ON THE NUMBER OF p-HYPERGEOMETRIC SOLUTIONS OF *KZ* EQUATIONS

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Abstract. It is known that solutions of the *KZ* equations can be written in the form of multidimensional hypergeometric integrals. In 2017 in a joint paper of the author with V. Schechtman the construction of hypergeometric solutions was modified, and solutions of the *KZ* equations modulo a prime number p were constructed. These solutions modulo p, called the p-hypergeometric solutions, are polynomials with integer coefficients. A general problem is to determine the number of independent p-hypergeometric solutions and understand the meaning of that number.

In this paper we consider the *KZ* equations associated with the space of singular vectors of weight $n-2r$ in the tensor power $W^{\otimes n}$ of the vector representation of $\mathfrak{sl}_2$. In this case the hypergeometric solutions of the *KZ* equations are given by r-dimensional hypergeometric integrals. We consider the module of the corresponding p-hypergeometric solutions, determine its rank, and show that the rank equals the dimension of the space of suitable square integrable differential r-forms.

Contents

1. Introduction 2
2. $\mathfrak{sl}_2$ *KZ* equations 3
2.1. Definition of equations 3
2.2. Gauge transformation 4
2.3. Solutions of *KZ* equations over $\mathbb{C}$ 4
2.4. p-integrals 5
2.5. *KZ* equations modulo p 5
2.6. Module of p-hypergeometric solutions 6
2.7. More general choice of the numbers M and c 6
3. Pairs (p, q) of type 1 7
3.1. Numbers M and c 7
3.2. Main Theorem 8
4. Leading term and leading index 9
4.1. Leading term of a polynomial 9
4.2. Leading index of a vector of $W^{\otimes n}[n-2r]_p$ 10
4.3. Index $\{m_1, \dots, m_r\}$ 10

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4.4. Special summand 10
4.5. Proof of Theorem 3.5 12
4.6. Leading terms and eigenvectors 12
5. Solutions and a Cartier map 13
5.1. Square integrability criterion 13
5.2. Square integrable differential r-forms 14
5.3. Schur polynomials 15
5.4. Cartier map 16
References 17

1. Introduction

The Knizhnik-Zamolodchikov (KZ) equations is a system of differential equations for conformal blocks in conformal field theory, see [KZ]. Versions of KZ equations appear in mathematical physics, representation theory, enumerative geometry, algebraic geometry, theory of special functions, see for example [EFK, MO, B].

It is known that solutions of the KZ equations can be written in the form of multidimensional hypergeometric integrals, see [SV1]. Relatively recently, the construction of these hypergeometric solutions was modified, and solutions of the KZ equations modulo a prime number p were constructed in [SV2]. These solutions modulo p, called the p-hypergeometric solutions, are vector-valued polynomials with integer coefficients. The general problem is to understand how these polynomials reflect the remarkable properties of the KZ equations and their hypergeometric solutions.

In this paper we address the particular problem to determine the number of independent p-hypergeometric solutions and understand the meaning of that number. We consider the KZ equations associated with the space of singular vectors of weight $n-2r$ in the tensor power $W^{\otimes n}$ of the vector representation of $\mathfrak{sl}_2$. In this case the hypergeometric solutions of the KZ equations are given by r-dimensional hypergeometric integrals. We consider the module of the corresponding p-hypergeometric solutions, determine its rank, and show that the rank equals the dimension of the space of suitable square integrable differential r-forms.

The KZ equations depend on a parameter $q \in \mathbb{C}^\times$. In this paper we assume that q is a prime number less than p, and the pair (p, q) satisfies certain conditions (the pair is of type 1).

On p-hypergeometric solutions and, more generally, on the solutions of the KZ equations modulo p^s see [SlV, V4, V5, V6, V7, V8, VZ1, VZ2].

In Section 2 we define the KZ equations, recall the construction of solutions in the form of hypergeometric integrals and the construction of p-hypergeometric solutions. We also introduce the module of p-hypergeometric solutions. The main result of the paper is Theorem 3.5, formulated in Section 3 and proved in Section 4. Theorem 3.5 determines the rank of the module of p-hypergeometric solutions.

In Section 5 we construct a suitable Cartier map, which relates the hypergeometric solutions and p-hypergeometric solutions. As a result of this construction, we interpret the rank of the module of p-hypergeometric solutions as the dimension of some vector space of square integrable differential r-forms on $\mathbb{P}^r$, the r-th direct power of the complex projective line. Previously this result was known for $r = 1$, see [SlV].

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2. $\mathfrak{sl}_2$ *KZ* equations

2.1. Definition of equations. Consider the complex Lie algebra $\mathfrak{sl}_2$ with generators e, f, h and relations $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$. Consider the complex vector space W with basis w_1, w_2 and the $\mathfrak{sl}_2$-action,

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The $\mathfrak{sl}_2$-module $W^{\otimes n}$ has a basis labeled by subsets $J \subset \{1, \dots, n\}$,

$$V_J = w_{j_1} \otimes \dots \otimes w_{j_n},$$

where $j_i = 1$ if $j_i \notin J$ and $j_i = 2$ if $j_i \in J$.

Consider the weight decomposition of $W^{\otimes n}$ into eigenspaces of h, $W^{\otimes n} = \sum_{r=0}^n W^{\otimes n}[n-2r]$. The vectors V_J with $|J| = r$ form a basis of $W^{\otimes n}[n-2r]$. Denote $\mathcal{J}_r$ the set of all r-element subsets of $\{1, \dots, n\}$.

Define the space of singular vectors of weight $n - 2r$,

$$\operatorname{Sing} W^{\otimes n}[n-2r] = \{w \in W^{\otimes n}[n-2r] \mid ew = 0\}.$$

This space is nonempty if and only if $n \geqslant 2r$. We assume that $n \geqslant 2r$. Then

$$\dim \operatorname{Sing} W^{\otimes n}[n-2r] = \dim W^{\otimes n}[n-2r] - \dim W^{\otimes n}[n-2r+2] = \binom{n}{r} - \binom{n}{r-1}.$$

Let $w = \sum_{J \in \mathcal{J}_r} c_J V_J \in W^{\otimes n}[n-2r]$. Then $w \in \operatorname{Sing} W^{\otimes n}[n-2r]$, if and only if its coefficients satisfy the system of linear equations labeled by $r-1$-element subsets $K \subset \{1, \dots, n\}$,

$$\sum_{j \notin K} c_{K \cup \{j\}} \;=\; 0. \tag{2.1}$$

Define the Casimir element $\Omega = \frac{1}{2} h \otimes h + e \otimes f + f \otimes e \,\in\, \mathfrak{sl}_2 \otimes \mathfrak{sl}_2$, and the linear operators on $W^{\otimes n}$ depending on parameters $z = (z_1, \dots, z_n)$,

$$H_m(z) = \sum_{\substack{j=1 \\ j \neq m}}^{n} \frac{\Omega_{mj}}{z_m - z_j}\,, \qquad m = 1, \dots, n,$$

where $\Omega_{mj} : W^{\otimes n} \to W^{\otimes n}$ is the Casimir operator acting in the mth and jth tensor factors. The operators $H_m(z)$ are called the Gaudin Hamiltonians. Denote

$$\partial_m = \frac{\partial}{\partial z_m}\,, \qquad m = 1, \dots, n.$$

For any nonzero number $q \in \mathbb{C}^\times$ and $1 \leqslant m, l \leqslant n$, we have

$$[q\partial_m - H_m(z_1, \dots, z_n), q\partial_l - H_l(z_1, \dots, z_n)] = 0, \tag{2.2}$$

and for any $x \in \mathfrak{sl}_2$ and $m = 1, \dots, n$, we have

$$[H_m(z_1, \dots, z_n), x \otimes 1 \otimes \dots \otimes 1 + \dots + 1 \otimes \dots \otimes 1 \otimes x] = 0. \tag{2.3}$$

The system of differential equations

$$(q\partial_m - H_m(z_1,\dots,z_n))\tilde I(z_1,\dots,z_n), \qquad m=1,\dots,n, \tag{2.4}$$

on a $W^{\otimes n}$-valued function $\tilde I(z_1,\dots,z_n)$ is called the *KZ* equations, see [KZ, EFK].

