# On harmonic symmetries for locally conformally Kähler manifolds

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#### Abstract

In this article, we study harmonic symmetries on the compact locally conformally Kähler manifold M of  $\dim_{\mathbb{C}} = n$ . The space of harmonic symmetries is a subspace of harmonic differential forms which defined by the kernel of a certain Laplacian-type operator  $\Box$ . We observe that the spaces  $\ker(\Box) \cap \Omega^l = \{0\}$  for any  $|l - n| \geq 2$  and  $\ker \Delta_{\bar{\partial}} \cap P^{k,n-1-k} \cap \ker(i_{\theta^{\sharp}}) \cong \ker(\Box^{k,n-1-k})$ ,  $\ker \Delta_{\bar{\partial}} \cap P^{k,n-k} \cong \ker(\Box^{k,n-k})$ . Furthermore, suppose that M is a Vaisman manifold, we prove that (i)  $\alpha$  is (n - 1)-form in  $\ker(\Box)$  if only if  $\alpha$  is a transversally harmonic and transversally effective  $\mathcal{V}$ -foliate form; (ii)  $\alpha$  is a (p, n - p)-form in  $\ker(\Box^{p,n-p})$  if only if there are two forms  $\beta_1 \in S^{p-1,n-p}$ and  $\beta_2 \in S^{p,n-p-1}$  such that  $\alpha = \theta^{1,0} \wedge \beta_1 + \theta^{0,1} \wedge \beta_2$ .

Keywords. LCK manifold, Vaisman manifold, harmonic symmetries, Hodge theory

# **1** Introduction

On a compact complex manifold, one can consider two different kinds of invariants: the topological ones of the underlying (real) manifold and the complex ones. Among the first ones, a fundamental role is played by de Rham cohomology, and among the second ones, we recall the Dolbeault, Bott–Chern and Aeppli cohomologies. The dimensions of Dolbeault, Bott-Chern and Aeppli cohomologies which are bounded from below by topological quantities, such as Betti numbers (see [1, 2]). But there are comparatively few reverse inequalities, and these are desirable for showing that a complex manifold has non-trivial topology, or conversely that a smooth manifold does not have a complex structure.

We denote by M a compact Hermitian manifold M, that is, M is a compact complex manifold with compatible metric. In [24, 25], Wilson described some topological and geometric inequalities for M. They are expressed in terms of the kernel of a certain Laplacian-type operator

$$\Box = \Delta_{\partial} + \Delta_{\bar{\partial}} + \Delta_{\tau} + \Delta_{\bar{\tau}} + \Delta_{\lambda} + \Delta_{\bar{\lambda}}.$$

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This is a real operator, and the last four summands are all order zero. The kernel of this second-order selfadjoint elliptic operator  $\Box$  determines a subspace of  $\Delta_d$ -harmonic forms that satisfies the Serre, Hodge, and conjugation dualities, generalizing the Kähler case. Moreover, there is an induced representation of  $\mathfrak{sl}(2,\mathbb{C})$  on the harmonic forms on ker( $\Box$ ), yielding a generalization of hard Lefschetz duality (see [24, Theorem 3.1]). The result relies on a generalization of the Kähler identities to the Hermitian setting [5, 10].

Define the space of  $\Box$ -harmonic forms in degree k by letting

$$\mathcal{H}^k_{\Box} = \ker(\Box) \cap \Omega^k$$

where  $\Omega^k$  denotes the space of k-forms. We denote by  $h_{\Box}^k$  the dimension of  $\mathcal{H}_{\Box}^k$ . It remains to further study what the dimensions of those subspaces tell us about a given Hermitian structure, and conversely, to determine what are the permissible numbers for a given complex structure.

In this article, we are interested in topological and complex analytic properties of compact LCK manifolds. A locally conformally Kähler (LCK) manifold is a Hermitian manifold whose metric is conformal to a Kähler metric in some neighbourhood of every point. This definition is equivalent to the existence of a global closed one-form  $\theta$  (called the **Lee** form) such that the fundamental two-form  $\omega$  satisfies  $d\omega = \theta \wedge \omega$ (see [7, 18, 21]). These manifolds appear naturally in complex geometry. Most examples of compact non-Kähler manifolds studied in complex geometry admit an LCK structure. An interesting example of such manifolds is offered by the Hopf manifolds [19], which are compact and have no Kähler metric at all. In many situations, the LCK structure becomes useful for the study of topology and complex geometry of an LCK manifold. Throughout our article, we always say that an LCK manifold M cannot admit any Kahler metric, that is, M is a non-Käherian manifold.

**Theorem 1.1** (=Theorem 3.5 and 3.7). Let  $(M, J, \theta)$  be a compact locally conformally Kähler manifold of  $\dim_{\mathbb{C}} = n$ . Then for any  $|l - n| \ge 2$ , we have

$$\mathcal{H}^l_{\square} = \{0\}.$$

Furthermore,

$$\ker \Delta_{\bar{\partial}} \cap P^{k,n-1-k} \cap \ker(i_{\theta^{\sharp}}) \cong \ker(\Box^{k,n-1-k}),$$
$$\ker \Delta_{\bar{\partial}} \cap P^{k,n-k} \cong \ker(\Box^{k,n-k}).$$

If (M, J) is a compact complex manifold, then dim ker $(\Box^{0,0}) = 1$  if and only if M is Kählerian (resp. dim ker $(\Box^{0,0}) = 0$  if and only if M is non-Kählerian) (see [24]). In [20], the author have given some sufficient conditions for a compact LCK manifold to be Kähler. Following Theorem 3.5, we can give a sufficient condition to such that a compact complex non-Kählerian manifold admits a Hermitian metric g which is not an LCK metric.

**Corollary 1.2.** Let (M, J, g) be a compact, complex, non-Kählerian manifold of  $\dim_{\mathbb{C}} = n$ . If there is a constant k,  $|k - n| \ge 2$ , such that  $h_{\square}^k \ge 1$ , then g is not an locally conformally Kähler metric.

Among the LCK manifolds, a distinguished class is the Vaisman manifold. The topology of compact Vaisman manifolds is very different from that of Kähler manifolds. In [16], the authors studied harmonic forms and Hodge decomposition on Vaisman and Sasakian manifolds. In [21], Vaisman established a Hodge decomposition theorem of  $\Delta_d$ -harmonic k-form for any  $k \leq n - 1$ . In [18], Tsukada obtained a complex version Hodge decomposition theorem on Vaisman manifold. But none of them give a Hodge decomposition of  $\Delta_d$ -harmonic n-form. Here, we study the Hodge decomposition of the forms in ker  $\Box \cap$  $\Omega^k$ , k = n - 1, n, on Vaisman manifold. We will develop a version of Hodge theory on the Vaisman manifold, hence we can give a Hodge decomposition of the  $\Delta_d$ -harmonic n-forms.

