \otimes E) = 0 \quad \textit{for any} \quad p + q \geq n + r.$$

Proof. Let $\pi_1 : \mathbb{P}(E_c^*) \to X_c$ be the dual projective bundle of E and $\mathcal{O}_{\mathbb{P}(E^*)}(1)$ be the tautological line bundle. According to Le Potier isomorphism Theorem 3.5, we have

$$H^q(X_c, \Omega^p(\log D) \otimes E) \simeq H^q(\mathbb{P}(E_c^*), \Omega^p_{\mathbb{P}(E_c^*)}(\log \pi^*D) \otimes \mathcal{O}_{\mathbb{P}(E^*)}(1)).$$

By the curvature calculation, if E is Nakano positive, then the line bundle $\mathcal{O}_{\mathbb{P}(E^*)}(1)$ is a positive line bundle. On the other hand, it is easy to see that π^*D is also a simple normal divisor. If necessary, we can choose smaller c to shrink X_c. Therefore, we get the desired result. □

Definition 3.7 (k-positive line bundle)**.** Let X be a complex manifold and $F \to X$ be a holomorphic line bundle over X. F is called k-positive ($1 \le k \le n$) if there exists a smooth Hermitian metric h^F on F such that the curvature form $\sqrt{-1}\Theta(F, h^F)$ is semi-positive everywhere and has at least $n-k+1$ positive eigenvalues at every point. A divisor D on X is called k-positive if its associated line bundle $\mathcal{O}(D)$ is k-positive.

Corollary 3.8. *Let D be a k-positive simple normal crossing divisor on an n-dimensional weakly pseudoconvex Kähler manifold X. For each real number c and the corresponding sublevel set X_c, we have the next vanishing theorem,*

$$H^q(X_c, \Omega^p(\log D)) = 0 \quad \text{for any} \quad p+q \ge n+k.$$

4. Global vanishing theorem

One of advantages to study the cohomology groups on weakly pseudoconvex manifolds is that we can investigate the corresponding higher direct images. Let $f : X \to S$ be a proper surjective morphism from a Kähler manifold X to a reduced and irreducible complex space S. Let $W \subset S$ be any Stein open subset, we put $V = f^{-1}(W)$. Then V is a holomorphically convex Kähler manifold. Let $\mathcal{F}$ be a coherent sheaf on V. Then $f^* : H^q(V, \mathcal{F}) \to H^0(W, R^q f_* \mathcal{F})$ is an isomorphism of topological vector space for every $q \ge 0$. From now on, unless otherwise mentioned, X denotes a complex manifold of dimension n. As a direct corollary of Theorem 3.4, we obtain

Corollary 4.1. *Let $f : X \to S$ be a proper holomorphic morphism from a Kähler manifold X onto the reduced and irreducible complex space S. Let D be an simple normal crossing divisor for which $f|_D$ is proper. And let F be a positive holomorphic line bundle on X, then*

$$R^q f_*(\Omega_X^p(\log D) \otimes F) = 0 \quad \text{for any} \quad p+q \ge n+1.$$

On holomorphically convex Kähler manifolds, the global vanishing can be deduced from the local vanishing.

Corollary 4.2. *Let X be a holomorphically convex Kähler manifold and F is a positive line bundle on X. Let D be a simple normal crossing divisor on X. We have*

$$H^q(X, \Omega_X^p(\log D) \otimes F) = 0 \quad \text{for any} \quad p+q \ge n+1.$$

Proof. Let $f : X \to S$ be the Remmert reduction, see Remark 2.13. For the coherent sheaf $\mathcal{G} := \Omega_X^p(\log D) \otimes F$, we have the Leray spectral sequence

$$H^p(S, R^q f_* \mathcal{G}) \Rightarrow H^{p+q}(X, \mathcal{G}).$$

Since S be a normal Stein space, owing to Cartan's theorem, we have the vanishing

$$H^q(S, R^0 f_*(\Omega_X^p(\log D) \otimes F)) = 0$$

for any $q \geq 1$. On the other hand, as Corollary 4.1, the local vanishing implies the vanishing of high direct image sheaf $R^q f_*(\Omega_X^p(\log D) \otimes F) = 0$ for any $p+q \geq n+1$. This two facts yield the vanishing $H^q(X, \Omega_X^p(\log D) \otimes F) = 0$ for any $p+q \geq n+1$. □

Now we focus on the weakly pseudoconvex Kähler manifolds. Firstly we have

Theorem 4.3. *Let X be a weakly pseudoconvex Kähler manifold and F is a positive line bundle on X. Let D be a simple normal crossing divisor on X. We have*

$$H^q(X, \Omega_X^n(\log D) \otimes F) = H^q(X, K_X \otimes \mathcal{O}(D) \otimes F) = 0,$$

for any $q \geq 2$.