By property (2.3) the Gaudin Hamiltonians preserve every space $\operatorname{Sing} W^{\otimes n}[n-2r]$. Hence the system of *KZ* equations can be considered with values in any particular space $\operatorname{Sing} W^{\otimes n}[n-2r]$.

2.2. **Gauge transformation.** If $\tilde I(z)$ satisfies the *KZ* equations (2.4), then the function $I(z)$, defined by

$$\tilde I(z) = I(z)\prod_{1\leqslant i<j\leqslant n}(z_i-z_j)^{1/2q}\,, \tag{2.5}$$

satisfies the equations

$$\Big(q\partial_m - \sum_{\substack{j=1\\ j\neq m}}^{n}\frac{\Omega_{mj}-1/2}{z_m-z_j}\Big)I(z_1,\dots,z_n), \qquad m=1,\dots,n, \tag{2.6}$$

which we also call the *KZ* equations.

The linear operator $\Omega-\frac12 : W^{\otimes 2}\to W^{\otimes 2}$ acts as follows:

$$\begin{aligned} &w_1\otimes w_1\mapsto 0, && w_2\otimes w_2\mapsto 0,\\ &w_1\otimes w_2\mapsto -w_1\otimes w_2+w_2\otimes w_1, && w_2\otimes w_1\mapsto w_1\otimes w_2-w_2\otimes w_1. \end{aligned} \tag{2.7}$$

We shall consider the *KZ* equations (2.6) with values in $\operatorname{Sing} W^{\otimes n}[n-2r]$ over the field of complex numbers, and we shall also consider the *KZ* equations (2.6) modulo p, where p is a prime number.

2.3. **Solutions of *KZ* equations over $\mathbb{C}$.** Denote $t=(t_1,\dots,t_k)$. Define the master function

$$\Phi(t,z) = \prod_{1\leqslant i\leqslant j\leqslant r}(t_i-t_j)^{2/q}\prod_{s=1}^{n}\prod_{i=1}^{r}(t_i-z_s)^{-1/q}. \tag{2.8}$$

For any function $F(t_1,\dots,t_r)$, denote $\operatorname{Sym}_t[F(t_1,\dots,t_r)] = \sum_{\sigma\in S_r}F(t_{\sigma_1},\dots,t_{\sigma_r})$. For $J=\{j_1,\dots,j_r\}\in\mathcal{J}_r$ define the weight function

$$W_J(t,z) = \operatorname{Sym}_t\Big[\prod_{i=1}^{r}\frac{1}{t_i-z_{j_i}}\Big].$$

For example,

$$W_{\{3\}} = \frac{1}{t_1-z_3}, \qquad W_{\{4,5\}} = \frac{1}{t_1-z_4}\frac{1}{t_2-z_5} + \frac{1}{t_2-z_4}\frac{1}{t_1-z_5}\,.$$

The function

$$W(t,z) = \sum_{J\in\mathcal{J}_r} W_J(t,z)V_J \tag{2.9}$$

is called the $W^{\otimes n}[n-2r]$-weight vector-function.

Consider the $W^{\otimes n}[n-2r]$-valued function

$$I^{(\gamma)}(z_1,\dots,z_n)=\int_{\gamma(z)}\Phi(t,z)\,W(t,z)\,dt_1\wedge\dots\wedge dt_r, \tag{2.10}$$

where $\gamma(z)$ in $\{z\}\times\mathbb{C}^r_t$ is a horizontal family of r-dimensional cycles of the twisted homology defined by the multivalued function $\Phi(t,z)$, see [CF, SV1, DJMM, V1, V3]. The cycles $\gamma(z)$ are r-dimensional analogs of Pochhammer double loops.

Theorem 2.1. *The function $I^{(\gamma)}(z)$ takes values in* $\operatorname{Sing}W^{\otimes n}[n-2r]$ *and satisfies the KZ equations* (2.6).

This theorem and its generalizations can be found, for example, in [CF, DJMM, SV1].

The solutions in Theorem 2.1 are called the hypergeometric solutions of the *KZ* equations.

Theorem 2.2 ([V1, Theorem 12.5.5])**.** *If $q\in\mathbb{C}^\times$ is generic, then all solutions of the KZ equations* (2.6) *have this form.*

For special values of the parameter q the space of the hypergeometric solutions of the *KZ* equations may span only a proper subspace of the space of all solutions, see in [FSV1, FSV2] the discussion of the relations of this subspace of solutions and conformal blocks in conformal field theory.

2.4. **p-integrals.** Let p be a positive integer. Let $f(t_1,\dots,t_k)=\sum_{d_1,\dots,d_k}c_{d_1,\dots,d_k}t_1^{d_1}\dots t_k^{d_k}$ be a polynomial. Let $l=(l_1,\dots,l_r)\in\mathbb{Z}^r_{>0}$. The coefficient $c_{l_1p-1,\dots,l_rp-1}$ will be called the p-integral of $f(t_1,\dots,t_k)$ over the cycle $\{l_1,\dots,l_r\}_p$ and is denoted by

$$\int_{\{l_1,\dots,l_r\}_p} f(t_1,\dots,t_k)\,dt_1\dots dt_r\,.$$

2.5. ***KZ* equations modulo p.** Let p and q be prime numbers, $p>q$. We consider the system of *KZ* equations (2.6) with parameter q modulo p. Namely, we look for $W^{\otimes n}[n-2r]$-valued polynomials in $z_1,\dots,z_n$ with integer coefficients, which satisfy system (2.6) modulo p and with values in the subspace $\operatorname{Sing}W^{\otimes n}[n-2r]$ modulo p. More precisely, let $I(z_1,\dots,z_n)=\sum_{J\in\mathcal{J}_r}I_J(z_1,\dots,z_n)V_J$ with $I_J(z_1,\dots,z_n)\in\mathbb{Z}[z_1,\dots,z_n]$. We request that

- for any $m=1,\dots,n$, the rational function $(q\partial_m-H_m(z))I(z)$ can be written as a ratio of two polynomials with integer coefficients such that the denominator is nonzero modulo p, while the numerator is zero modulo p;
- the coefficients $I_J(z_1,\dots,z_n)$ of the polynomial $I(z)$ satisfy equations (2.1) modulo p.

A construction of such solutions was presented in [SV2].

Let M, c be the least positive integers such that

$$M\equiv-\frac{1}{q},\qquad c\equiv\frac{2}{q}\qquad(\text{mod } p). \tag{2.11}$$

Let $t=(t_1,\dots,t_r)$, $z=(z_1,\dots,z_n)$. Define the master polynomial,

$$\Phi_p(t,z)=\prod_{1\leqslant i<j\leqslant r}(t_i-t_j)^c\prod_{i=1}^r\prod_{s=1}^n(t_i-z_s)^M. \tag{2.12}$$

Recall the weight vector-function $W(t,z)$ in (2.9). The function $\Phi_p(t,z)W(t,z)$ is a $W^{\otimes n}[n-2r]$-valued polynomial in t,z. Let $(l_1,\dots,l_r)\in\mathbb{Z}^r_{>0}$. Denote

$$I^{(l_1,\dots,l_r)}(z)=\int_{\{l_1,\dots,l_r\}_p}\Phi_p(t,z)W(t,z)\,dt_1\dots dt_r\,. \tag{2.13}$$

This is a $W^{\otimes n}[n-2r]$-valued polynomial in z.

Theorem 2.3 ([SV2]). *For any positive integers $(l_1,\dots,l_r)$, the polynomial $I^{(l_1,\dots,l_r)}(z)$ is a solution of the KZ equations modulo p with values in $\operatorname{Sing}W^{\otimes n}[n-2r]$ modulo p.*

We call the polynomials $I^{(l_1,\dots,l_r)}(z)$ the *p-hypergeometric* solutions.