**Theorem 1.3** (=Theorem 3.14 and 3.16). Let  $(M, J, \theta)$  be a compact Vaisman manifold of  $\dim_{\mathbb{C}} = n$ . Then

$$\ker(\Box^{p,n-1-p}) \cong \mathcal{S}^{p,n-1-p}(\mathcal{V})$$
$$\mathcal{H}^n_d \cong \mathcal{H}^n_{\Box} \cong \bigoplus_{k=0}^n \mathcal{H}^{k,n-k}_{\bar{\partial}},$$

where  $S^{p,q}$  denote the vector space of transversally harmonic and transversally effective  $\mathcal{V}$ -foliate (p,q)forms. Furthermore, the following four conditions are equivalent: (i)  $\alpha \in \ker(\square^{p,n-p})$ , (ii)  $\alpha \in \ker \Delta_{\bar{\partial}} \cap \Omega^{p,n-p}$ , (iii)  $\alpha \in \ker \Delta_d \cap \Omega^{p,n-p}$ ,

(iv) there are  $\beta_1 \in S^{p-1,n-p}$  and  $\beta_2 \in S^{p,n-p-1}$  such that

$$\alpha = \theta^{1,0} \wedge \beta_1 + \theta^{0,1} \wedge \beta_2.$$

### 2 **Preliminaries**

#### 2.1 Harmonic symmetries

Throughout this section (M, J, g) will denote a compact Hermitian manifold of complex dimension  $n \ge 2$ , with fundamental two-form defined by  $\omega(X, Y) = g(X, JY)$ . The space of all smooth (p, q)-forms (resp. k-forms) on M denoted  $\Omega^{p,q}(M)$  (resp.  $\Omega^k(M)$ . Let  $\langle, \rangle$  denote the pointwise inner product. The global inner product is defined

$$(\alpha,\beta) = \int_M \langle \alpha,\beta \rangle dV,$$

where  $dV = \frac{\omega^n}{n!}$ . Another important operator is the operator L of type (1, 1) defined by

$$L(\cdot) = \omega \wedge \cdot,$$

and its adjoint  $\Lambda = *^{-1}L*$ :

$$\langle \alpha, \Lambda \beta \rangle = \langle L\alpha, \beta \rangle.$$

**Definition 2.1.** A differential k-form  $\alpha_k$  with  $k \leq n$  is called primitive, i.e.,  $\alpha_k \in P^k(M)$ , if it satisfies the two equivalent conditions: (i)  $\Lambda \alpha_k = 0$ ; (ii)  $L^{n-k+1}\alpha_k = 0$ .

One can also prove the following very important result: every k-form  $\alpha$  on M has a unique decomposition of the form

$$\alpha = \sum_{r \ge \max(0,k-n)} L^r \beta_{k-2r}$$

where all of the  $\beta_j$  are primitive forms of a corresponding degree and we denote  $\beta_j = 0$  for  $j \notin [0, n]$ .

Let  $\lambda = [\partial, L] = (\partial \omega \wedge \cdot)$ , so that  $\overline{\lambda} = [\overline{\partial}, L] = (\overline{\partial} \omega \wedge \cdot)$ . The operator  $\overline{\lambda}$  has bidegree (1, 2) and governs the symplectic condition: a Hermitian manifold is Kähler if and only if  $\overline{\lambda} = 0$ . In [5], Demailly derives a set of Hermitian identities which generalize the Kähler identities. Consider the zero-order torsion operator  $\tau := [\Lambda, \lambda]$  of bidegree (1, 0). Demailly shown

$$\begin{split} [\Lambda,\partial] &= -i(\partial^* + \tau^*), \\ [\Lambda,\partial] &= i(\bar{\partial}^* + \bar{\tau}^*), \\ [L,\bar{\partial}^*] &= -i(\partial + \tau), \\ [L,\partial^*] &= i(\bar{\partial} + \bar{\tau}) \end{split}$$
(2.1)

with Kähler identities recovered in the case  $\tau=0$  and

$$\begin{split} &[\Lambda,\tau] = -2i\bar{\tau}^{*}, \ [L,\bar{\tau}] = 3\bar{\lambda}, \\ &[\Lambda,\bar{\tau}] = 2i\tau^{*}, \ [L,\tau] = 3\lambda, \\ &[L,\tau^{*}] = -2i\bar{\tau}, \ [\Lambda,\bar{\tau}^{*}] = -3\bar{\lambda}^{*}, \\ &[L,\bar{\tau}^{*}] = 2i\tau, \ [\Lambda,\tau^{*}] = -3\lambda^{*}. \end{split}$$

$$(2.2)$$

We also have (see [24])

$$[\Lambda, \lambda] = \tau, \ [L, \lambda] = 0,$$
  

$$[\Lambda, \bar{\lambda}] = \bar{\tau}, \ [L, \bar{\lambda}] = 0,$$
  

$$[L, \lambda^*] = -\tau^*, \ [\Lambda, \lambda^*] = 0,$$
  

$$[L, \bar{\lambda}^*] = -\bar{\tau}^*, \ [\Lambda, \bar{\lambda}^*] = 0.$$
  
(2.3)

For any operator  $\delta$ , let  $\Delta_{\delta} = [\delta, \delta^*]$ , and let  $\Delta_{\delta}^{p,q}$  denote the restriction to  $\Omega^{p,q}$ .

**Proposition 2.2.** ([24, Corollary 2.1]) For any Hermitian manifold there is an induced representation of  $\mathfrak{sl}(2,\mathbb{C})$  on the space ker $(\Delta_{\tau} + \Delta_{\bar{\tau}} + \Delta_{\lambda} + \Delta_{\bar{\lambda}})$ .

The operator  $\Delta_{\tau} + \Delta_{\bar{\tau}} + 3\Delta_{\lambda} + 3\Delta_{\bar{\lambda}}$  is the one that commutes directly with L and  $\Lambda$  on all forms (see [25, Equation (4)]).Noting that  $\ker(\Delta_{\tau} + \Delta_{\bar{\tau}} + \Delta_{\lambda} + \Delta_{\bar{\lambda}}) = \ker(\Delta_{\tau} + \Delta_{\bar{\tau}} + 3\Delta_{\lambda} + 3\Delta_{\bar{\lambda}})$ . Hence, we get

$$\begin{split} [L, \Delta_{\tau} + \Delta_{\bar{\tau}} + \Delta_{\lambda} + \Delta_{\bar{\lambda}}]|_{\ker(\Delta_{\tau} + \Delta_{\bar{\tau}} + \Delta_{\lambda} + \Delta_{\bar{\lambda}})} &= 0, \\ [\Lambda, \Delta_{\tau} + \Delta_{\bar{\tau}} + \Delta_{\lambda} + \Delta_{\bar{\lambda}}]|_{\ker(\Delta_{\tau} + \Delta_{\bar{\tau}} + \Delta_{\lambda} + \Delta_{\bar{\lambda}})} &= 0. \end{split}$$

Lemma 2.3. For any Hermitian manifold,

$$\ker(\lambda) \cap \ker(\bar{\lambda}) \cap P^k \subset \ker(\Delta_\tau + \Delta_{\bar{\tau}} + \Delta_\lambda + \Delta_{\bar{\lambda}}) \cap \Omega^k$$

*Proof.* We denote by  $\alpha$  a primitive k-form in ker $(\lambda) \cap \text{ker}(\overline{\lambda})$ . By the definition of  $\tau, \overline{\tau}$ , we have

$$\tau \alpha = [\Lambda, \lambda] \alpha = 0, \ \bar{\tau} \alpha = [\Lambda, \bar{\lambda}] \alpha = 0.$$

Therefore,

$$\begin{split} \tau^* \alpha &= -\frac{i}{2} [\Lambda, \bar{\tau}] \alpha = 0, \; \bar{\tau}^* \alpha = \frac{i}{2} [\Lambda, \tau] \alpha = 0, \\ \lambda^* \alpha &= -\frac{1}{3} [\Lambda, \tau^*] \alpha = 0, \; \bar{\lambda}^* \alpha = -\frac{1}{3} [\Lambda, \bar{\tau}^*] \alpha = 0. \end{split}$$

Hence,  $\alpha \in \ker(\Delta_{\tau} + \Delta_{\bar{\tau}} + \Delta_{\lambda} + \Delta_{\bar{\lambda}}) \cap \Omega^k$ .

Wilison considered the following positive definite self-adjoint elliptic operator of order two:

$$\Box = \Delta_{\partial} + \Delta_{\bar{\partial}} + \Delta_{\tau} + \Delta_{\bar{\tau}} + \Delta_{\lambda} + \Delta_{\bar{\lambda}}.$$

Let  $\Box^{p,q}$  denote the restriction to  $\Omega^{p,q}$ .