At present, we can not prove the global vanishing of $H^1(X, K_X \otimes \mathcal{O}(D) \otimes F)$. We will deal with it later.

Proof. According to Sard's theorem, we can choose a sequence $\{c_v\}_{v=0,1,\cdots}$ of real numbers such that

(1) $c_v < c_{v+1}$ and $\lim_{v\to\infty} c_v = +\infty$;
(2) the bondary ∂X_v of $X_v = \{x \in X : \Phi(x) < c_v\}$ is smooth for any v.

So $\mathfrak{X} = \{X_v\}_{v\geq 0}$ is a covering of X. For any v, we set $\mathfrak{X}_v = \{X_k\}_{k\leq v}$, here k are non negative integers. Then $\mathfrak{X}_v$ is a covering of X_v. By the vanishing Theorem 3.4 on each sublevel set X_v, this covering $\mathfrak{X}$ (resp. $\mathfrak{X}_V$) is the Leray covering of the sheaf $\Omega^n(\log D) \otimes F$ on X (resp. X_v). Therefore we have, for any $q \geq 1$ and $v \geq 0$,

$$H^q(X, \Omega^n(\log D) \otimes F) = \check{H}^q(\mathfrak{X}, \Omega^n(\log D) \otimes F)$$

and

$$H^q(X_v, \Omega^n(\log D) \otimes F) = \check{H}^q(\mathfrak{X}_v, \Omega^n(\log D) \otimes F) = 0.$$

The right cohomology groups are Čech cohomology.

For any $q \geq 1$, the q-cocycle $\sigma \in Z^q(\mathfrak{X}, \Omega^n(\log D) \otimes F)$ and set σ_v be the restriction of σ to $\mathfrak{X}_v$. Then it is obviously that $\sigma_v \in Z^q(\mathfrak{X}_v, \Omega^n(\log D) \otimes F)$. According to the local vanishing, there is a $(q-1)$-cochain $\alpha_v \in C^{q-1}(\mathfrak{X}_v, \Omega^n(\log D) \otimes F)$ such that $\delta\alpha_v = \sigma_v$. The notation δ here is the Čech differential. As an element of $C^{q-1}(\mathfrak{X}_{v-1}, \Omega^n(\log D) \otimes F)$, we have $\delta\alpha_v = \delta\alpha_{v-1}$, and hence $\alpha_v - \alpha_{v-1} \in Z^{q-1}(\mathfrak{X}_{v-1}, \Omega^n(\log D) \otimes F)$.

By assumption $q \geq 2$, so there is a $(q-2)$-cochain $\beta_{v-1} \in C^{q-2}(\mathfrak{X}_{v-1}, \Omega^n(\log D) \otimes F)$ such that $\delta\beta_{v-1} = \alpha_v - \alpha_{v-1}$ on X_{v-1}. We now can define $\alpha \in C^{q-1}(\mathfrak{X}, \Omega^n(\log D) \otimes F)$ as follows, on each X_v,

$$\begin{aligned}\alpha =& \alpha_v - \delta(\sum_{k<v} \beta_k) \\ =& \alpha_v - \delta(\beta_{v-1}) - \delta(\beta_{v-2}) \cdots - \delta(\beta_1).\end{aligned}$$

On X_{v+1}, similarly we have

$$\begin{aligned}\alpha =&\alpha_{v+1} - \delta(\sum_{k<v+1} \beta_k) \\ =&\alpha_{v+1} - \delta(\beta_v) - \delta(\beta_{v-1}) \cdots - \delta(\beta_1).\end{aligned}$$

According to the definition of β_v, we have $\alpha_{v+1} - \delta(\beta_v) = \alpha_v$ on X_v. It follows that α is well defined. Finally, on each X_v,

$$\delta\alpha = \delta\alpha_v - \delta\delta(\sum_{k<v} \beta_k) = \delta\alpha_v = \sigma_v.$$

