

THE B-B-G TRANSFER PRINCIPLE FOR SIGNATURE FOUR

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ABSTRACT. We show how the elliptic function dn_2 of Shen leads to the signature four transfer principle of Berndt, Bhargava and Garvan.

Berndt, Bhargava and Garvan in [1995] established a Transfer Principle by which to pass from the classical theory of elliptic functions to the Ramanujan theory of elliptic functions in signature four. The essence of their transfer principle is contained in a pair of identities that relate the ‘classical’ hypergeometric function $F_2 = F(\frac{1}{2}, \frac{1}{2}; 1; \bullet)$ to the ‘signature four’ hypergeometric function $F_4 = F(\frac{1}{4}, \frac{3}{4}; 1; \bullet)$: explicitly, if $0 < x < 1$ then

$$\sqrt{1+x} F_4(x^2) = F_2(2x/(1+x))$$

and

$$\sqrt{1+x} F_4(1-x^2) = \sqrt{2} F_2((1-x)/(1+x)).$$

These identities are derived in [1995] from hypergeometric identities recorded in the second notebook of Ramanujan: the first is an identity of Kummer, while the second also involves an identity due to Gauss.

Shen in [2014] revealed an actual elliptic function that naturally resides in signature four: his function dn_2 involves incomplete integrals of the hypergeometric function $F(\frac{1}{4}, \frac{3}{4}; \frac{1}{2}; \bullet)$; its construction is motivated by the way in which the classical Jacobian elliptic functions may be developed from $F(\frac{1}{2}, \frac{1}{2}; \frac{1}{2}; \bullet)$. Among the results in [2014] are identifications of the fundamental periods of dn_2 in terms of the hypergeometric function F_4 ; these identifications are established by explicit integral calculations. Also included in [2014] are formulae for dn_2 and its companion functions cn_2 (which is elliptic) and sn_2 (which is not) in terms of the classical Jacobian functions dn , cn and sn to a related modulus; these formulae are established with the aid of theta functions.

In this paper, we reconsider the elliptic function dn_2 of Shen and thereby forge a new route to the signature four transfer principle of Berndt, Bhargava and Garvan. Our reconsideration of dn_2 takes place entirely within the realm of elliptic functions, without the use of theta functions as intermediaries. The title of our paper notwithstanding, our primary aim is not to offer a new proof of the transfer principle: after all, the original proof by manipulation of hypergeometric functions is arguably more direct than a proof based on elliptic functions; rather, it is to reaffirm the status of dn_2 as a natural elliptic function within the signature four theory.

The organization of this paper is as follows. In Section 1 we introduce the Shen elliptic function dn_2 of modulus κ and identify its coperiodic Weierstrass function p_κ in terms of its invariants. In Section 2 we identify the fundamental periods of dn_2 and p_κ in terms of the signature four hypergeometric function F_4 . In Section 3 we identify these fundamental periods in terms of the classical hypergeometric function F_2 . Finally, in Section 4 we compare these two identifications of the periods, deducing the pair of identities that we displayed in our opening paragraph; these identities then yield the relationship between the signature four base q_4 and the classical base q on which rests the Berndt-Bhargava-Garvan transfer principle.

1. THE ELLIPTIC FUNCTION dn_2

Fix $\kappa \in (0, 1)$ as modulus, with corresponding (acute) modular angle $\alpha \in (0, \frac{1}{2}\pi)$ defined by $\sin \alpha = \kappa$ and with complementary modulus $\lambda \in (0, 1)$ defined by $\lambda = (1 - \kappa^2)^{1/2}$. The rule

$$f(T) = \int_0^T F\left(\frac{1}{4}, \frac{3}{4}; \frac{1}{2}; \kappa^2 \sin^2 t\right) dt$$

defines a strictly increasing bijection $f : \mathbb{R} \rightarrow \mathbb{R}$. We write $\phi : \mathbb{R} \rightarrow \mathbb{R}$ for its inverse: thus, if $u \in \mathbb{R}$ then

$$u = \int_0^{\phi(u)} F\left(\frac{1}{4}, \frac{3}{4}; \frac{1}{2}; \kappa^2 \sin^2 t\right) dt.$$

A subsidiary angular function with range $[-\alpha, \alpha]$ is then defined as the composite

$$\psi = \arcsin(\kappa \sin \phi).$$

Now, the function

$$d_\kappa = \cos \psi : \mathbb{R} \rightarrow \mathbb{R}$$

has range $[\cos \alpha, 1] = [\lambda, 1]$ and satisfies the following initial value problem.