**Theorem 2.4.** ([24, Theorem 3.1]) Let  $(M, J, \omega)$  be a compact Hermitian manifold of complex dimension n. For any  $0 \le k \le 2n$ , there is an orthogonal direct sum decomposition

$$\ker(\Box) \cap \Omega^k = \bigoplus_{p+q=k} \ker(\Box^{p,q})$$

For all  $0 \le p, q \le n$ , the following dualities hold: (1) (Complex conjugation). We have equalities

$$\ker(\Box^{p,q}) = \overline{\ker(\Box^{q,p})}.$$

(2) (Hodge duality). The Hodge \*-operator induces isomorphisms

$$*: \ker(\square^{p,q}) \to \ker(\square^{n-q,n-p}).$$

(3) (Serre duality). There are isomorphisms

$$\ker(\Box^{p,q}) \cong \ker(\Box^{n-p,n-q}).$$

The operator  $\{L, \Lambda, H\}$  define a finite dimensional representation of  $\mathfrak{sl}(2, \mathbb{C})$  on ker $(\Box)$ . Moreover, for every  $0 \le p, q \le n$ , the maps

$$L^{n-p-q}$$
: ker( $\Box^{p,q}$ )  $\rightarrow$  ker( $\Box^{n-q,n-p}$ )

are isomorphisms.

The next result follows from some well established facts about  $sl(2, \mathbb{C})$  representations.

**Corollary 2.5.** ([24, Corollary 3.2]) There is an orthogonal direct sum decomposition

$$\ker(\Box^{p,q}) = \bigoplus_{j \ge 0} L^j(\ker(\Box^{p-j,q-j}))_{prim}$$

where

$$(\ker(\Box^{r,s}))_{prim} := \ker(\Box^{r,s}) \cap P^{r,s}$$

**Proposition 2.6.** Let  $(M, J, \omega)$  be a compact complex manifold of  $\dim_{\mathbb{C}} = n$ . Then

$$\ker(\Box^{n,0}) = \ker(\Delta_{\bar{\partial}}) \cap \Omega^{n,0},$$
$$\ker(\Box^{0,n}) = \ker(\Delta_{\bar{\partial}}) \cap \Omega^{0,n}.$$

*Proof.* We only proof the case of (n, 0)-forms. We now denote by  $\alpha^{n,0}$  a (n, 0)-form in ker $(\Delta_{\bar{\partial}})$ . Therefore,  $\bar{\partial}\alpha^{n,0} = 0$ . At first, we observe that

$$\Lambda \alpha^{n,0} = \lambda \alpha^{n,0} = \bar{\lambda} \alpha^{n,0} = 0.$$

Following Lemma 2.3, we have

$$\Omega^{n,0} = \ker(\Delta_{\tau} + \Delta_{\bar{\tau}} + \Delta_{\lambda} + \Delta_{\bar{\lambda}}) \cap \Omega^{n,0}.$$

It's easy to see  $\partial \alpha^{n,0} = 0$ . Following the first identity on (2.1), we have

$$\partial^* \alpha^{n,0} = (i[\Lambda, \bar{\partial}] - \tau^*) \alpha^{n,0} = 0.$$

Therefore,

$$\ker(\Delta_{\bar{\partial}}) \cap \Omega^{n,0} \subset \ker(\Box^{n,0}) \subset \ker(\Delta_{\bar{\partial}}) \cap \Omega^{n,0},$$

that is,  $\ker(\Delta_{\bar{\partial}}) \cap \Omega^{n,0} \cong \ker(\Box^{n,0}).$ 

#### 2.2 Morse-Novikov cohomology

Let *M* be a smooth manifold and  $\theta$  a real valued closed one form on *M*. Define  $d_{\theta} : \Omega^{k}(M) \to \Omega^{k+1}(M)$ as  $d_{\theta}\alpha = d\alpha + \theta \wedge \alpha$  for  $\alpha \in \Omega^{p}(M)$ . Then we have a complex

$$\ldots \to \Omega^{k-1}(M) \xrightarrow{d_{\theta}} \Omega^{k}(M) \xrightarrow{d_{\theta}} \Omega^{k+1}(M) \to \ldots$$

whose cohomology  $H^k(M, \theta) = H^k(\Omega^*(M), d_{\theta})$  is called the *k*-th Morse-Novikov cohomology group of M with respect to  $\theta$ .  $H^k(M, \theta)$  only depends on the de Rham cohomology class of  $\theta$ . This cohomology shares many properties with the ordinary de Rham cohomology (see [4, 14, 15, 17]).

We can also define an operator  $d_{\theta}^*$  as the formal adjoint of  $d_{\theta}$  with respect to metric g. Further,  $\Delta_{\theta} = d_{\theta}d_{\theta}^* + d_{\theta}^*d_{\theta}$  is the corresponding Laplacian. These operators are lower-order perturbations of the corresponding operators in the usual Hodge-de Rham theory. We denote by  $\mathcal{H}^k(M, \theta)$  the space of  $\Delta_{\theta}$ -harmonic forms. The space  $\mathcal{H}^k(M, \theta)$  is isomorphic to  $H^k(M, \theta)$ . Let  $\theta^{\sharp}$  the dual vector field of  $\theta$ defined by  $g(\theta^{\sharp}, \cdot) = \theta(\cdot)$ .

**Lemma 2.7.** ([3, Lemma 4.4]) For any k-form  $\alpha$ , we have

$$*i_{\theta^{\sharp}}\alpha = (-1)^{k-1}\theta \wedge *\alpha.$$

We denote by

$$\mathcal{S}^{k}(M,\theta) = \{ \alpha \in \Omega^{k}(M) : \Delta_{d}\alpha = 0, \theta \land \alpha = 0, \theta \land *\alpha = 0 \}$$

a subspace of  $\mathcal{H}^k(M,\theta)$  on a closed manifold. We then have the following vanishing theorem.

**Theorem 2.8.** Let M be a n-dimensional closed Riemannian manifold,  $\theta$  a smooth 1-form in M. If  $\alpha$  is smooth k-form in  $S^k(M, \theta)$ ,  $0 \le k \le n$ , then either  $\theta = 0$  or  $\alpha = 0$ .

*Proof.* We also assume that  $\alpha$  is a non-zero  $\Delta_d$ -harmonic k-form, i.e.,  $(d + d^*)\alpha = 0$ . Following Lemma 2.7, the equation  $\theta \wedge *\alpha = 0$  is equivalent to  $i_{\theta^{\sharp}}\alpha = 0$ . Then we have

$$0 = i_{\theta^{\sharp}}(\theta \wedge \alpha) = |\theta|^2 \alpha.$$

We denote by  $Z^{c}(\alpha)$  the complement of the zero of  $\alpha$ . By unique continuation of the elliptic equation  $(d + d^{*})\alpha = 0$ ,  $Z^{c}(\alpha)$  is either empty or dense. Therefore, the vector field  $\theta^{\sharp}$  is zero along  $Z^{c}(\alpha)$ . The set  $Z^{c}(\alpha)$  is empty which is equivalent to  $\alpha = 0$  on M. If  $Z^{c}(\alpha)$  is dense, then  $\theta^{\sharp} = 0$  almost everywhere on M. Since  $\theta^{\sharp}$  is smooth,  $\theta^{\sharp} = 0$ , i.e.,  $\theta = 0$  on M.

## 3 Harmonic symmetries on locally conformally Kähler manifold

#### 3.1 Locally conformally Kähler manifold

In this section, we first give the necessary definitions and properties of locally conformally Kähler (LCK) manifolds.