Hence we have $\delta\alpha = \sigma$ and this yields the vanishing of cohomology groups. □

Now we give the proof of the vanishing of $H^1(X, K_X \otimes \mathcal{O}_X(D) \otimes F)$ on weakly pseudoconvex Kähler manifold. The key method is a Runge-type approximation which have been used in [Naka70, Naka73, Kaza73, Take81, OhTa81]. For any real number pair $c_1 < c_2$, let $X_1 := \{x \in X : \Phi(x) < c_1\}$ and $X_2 := \{x \in X : \Phi(x) < c_2\}$. Set $Y_1 := X_1 \backslash D$ and $Y_2 := X_2 \backslash D$. As soon as we have the fixed pair (X_1, X_2) and (Y_1, Y_2). We can choose any smooth Hermitian metric $\{h_1 = e^{-\phi_1}\}$ on $\mathcal{O}(D)$, and the canonical singular metric $\{h_2 = e^{-\phi_2}\}$ on $\mathcal{O}(D)$, here locally $\phi_2 = \sum_i \log |g_i|^2$, where g_i be the generator of D_i. Let (F, h^F) be the fixed positive line bundle, we construct a new metric h_δ on the $F_D := \mathcal{O}(D) \otimes F$,

$$h_\delta := h^F \prod_{i=1} \left(\log^2(\|\sigma_i\|_i^2)\right)^{\frac{\kappa}{2}} e^{-\delta\phi_1} e^{-(1-\delta)\phi_2} \quad (0 < \delta < 1).$$

Here the constant κ and δ are to be determined later. According to this construction, we know the associated multiplier ideal sheaf $\mathscr{I}(h_\delta) = \mathcal{O}_{X_2}$ on X_2 because $0 < \delta < 1$ and the logarithmic part does not affect the integration.

On Y_2, h_δ is smooth and the curvature forms

$$\sqrt{-1}\Theta_{F_D, h_\delta} = \sqrt{-1}\Theta_{F, h^F} + (-\kappa\sqrt{-1}\partial\overline{\partial} \log(\log^2 \|\sigma_i\|_i^2)) + \sqrt{-1}\delta\partial\overline{\partial}\phi_1.$$

Set $L^{n,0}(Y_2, F_D, h_\delta)$ be the set of F_D-valued $(n, 0)$-form on Y_2 with a finite L^2 norms with respect to h_δ, this norm is independent on Kähler metric since we are focusing on the $(n, 0)$-forms. We Set $\mathcal{A}^{n,0}(Y_2, F_D, h_\delta) := \ker \overline{\partial} \cap L^{n,0}(Y_2, F_D, h_\delta)$ and moreover we have the isomorphism

$$H^0(X_2, K_X \otimes F \otimes \mathcal{O}(D)) = H^0(X_2, K_X \otimes F \otimes \mathcal{O}(D) \otimes \mathscr{I}(h_\delta)) \simeq \mathcal{A}^{n,0}(Y_2, F_D, h_\delta)$$

because of the L^2 extension property of holomorphic $(n, 0)$-forms. On X_1, we have the similar isomorphism. We define $\mathcal{A}^{n,0}(\overline{Y}_1, F_D, h_\delta)$ the set of holomorphic $(n, 0)$-forms with values in the bundle $F \otimes \mathcal{O}(D)$ in the neighborhoods of Y_1 in Y_2 (not the neighborhoods in X_2). Similarly, we denote by $H^0(\overline{X}_1, K_X \otimes F \otimes \mathcal{O}(D))$ the set of holomorphic section of $K_X \otimes F \otimes \mathcal{O}(D)$ in the neighborhoods of X_1 in X_2. According to the above isomorphism, we have $\mathcal{A}^{n,0}(\overline{Y}_1, F_D, h_\delta) = H^0(\overline{X}_1, K_X \otimes F \otimes \mathcal{O}(D))$. Our key step is to show that the restriction map

$$\mathcal{A}^{n,0}(Y_2, F_D, h_\delta) \to \mathcal{A}^{n,0}(\overline{Y}_1, F_D, h_\delta).$$

has the dense image with respect to the L^2 norms.