In this paper we determine the number of independent p-hypergeometric solutions under the assumption that (p,q) is a pair of type 1.

Remark. The symmetric group S_n acts on the polynomials in $z_1,\dots,z_n$ with values in $W^{\otimes n}$ by permuting the tensor factors and the variables simultaneously,

$$\sigma(p(z_1,\dots,z_n)\,v_1\otimes\dots\otimes v_n)=(p(z_{\sigma(1)},\dots,z_{\sigma(n)})\,v_{\sigma^{-1}(1)}\otimes\dots\otimes v_{\sigma^{-1}(n)},\qquad \sigma\in S_n\,.$$

It is clear that for any positive integers $(l_1,\dots,l_r)$, the polynomial $I^{(l_1,\dots,l_r)}(z)$ is symmetric with respect to the S_n-action.

2.6. **Module of p-hypergeometric solutions.** Denote $\mathbb{Z}[z^p]=\mathbb{Z}[z_1^p,\dots,z_n^p]$. Let

$$\mathcal{M}\;=\;\Big\{\sum_{l_1,\dots,l_r}f_{l_1,\dots,l_r}(z)I^{(l_1,\dots,l_r)}(z)\mid f_{l_1,\dots,l_r}(z)\in\mathbb{Z}[z^p]\Big\}, \tag{2.14}$$

be the $\mathbb{Z}[z^p]$-module generated by the p-hypergeometric solutions $I^{(l_1,\dots,l_r)}(z)$ of Theorem 2.3. Every element of $\mathcal{M}$ is a solution of the KZ equations modulo p with values in $\operatorname{Sing}W^{\otimes n}[n-2r]$ modulo p. Indeed the KZ equations (2.6) are linear and $\frac{\partial z_i^p}{\partial z_j}\equiv 0\pmod p$ for all i,j.

We say that two elements $I,I'\in\mathcal{M}$ are equivalent and write $I\sim I'$, if $I-I'$ is divisible by p. Then the set $\mathcal{M}/(\sim)$ of equivalence classes is an $\mathbb{F}_p[z^p]$-module.

2.7. **More general choice of the numbers M and c.** For $s=1,\dots,n$ fix a positive integer M_s such that

$$M_s\equiv-\frac{1}{q}\pmod p.$$

Fix a positive integer c' such that

$$c'\equiv\frac{2}{q}\pmod p.$$

Define the master polynomial,

$$\Phi_p(t,z;\vec M,c')=\prod_{1\leqslant i<j\leqslant r}(t_i-t_j)^{c'}\prod_{i=1}^r\prod_{s=1}^n(t_i-z_s)^{M_s}.$$

Recall the weight vector-function $W(t,z)$ in (2.9). The function $\Phi_p(t,z;\vec M,c')W(t,z)$ is a $W^{\otimes n}[n-2r]$-valued polynomial in t,z with integer coefficients. Let $l=(l_1,\dots,l_r)\in\mathbb{Z}^r_{>0}$. Denote

$$I^{(l_1,\dots,l_r)}(z;\vec M,c')=\int_{\{l_1,\dots,l_r\}_p}\Phi_p(t,z;\vec M,c')W(t,z)\,dt_1\dots dt_r\,.$$

This is a $W^{\otimes n}[n-2r]$-valued polynomial in z with integer coefficients.

Theorem 2.4 ([SV2])**.** *For any positive integers $(l_1,\dots,l_r)$, the polynomial $I^{(l_1,\dots,l_r)}(z;\vec M,c')$ is a solution of the KZ equations modulo p with values in* $\operatorname{Sing}W^{\otimes n}[n-2r]$ *modulo p.*

Theorem 2.5. *For any positive integers $(l_1,\dots,l_r)$, the polynomial $I^{(l_1,\dots,l_r)}(z;\vec M,c')$ belongs to the module $\mathcal{M}$ of p-hypergeometric solutions.*

For $l=1$ this statement follows from [SlV, Theorem 3.1].

Proof. Let M,c be defined in (2.11). Then $c'=c+d_0p$, $M_s=M+d_sp$, $s=1,\dots,n$, where $d_0,\dots,d_n$ are nonnegative integers. Then we have modulo p,

$$\Phi_p(t,z;\vec M,c')\equiv\Phi_p(t,z)\prod_{1\leqslant i<j\leqslant r}(t_i^p-t_j^p)^{d_0}\prod_{i=1}^r\prod_{s=1}^n(t_i^p-z_s^p)^{d_s}\pmod p.$$

This formula implies the theorem. □

3. Pairs (p,q) of type 1

3.1. **Numbers M and c.** Let p,q be prime numbers, $p>q$. Let k be the minimal positive integer such that $q|(kp-1)$. We have $1\leqslant k<q$.

We say that the pair (p,q) is of type 1 or type 2 if

$$1\leqslant k\leqslant q/2\qquad\text{or}\qquad q/2<k<q,\tag{3.1}$$

respectively.

The pairs $(p,2)$ are all of type 1.

The pairs $(p,3)$ are of type 1 if $p=3m+1$ and of type 2 if $p=3m+2$.

Let $p=mq+s,\ s\in\{1,\dots,q-1\}$. Then the minimal positive integer k such that $q|(kp-1)$ belongs to $\{1,\dots,q-1\}$ and is determined by the property $ks\equiv1\pmod q$. Hence half of the values of s gives pairs (p,q) of type 1 and half of the values of s gives pairs (p,q) of type 2.

In the rest of the paper *we always assume that (p,q) is of type 1.*

Lemma 3.1. *The integer $q-2k$ is the least positive integer m such that $q|(mp+2)$.*

Proof. We have $q|(kp-1)$. Hence $q|(qp-2(kp-1))$, and $qp-2(kp-1)=(q-2k)p+2$. We also have $0\leqslant q-2k\leqslant q-2$. □

Lemma 3.2. *The integers*

$$M=\frac{kp-1}{q}\,,\qquad c=\frac{(q-2k)p+2}{q}\tag{3.2}$$

are the least positive integers satisfying the congruences (2.11)*. The integer c is odd and*

$$2M + c = p. \tag{3.3}$$

□

Example. If $q = 2$, then $M = \frac{p-1}{2}$, $c = 1$. The case $q = 2$ is the only case in which c does not depend on p.

Lemma 3.3. *For positive integers $(l_1, \dots, l_r)$ the polynomial $I^{(l_1,\dots,l_r)}(z)$ equals zero, if $l_1, \dots, l_r$ are not pairwise distinct.*

Proof. The integer c is odd. Hence the polynomial $\Phi_p(t,z)W(t,z)$ is skew-symmetric with respect to permutations of $t_1, \dots, t_r$. □

In what follows *we always assume*

$$n = qg + 2r - 1 \tag{3.4}$$

for some positive integer g.

Lemma 3.4. *Let q, k, g be fixed. Let p be large enough so that*

$$M - g = \frac{kp-1}{q} - g = \frac{kp-1-gq}{q} \geqslant 0\,. \tag{3.5}$$

Let $l_1 > \dots > l_r \geqslant 1$. Then the inequality

$$kg + r - 1 \geqslant l_1 \tag{3.6}$$

is necessary for $I^{(l_1,\dots,l_r)}(z)$ to be nonzero. Hence there are at most $\binom{kg+r-1}{r}$ tuples $(l_1 > \dots > l_r \geqslant 1)$ such that $I^{(l_1,\dots,l_r)}(z)$ is nonzero.

Proof. For $i = 1, \dots, r$ we have

$$\begin{aligned} \deg_{t_i} \Phi_p(t,z)W(t,z) &= nM + (r-1)c - 1 \\ &= (gq + 2r - 1)\frac{kp-1}{q} + (r-1)\frac{(q-2k)p+2}{q} - 1 \\ &= (kg + r - 1)p - 1 + \frac{kp-1-gq}{q}\,. \end{aligned} \tag{3.7}$$

This proves the lemma. □

3.2. **Main Theorem.** For a polynomial F in some variables with integer coefficients, denote by $[F]_p$ the polynomial F whose integer coefficients are projected to $\mathbb{F}_p$.