**Definition 3.1.** Let  $(M, \omega)$  be a complex Hermitian manifold of dim<sub>C</sub> M = n, with

$$d\omega = \theta \wedge \omega.$$

where  $\theta$  is a closed 1-form. Then M is called an LCK manifold.

Therefore  $\omega$  is  $(d-\theta)$ -closed. The Morse–Novikov cohomology class  $[\omega]$  of  $\omega$  is called the Morse–Novikov class of M (see [4, 11, 15, 17]). This notion is similar to the notion of a Kähler class of a Kähler manifold.

**Lemma 3.2.** Let  $(M, J, \theta)$  be a compact locally conformally Kähler manifold of  $\dim_{\mathbb{C}} = n$ ,  $\alpha$  a smooth k-form in  $\ker(\Delta_{\tau} + \Delta_{\bar{\tau}} + \Delta_{\lambda} + \Delta_{\bar{\lambda}})$ . Then for any  $|k - n| \ge 2$ , we have

$$\theta \wedge \alpha = 0, \ \theta \wedge *\alpha = 0.$$

*Proof.* By the definitions of  $\lambda$  and  $\overline{\lambda}$ , we obtain that

$$\lambda + \bar{\lambda} = (\partial \omega + \bar{\partial} \omega) \wedge (\cdot) = \omega \wedge \theta \wedge (\cdot).$$

For simply, we let  $k \leq n-2$ . The case  $k \geq n+2$  follows by the Poincare duality as the operator  $*: \Omega^k \to \Omega^{2n-k}$  commutes with  $\ker(\Delta_{\tau} + \Delta_{\bar{\tau}} + \Delta_{\lambda} + \Delta_{\bar{\lambda}})$ . Since  $\alpha \in \ker(\Delta_{\tau} + \Delta_{\bar{\tau}} + \Delta_{\lambda} + \Delta_{\bar{\lambda}}) \cap \Omega^k$ ,  $\alpha$  satisfies

$$\lambda \alpha = \bar{\lambda} \alpha = 0,$$

Therefore, we get

$$\omega \wedge (\theta \wedge \alpha) = 0.$$

We then have

$$L^{n-k-1}(\theta \wedge \alpha) = 0$$

Since the map  $L^{n-k-1}: \Omega^{k+1} \to \Omega^{2n-k-1}$  is bijective for  $k+1 \le n-1$  (see [13]), we get

$$\theta \wedge \alpha = 0$$

For any k-form  $\alpha$  on M,  $(k \le n-2)$ , there exists a k-form  $\beta$  such that as

$$*\alpha = L^{n-k}\beta,$$

Noting that  $(\Delta_{\tau} + \Delta_{\bar{\tau}} + \Delta_{\lambda} + \Delta_{\bar{\lambda}})(*\alpha) = 0$  and

$$[L, \Delta_{\tau} + \Delta_{\bar{\tau}} + \Delta_{\lambda} + \Delta_{\bar{\lambda}}]|_{\ker(\Delta_{\tau} + \Delta_{\bar{\tau}} + \Delta_{\lambda} + \Delta_{\bar{\lambda}})} = 0.$$

Therefore, we have

$$0 = (\Delta_{\tau} + \Delta_{\bar{\tau}} + \Delta_{\lambda} + \Delta_{\bar{\lambda}})(*\alpha) = L^{n-k}(\Delta_{\tau} + \Delta_{\bar{\tau}} + \Delta_{\lambda} + \Delta_{\bar{\lambda}})\beta$$

Since the map  $L^{n-k}: \Omega^k \to \Omega^{2n-k}$  is bijective for  $k \leq n$  (see [13]),

$$(\Delta_{\tau} + \Delta_{\bar{\tau}} + \Delta_{\lambda} + \Delta_{\bar{\lambda}})\beta = 0.$$

By a similar way, we also have

$$\theta \wedge \beta = 0.$$

Hence

$$\theta \wedge *\alpha = L^{n-k}(\theta \wedge \beta) = 0$$

**Corollary 3.3.** Let  $(M, J, \theta)$  be a compact locally conformally Kähler manifold of  $\dim_{\mathbb{C}} = n$ . If  $\theta$  is  $\Delta_d$ -harmonic, then for any  $|k - n| \ge 2$ , we have

$$\ker(\Delta_{\tau} + \Delta_{\bar{\tau}} + \Delta_{\lambda} + \Delta_{\bar{\lambda}}) \cap \Omega^{k} = \{0\}.$$

In particular, if  $(M, J, \theta)$  is a Vaisman manifold, i.e.,  $\nabla \theta = 0$ , then  $\ker(\Delta_{\tau} + \Delta_{\bar{\tau}} + \Delta_{\lambda} + \Delta_{\bar{\lambda}}) \cap \Omega^k = \{0\}$ .

*Proof.* We denote by  $\alpha$  a k-form in ker $(\Delta_{\tau} + \Delta_{\bar{\tau}} + \Delta_{\lambda} + \Delta_{\bar{\lambda}}) \cap \Omega^k$ . By Lemma 3.2, we have  $\theta \wedge \alpha = 0$ . Therefore, we have

$$i_{\theta^{\sharp}}(\theta \wedge \alpha) = \alpha |\theta|^2 = 0.$$

By unique continuation of the elliptic equation  $(d + d^*)\theta = 0$ ,  $Z^c(\theta)$  is dense or empty. Since  $\theta$  is non-zero,  $Z^c(\theta)$  is dense. Therefore, the k-form  $\alpha$  is zero along  $Z^c(\theta)$ , i.e.,  $\alpha = 0$  almost everywhere on M. Since  $\alpha$  is smooth,  $\alpha = 0$  all over M.

**Remark 3.4.** The Lee form  $\theta$  is co-closed with respect to g if only if the metric g is Gauduchon, i.e.,  $\partial \bar{\partial} \omega^{n-1} = 0$  ([9, pp. 502]). A classical result of Gauduchon [8] states that every Hermitian metric is conformal to a Gauduchon metric, which is unique up to rescaling when  $n \ge 2$ .

The Hermitian metric on general LCK manifold could not always Gauduchon. But when we consider the space  $\mathcal{H}^k_{\Box}$ , we can also prove the following vanishing theorem.

**Theorem 3.5.** Let  $(M, J, \theta)$  be a compact locally conformally Kähler manifold of  $\dim_{\mathbb{C}} = n$ . Then for any  $|k - n| \ge 2$ , we have

$$\mathcal{H}^k_{\Box} = \{0\}.$$

*Proof.* For any k-form  $\alpha \in \ker(\Box)$  on M,  $(|k - n| \ge 2)$ , following Lemma 3.2, we get

$$\theta \wedge \alpha = \theta \wedge *\alpha = 0.$$

Hence following Theorem 2.8, we get either  $\theta = 0$  or  $\alpha = 0$ . Since M is non-Kählerian, there are some points in M such that  $\theta \neq 0$ . Therefore,  $\alpha \equiv 0$ , i.e., ker $(\Box) \cap \Omega^k = \{0\}$ .

We now study the relationship between  $\ker(\Delta_{\tau} + \Delta_{\bar{\tau}} + \Delta_{\lambda} + \Delta_{\bar{\lambda}}) \cap \Omega^{n-1}$  and  $\ker(\Delta_{\tau} + \Delta_{\bar{\tau}} + \Delta_{\lambda} + \Delta_{\bar{\lambda}}) \cap \Omega^n$ . The following result is very important in the study of Vaisman manifold.