Construction 4.4 (Smooth increasing convex function)**.** We take a $\mathcal{C}^\infty$ increasing convex function $\tau(t)$ such that:

(1) $\tau(t) : (-\infty, +\infty) \to (-\infty, +\infty)$,
(2) $\tau(t) = 0$ if $t \leq \frac{1}{c_2 - c_1}$ and $\tau(t) > 0$ when $t > \frac{1}{c_2 - c_1}$,
(3) $\int\limits_0^{+\infty} \sqrt{\tau''(t)}dt = +\infty$.

We set $\Psi = \tau(\frac{1}{c_2 - \Phi})$, it is a psh exhaustion function on X_2 and $\Psi \equiv 0$ on X_1 by the construction.

Construction 4.5. For each non-negative integers $m \geq 0$, we define new metric on $F_D = F \otimes \mathcal{O}(D)$ by

$$h_{\delta_1} := h_\delta e^{-\Psi},$$
$$h_{\delta_m} := h_\delta e^{-m\Psi}.$$

We define a complete Kähler metric $\overline{\omega}$ on Y_2 by

$$\overline{\omega} := \omega_{c_2,p} + \sqrt{-1}\partial\overline{\partial}\Psi.$$

Here $\omega_{c_2,p}$ is the Poincaré type metric along D, recall that

$$\omega_{c_2,p} = k_{c_2}\omega_{c_2} - \frac{1}{2}\sum_i \sqrt{-1}\partial\overline{\partial}\log\log^2\|\sigma_i\|_i^2$$

just like there in Definition 3.1. We choose the big positive constant k_{c_2} in order to ensure $\omega_{c_2,p}$ is positive on Y_2. By Proposition 2.9 and the above Construction 4.4, one knows that $\overline{\omega}$ is complete on Y_2.

Remark 4.6. We define a new curvature form $\sqrt{-1}\Theta_m := \sqrt{-1}\Theta_{F_D,h_\delta} + \sqrt{-1}\partial\overline{\partial}m\Psi$. We want to compare it with $\overline{\omega}$. One obtains

$$\overline{\omega} = k_{c_2}\omega_{c_2} - \frac{1}{2}\sum_i \sqrt{-1}\partial\overline{\partial}\log\log^2\|\sigma_i\|_i^2 + \sqrt{-1}\partial\overline{\partial}\Psi, \tag{4.1}$$

and

$$\sqrt{-1}\Theta_m = \sqrt{-1}\Theta_{F,h^F} - \kappa\sqrt{-1}\partial\overline{\partial}\log(\log^2\|\sigma_i\|_i^2) + \sqrt{-1}\delta\partial\overline{\partial}\phi_1 + \sqrt{-1}\partial\overline{\partial}m\Psi. \tag{4.2}$$

We know $\sqrt{-1}\Theta_{F,h^F}$ is positive with respect to ω_{c_2}. So we can arrange κ, δ small enough such that the eigenvalues of $\sqrt{-1}\Theta_m$ with respect to $\overline{\omega}$ are all positive on the whole Y_2. More specifically, at each point $x \in Y_2$, we may choose a coordinate system which diagonalize simultaneously the forms $\overline{\omega}$ and $\sqrt{-1}\Theta_m$, in such a way that

$$\overline{\omega}(x) = \sqrt{-1} \sum_{1 \le j \le n} dz_j \wedge d\overline{z}_j, \quad \sqrt{-1}\Theta_m(x) = \sqrt{-1} \sum_{1 \le j \le n} \gamma_j dz_j \wedge d\overline{z}_j.$$

We want there exists a positive constant ϵ such that $\gamma_j > \epsilon$ hold on Y_2 for each γ_j.

Definition 4.7 (Inner product)**.** For any non-negative integer m and any $\varphi, \psi \in L^{n,0}(Y_2, F_D, h_{\delta_m})$, we define the inner product

$$(\varphi, \psi)_m := \int_{Y_2} \langle \varphi, \psi \rangle_{\overline{\omega}} h_{\delta_m} dV = \int_{Y_2} \langle \varphi, \psi \rangle_{\overline{\omega}} h_\delta e^{-m\Psi} dV.$$

and $\|\varphi\|_m^2 = (\varphi, \varphi)_m$. We denote the adjoint operator of $\overline{\partial}$ in $L^{n,q}(Y_2, F_D, h_{\delta_m})$ by $\overline{\partial}_m^*$.