Let

$$f(z) = \sum_{d_1,\dots,d_n} a_{d_1,\dots,d_n} z_1^{d_1} \dots z_n^{d_n}$$

be a $W^{\otimes n}[n-2r]$-valued polynomial in z with integer coefficients. Here each $a_{d_1,\dots,d_n}$ is a linear combination of basis vectors $\{V_J \mid J \in \mathcal{J}_r\}$ with integer coefficients. Denote by $[f(z)]_p$ the polynomial

$$[f(z)]_p = \sum_{d_1,\dots,d_n} [a_{d_1,\dots,d_n}]_p\, z_1^{d_1} \dots z_n^{d_n}\,,$$

where each $[a_{d_1,\dots,d_n}]_p$ is the linear combination $a_{d_1,\dots,d_n}$ whose integer coefficients are projected to $\mathbb{F}_p$.

Denote by $W^{\otimes n}[n-2r]_p$ the vector space over $\mathbb{F}_p$ of linear combinations of symbols $\{V_J \mid J \in \mathcal{J}_r\}$ with coefficients in $\mathbb{F}_p$. Define the subspace $\operatorname{Sing} W^{\otimes n}[n-2r]_p \subset W^{\otimes n}[n-2r]_p$ as the subspace of all vectors $\sum_{J\in\mathcal{J}_r} c_J V_J$ whose coefficients satisfy equations (2.1).

Theorem 3.5. *Let (p, q) be of type 1. Let inequality (3.5) hold. Then for any $(l_1, \dots, l_r)$ such that $kg + r - 1 \geqslant l_1 > \dots > l_r \geqslant 1$, the polynomial $[I^{(l_1,\dots,l_r)}(z)]_p$ is nonzero. The polynomials*

$$\{[I^{(l_1,\dots,l_r)}(z)]_p \mid kg + r - 1 \geqslant l_1 > \dots > l_r \geqslant 1\}$$

are linear independent over $\mathbb{F}_p[z]$, that is, if

$$\sum_{kg+r-1\geqslant l_1>\dots>l_r\geqslant 1} f_{l_1,\dots,l_r}(z)[I^{(l_1,\dots,l_r)}(z)]_p = 0$$

for some $f_{l_1,\dots,l_r}(z) \in \mathbb{F}_p[z]$, then all $f_{l_1,\dots,l_r}(z)$ are equal to zero.

The theorem is proved in Section 4.5.

Corollary 3.6. *Let (p, q) be of type 1. Let inequality (3.5) hold. Then the $\mathbb{F}_p[z^p]$-module $\mathcal{M}/(\sim)$ is free of rank $\binom{kg+r-1}{r}$ with a basis $[I^{(l_1,\dots,l_r)}(z)]_p$, where $kg + r - 1 \geqslant l_1 > \dots > l_r \geqslant 1$.* □

4. Leading term and leading index

4.1. **Leading term of a polynomial.** Denote by $\succ$ the lexicographical ordering of monomials $z_1^{d_1} \dots z_n^{d_n}$, where $d_1, \dots, d_n \in \mathbb{Z}_{\geqslant 0}$. Thus $z_1 \succ z_2 \succ \dots \succ z_{n-1} \succ z_n$ and so on. For example, $z_1^2 z_2^2 z_3^2 \succ z_1^2 z_2 z_4^5$.

For a nonzero polynomial

$$f(z) = \sum_{d_1,\dots,d_n} a_{d_1,\dots,d_n} z_1^{d_1} \dots z_n^{d_n}$$

consider the summand $a_{d_1,\dots,d_n} z_1^{d_1} \dots z_n^{d_n}$ corresponding to the lexicographically largest monomial $z_1^{d_1} \dots z_n^{d_n}$ entering $f(z)$ with a nonzero coefficient $a_{d_1,\dots,d_n}$. We call this summand the leading term of $f(z)$, the corresponding $a_{d_1,\dots,d_n}$ – the leading coefficient, and $z_1^{d_1} \dots z_n^{d_n}$ – the leading monomial.

For example, let $I^{(l_1,\dots,l_r)}(z)$ be one of the p-hypergeometric solutions of Theorem 2.3. Then

$$[I^{(l_1,\dots,l_r)}(z)]_p = \sum_{d_1,\dots,d_n} [a_{d_1,\dots,d_n}]_p \, z_1^{d_1} \dots z_n^{d_n} ,$$

with $[a_{d_1,\dots,d_n}]_p \in \operatorname{Sing} W^{\otimes n}[n-2r]_p$. If $[I^{(l_1,\dots,l_r)}(z)]_p$ is nonzero, then it has the leading term, monomial, and coefficient. The leading coefficient is a nonzero vector of $\operatorname{Sing} W^{\otimes n}[n-2r]_p$.

4.2. **Leading index of a vector of $W^{\otimes n}[n-2r]_p$.** We order $\{V_J \mid J \in \mathcal{J}_r\}$, the basis vectors of $W^{\otimes n}[n-2r]_p$, lexicographically with the largest of them being $V_{\{1,2,\dots,r\}} = w_2 \otimes \dots \otimes w_2 \otimes w_1 \otimes \dots \otimes w_1$ and the smallest of them being $V_{\{n-r+1,n-r+2,\dots,n\}} = w_1 \otimes \dots \otimes w_1 \otimes w_2 \otimes \dots \otimes w_2$.

Every nonzero vector $w \in W^{\otimes n}[n-2r]_p$ is a linear combination of the basis vectors. Let $V_{\{m_1,\dots,m_r\}}$ be the largest of the basis vectors entering w with a nonzero coefficient. We call the index $\{m_1,\dots,m_r\}$ the leading index of w.

4.3. **Index $\{m_1,\dots,m_r\}$.** Given integers $(l_1,\dots,l_r)$, consider the system of inequalities for integers $m_1,\dots,m_r$,

$$(m_i-1)M \;\leqslant\; nM+(r-i)c-l_ip \;<\; m_iM, \qquad i=1,\dots,r. \tag{4.1}$$

Clearly, these inequalities uniquely determine the integers $m_1,\dots,m_r$.

Lemma 4.1. *Let inequality (3.5) hold. Let $kg+r-1 \geqslant l_1 > \dots > l_r \geqslant 1$. Then $m_i+2 \leqslant m_{i+1}$ for $i=1,\dots,r-1$, and $1 \leqslant m_1$, $m_r < n$.*

Proof. For $i=1,\dots,r-1$ we have $(m_i-1)M \leqslant nM+(r-i)c-l_ip = nM+(r-i-1)c+c-l_{i+1}p+(l_{i+1}-l_i)p \leqslant nM+(r-i-1)c+c-l_{i+1}p-p = nM+(r-i-1)c-l_{i+1}p-2M < (m_{i+1}-2)M$. Hence $m_{i+1} > m_i+1$. This implies that $m_i+2 \leqslant m_{i+1}$.