Theorem 1. *The function d_κ has initial value $d_\kappa(0) = 1$ and satisfies the differential equation*

$$(d'_\kappa)^2 = 2(1 - d_\kappa)(d_\kappa^2 - \lambda^2).$$

Proof. The initial value is clear: $\phi(0) = 0$ so that $\psi(0) = 0$ and therefore $d(0) = 1$; here and below, we drop the subscript κ when convenient. From $d = \cos \psi$ follows $d' = -(\sin \psi)\psi'$; from $\sin \psi = \kappa \sin \phi$ follows $(\cos \psi)\psi' = \kappa(\cos \phi)\phi'$; and from $f \circ \phi = \operatorname{id}$ follows

$$\phi' = \frac{1}{f' \circ \phi} = \frac{1}{F\left(\frac{1}{4}, \frac{3}{4}; \frac{1}{2}; \kappa^2 \sin^2 \phi\right)} = \frac{1}{F\left(\frac{1}{4}, \frac{3}{4}; \frac{1}{2}; \sin^2 \psi\right)} = \frac{\cos \psi}{\cos \frac{1}{2}\psi}$$

on account of the standard hypergeometric identity

$$F\left(\frac{1}{4}, \frac{3}{4}; \frac{1}{2}; \sin^2 \psi\right) = \frac{\cos \frac{1}{2}\psi}{\cos \psi}$$

for which we refer to item (11) on page 101 in Volume 1 of the compendious Bateman Manuscript Project [1953]. Thus

$$d' = -\sin \psi \left(\kappa \frac{\cos \phi}{\cos \psi} \right) \frac{\cos \psi}{\cos \frac{1}{2}\psi} = -2 \sin \frac{1}{2}\psi (\kappa \cos \phi)$$

and so

$$(d')^2 = 4 \sin^2 \frac{1}{2}\psi (\kappa^2 - \sin^2 \psi) = 2(1 - \cos \psi) (\kappa^2 - 1 + \cos^2 \psi).$$

□

The solution to this initial value problem is readily identifiable in Weierstrassian terms.

Theorem 2. *The function $d_\kappa : \mathbb{R} \rightarrow \mathbb{R}$ satisfies*

$$(1 - d_\kappa)\left(\frac{1}{3} + p_\kappa\right) = \frac{1}{2}\kappa^2$$

where $p_\kappa = \wp(\bullet; g_2, g_3)$ is the Weierstrass function with invariants

$$g_2 = \lambda^2 + \frac{1}{3} \text{ and } g_3 = \frac{1}{3}\lambda^2 - \frac{1}{27}.$$

Proof. Either verify that the function

$$p = -\frac{1}{3} + \frac{\frac{1}{2}\kappa^2}{1 - d}$$

has a pole at 0 and satisfies the differential equation

$$(p')^2 = 4p^3 - \left(\lambda^2 + \frac{1}{3}\right)p - \left(\frac{1}{3}\lambda^2 - \frac{1}{27}\right)$$

or apply the argument that is to be found on page 453 in the classic treatise [1927] of Whittaker and Watson. \square

Thus, d_κ is the restriction to \mathbb{R} of an elliptic function: this is the elliptic function dn_2 of Shen, given by

$$\text{dn}_2 = 1 - \frac{\frac{1}{2}\kappa^2}{\frac{1}{3} + p_\kappa}.$$

The elliptic function dn_2 and the Weierstrass function p_κ are evidently coproperiodic. We shall write $(2\omega_\kappa, 2\omega'_\kappa)$ for their shared fundamental pair of periods such that $\omega_\kappa > 0$ and $-i\omega'_\kappa > 0$. In the next two sections, we shall develop hypergeometric expressions for these periods. To close the present section, it is convenient to record the midpoint values of the Weierstrass function p_κ : in decreasing order, these zeros of the cubic

$$4e^3 - (\lambda^2 + \frac{1}{3})e - (\frac{1}{3}\lambda^2 - \frac{1}{27})$$

are readily checked to be

$$\begin{aligned} e_1 &= p_\kappa(\omega_\kappa) = \frac{1}{6} + \frac{1}{2}\lambda \\ e_2 &= p_\kappa(\omega_\kappa + \omega'_\kappa) = \frac{1}{6} - \frac{1}{2}\lambda \\ e_3 &= p_\kappa(\omega'_\kappa) = -\frac{1}{3}. \end{aligned}$$

2. FUNDAMENTAL PERIODS IN TERMS OF F_4

The very definition of dn_2 as an extension of d_κ provides immediate access to the real half-period ω_κ of dn_2 and p_κ .

Theorem 3. $\omega_\kappa = \frac{1}{2}\pi F(\frac{1}{4}, \frac{3}{4}; 1; \kappa^2)$.