The next lemma is very important for our proof.

Lemma 4.8 (Uniform estimate)**.** *There exist a positive constant M which is independent to m such that for any $m \ge 0$ and $0 \le q \le n$, we have the estimate*

$$\|\varphi\|_m^2 \le M(\|\overline{\partial}\varphi\|_m^2 + \|\overline{\partial}_m^*\varphi\|_m^2)$$

provided $\varphi \in D_{\overline{\partial}}^{n,q} \cap D_{\overline{\partial}_m^*}^{n,q} \subset L^{n,q}(Y_2, F_D, h_{\delta_m})$. Here $D_{\overline{\partial}}^{n,q}$ is the domain of definition of $\overline{\partial}$ in $L^{n,q}(Y_2, F_D, h_{\delta_m})$, and $D_{\overline{\partial}_m^*}^{n,q}$ is similar.

Proof. Since $\overline{\omega}$ is a complete Kähler metric on Y_2, the classical Bochner–Kodaira–Nakano identity shows

$$\Delta'' = \Delta' + [i\Theta_m, \Lambda_{\overline{\omega}}].$$

If $\varphi \in \mathcal{C}_0^\infty(Y_2, \Lambda^{n,q}T^*Y \otimes F_D)$ be a smooth compact supported F_D-valued (n,q)-form. We have

$$\|\overline{\partial}\varphi\|_m^2 + \|\overline{\partial}_m^*\varphi\|_m^2 \ge \int_{Y_2} \langle [i\Theta_m, \Lambda_{\overline{\omega}}]\varphi, \varphi \rangle_{\overline{\omega}} h_\delta e^{-m\Psi} dV. \tag{4.3}$$

By the above Remark 4.6, we have

$$\overline{\omega} = k_{c_2}\omega_{c_2} - \frac{1}{2} \sum_i \sqrt{-1}\partial\overline{\partial} \log \log^2 \|\sigma_i\|_i^2 + \sqrt{-1}\partial\overline{\partial}\Psi, \tag{4.4}$$

and

$$\sqrt{-1}\Theta_m = \sqrt{-1}\Theta_{F,h^F} - \kappa\sqrt{-1}\partial\overline{\partial} \log(\log^2 \|\sigma_i\|_i^2) + \sqrt{-1}\delta\partial\overline{\partial}\phi_1 + \sqrt{-1}\partial\overline{\partial} m\Psi.$$

We can diagonalize simultaneously the Hermitian forms $\overline{\omega}$ and $\sqrt{-1}\Theta_m$. We know $\sqrt{-1}\Theta_{F,h^F}$ is positive with respect to ω_{c_2}, i.e., there exists a constant ϵ_1 such that all eigenvalues of $\sqrt{-1}\Theta_{F,h^F}$ with respect to ω_{c_2} are bigger than ϵ_1 on Y_2. If we let $\kappa = \frac{\epsilon_1}{4k_{c_2}}$, then

$$\begin{aligned}\sqrt{-1}\Theta_m &= \sqrt{-1}\Theta_{F,h^F} - \frac{\epsilon_1}{4k_{c_2}}\sqrt{-1}\partial\overline{\partial}\log(\log^2\|\sigma_i\|_i^2) + \sqrt{-1}\delta\partial\overline{\partial}\phi_1 + \sqrt{-1}\partial\overline{\partial}m\Psi \\ &\geq \epsilon_1\omega_{c_2} - \frac{\epsilon_1}{4k_{c_2}}\sqrt{-1}\partial\overline{\partial}\log(\log^2\|\sigma_i\|_i^2) + \sqrt{-1}\delta\partial\overline{\partial}\phi_1 + \sqrt{-1}\partial\overline{\partial}m\Psi \\ &= \frac{\epsilon_1}{2k_{c_2}}(k_{c_2}\omega_{c_2} - \frac{1}{2}\sum_i \sqrt{-1}\partial\overline{\partial}\log\log^2\|\sigma_i\|_i^2) + (\frac{\epsilon_1}{2}\omega_{c_2} + \sqrt{-1}\delta\partial\overline{\partial}\phi_1) + \sqrt{-1}\partial\overline{\partial}m\Psi.\end{aligned}$$