The inequality $1 \leqslant m_1$ follows from the inequality $l_1p \leqslant nM+(r-1)c$, which is true since

$$l_1p \leqslant (kg+r-1)p \leqslant (kg+r-1)p + \frac{kp-1-gq}{q} = nM+(r-1)c,$$

see (3.7). We also have $M+2c = p \leqslant l_rp \leqslant (n-m_r+1)M$. Hence $m_r < n$. □

Lemma 4.2. *Let $(l_1,\dots,l_r)$ and $(l'_1,\dots,l'_r)$ be two distinct tuples of integers such that $kg+r-1 \geqslant l_1 > \dots > l_r \geqslant 1$ and $kg+r-1 \geqslant l'_1 > \dots > l'_r \geqslant 1$. Let $\{m_1,\dots,m_r\}$ and $\{m'_1,\dots,m'_r\}$ be the corresponding sets defined by (4.1). Then $\{m_1,\dots,m_r\} \neq \{m'_1,\dots,m'_r\}$.*

Proof. Let $l_i > l'_i$ for some i. Then $(m_i-1)M \leqslant nM+(r-i)c-l_ip = nM+(r-i)c-l'_ip+(l'_i-l_i)p \leqslant nM+(r-i)c-l'_ip-p = nM+(r-i)c-l'_ip-2M-c < nM+(r-i)c-l'_ip-2M < (m'_i-2)M$. Hence $m'_i > m_i+1$. □

4.4. **Special summand.** We have

$$\begin{aligned}\prod_{1\leqslant i<j\leqslant r}(t_i-t_j)^c \;&=\; \prod_{1\leqslant i<j\leqslant r}\Big(\sum_{a_{ij}+a_{ji}=c}\binom{c}{a_{ji}}(-1)^{a_{ji}}t_i^{a_{ij}}t_j^{a_{ji}}\Big)\\ &=\; \sum_{(a_{ij})\in A} t_1^{\sum_{j\neq 1}a_{1j}}\cdots t_r^{\sum_{j\neq r}a_{rj}}\prod_{i<j}(-1)^{a_{ji}}\binom{c}{a_{ji}},\end{aligned}$$

where $A = \{(a_{ij})_{1\leqslant i,j\leqslant r, i\neq j} \mid a_{ij}\in\mathbb{Z}_{\geqslant 0},\ a_{ij}+a_{ji}=c \ \text{for every}\ \ i\neq j\}$. Hence

$$[I^{(l_1,\dots,l_r)}(z)]_p = [\int_{\{l_1,\dots,l_r\}_p}\Phi_p(t,z)W(t,z)dt_1\dots dt_r]_p \;= \tag{4.2}$$

$$= \sum_{(a_{ij})\in A}\Big(\prod_{i<j}(-1)^{a_{ji}}\binom{c}{a_{ji}}\Big)[\int_{\{l_1,\dots,l_r\}_p}W(t,z)\prod_{i=1}^r t_i^{\sum_{j\neq i}a_{ij}}\prod_{s=1}^n(t_i-z_s)^M dt_1\dots dt_r]_p.$$

This sum contains the special summand

$$S(z) := [\int_{\{l_1,\dots,l_r\}_p} W(t,z)\prod_{i=1}^{r} t_i^{(r-i)c}\prod_{s=1}^{n}(t_i-z_s)^M dt_1\dots dt_r]_p$$

corresponding to the collection (a_{ij}) such that $a_{ij}=c$ for $i<j$ and $a_{ij}=0$ for $i>j$.

For $1\leqslant u\leqslant nM$, denote

$$z(u) := z_s\,, \qquad \text{if} \quad (s-1)M<u\leqslant sM. \tag{4.3}$$

Thus $z(1)=z(2)=\dots=z(M)=z_1$, $z(M+1)=z_2$ and so on.

Lemma 4.3. *The vector-polynomial $S(z)$ is nonzero. The leading monomial of $S(z)$ equals*

$$z^{l_1,\dots,l_r} := \prod_{i=1}^{r}\ \prod_{u=1}^{nM+(r-i)c-l_ip} z(u). \tag{4.4}$$

Denote by $C_{l_1,\dots,l_r}$ the leading coefficient of $S(z)$. Then the leading index of the vector $C_{l_1,\dots,l_r}$ equals $\{m_1,\dots,m_r\}$ determined by inequalities (4.1). Moreover, the leading monomial of any other nonzero summand in (4.2) is lexicographically smaller than $z^{l_1,\dots,l_r}$. Thus $C_{l_1,\dots,l_r}z^{l_1,\dots,l_r}$ is the leading term of $[I^{(l_1,\dots,l_r)}(z)]_p$.

Proof. We rewrite $z^{l_1,\dots,l_r}$ using the integers $m_1,\dots,m_r$,

$$z^{l_1,\dots,l_r} = \prod_{i=1}^{r} z_1^M\cdots z_{m_i-1}^M\, z_{m_i}^{(n-m_i+1)M+(r-i)c-l_ip}. \tag{4.5}$$

Let $z^{l_1,\dots,l_r}=z_1^{d_1}\dots z_n^{d_n}$ for some $d_1,\dots,d_n$. Then

$$\begin{aligned} d_{m_i} &= (n-m_i+1)M+(r-i)(c+M)-l_ip, && i=1,\dots,r;\\ d_j &= rM, && j=1,\dots,m_1-1;\\ d_j &= (r-i)M, \quad j=m_i+1,\dots,m_{i+1}-1, && i=1,\dots,r-1;\\ d_j &= 0 && j=m_r+1,\dots,n. \end{aligned} \tag{4.6}$$

It is clear that $z^{l_1,\dots,l_r}$ is the lexicographically maximal monomial which can be produced by $S(z)$. Let $C_{l_1,\dots,l_r}$ be the coefficient of $z^{l_1,\dots,l_r}$ in $S(z)$. Then $C_{l_1,\dots,l_r}=\sum_{J\in\mathcal{J}_r}c_JV_J$ with $c_J\in\mathbb{F}_p$. It is clear that $\{m_1,\dots,m_r\}$ is the leading index of $C_{l_1,\dots,l_r}$, if $c_{\{m_1,\dots,m_r\}}$ is nonzero, but

$$c_{\{m_1,\dots,m_r\}} = \prod_{i=1}^{r}(-1)^{(n-m_i+1)M+(r-i)c-l_ip}\binom{M-1}{(n-m_i+1)M+(r-i)c-l_ip}, \tag{4.7}$$

which is nonzero due to (3.3) and (4.1). Thus $S(z)$ is nonzero, its leading term equals $C_{l_1,\dots,l_r}z^{l_1,\dots,l_r}$, and the leading index of $C_{l_1,\dots,l_r}$ is $\{m_1,\dots,m_r\}$. It remains to prove that the leading monomial of any other nonzero summand in (4.2) is lexicographically smaller than $z^{l_1,\dots,l_r}$.

Let $\tilde S(z)$ be any other summand in (4.2),

$$\tilde S(z) := \Big(\prod_{i<j}(-1)^{a_{ji}}\binom{c}{a_{ji}}\Big)[\int_{\{l_1,\dots,l_r\}_p} W(t,z)\prod_{i=1}^{r} t_i^{\sum_{j\neq i}a_{ij}}\prod_{s=1}^{n}(t_i-z_s)^M dt_1\dots dt_r]_p.$$

It is clear that the lexicographically maximal monomial which can be produced by $\tilde{S}(z)$ equals

$$\tilde{z}^{\,l_1,\dots,l_r} := \prod_{i=1}^{r} \prod_{u=1}^{nM-l_ip+\sum_{j\neq i} a_{ij}} z(u). \tag{4.8}$$

Assume that $\sum_{j\neq 1} a_{1j} < (r-1)c$. Let s be the least index such that the maximal power of z_s dividing $\prod_{u=1}^{nM-l_1p+\sum_{j\neq 1} a_{1j}} z(u)$ is strictly smaller than the maximal power of z_s (which we denote by b) dividing $\prod_{u=1}^{nM+(r-1)c-l_1p} z(u)$. Then the maximal power of z_s dividing $\prod_{i=1}^{r}\prod_{u=1}^{nM-l_ip+\sum_{j\neq i} a_{ij}} z(u)$ is strictly smaller than $b+(r-1)M$. This implies that $\tilde{z}^{\,l_1,\dots,l_r}$ is lexicographically smaller than $z^{l_1,\dots,l_r}$.

Thus, a summand $\tilde{S}(z)$ must have $\sum_{j\neq 1} a_{ij} = (r-1)c$ in order to have a monomial as large lexicographically as $z^{l_1,\dots,l_r}$. This means that $a_{1j}=c$ and $a_{j1}=0$ for $j\neq 1$.