**Proposition 3.6.** Let  $(M, J, \theta)$  be a compact locally conformally Kähler manifold of  $\dim_{\mathbb{C}} = n$ . If a (n-1)-form  $\alpha \in \ker(\Delta_{\tau} + \Delta_{\bar{\tau}} + \Delta_{\lambda} + \Delta_{\bar{\lambda}})$ , then we have

$$(\Delta_{\tau} + \Delta_{\bar{\tau}} + \Delta_{\lambda} + \Delta_{\bar{\lambda}})(\theta \wedge \alpha) = 0,$$
$$\theta \wedge *\alpha = 0.$$

*Proof.* In order to get  $(\Delta_{\tau} + \Delta_{\bar{\tau}} + \Delta_{\lambda} + \Delta_{\bar{\lambda}})(\theta \wedge \alpha) = 0$ , by Lemma 2.3, we only need to prove that  $\Lambda(\theta \wedge \alpha) = 0$ ,  $\lambda(\theta \wedge \alpha) = \bar{\lambda}(\theta \wedge \alpha) = 0$ . Since  $\alpha \in \ker(\Delta_{\tau} + \Delta_{\bar{\tau}} + \Delta_{\lambda} + \Delta_{\bar{\lambda}})$ , we have

$$(\lambda + \overline{\lambda})\alpha = \omega \wedge (\theta \wedge \alpha) = 0,$$

i.e.,  $\Lambda(\theta \wedge \alpha) = 0$ . We also get

$$\lambda(\theta \wedge \alpha) = \theta^{1,0} \wedge \omega \wedge (\theta \wedge \alpha) = 0,$$
  
$$\bar{\lambda}(\theta \wedge \alpha) = \theta^{0,1} \wedge \omega \wedge (\theta \wedge \alpha) = 0.$$

For any (n-1)-form  $\alpha$  on M, there exists a (n-1)-form  $\beta$  such that

$$*\alpha = L\beta.$$

By a similar way in Lemma 3.2, we get

$$(\Delta_{\tau} + \Delta_{\bar{\tau}} + \Delta_{\lambda} + \Delta_{\bar{\lambda}})\beta = 0.$$

We then have

$$\lambda(\beta) = \omega \wedge \theta^{1,0} \wedge \beta = 0, \ \bar{\lambda}(\beta) = \omega \wedge \theta^{0,1} \wedge \beta = 0.$$

Hence

$$0 = (\lambda + \bar{\lambda})\beta = \omega \wedge (\theta \wedge \beta) = \theta \wedge *\alpha.$$

The next result is in regards to the k-forms (k = n, n - 1) in  $ker(\Box)$ .

**Theorem 3.7.** Let  $(M, J, \theta)$  be a compact locally conformally Kähler manifold of  $\dim_{\mathbb{C}} = n$ . We then have

$$\ker \Delta_{\bar{\partial}} \cap P^{k,n-1-k} \cap \ker(i_{\theta^{\sharp}}) \cong \ker(\Box^{k,n-1-k}).$$
$$\ker \Delta_{\bar{\partial}} \cap P^{k,n-k} \cong \ker(\Box^{k,n-k}).$$

*Proof.* First, we prove that

$$P^{k,n-1-k} \cap \ker(i_{\theta^{\sharp}}) \subset \ker(\Delta_{\tau} + \Delta_{\bar{\tau}} + \Delta_{\lambda} + \Delta_{\bar{\lambda}}) \cap \Omega^{k,n-1-k}.$$

We denote by  $\alpha$  a primitive (k, n-1-k)-form. Hence we have ([13, Proposition 1.2.31])

$$*\alpha = (-1)^{\frac{n(n-1)}{2}} (\sqrt{-1})^{2k+1-n} (L\alpha).$$

Since  $\alpha \in \ker(i_{\theta^{\sharp}})$ , we get

$$0 = i_{\theta^{\sharp}} \alpha = *(\theta \wedge *\alpha) = *(\theta \wedge L\alpha),$$

i.e.,

$$0 = \theta^{1,0} \wedge L\alpha = \lambda(\alpha), \ 0 = \theta^{0,1} \wedge L\alpha = \overline{\lambda}(\alpha).$$

Following Lemma 2.3, we obtain that

$$\alpha \in \ker(\Delta_{\tau} + \Delta_{\bar{\tau}} + \Delta_{\lambda} + \Delta_{\bar{\lambda}}) \cap \Omega^{k, n-1-k}$$

Noting that (see [6, Chapter VI, Corollary 6.15], [25, Equation (9)])

$$\Delta_{\partial} + [\partial, \tau^*] = \Delta_{\bar{\partial}} + [\bar{\partial}, \bar{\tau}^*].$$

We also assume that  $\alpha \in \ker \Delta_{\bar{\partial}}$ . Then we observe that

$$([\partial, \tau^*]\alpha, \alpha) = (\tau^* \partial \alpha, \alpha) = (\partial \alpha, \tau \alpha) = 0,$$
  
$$([\bar{\partial}, \bar{\tau}^*]\alpha, \alpha) = (\bar{\tau}^* \bar{\partial} \alpha, \alpha) = (\bar{\partial} \alpha, \bar{\tau} \alpha) = 0.$$

Combing above identities, we get

$$(\Delta_{\partial}\alpha, \alpha) = 0,$$

i.e.,  $\Delta_{\partial} \alpha = 0$ . Therefore,

$$\ker \Delta_{\bar{\partial}} \cap P^{k,n-1-k} \cap \ker(i_{\theta^{\sharp}}) \subset \ker(\Box^{k,n-1-k})$$

On the other hand, we let  $\alpha \in \ker(\Box^{k,n-1-k})$ . Following Corollary 2.5, we get

$$\alpha = \alpha_0 + \sum_{j \ge 1} L^j \alpha_j,$$

where  $\alpha_i$  is a primitive (k - i, n - 1 - k + i)-form in ker( $\Box$ ) for all  $i \ge 0$ . Following vanishing theorem 3.5, we get  $\alpha_j = 0$  for all  $j \ge 1$ . Therefore,  $\alpha$  is primitive. Following Proposition 3.6, we also have  $\theta \wedge *\alpha = 0$ . Therefore,

$$\ker(\Box^{k,n-1-k}) \subset \ker \Delta_{\bar{\partial}} \cap P^{k,n-1-k} \cap \ker(i_{\theta^{\sharp}}).$$

Hence, we get  $\ker \Delta_{\bar{\partial}} \cap P^{k,n-1-k} \cap \ker(i_{\theta^{\sharp}}) \cong \ker(\Box^{k,n-1-k}).$ 

Next, we consider the case of (k, n - k)-forms. We denote by  $\alpha$  a (k, n - k)-form in ker  $\Delta_{\bar{\partial}} \cap P^{k, n-k}$ . Noting that  $\Lambda \alpha = 0$ , i.e.,  $L \alpha = 0$ . Therefore,

$$\lambda \alpha = \theta^{1,0} \wedge L \alpha = 0, \ \bar{\lambda} \alpha = \theta^{0,1} \wedge L \alpha = 0.$$

Following Lemma 2.3, we also have  $(\Delta_{\tau} + \Delta_{\bar{\tau}} + \Delta_{\lambda} + \Delta_{\bar{\lambda}})\alpha = 0$ . Using the identities in (2.1), we get

$$\begin{split} \partial^* \alpha &= (i[\Lambda,\bar{\partial}] - \tau^*) \alpha = 0, \\ \partial \alpha &= (i[L,\bar{\partial}^*] - \tau) \alpha = 0. \end{split}$$

Therefore,

$$\ker \Delta_{\bar{\partial}} \cap P^{k,n-k} \subset \ker(\Box^{k,n-k}).$$

On the other hand, we let  $\alpha \in \ker(\Box^{k,n-k})$ . By a similar way, we also obtain that  $\alpha$  is primitive. Therefore,

$$\ker(\Box^{k,n-k}) \subset \ker \Delta_{\bar{\partial}} \cap P^{k,n-k}.$$

Hence, we get  $\ker \Delta_{\bar{\partial}} \cap P^{k,n-k} \cong \ker(\Box^{k,n-k})$ .