Compare this with formula (4.4), we can arrange δ small enough such that the eigenvalues of $\sqrt{-1}\Theta_m$ with respect to $\overline{\omega}$ are all positive on the whole Y_2. Hence there exist a positive constant M_0 on Y_2, independent to m, so that

$$\langle[i\Theta_m, \Lambda_{\overline{\omega}}]\varphi, \varphi\rangle_{\overline{\omega}} \geq M_0|\varphi|^2.$$

So if we plug this back into formula (4.3) above, as a consequence, we get the desired uniform estimate

$$\|\varphi\|_m^2 \leq M(\|\overline{\partial}\varphi\|_m^2 + \|\overline{\partial}_m^*\varphi\|_m^2)$$

for $\varphi \in \mathcal{C}_0^\infty(Y_2, \lambda^{n,q}T^*Y \otimes F_D)$. Since the metric $\overline{\omega}$ is complete, the above estimate still holds provided $\varphi \in D_{\overline{\partial}}^{n,q} \cap D_{\overline{\partial}_m^*}^{n,q} \subset L^{n,q}(Y_2, F_D, h_{\delta_m})$. □

Lemma 4.9 (Approximation lemma)**.** *If $\varphi \in \mathcal{A}^{n,0}(\overline{Y}_1, F_D, h_\delta)$, then for any $\varepsilon > 0$, there exist a $\tilde{\varphi} \in \mathcal{A}^{n,0}(Y_2, F_D, h_\delta)$ such that $\|\tilde{\varphi}|_{Y_1} - \varphi\|_0^2 < \epsilon$.*

Proof. It suffices to show that if $u \in \overline{\mathcal{A}^{n,0}(\overline{Y}_1, F_D, h_\delta)} \subset L^{n,0}(Y_1, F_D, h_\delta)$ and

$$(u, f)_{Y_1} = \int_{Y_1}\langle u, f\rangle_{\overline{\omega}} h_\delta dV = 0 \tag{4.5}$$

for any $f \in \mathcal{A}^{n,0}(Y_2, F_D, h_\delta)$. Then we have

$$(u, g)_{Y_1} = \int_{Y_1}\langle u, f\rangle_{\overline{\omega}} h_\delta dV = 0 \tag{4.6}$$

provided $g \in \mathcal{A}^{n,0}(\overline{Y}_1, F_D, h_\delta)$.

We change the definition of u by setting $u = 0$ on $Y_2\backslash Y_1$ and remain unchanged on Y_1, we denote it by u'. Since $\Psi \equiv 0$ on X_1 and therefore the above equality (4.5) implies

$$u' \perp \{L^{n,0}(Y_2, F_D, h_{\delta_m}) \cap \ker\overline{\partial}\}$$

for each m. Then we obtain $u' \in \overline{\mathrm{Im}\overline{\partial}_m^*} \subset L^{n,0}(Y_2, F_D, h_{\delta_m})$. According to the uniform estimate Lemma 4.8, we know

$$\overline{\mathrm{Im}\overline{\partial}_m^*} = \mathrm{Im}\overline{\partial}_m^*.$$

According to [Hörm65], we acquire $u' = \overline{\partial}_m^* v_m$ for some $v_m \in L^{n,1}(Y_2, F_D, h_{\delta_m})$ with estimate

$$\|v_m\|_m^2 \leq C_1\|u'\|_m^2 \leq C_1\|u'\|_0^2.$$

We set $w_m = e^{-m\Psi} v_m$ which yields $\overline{\partial}^* w_m = \overline{\partial}_m^* v_m = u'$. Thus one obtain

$$\|w_m\|_0^2 \le \|w_m\|_{-m}^2 = \|v_m\|_m^2 \le C_1 \|u'\|_0^2.$$

Hence $\{w_m\}$ has a subsequence which is weakly convergent in $L^{n,1}(Y_2, F_D, h_\delta)$, we denote the weak limit by w. On the other hand, for every $\epsilon > 0$, by the inequality $\|w_m\|_{-m}^2 \le C_1 \|u'\|_0^2$, we have the inequality

$$\int_{\{x \in Y_2 : \Psi > \epsilon\}} e^{m\Psi} \langle w_m, w_m \rangle_{\overline{\omega}} \, dV \le C_1 \|u'\|_0^2.$$