Now take any summand $\tilde{S}(z)$ in (4.2) with $\sum_{j\neq 1} a_{1j} = (r-1)c$. In a similar way we show that $\tilde{S}(z)$ must have $\sum_{j\neq 2} a_{2j} = (r-2)c$ in order to have a monomial as large lexicographically as $z^{l_1,\dots,l_r}$. Repeating this reasoning we conclude that the special summand $S(z)$ is the only summand in (4.2) which may have the monomial $z^{l_1,\dots,l_r}$ with a nonzero coefficient; and no summands in (4.2) may have a monomial larger than $z^{l_1,\dots,l_r}$. Lemma 4.3 is proved. □

4.5. **Proof of Theorem 3.5.** Let $[I^{(l_1,\dots,l_r)}(z)]_p$ be one of the vector-polynomials of Theorem 3.5. Then $[I^{(l_1,\dots,l_r)}(z)]_p$ has the leading term $C_{l_1,\dots,l_r} z^{l_1,\dots,l_r}$ described in Lemma 4.3. Also the leading index $\{m_1,\dots,m_r\}$ of $C_{l_1,\dots,l_r}$ is determined by $l_1,\dots,l_r$ in (4.1). Let $f_{l_1,\dots,l_r}(z)\in\mathbb{F}_p[z]$. Then the leading term of the vector-polynomial $f_{l_1,\dots,l_r}(z)[I^{(l_1,\dots,l_r)}(z)]_p$ equals the product of leading terms of $f_{l_1,\dots,l_r}(z)$ and $[I^{(l_1,\dots,l_r)}(z)]_p$. Moreover, the leading index of the leading coefficient of $f_{l_1,\dots,l_r}(z)[I^{(l_1,\dots,l_r)}(z)]_p$ equals the leading index $\{m_1,\dots,m_r\}$ of the leading coefficient of $[I^{(l_1,\dots,l_r)}(z)]_p$.

Consider a sum $\sum_{kg+r-1\geqslant l_1>\dots>l_r\geqslant 1} f_{l_1,\dots,l_r}(z)[I^{(l_1,\dots,l_r)}(z)]_p$ as in Theorem 3.5. Then all nonzero summands have different leading indices due to the previous remark and Lemma 4.2. This implies that such a sum is not zero if it has nonzero summands. Theorem 3.5 is proved.

Example. Let $q=2$, $n=5$, $r=2$. Then Theorem 3.5 says that for any odd prime p there is just one p-hypergeometric solution $[I^{(2,1)}(z_1,\dots,z_5)]_p$. This polynomial is homogeneous of degree $2p-4$ and takes values in $\operatorname{Sing} W^{\otimes 5}[1]_p$. The leading term of $[I^{(2,1)}(z_1,\dots,z_5)]_p$ is $C_{2,1}z_1^{p-2}z_2^{(p-1)/2}z_3^{(p-3)/2}$ where the leading coefficient $C_{2,1}\in W^{\otimes 5}[1]_p$ has the leading index $\{m_1,m_2\}=\{1,3\}$.

Notice that the space $\operatorname{Sing} W^{\otimes 5}[1]$ has dimension 5.

4.6. **Leading terms and eigenvectors.** Let $I(z)\in \operatorname{Sing} W^{\otimes n}[n-2r]_p\otimes\mathbb{F}_p[z]$ be a polynomial solution of the KZ equations (2.6) with a positive integer parameter q (not necessarily a p-hypergeometric solution). Let $Cz_1^{d_1}\dots z_n^{d_n}$ be its leading term, $C\in\operatorname{Sing} W^{\otimes n}[n-2r]_p$. Then we have modulo p,

$$\sum_{\ell=j+1}^{n} \bar{\Omega}_{j,\ell}\, C \equiv q d_j C, \qquad j=1,\dots,n-1, \qquad d_n\equiv 0, \tag{4.9}$$

see [V7, Lemma 5.1]. Thus the leading coefficient C is an eigenvector of the linear operators $\bar\Omega_j = \sum_{\ell=j+1}^{n} \bar\Omega_{j,\ell}$, $j = 1, \dots, n-1$, with prescribed eigenvalues.

An eigenbasis of the operators $\bar\Omega_j$, $j = 1, \dots, n-1$, on $\operatorname{Sing} W^{\otimes n}[n-2r]_p$ is formed by the so-called iterated singular vectors, for example see [V2]. Such an iterated vector is determined by its leading index.

If p is large enough with respect to n, then the vectors of that eigenbasis are separated by eigenvalues.

Let $[I^{(l_1,\dots,l_r)}(z)]_p$ be one of the p-hypergeometric solutions of Theorem 3.5. Let $C_{l_1,\dots,l_r} z_1^{d_1} \dots z_n^{d_n}$ be its leading term, where d_j are defined in (4.6). Let $\{m_1, \dots, m_r\}$ be the leading index of $C_{l_1,\dots,l_r}$. If p is large enough with respect to n, then $C_{l_1,\dots,l_r}$ is the iterated singular vector with leading index $\{m_1, \dots, m_r\}$. It is the eigenvector of the operators $\bar\Omega_j$, $j = 1, \dots, n-1$, with eigenvalues defined by formula (4.9).

5. Solutions and a Cartier map

In this section we discuss how the two objects:

- the set of indices $(l_1, \dots, l_r)$ with $kg+r-1 \geqslant l_1 > \dots > l_r \geqslant 1$, appearing in Theorem 3.5,
- and the number of such indices $\binom{kg+r-1}{r}$, appearing in Corollary 3.6,

are related to the integrand $\Phi(t,z)W(t,z)dt_1 \wedge \dots \wedge dt_r$ of the integral representation of complex hypergeometric solutions of the KZ equations, appearing in Theorems 2.1 and 2.2.

Recall that the complex hypergeometric solutions with values in $\operatorname{Sing} W^{\otimes n}[n-2r]$ are given by the formulas:

$$I^{(\gamma)}(z) = \int_{\gamma(z)} \Phi(t,z)W(t,z)\, dt_1 \wedge \dots \wedge dt_r\,,$$
$$\Phi(t,z) = \prod_{1\leqslant i<j\leqslant r} (t_i - t_j)^{2/q} \prod_{i=1}^{r} \prod_{s=1}^{n} (t_i - z_s)^{-1/q},$$
$$W(t,z) = \sum_{1\leqslant i_1<\dots<i_r\leqslant n} W_{i_1,\dots,i_r}(t,z)\, V_{i_1,\dots,i_r}\,, \quad W_{i_1,\dots,i_r}(t,z) = \operatorname{Sym}_{t_1,\dots,t_r} \prod_{j=1}^{r} \frac{1}{t_j - z_{i_j}}\,,$$

see Section 2.3, while the corresponding p-hypergeometric solutions are given by the formulas:

$$I^{(l_1,\dots,l_r))}(z) = \int_{\{l_1,\dots,l_r\}_p} \Phi_p(t,z)W(t,z)\, dt_1 \dots dt_r\,,$$
$$\Phi_p(t,z) = \prod_{1\leqslant i<j\leqslant r} (t_i - t_j)^{c} \prod_{i=1}^{r} \prod_{s=1}^{n} (t_i - z_s)^{M}, W(t,z) = \sum_{J\in\mathcal{J}_r} W_J(t,z)V_J,$$

see Section 2.5.

5.1. **Square integrability criterion.** Let M be a complex manifold of complex dimension r. If f is a meromorphic function on M and S is an irreducible subvariety of M, then the order of f along S, $\operatorname{ord}_S(f)$, is the coefficient of the exceptional divisor of the blow up of S at the divisor of f. This notion generalizes to the setting where f has only finitely many determinations, which means that f becomes univalued on a finite (possibly ramified) cover of M. Then f has at a generic point of the exceptional divisor a fractional order.

Let ω be a multivalued meromorphic r-form on M with only finitely many determinations and let S be an irreducible subvariety of M. Write ω at some point s of S as $f\omega_0$ where ω_0

is d-form on M that is nonzero at s and f is multivalued meromorphic at s. The logarithmic order of ω along S is $\operatorname{codim}(S) + \operatorname{ord}_{S;s}(f)$. This only depends on ω and S.