Proof. With

$$I = \int_0^{\frac{1}{2}\pi} F(\frac{1}{4}, \frac{3}{4}; \frac{1}{2}; \kappa^2 \sin^2 t) dt$$

it may be verified by integration that

$$\phi(u + 2I) = \phi(u) + \pi$$

so that

$$\psi(u + 2I) = -\psi(u)$$

and

$$d(u + 2I) = \cos \psi(u + 2I) = \cos \psi(u) = d(u).$$

This shows that dn_2 has $2I$ as a period, which is easily seen to be least positive. Finally, expansion of the hypergeometric integrand and termwise integration show that

$$\int_0^{\frac{1}{2}\pi} F(\frac{1}{4}, \frac{3}{4}; \frac{1}{2}; \kappa^2 \sin^2 t) dt = \frac{1}{2}\pi F(\frac{1}{4}, \frac{3}{4}; 1; \kappa^2).$$

\square

Access to the imaginary half-period ω'_κ of dn_2 and p_κ is facilitated by investigating the relationship between the primary Weierstrass function

$$p_\kappa = \wp(\bullet; \omega_\kappa, \omega'_\kappa) = \wp(\bullet; g_2, g_3)$$

and the auxiliary Weierstrass function

$$q_\kappa = \wp(\bullet; \omega_\kappa, \frac{1}{2}\omega'_\kappa) = \wp(\bullet; h_2, h_3)$$

that results when its imaginary period is halved. Here, the invariants h_2 and h_3 of q_κ are related to the invariants g_2 and g_3 of p_κ by

$$\begin{aligned} h_2 &= -4g_2 + 60p_\kappa(\omega'_\kappa)^2 \\ h_3 &= 8g_3 + 56p_\kappa(\omega'_\kappa)^3. \end{aligned}$$

This is a quite general consequence of the halving of a Weierstrassian period, for the proof of which we refer to Section 9.8 of [1989].

Theorem 4. $\omega'_\kappa = i\sqrt{2}\frac{1}{2}\pi F(\frac{1}{4}, \frac{3}{4}; 1; 1 - \kappa^2)$.

Proof. When the invariants of p_κ as displayed in Theorem 2 and the subsequent evaluation $p_\kappa(\omega'_\kappa) = -1/3$ are taken into account, we find that q_κ has invariants

$$h_2 = \frac{4}{3} + 4\kappa^2 = (i\sqrt{2})^4(\kappa^2 + \frac{1}{3})$$

$$h_3 = \frac{8}{27} - \frac{8}{3}\kappa^2 = (i\sqrt{2})^6(\frac{1}{3}\kappa^2 - \frac{1}{27}).$$

By a further consultation of Theorem 2 (but for the complementary modulus) in conjunction with the homogeneity relation for \wp functions, we deduce that q_κ is related to the Weierstrass function p_λ of complementary modulus according to the rule

$$q_\kappa(z) = -2p_\lambda(i\sqrt{2}z).$$

Now on the one hand q_κ has fundamental half-periods ω_κ and $\frac{1}{2}\omega'_\kappa$, while on the other hand p_λ has fundamental half-periods ω_λ and ω'_λ . In light of the above rule by which q_κ and p_λ are related, we see that

$$\omega'_\kappa = i\sqrt{2}\omega_\lambda.$$

It only remains to invoke Theorem 3 (for the complementary modulus) and recall that $\lambda^2 = 1 - \kappa^2$. \square

3. FUNDAMENTAL PERIODS IN TERMS OF F_2

In order to obtain equivalent expressions for ω_κ and ω'_κ in terms of the hypergeometric function $F(\frac{1}{2}, \frac{1}{2}; 1; \bullet)$ we shall reformulate the Weierstrass function p_κ in terms of classical Jacobian elliptic functions.

Recall from page 505 of [1927] that if the Weierstrass function p has real midpoint values $e_1 > e_2 > e_3$ then

$$p(z) = e_3 + \frac{e_1 - e_3}{\text{sn}^2[z(e_1 - e_3)^{1/2}]}$$

where $\text{sn} = \text{sn}(\bullet, k)$ is the Jacobian sine function with modulus $k \in (0, 1)$ given by

$$k^2 = \frac{e_2 - e_3}{e_1 - e_3}$$

and its square sn^2 has fundamental periods $(2K, 2iK')$ given by

$$K = \frac{1}{2}\pi F(\frac{1}{2}, \frac{1}{2}; 1; k^2) \quad \text{and} \quad K' = \frac{1}{2}\pi F(\frac{1}{2}, \frac{1}{2}; 1; 1 - k^2).$$

Accordingly, p itself has fundamental half-periods

$$\frac{K}{(e_1 - e_3)^{1/2}} \quad \text{and} \quad i\frac{K'}{(e_1 - e_3)^{1/2}}.$$