Thus we have

$$e^{m\epsilon} \int_{\{x \in Y_2 : \Psi > \epsilon\}} \langle w_m, w_m \rangle_{\overline{\omega}} \, dV \le C_1 \|u'\|_0^2$$

for each m. It follows that $\int_{\{x \in Y_2 : \Psi > \epsilon\}} \langle w_m, w_m \rangle_{\overline{\omega}} \, dV$ tends to zero and hence $w_m \to 0$ almost everywhere in $\{x \in Y_2 : \Psi > \epsilon\}$. As a consequence, the weak limit $w = 0$ on $\{x \in Y_2 : \Psi > \epsilon\}$ for every $\epsilon > 0$. In summary, we have

$$\mathrm{supp} w \subseteq \overline{Y}_1 \quad \text{and} \quad \overline{\partial}^* w = u'.$$

For any open neighborhood H_1 of X_1 in X_2. We can take a $\mathcal{C}^\infty$ function ζ on X_2 satisfying $0 \le \zeta \le 1$, supp $\zeta \subseteq H_1$ and $\zeta = 1$ on X_1. For these $g \in \mathcal{A}^{n,0}(\overline{Y}_1, F_D, h_\delta)$, we still have $\overline{\partial}(\zeta g) = 0$ on Y_1. And we arrange H_1 very close to X_1 such that g is defined on $H_1 \backslash D$. Hence ζg is defined on Y_2 and obviously belong in $L^{n,0}(Y_2, F_D, h_\delta)$. So

$$\begin{aligned}
(u, g)_{Y_1} &= \int_{Y_1} \langle u, g \rangle_{\overline{\omega}} \, dV = \int_{Y_2} \langle u', \zeta g \rangle_{\overline{\omega}} \, dV \\
&= \int_{Y_2} \left\langle \overline{\partial}^* w, \zeta g \right\rangle_{\overline{\omega}} dV \\
&= \int_{Y_2} \langle w, \overline{\partial}(\zeta g) \rangle_{\overline{\omega}} \, dV \\
&= 0.
\end{aligned}$$

This confirms the equality (4.6) and therefore completes the proof of Lemma 4.9. □

Definition 4.10 (Semi-norms)**.** Let h be any smooth metric of $F_D = F \otimes \mathcal{O}(D)$ on the whole X. For a fixed real number c, the sublevel set X_c is relatively compact in X. Let K be a compact subset of X_c, we set

$$|\varphi|_K := \sup_{x \in K} \sqrt{\langle \varphi, \varphi \rangle_\omega h(x)}$$

for $\varphi \in H^0(X_c, K_X \otimes F_D)$, where $\langle \varphi, \varphi \rangle_\omega h(x)$ be the pointwise norms and it is independent to ω because φ is a $(n, 0)$-form.

We can find two positive constants M_1 and M_2 such that $M_1 \le h \le M_2 h_\delta$ on Y_c, here M_1 and M_2 are constants depend on Y_c. So using Cauchy's integral formula in each local coordinate U_i with $U_i \cap K \neq \emptyset$, we have

$$\begin{aligned}|\varphi|^2_{U_i\cap K} &\le M_3 \int_{U_i\cap K} |\varphi|^2 h dV \\ &\le M_2 M_3 \int_{U_i \cap K} |\varphi|^2 h_\delta dV \\ &\le M_2 M_3 \|\varphi\|_0^2.\end{aligned}$$

This shows we can find a positive constant M depends on X_c such that

$$|\varphi|_K \le M \|\varphi\|_0.$$

In summary, we get the desired approximation.

Lemma 4.11. *Let $X_1 \subset X_2$ be the pair of sublevel set. Then for any holomorphic section $\varphi \in H^0(\overline{X}_1, K_X \otimes F \otimes \mathcal{O}(D))$ and for any $\epsilon > 0$, there exists a section $\tilde{\varphi} \in H^0(X_2, K_X \otimes F \otimes \mathcal{O}(D))$ such that $|\tilde{\varphi} - \varphi|_{X_1} < \epsilon$.*

Now we can proof the vanishing of $H^1(X, K_X \otimes \mathcal{O}_X(D) \otimes F)$.