#### 3.2 Vaisman manifold

Among the LCK manifolds, a distinguished class is the following.

**Definition 3.8.** ([21] and [22, Definition 3.7]) An LCK manifold  $(M, J, \theta)$  is called Vaisman if  $\nabla \theta = 0$ , where  $\nabla$  is the Levi–Civita connection of the metric  $g(\cdot, \cdot) = \omega(J \cdot, \cdot)$ . If  $\theta \neq 0$ , then after rescaling, we may always assume that  $|\theta| = 1$ .

Before proof our results, we recall the decomposition of harmonic forms on a compact Vaisman manifold [18, 21]. We denote by  $\theta^{\sharp}$  (resp.  $(J\theta)^{\sharp}$  the dual vector field of  $\theta$  (resp.  $J\theta$ ) with respect to metric g. Let  $\mathcal{D}^1$  (resp.  $\mathcal{D}^2$ ) be the 1-dimensional distribution spanned by the Lee field  $\theta^{\sharp}$  (resp. by the anti-Lee field  $(J\theta)^{\sharp}$ ). We set  $\mathcal{V} = \mathcal{D}^1 \oplus \mathcal{D}^2$  and call it the vertical foliation. In any foliated chart the metric of M can be expressed as

$$ds^{2} = g_{a\bar{b}}dz^{a} \otimes d\bar{z}^{b} + (\theta - iJ\theta) \otimes (\theta + iJ\theta).$$
(3.1)

The direct sum decomposition  $TM = \mathcal{V} \oplus \mathcal{V}^{\perp}$  produces a corresponding decomposition of the differential forms on M into sums of bihomogeneous forms of type (p, q), where p is the transversal degree and q the leaf degree. This, moreover, decomposes the exterior differentiation operator as

$$d = d' + d'' + \partial \tag{3.2}$$

where d' has type (1,0), d" has type (0,1) and  $\partial$  has type (2,-1). The Hodge \*-operator of (M,g) acts homogeneously and (3.2) implies a decomposition of the corresponding adjoint operators

$$\delta = \delta' + \delta'' + \tilde{\partial},\tag{3.3}$$

where  $\delta, \delta', \delta'', \tilde{\partial}$  are the adjoint operator of  $d, d', d'', \partial$  respectively. If \*' denotes the Hodge \* of the transversal part of the metric g of M given by (3.1), we have

$$*\alpha = -J\theta \wedge \theta \wedge *'\alpha,$$

where  $\alpha$  is a  $\mathcal{V}$ -foliation form.

4

We also denote by  $\omega' := -dJ\theta = 2i\partial\theta^{0,1}$  form of transverse (Kählerian) part of the metric. For simple set  $L' = \omega' \wedge \cdot$  and  $\Lambda' = i_{\omega'}$ . Let us define

$$S^{k}(\mathcal{V}) = \{ \alpha \in \Omega^{k}(\mathcal{V}) : \Delta' \alpha = 0, \ \Lambda' \alpha = 0 \}, \ k \le n - 1$$
$$S^{p,q}(\mathcal{V}) = \{ \alpha \in \Omega^{p,q}(\mathcal{V}) : \Delta' \alpha = 0, \ \Lambda' \alpha = 0 \}, \ p + q \le n - 1,$$

where  $\Delta' = d'\delta' + \delta'd'$ ,  $\Omega^k(\mathcal{V})$  (resp.  $\Omega^{p,q}(\mathcal{V})$ ) is the set of  $\mathcal{V}$ -foliate k-forms (resp. (p,q)-forms) and denote  $s_k = \dim \mathcal{S}^k(\mathcal{V})$  (resp.  $s_{p,q} = \dim \mathcal{S}^{p,q}(\mathcal{V})$ ). Clearly

$$S^{k}(\mathcal{V}) = \bigoplus_{p+q=k} S^{p,q}(\mathcal{V}).$$

We denote by  $H^{p,q}_{\bar{\partial}}(M)$  the Dolbeault cohomology group of type (p,q) and put  $h^{p,q}(M) = \dim H^{p,q}_{\bar{\partial}}(M)$ .

**Theorem 3.9.** ([21, Theorem 4.1]) Let M be a compact Vaisman manifold of  $\dim_{\mathbb{C}} = n$ . Then, an k-form  $\alpha$  of M with  $0 \le k \le n - 1$  is  $\Delta_d$ -harmonic iff

$$\alpha = \beta + \theta \wedge \gamma,$$

where  $\beta$ ,  $\gamma$  are transversally harmonic and transversally effective foliate forms. In particular,

$$H^k(M) \cong S^k(\mathcal{V}) \oplus S^{k-1}(\mathcal{V}), \ k \le n-1.$$

It is known that on any Vaisman manifold, the following formula holds, [7, 21]

$$\omega = \theta \wedge J\theta - dJ\theta. \tag{3.4}$$

We then have

**Lemma 3.10.** Let  $(M, J, \theta)$  be a compact Vaisman manifold of  $\dim_{\mathbb{C}} = n$ . If a k-form  $\alpha$  with  $0 \le k \le n-1$  is  $\Delta_d$ -harmonic, then

$$\Lambda(\theta \wedge \alpha) = 0.$$

*Proof.* Following Theorem 3.9, there exist two forms  $\beta$ ,  $\gamma$  such that

$$\alpha = \beta + \theta \wedge \gamma,$$

where  $\beta$ ,  $\gamma$  are transversally harmonic and transversally effective foliate forms. Therefore,

$$\Lambda(\theta \land \alpha) = \Lambda(\theta \land \beta)$$
  
= \*(\omega \land \*(\theta \land \beta))  
= (\pm ) \* (\omega \land J\theta \land \*'\beta)  
= (\pm ) \* (\omega J\theta \land J\theta \land \*'\beta).

Noticing that  $\beta$  satisfies  $\Lambda'\beta = 0$ , i.e.,  $dJ\theta \wedge *'\beta = 0$ . We then have  $\Lambda(\theta \wedge \alpha) = 0$ .

We now want to obtain a complex version of Theorem 3.9. As usual we let  $\Delta_{\bar{\partial}} = \bar{\partial}^* \bar{\partial} + \bar{\partial} \bar{\partial}^*$  be the complex Laplacian.

**Theorem 3.11.** ([18, Theorem 3.2]) Let  $(M, J, \theta)$  be a compact Vaisman manifold of  $\dim_{\mathbb{C}} = n$ . Then any (p,q)-form  $\alpha$  on M,  $0 \le p + q \le n - 1$  satisfies  $\Delta_{\bar{\partial}} \alpha = 0$  iff

$$\alpha = \beta + \theta^{0,1} \wedge \gamma,$$

where  $\beta$  and  $\gamma$  are transversally harmonic and transversally effective  $\mathcal{V}$ -foliate forms. In particular,

$$H^{p,q}_{\bar{\partial}}(M) \cong S^{p,q}(\mathcal{V}) \oplus S^{p,q-1}(\mathcal{V}), \ k \le n-1.$$

**Theorem 3.12.** ([18, Theorem 3.5]) On a compact Vaisman manifold M of  $\dim_{\mathbb{C}} = n$ , for any  $0 \le k \le 2n$ , we have

$$b^{k}(M) = \sum_{p+q=k} h^{p,q}(M).$$

Furthermore,

$$b^n(M) = 2s_{n-1}.$$

*Proof.* Following Theorem 3.9 and 3.11, we get

$$H^{k}(M) \cong \bigoplus_{p+q=k} S^{p,q}(\mathcal{V}) \oplus \bigoplus_{p+q=k} S^{p,q-1}(\mathcal{V}) \cong \bigoplus_{p+q=k} H^{p,q}_{\bar{\partial}}(M).$$