Theorem 5. *The half-periods ω_κ and ω'_κ of dn_2 and p_κ are given by*

$$\sqrt{\frac{1+\lambda}{2}} \omega_\kappa = \frac{1}{2}\pi F(\frac{1}{2}, \frac{1}{2}; 1; \frac{1-\lambda}{1+\lambda})$$

and

$$\sqrt{\frac{1+\lambda}{2}} \omega'_\kappa = i\frac{1}{2}\pi F(\frac{1}{2}, \frac{1}{2}; 1; \frac{2\lambda}{1+\lambda}).$$

Proof. Apply to $p = p_\kappa$ the foregoing recollections. As noted after Theorem 2, p_κ has midpoint values

$$e_1 = \frac{1}{6} + \frac{1}{2}\lambda, \quad e_2 = \frac{1}{6} - \frac{1}{2}\lambda, \quad e_3 = -\frac{1}{3}$$

so that

$$k^2 = \frac{e_2 - e_3}{e_1 - e_3} = \frac{1 - \lambda}{1 + \lambda}$$

and

$$1 - k^2 = \frac{2\lambda}{1 + \lambda}$$

while

$$(e_1 - e_3)^{1/2} = \sqrt{\frac{1+\lambda}{2}}.$$

□

Of course, we may also use the assembled information to express the elliptic function dn_2 in terms of the classical Jacobian elliptic functions to modulus k . Thus, the relation

$$p_\kappa(z) = -\frac{1}{3} + \frac{\frac{1}{2}(1 + \lambda)}{\text{sn}^2\left[z\left(\frac{1}{2}(1 + \lambda)\right)^{1/2}\right]}$$

may be recast as

$$\text{dn}_2(z) = 1 - (1 - \lambda) \text{sn}^2\left[\left(\frac{1}{2}(1 + \lambda)\right)^{1/2}z\right];$$

equivalently, it may be recast either in terms of the Jacobian cosine function cn as

$$\text{dn}_2(z) = \lambda + (1 - \lambda) \text{cn}^2\left[\left(\frac{1}{2}(1 + \lambda)\right)^{1/2}z\right]$$

or in terms of the Jacobian ‘delta amplitude’ dn as

$$\text{dn}_2(z) = -\lambda + (1 + \lambda) \text{dn}^2\left[\left(\frac{1}{2}(1 + \lambda)\right)^{1/2}z\right].$$

Incidentally, it may be checked that the Jacobian modulus k equals $\tan \frac{1}{2}\alpha$.

4. THE TRANSFER PRINCIPLE

All the pieces are in place: we are now in a position to deduce the hypergeometric identities that opened our paper.

Theorem 6. *If $0 < \lambda < 1$ then*

$$\sqrt{1 + \lambda} F\left(\frac{1}{4}, \frac{3}{4}; 1; 1 - \lambda^2\right) = \sqrt{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{1 - \lambda}{1 + \lambda}\right)$$

and

$$\sqrt{1 + \lambda} F\left(\frac{1}{4}, \frac{3}{4}; 1; \lambda^2\right) = F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{2\lambda}{1 + \lambda}\right).$$

Proof. Direct comparison of Theorem 3 with the first formula of Theorem 5 yields

$$\sqrt{1 + \lambda} F\left(\frac{1}{4}, \frac{3}{4}; 1; \kappa^2\right) = \sqrt{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{1 - \lambda}{1 + \lambda}\right)$$

while direct comparison of Theorem 4 with the second formula of Theorem 5 yields

$$\sqrt{1 + \lambda} F\left(\frac{1}{4}, \frac{3}{4}; 1; 1 - \kappa^2\right) = F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{2\lambda}{1 + \lambda}\right).$$

□

As in [1995] these hypergeometric identities entail a connexion between the base q_4 that is appropriate to the signature four elliptic theory and the base q that is appropriate to the classical elliptic theory. To be explicit, Theorem 6 implies that

$$\frac{F\left(\frac{1}{4}, \frac{3}{4}; 1; 1 - \lambda^2\right)}{F\left(\frac{1}{4}, \frac{3}{4}; 1; \lambda^2\right)} = \sqrt{2} \frac{F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{1 - \lambda}{1 + \lambda}\right)}{F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{2\lambda}{1 + \lambda}\right)}$$

whence

$$q_4(\lambda^2) := \exp\left\{-\pi\sqrt{2} \frac{F\left(\frac{1}{4}, \frac{3}{4}; 1; 1 - \lambda^2\right)}{F\left(\frac{1}{4}, \frac{3}{4}; 1; \lambda^2\right)}\right\}$$

and

$$q\left(\frac{2\lambda}{1+\lambda}\right) := \exp\left\{-\pi \frac{F\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - \frac{2\lambda}{1+\lambda}\right)}{F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{2\lambda}{1+\lambda}\right)}\right\}$$

satisfy the relation

$$q_4(\lambda^2) = q\left(\frac{2\lambda}{1+\lambda}\right)^2.$$

The signature four transfer principle now follows exactly as in [1995].

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