Theorem 4.12. *Let X be a weakly pseudoconvex Kähler manifold and F is a positive line bundle on X. Let D be a simple normal crossing divisor on X. We have*

$$H^1(X, K_X \otimes \mathcal{O}_X(D) \otimes F) = 0.$$

Proof. Recall we have proved that for any real number c, the local vanishing of $H^1(X, K_X \otimes \mathcal{O}_X(D) \otimes F)$, i.e., $H^1(X_c, K_X \otimes F \otimes \mathcal{O}(D)) = 0$. Let $\mathfrak{X} = \{X_v\}_{v\ge 0}$ and $\mathfrak{X}_v = \{X_k\}_{k\le v}$ be the covering of X and X_v respectively. Moreover $\mathfrak{X}$ and $\mathfrak{X}_v$ are the Leray covering for the sheaf $K_X \otimes F \otimes \mathcal{O}(D)$ on X and X_v respectively. We have

$$H^1(X, K_X \otimes F \otimes \mathcal{O}(D)) = \check{H}^1(\mathfrak{X}, K_X \otimes F \otimes \mathcal{O}(D)),$$

and

$$H^1(X_v, K_X \otimes F \otimes \mathcal{O}(D)) = \check{H}^1(\mathfrak{X}_v, K_X \otimes F \otimes \mathcal{O}(D)) = 0,$$

for each v.

For any 1-cocycle $\sigma \in Z^1(\mathfrak{X}, K_X \otimes \mathcal{O}(D) \otimes F)$, let σ_v be the restriction of σ to X_v. Then it is obviously that $\sigma_v \in Z^1(\mathfrak{X}_v, K_X \otimes \mathcal{O}(D) \otimes F)$. According to the local vanishing, there is a 0-cochain $\alpha_v \in C^0(\mathfrak{X}_v, K_X \otimes \mathcal{O}(D) \otimes F)$ such that $\delta\alpha_v = \sigma_v$. As an element of $C^0(\mathfrak{X}_{v-1}, K_X \otimes \mathcal{O}(D) \otimes F)$, we have $\delta\alpha_v = \delta\alpha_{v-1}$, and hence $\alpha_v - \alpha_{v-1} \in Z^0(\mathfrak{X}_{v-1}, K_X \otimes \mathcal{O}(D) \otimes F)$, i.e., $\alpha_v - \alpha_{v-1} \in \Gamma(X_{v-1}, K_X \otimes \mathcal{O}(D) \otimes F)$. Now by the approximation Lemma 4.11, for any $\epsilon > 0$ we can find a $\gamma \in \Gamma(X_v, K_X \otimes \mathcal{O}(D) \otimes F)$ so as to

$$|\alpha_v - \alpha_{v-1} - \gamma|_{\overline{X}_{v-2}} < \epsilon.$$

Therefore, inductively we have a sequence $\{\lambda_v\}_{v\ge 1}$ such that

(1) $\lambda_v \in C^0(\mathfrak{X}_v, K \otimes \mathcal{O}(D) \otimes F)$ and $\lambda_1 = \alpha_1$,

(2) $\delta\lambda_v = \sigma_v$,

(3) $|\lambda_{v+1} - \lambda_v|_{\overline{X}_{v-1}} < \frac{1}{2^v}$.

As a consequence, for any v,

$$\lim_{u \geq v} \lambda_u = \lambda_v + \sum_{k \geq v} (\lambda_{k+1} - \lambda_k)$$

defines an element of $C^0(\mathfrak{X}_v, K \otimes \mathcal{O}(D) \otimes F)$. And

$$\lim_{u \geq v+1} \lambda_u = \lambda_{v+1} + \sum_{k \geq v+1} (\lambda_{k+1} - \lambda_k)$$

defines the same element as $\lim_{u \geq v} \lambda_u$ when restrict to $C^0(\mathfrak{X}_v, K_X \otimes \mathcal{O}(D) \otimes F)$. Thus we can define an element λ of $C^0(\mathfrak{X}, K_X \otimes \mathcal{O}(D) \otimes F)$ by $\lambda = \lim_{v \to \infty} \lambda_v$. For any v,

$$\delta(\lim_{u \geq v} \lambda_u) = \lim_{u \geq v} \delta\lambda_u = \sigma_v.$$

Hence we have $\delta\lambda = \sigma$ and the proof is complete. □

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