Therefore,  $b^k = \sum_{p+q=k} h^{p,q}$  for  $k \le n-1$ . By Poincaré and Serre duality it is also true for  $k \ge n+1$ . As for k = n, one uses the following formula for the Euler characteristic:

$$\chi(M) = \sum_{k=0}^{2n} (-1)^k b^k(M) = \sum_{p+q=0}^{2n} (-1)^{p+q} h^{p,q}(M).$$

Noting that the Euler number  $\chi(M)$  of compact Vaisman manifold is zero. Since  $b^k = s_k + s_{k-1}$  for  $k \leq n-1$  and  $b_k = b_{2n-k}$ , we then have

$$0 = \sum_{k=0}^{2n} (-1)^k b^k$$
  
=  $2 \sum_{k \le n-1} (-1)^k b^k + (-1)^n b^n$   
=  $2 \sum_{k \le n-1} (-1)^k s_k - 2 \sum_{k \le n-2} (-1)^k s_k + (-1)^n b^n$   
=  $(-1)^n (b^n - 2s_{n-1}).$ 

Hence we get  $b^n = 2s_{n-1}$ .

**Proposition 3.13.** Let  $(M, J, \theta)$  be a compact Vaisman manifold of  $\dim_{\mathbb{C}} = n$ . If a (k, n - k - 1)-form  $\alpha \in \ker(\Box^{k, n-1-k})$ , then

$$\Box(\theta \wedge \alpha) = 0.$$

In particular,

$$\Box(\theta^{1,0} \wedge \alpha) = 0, \ \Box(\theta^{0,1} \wedge \alpha) = 0.$$

*Proof.* Noting that  $(\Delta_{\tau} + \Delta_{\bar{\tau}} + \Delta_{\lambda} + \Delta_{\bar{\lambda}})\alpha = 0$ . Following Proposition 3.6, we get

$$(\Delta_{\tau} + \Delta_{\bar{\tau}} + \Delta_{\lambda} + \Delta_{\bar{\lambda}})(\theta \wedge \alpha) = 0.$$

Hence we have

$$\tau^*(\theta \wedge \alpha) = \bar{\tau}^*(\theta \wedge \alpha) = 0,$$

On harmonic symmetries for locally conformally Kähler manifolds

$$\tau(\theta \wedge \alpha) = \bar{\tau}(\theta \wedge \alpha) = 0.$$

Now we begin to prove that

$$(\Delta_{\partial} + \Delta_{\bar{\partial}})(\theta \wedge \alpha) = 0.$$

Noting that  $\nabla \theta = 0$ . Following [12, Corollary 3.6] and [23, Proposition 2.5 and Corollary 2.9], for any  $\alpha \in \ker(\Box^{k,n-1-k}) \subset \ker(\Delta_d)$ , we have

$$\Delta_d(\theta \wedge \alpha) = 0,$$

i.e.,  $d(\theta \wedge \alpha) = d^*(\theta \wedge \alpha) = 0$ . Following the identities in (2.2), we have

$$([\Lambda,\bar{\partial}] - [\Lambda,\partial])(\theta \wedge \alpha) = -i(\partial^* + \tau^*)(\theta \wedge \alpha) - i(\bar{\partial}^* + \bar{\tau}^*)(\theta \wedge \alpha)$$
$$= -id^*(\theta \wedge \alpha) = 0.$$

Following  $\Lambda(\theta \wedge \alpha) = 0$  (see Lemma 3.10) and the fact  $0 = d\theta = \partial \theta^{0,1} + \bar{\partial} \theta^{1,0}$ ,  $\partial \theta^{1,0} = \bar{\partial} \theta^{0,1} = 0$ , we then have

$$\begin{split} ([\Lambda,\bar{\partial}] - [\Lambda,\partial])(\theta \wedge \alpha) &= \Lambda((\bar{\partial} - \partial)\theta \wedge \alpha) \\ &= \Lambda((\bar{\partial}\theta^{1,0} - \partial\theta^{0,1}) \wedge \alpha) \\ &= 2\Lambda(\bar{\partial}\theta \wedge \alpha) \\ &= 2[\Lambda,\bar{\partial}](\theta \wedge \alpha). \end{split}$$

Therefore, we have

$$[\Lambda,\bar{\partial}](\theta \wedge \alpha) = [\Lambda,\partial](\theta \wedge \alpha) = 0$$

Following the identities in (2.1) and Proposition 3.6, we have

$$\partial^*(\theta \wedge \alpha) = i([\Lambda, \partial] - \tau^*)(\theta \wedge \alpha) = 0,$$
  
$$\bar{\partial}^*(\theta \wedge \alpha) = -i([\Lambda, \partial] - \bar{\tau}^*)(\theta \wedge \alpha) = 0$$

Next, using the identities in (2.1) again, we have

$$\partial(\theta \wedge \alpha) = i([L, \partial^*] - \tau)(\theta \wedge \alpha) = 0,$$
  
$$\bar{\partial}(\theta \wedge \alpha) = -i([L, \partial^*] - \bar{\tau})(\theta \wedge \alpha) = 0$$

Here we use the fact  $\lambda(\alpha) = \omega \wedge \theta^{1,0} \wedge \alpha = 0$ ,  $\overline{\lambda}(\alpha) = \omega \wedge \theta^{0,1} \wedge \alpha = 0$ . Hence, we have

$$\Box(\theta \wedge \alpha) = 0.$$

Therefore,  $\Box(\theta^{1,0} \wedge \alpha) = 0$ ,  $\Box(\theta^{0,1} \wedge \alpha) = 0$ .

We now study the spaces  $\ker(\Box^{p,n-p-1})$   $(0 \le p \le n-1)$ . We prove that  $\ker(\Box^{p,n-p-1})$  is actually  $S^{p,n-1-p}(\mathcal{V})$ .

**Theorem 3.14.** Let  $(M, J, \theta)$  be a compact Vaisman manifold of  $\dim_{\mathbb{C}} = n$ . Then

 $\ker(\Box^{p,n-1-p}) \cong \mathcal{S}^{p,n-1-p}(\mathcal{V}),$ 

where  $S^{p,q}$  denote the vector space of transversally harmonic and transversally effective V-foliate (p,q)-forms.

*Proof.* Following Theorem 3.7, we only need prove that

 $\ker \Delta_{\bar{\partial}} \cap P^{p,n-1-p} \cap \ker(i_{\theta^{\sharp}}) \cong \mathcal{S}^{p,n-1-p}(\mathcal{V}).$ 

We denote by  $\alpha$  an  $\mathcal{V}$ -foliate transversally (p, n-1-p)-form in  $\mathcal{S}^{p,n-p-1}$ . Noting that  $\Lambda' \alpha = 0$ , i.e.,

 $0 = dJ\theta \wedge *'\alpha,$ 

and  $i_{\theta^{\sharp}} \alpha = 0$ . Since  $\omega = \theta \wedge J\theta - dJ\theta$ , we get

$$\Lambda \alpha = - *^{-1} L * \alpha$$
  
= - \*<sup>-1</sup> (\omega \lambda \*'\alpha \lambda \theta \lambda \theta)  
= - \*<sup>-1</sup> (dJ\theta \lambda \*'\alpha \lambda \theta \lambda \theta) = 0.