Suppose that $D \subset M$ is a hypersurface which is arrangement-like in the sense that D can be covered by analytic coordinate charts of M on which D is given by a product of linear forms in the coordinates, see [STV]. It is clear that D then comes with a natural partition into connected, locally closed submanifolds, its strata.

We say that a stratum S is decomposable if at a generic point $s \in S$ the germ D_s of the hyperplane arrangement can be decomposed into the disjoint sum $A_1 \cup A_2$ of two germs of hyperplane arrangements and, after a suitable linear coordinate change, defining equations for A_1 and A_2 have no common variables. We say that a stratum S is dense if it is of codimension 1 or if it is not decomposable, see [V1, STV].

We have the following trivial observation.

Proposition 5.1 ([LV]). *Suppose M is compact and ω is a meromorphic multivalued r-form on M with only finitely many determinations and whose polar set is contained in an arrangement-like hypersurface D. Then the following properties are equivalent:*

(i) *the form ω is square integrable in the sense that $\int_M \omega \wedge \bar{\omega}$ converges;*
(ii) *the form ω has positive logarithmic order along any dense stratum of D;*
(iii) *if $\tilde{M} \to M$ is a proper surjective map with $\tilde{M}$ a complex manifold of the same dimension as M and such that ω becomes a univalued d-form on $\tilde{M}$, then the latter form is regular.*

5.2. **Square integrable differential r-forms.** Let $t = (t_1, \dots, t_r)$ be coordinates on $\mathbb{C}^r \subset (\mathbb{P}^1)^r$ and $z = (z_1, \dots, z_n)$ distinct parameters.

Consider a differential r-form

$$\begin{aligned} \omega \;&=\; P(t)\,\Phi(t,z)^k dt \\ &=\; P(t) \cdot \prod_{i=1}^{r}\prod_{s=1}^{n}(t_i - z_s)^{-k/q} \cdot \prod_{1\leqslant i<j\leqslant n}(t_i - t_j)^{2k/q} \cdot dt_1 \wedge \dots \wedge dt_r \end{aligned} \tag{5.1}$$

on $\mathbb{C}^r \subset (\mathbb{P}^1)^r$. Here k, $0 < k < q$, is a positive integer; $dt = dt_1 \wedge \dots \wedge dt_r$; and $P(t)$ is a polynomial in t.

Let $n = qg + 2r - 1$ where g is some positive integer, cf. (3.4).

Theorem 5.2. *The form ω is square integrable on $(\mathbb{P}^1)^r$ if and only if*

$$\deg_{t_i} P \leqslant kg - 1 \qquad \text{for all} \qquad i = 1, \dots, r. \tag{5.2}$$

Proof. Denote by D the support of the divisor of ω. The support D lies in the union of hypersurfaces defined by $t_i = z_s$, $t_i = \infty$, and $t_i = t_j$ for $i < j$. This union is clearly arrangement-like. Its dense strata of codimension l are:

(i) diagonals in $(\mathbb{P}^1)^r$ defined by letting $l \geqslant 2$ coordinates to coalesce;
(ii) loci in $(\mathbb{P}^1)^r$ defined by setting $l \geqslant 1$ coordinates equal to ∞;
(iii) loci in $(\mathbb{P}^1)^r$ defined by setting $l \geqslant 1$ coordinates equal to some z_s.

Let S be defined by the equations $t_{i_1} = \dots = t_{i_l}$ for some $1 \leqslant i_1 < \dots < i_l \leqslant r$. Then the logarithmic order of ω along S is $\geqslant l - 1 + \binom{l}{2} \cdot 2k/q > 0$.

Let S be defined by the equations $t_{i_1} = \cdots = t_{i_l} = z_s$ for some $1 \leqslant i_1 < \cdots < i_l \leqslant r$. Then the logarithmic order ω along S is $\geqslant\ l - lk/q + \binom{l}{2} \cdot 2k/q = l(1-k/q) + \binom{l}{2} \cdot 2k/q > 0$.

Let S be defined by the equation $t_{i_1} = \infty$ for some i_1. In the coordinates $u_1, \dots, u_r$, $u_i = 1/t_i$ for $i = 1, \dots, r$, we have

$$\begin{aligned}(5.3)\qquad \omega \ &= \ (-1)^r \Big(P(1/u_1, \dots, 1/u_r, z) \cdot \prod_{i=1}^{r} u_r^{\deg_{t_i} P} \Big) \cdot \prod_{i=1}^{r} \prod_{s=1}^{n} (1 - z_s u_i)^{-k/q} \cdot \\ &\cdot \prod_{1 \leqslant i < j \leqslant n} (u_j - u_i)^{2k/q} \cdot \prod_{i=1}^{r} u_r^{-\deg_{t_i} P + nk/q - (r-1)2k/q - 2} \cdot du_1 \wedge \cdots \wedge du_r\end{aligned}$$

Hence the logarithmic order of ω along S equals

$$\begin{aligned} 1 + nk/q - (r-1)2k/q - 2 - \deg_{t_{i_1}} P \ &= \ (n - 2(r-1))k/q - 1 - \deg_{t_{i_1}} P \\ &= \ kg - 1 - \deg_{t_{i_1}} P + k/q. \end{aligned}$$

Hence the logarithmic order along S is positive if and only if $\deg_{t_{i_1}} P \leqslant kg - 1$.

Let S be defined by the equations $t_{i_1} = \cdots = t_{i_l} = \infty$ for some $1 \leqslant i_1 < \cdots < i_l \leqslant r$. It follows from (5.3) that the logarithmic order along S is positive if the logarithmic order is positive along every hyperplpane defined by the equation $t_{i_j} = \infty$ for $j = 1, \dots, l$. The theorem is proved. □

5.3. **Schur polynomials.** A sequence of integers $a = (a_1 \geqslant a_2 \geqslant \cdots \geqslant a_r \geqslant 0)$ is called a partition. For a partition a the polynomial $m_a(t) = \mathrm{Sym}_t\, t_1^{a_1} \dots t_r^{a_r}$ is called a symmetric monomial function. The polynomial

$$s_{(a_1,\dots,a_r)}(t) = \frac{\det\,(t_i^{a_j + r - j})_{i,j=1,\dots,r}}{\prod_{1 \leqslant i < j \leqslant r} (t_i - t_j)}$$

is called a Schur polynomial. It is known that

$$(5.4)\qquad s_a(t) \ = \ \sum_{b \leqslant a} K_{a,b}\, m_{b_1,\dots,b_r}(t),$$

where $K_{a,b}$ are nonnegative integers, $K_{a,a} = 1$. The inequality $b \leqslant a$ means $\sum_{j=1}^{i} b_j \leqslant \sum_{j=1}^{i} a_j$ for all $i = 1, \dots, r$. The numbers $K_{a,b}$ are called the Kostka numbers.

For a positive integer d, denote

$$A(d) = \{(a_1, \dots, a_r) \mid d \geqslant a_1 \geqslant a_2 \geqslant \cdots \geqslant a_r \geqslant 0,\, a_i \in \mathbb{Z}\}.$$

Let $\mathcal{V}(d)$ be the free $\mathbb{Z}$-module with basis $\{m_a(t) \mid a \in A(d)\}$. The module $\mathcal{V}(d)$ has rank $\binom{d+r}{r}$. The set $\{s_a(t) \mid a \in A(d)\}$ of Schur polynomials is a basis of $\mathcal{V}(d)$ by formula (5.4).

Let us return to Theorem 5.2. Let $\mathcal{W}$ be the vector space of all differential r-forms $\omega = P(t)\Phi(t,z)^k dt$ such that $P(t)$ is a polynomial in $t_1, \dots, t_r$ symmetric with respect to permutations of $t_1, \dots, t_r$, and ω is square integrable on $(\mathbb{P}^1)^r$.