Therefore,  $\mathcal{S}^{p,n-1-p}(\mathcal{V}) \subset \ker \Delta_{\bar{\partial}} \cap P^{p,n-1-p} \cap \ker(i_{\theta^{\sharp}}).$ 

On the other hand, we denote by  $\alpha$  a (p, n - 1 - p)-form in ker $(\Box^{p,n-1-p})$ . Following Theorem 3.11, we obtain that

$$\alpha = \beta + \theta^{0,1} \wedge \gamma$$

where  $\beta$ ,  $\gamma$  are transversally effective  $\mathcal{V}$ -foliation forms. Following Proposition 3.6, we have

$$0 = i_{\theta^{\sharp}} \alpha = |\theta^{0,1}|^2 \gamma = \frac{1}{2} \gamma,$$

i.e.,  $\gamma = 0$ . Therefore,  $\alpha \in \mathcal{S}^{p,n-p-1}$ .

**Corollary 3.15.** ([18, Corollay 3.4]) On a compact Vaisman manifold  $(M, J, \theta)$ , there exists the isomorphism

$$H^{n,0}_{\bar{\partial}}(M) \cong H^{n-1,0}_{\bar{\partial}}(M)$$

where  $H^{p,q}_{\bar{\partial}}(M)$  is the Dolbeault cohomology group of type (p,q). Furthermore,

$$H^{n,0}_{\bar{\partial}}(M) \cong S^{n,0}(\mathcal{V}) \cong H^{n-1,0}_{\bar{\partial}}(M) \cong S^{n-1,0}(\mathcal{V}).$$

We begin to study the spaces  $\ker(\Box^{p,n-p})$   $(0 \le p \le n)$ .

**Theorem 3.16.** Let  $(M, J, \theta)$  be a compact Vaisman manifold of  $\dim_{\mathbb{C}} = n$ . Then we have

$$h_{\Box}^{p,n-p} = s_{p,n-p-1} + s_{p-1,n-p}$$

where  $s_k = \dim S^k$  (resp.  $s_{p,q} = \dim S^{p,q}$ ). Furthermore,

$$\ker \Delta_d \cap \Omega^n \cong \ker(\Box) \cap \Omega^n,$$
$$\ker(\Box^{k,n-k}) \cong \ker \Delta_{\bar{\partial}} \cap \Omega^{k,n-k}.$$

*Proof.* Let  $\alpha_1$  be a non-zero (p, n - p - 1)-form in  $S^{p,n-p-1}$  and  $\alpha_2$  a non-zero (p - 1, n - p)-form in  $S^{p-1,n-p}$ . Following Proposition 3.13,  $\theta^{1,0} \wedge \alpha_1$  and  $\theta^{0,1} \wedge \alpha_2$  are two non-zero differential forms in  $\ker(\Box^{p,q})$ . Therefore,

$$h_{\square}^{p,n-p} \ge s_{p,n-p-1} + s_{p-1,n-p}$$

It's easy to see  $\ker(\Box) \cap \Omega^n \subset \ker \Delta_d \cap \Omega^n$ . Following Theorem 3.12, we then have

$$b^n = 2s_{n-1} \ge \sum_{p=0}^n h_{\Box}^{p,n-p}.$$

Therefore,

$$2s_{n-1} \ge \sum_{p=0}^{n} s_{p,n-p-1} + \sum_{p=0}^{n} s_{p-1,n-p} = 2s_{n-1}.$$

Hence, all inequalities must be equalities. We obtain that

$$h_{\Box}^{p,n-p} = s_{p,n-p-1} + s_{p-1,n-p},$$

i.e., if  $\alpha \in \ker(\square^{p,n-p})$ , then there are  $\beta_1 \in S^{p-1,n-p}$  and  $\beta_2 \in S^{p,n-p-1}$  such that

$$\alpha = \theta^{1,0} \wedge \beta_1 + \theta^{0,1} \wedge \beta_2,$$

Following ker( $\Box$ )  $\cap \Omega^n = \bigoplus_{0 \le k \le n} \ker(\Box^{k,n-k})$  and  $b^n = 2s_{n-1} = \sum_{k=0}^n h_{\Box}^{k,n-k}$ , we then have

$$\ker \Delta_d \cap \Omega^n \cong \ker(\Box) \cap \Omega^n,$$

Following Theorem 3.12, we get

$$b^{n} = \sum_{k=0}^{n} h^{k,n-k} = \sum_{k=0}^{n} h_{\Box}^{k,n-k} = h_{\Box}^{n}$$

We then have

$$h^{k,n-k} = h_{\Box}^{k,n-k},$$

i.e.,

$$\ker(\Box^{k,n-k}) \cong \ker \Delta_{\bar{\partial}} \cap \Omega^{k,n-k}.$$

**Corollary 3.17.** Let  $(M, J, \theta)$  be a compact Vaisman manifold of  $\dim_{\mathbb{C}} = n \ge 2$ . (1) If the Betti number  $b^n = 0$ , then for any  $0 \le k \le 2n$ , we have

$$\mathcal{H}^k_{\square} = \{0\}$$

(2) If the Betti number  $b^n = 2$ , then

$$\mathcal{H}^{n,0}_{\bar{\partial}}(M) = \mathcal{H}^{n-1,0}_{\bar{\partial}}(M) = \{0\}.$$

*Proof.* If  $b^n = 0$ , then

$$\mathcal{H}^n_{\square} = \mathcal{H}^{n-1}_{\square} = 0$$

If  $b^n = 2$ , then following Corollary 3.15 and Theorem 3.12, we have

$$2h^{n-1,0} \le 2s_{n-1} = b^n$$

Therefore,  $h^{n-1,0}(M) \leq 1$ . When  $h^{n-1,0}(M) = 1$ , we denote by  $\alpha$  a non-zero (n - 1, 0)-form in  $\ker(\Box^{n-1,0})$ . Following Theorem 3.14,  $\alpha \in S^{n-1,0}(\mathcal{V})$ . Therefore,  $\theta^{1,0} \wedge \alpha$  and  $\theta^{0,1} \wedge \alpha$  are not zero. Then by Proposition 3.13, we get  $\theta^{1,0} \wedge \alpha \in \ker(\Box^{n,0}), \theta^{0,1} \wedge \alpha \in \ker(\Box^{n-1,1})$ . Therefore, we get

$$h_{\square}^{n,0} = h_{\square}^{n-1,0} = 1, \ h_{\square}^{n-1,1} \ge h_{\square}^{n-1,0} = 1.$$

Therefore,

$$b^n = \sum_{p+q=n} h_{\Box}^{p,q} \ge 2h_{\Box}^{n,0} + 2h_{\Box}^{n-1,1} \ge 4.$$

This contradicts the fact that  $b^n = 2$ . Hence  $h^{n-1,0}(M) = 0$ .

**Remark 3.18.** For larger number  $b^n$ , we cannot prove that  $s_{n-1,0}$  must be zero. For example, when n = 3,  $b^3 = 4$ , i.e.,  $s_2 = h_{\Box}^{1,1} + 2h_{\Box}^{2,0} = 2$ , we get

$$b^3 = 2h_{\Box}^{3,0} + 2h_{\Box}^{2,1} \ge 4h_{\Box}^{2,0}$$

Therefore,  $h_{\Box}^{2,0} = 0$  or 1. There are two cases as follows: (i)  $h_{\Box}^{1,1} = 2$ ,  $h_{\Box}^{0,2} = h_{\Box}^{2,0} = 0$ ,  $h_{\Box}^{2,1} = h_{\Box}^{1,2} = 2$  and  $h_{\Box}^{0,3} = h_{\Box}^{3,0} = 0$ ; (ii) $h_{\Box}^{1,1} = 0$ ,  $h_{\Box}^{0,2} = h_{\Box}^{2,0} = 1$ ,  $h_{\Box}^{2,1} = h_{\Box}^{1,2} = 1$  and  $h_{\Box}^{0,3} = h_{\Box}^{3,0} = 1$ .

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