Corollary 5.3. *The set $\{s_a(t)\Phi(t,z)^k dt \mid a \in A(kg-1)\}$ of differential r-forms on $(\mathbb{P}^1)^r$ is a basis of $\mathcal{W}$. The vector space $\mathcal{W}$ has dimension $\binom{kg+r-1}{r}$.* □

Notice that this binomial coefficient equals the rank of the module $\mathcal{M}/(\sim)$ in Corollary 3.6.

We introduce the following $\mathbb{F}_p[z]$-variant of the vector space $\mathcal{W}$. Let $\mathcal{W}_p[z]$ be the free $\mathbb{F}_p[z]$-module with basis $\mathcal{B}$ of formal algebraic differential r-forms

$$\omega_a = s_a(t)\Phi(t,z)^k dt, \qquad a \in A(kg-1).$$

Let $\mathcal{B}^* = \{\omega^a \,|\, a \in A(kg-1)\}$ be the collection of $\mathbb{F}_p[z]$-linear functions on $\mathcal{W}_p[z]$ such that $\langle \omega^a, \omega_b\rangle = \delta_{a,b}$ for all $a, b \in A(kg-1)$.

5.4. **Cartier map.** Assume that (p,q) is of type 1. Hence $q|(kp-1)$, $0 < k \leqslant q/2$.

We define a map which sends every differential form $\Phi(t,z)W_J(t,z)dt$, $J \in \mathcal{J}_r$, to $\mathcal{W}_p[z]$, that is, to a linear combination of differential r-forms ω_a, $a \in A(kg-1)$, with coefficients in $\mathbb{F}_p[z]$. We call this map the Cartier map. We have

$$\begin{aligned}
&\Phi(t,z)W_J(t,z)dt = \frac{\Phi(t,z)^{kp}}{\Phi(t,z)^{kp-1}} W_J(t,z)dt \\
&= \Phi(t,z)^{kp} \prod_{1\leqslant i<j\leqslant r} (t_i - t_j)^{(2-2kp)/q} \prod_{i=1}^{r}\prod_{s=1}^{n} (t_i - z_s)^{(kp-1)/q} W_J(t,z)dt \\
&= \frac{\Phi(t,z)^{kp}}{\prod_{1\leqslant i<j\leqslant r}(t_i-t_j)^p} \prod_{1\leqslant i<j\leqslant r} (t_i - t_j)^{(2+(q-2k)p)/q} \prod_{i=1}^{r}\prod_{s=1}^{n} (t_i - z_s)^{(kp-1)/q} W_J(t,z)dt \\
&= \frac{\Phi(t,z)^{kp}}{\prod_{1\leqslant i<j\leqslant r}(t_i-t_j)^p} \prod_{1\leqslant i<j\leqslant r} (t_i - t_j)^{c} \prod_{i=1}^{r}\prod_{s=1}^{n} (t_i - z_s)^{M} W_J(t,z)dt \\
&= \frac{\Phi(t,z)^{kp}}{\prod_{1\leqslant i<j\leqslant r}(t_i-t_j)^p} \Phi_p(t,z) W_J(t,z)dt
\end{aligned}$$

where c, M, $\Phi_p(t,z)$ are defined in (3.2) and (2.12).

Since (p,q) is of type 1, the integer c is odd. Hence the polynomial $\Phi_p(t,z)W_J(t,z)$ is skew-symmetric with respect to permutations of $t_1,\dots,t_r$. We expand $\Phi_p(t,z)W_J(t,z)$ with respect to the t-variables as follows:

$$\begin{aligned}
&\Phi_p(t,z)W_J(t,z) = \\
&= (t_1\dots t_r)^{p-1} \sum_{(a_1,\dots,a_r)\in A(kg-1)} c_J^{(a_1,\dots,a_r)}(z)\Big(\sum_{\sigma\in S_r} (-1)^{|\sigma|} t_{\sigma(1)}^{(a_1+r-1)p} t_{\sigma(1)}^{(a_2+r-2)p}\dots t_{\sigma(r)}^{a_r p}\Big) + {\sum}' \\
&\quad = (t_1\dots t_r)^{p-1} \sum_{(a_1,\dots,a_r)\in A(kg-1)} c_J^{(a_1,\dots,a_r)}(z) s_{(a_1,\dots,a_r)}(t_1^p,\dots,t_r^p) \prod_{1\leqslant i<j\leqslant r} (t_i^p - t_j^p) + {\sum}',
\end{aligned}$$

where ${\sum}'$ denotes the sum of the monomials $t_1^{d_1}\dots, t_r^{d_r}$ such that at least one of $d_1,\dots,d_r$ is not of the form $lp-1$ for some positive integer l; the coefficients $c_J^{(a_1,\dots,a_r)}(z)$ are suitable polynomials in z.

Returning to $\Phi(t,z)W_J(t,z)dt$ we write

$$\Phi(t,z)W_J(t,z)dt =$$
$$= \sum_{(a_1,\dots,a_r)\in A(kg-1)} c_J^{(a_1,\dots,a_r)}(z) s_{(a_1,\dots,a_r)}(t_1^p,\dots,t_r^p) \prod_{1\leqslant i<j\leqslant r} \frac{t_i^p-t_j^p}{(t_i-t_j)^p} \Phi(t,z)^{kp}(t_1\dots t_r)^{p-1}dt$$
$$+ \frac{\sum'}{\prod_{1\leqslant i<j\leqslant r}(t_i-t_j)^p}\Phi(t,z)^{kp}dt\,.$$

Notice that $\frac{t_i^p-t_j^p}{(t_i-t_j)^p}\equiv 1 \pmod p$. We define the *Cartier map* $\mathcal{C}$ by the formula

$$\mathcal{C}\ :\ \Phi(t,z)W_J(t,z)dt\ \mapsto \sum_{(a_1,\dots,a_r)\in A(kg-1)} [c_J^{(a_1,\dots,a_r)}(z)]_p\, s_{(a_1,\dots,a_r)}(t_1,\dots,t_r)\Phi(t,z)^k dt\,, \tag{5.5}$$

cf. [AH].

Recall the collection $\{\omega^{(a_1,\dots,a_r)}\mid (a_1,\dots,a_r)\in A(kg-1)\}$ of linear functions on $\mathcal{W}_p[z]$. We have

$$\big\langle \omega^{(a_1,\dots,a_r)},\, \mathcal{C}(\Phi(t,z)W_J(t,z)dt)\big\rangle = [c_J^{(a_1,\dots,a_r)}(z)]_p\,.$$

Theorem 5.4. *For any p-hypergeometric solution $[I^{(a_1+r,a_2+r-1,\dots,a_r+1)}(z)]_p$, $kg-1\geqslant a_1\geqslant a_2\geqslant\dots\geqslant a_r\geqslant 0$ we have*

$$[I^{(a_1+r,a_2+r-1,\dots,a_r+1)}(z)]_p\ =\ \sum_{J\in\mathcal{J}_r} [c_J^{(a_1,\dots,a_r)}(z)]_p V_J\,. \tag{5.6}$$

Proof. The proof follows from the definition of $I^{(a_1+r,a_2+r-1,\dots,a_r+1)}(z)$ in (2.13). □

Formula (5.6) can be reformulates as follows. For any integers $(l_1,\dots,l_r)$, $kg+r-1\geqslant l_1>\dots>l_r\geqslant 1$, the p-hypergeometric solution $[I^{(l_1,\dots,l_r)}(z)]_p$ is given by the formula

$$[I^{(l_1,\dots,l_r)}(z)]_p = \big\langle \omega^{(l_1-r,\,l_2-r+1,\,\dots,\,l_r-1)},\, \mathcal{C}(\Phi(t,z)W(t,z)dt)\big\rangle\,. \tag{5.7}$$

Notice that $\Phi(t,z)W(t,z)dt$ is the integrand of the integral representation of complex hypergeometric solutions of the *KZ* equations, see (2.10), while $[I^{(l_1,\dots,l_r)}(z)]_p$ is a solution of the *KZ* equations over $\mathbb{F}_p$.

Formula (5.7) for $r=1$ and two prime numbers $(p>q)$ not necessarily of type 1 is the subject of [SlV, Theorem 6.2].

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