

On the maximality of genus-3 nonhyperelliptic curves of Ciani type

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Abstract

In this paper, we study a Ciani curve $C : x^4 + y^4 + z^4 + rx^2y^2 + sy^2z^2 + tz^2x^2 = 0$ in positive characteristic $p \geq 3$. We will show that if C is superspecial, then its standard form is maximal or minimal over \mathbb{F}_{p^2} without taking its \mathbb{F}_{p^2} -form.

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1 Introduction

Throughout this paper, a curve always means a projective variety in positive characteristic $p \geq 3$ of dimension one. It is well-known that all nonsingular genus- g curves C defined over \mathbb{F}_q with $q = p^n$ satisfy the Hasse-Weil inequality

$$1 + q - 2g\sqrt{q} \leq \#C(\mathbb{F}_q) \leq 1 + q + 2g\sqrt{q},$$

where $C(\mathbb{F}_q)$ denotes the set of \mathbb{F}_q -rational points of C . Now, a nonsingular curve C is called *maximal* (resp. *minimal*) when the number of \mathbb{F}_q -rational points of C attains the upper (resp. lower) bound. Maximal curves have been investigated for their applications to coding theory. On the other hand, we call a curve C *superspecial* if $\text{Jac}(C)$ is isomorphic to the product of supersingular elliptic curves. This is equivalent to saying that the a -number of C is equal to g . The a -number of C is defined to be the dimension of $\text{Hom}(\alpha_p, \text{Jac}(C)[p])$ where α_p is the kernel of the Frobenius map on the additive group \mathbb{G}_a . And then, it is also known that any maximal or minimal curve over \mathbb{F}_{p^2} is superspecial, though a superspecial curve over \mathbb{F}_{p^2} is not necessarily maximal nor minimal.

In this paper, we study the maximality of genus-3 nonhyperelliptic curves of Ciani type, and we will call them Ciani curves briefly. A Ciani curve C is a plane quartic defined by the equation

$$C : x^4 + y^4 + z^4 + rx^2y^2 + sy^2z^2 + tz^2x^2 = 0,$$

which was studied by Ciani [3]. Brock [2, Theorem 3.15] studied the superspeciality of Ciani curves and their enumerations, using the result by Hashimoto [5] on the computations of the class numbers of quaternion unitary groups. Our main result is as below:

Theorem 1.1. Assume that a nonsingular curve

$$C : x^4 + y^4 + z^4 + rx^2y^2 + sy^2z^2 + tz^2x^2 = 0$$

is superspecial, then r, s and t belong to \mathbb{F}_{p^2} . Moreover C is maximal or minimal over \mathbb{F}_{p^2} .

See Corollary 4.6 for a condition determining whether C is maximal or minimal.

The remainder of this paper is structured as follows. In Section 2, we review several properties of Ciani curves. Proposition 2.5 gives us the classification of automorphism groups of a Ciani curve. In Section 3, we look into the structure of a Ciani curve C . In particular, we describe explicitly the elliptic curves appearing as quotients of C by involutions. In Section 4, we prove Theorem 1.1.

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2 Ciani curve

Let K be a perfect field of characteristic $p \geq 3$. In this section, we consider a nonhyperelliptic curve of genus 3 defined over K

$$C : x^4 + y^4 + z^4 + rx^2y^2 + sy^2z^2 + tz^2x^2 = 0, \quad (2.1)$$

which is called a *Ciani curve*. First of all, we discuss the singularity of a Ciani curve.

Lemma 2.1. The curve C is nonsingular if and only if $r, s, t \neq \pm 2$ and $r^2 + s^2 + t^2 - rst - 4 \neq 0$.

Proof. Put $F := x^4 + y^4 + z^4 + rx^2y^2 + sy^2z^2 + tz^2x^2$. Then, we have

$$\frac{\partial F}{\partial x} = 2x(2x^2 + ry^2 + tz^2), \quad \frac{\partial F}{\partial y} = 2y(2y^2 + rx^2 + sz^2), \quad \frac{\partial F}{\partial z} = 2z(2z^2 + sy^2 + tx^2).$$

To show the “if”-part, assume that $(x : y : z) \in \mathbb{P}^2$ is a singular point of C .

Firstly, we note that there is no point $(x : y : z)$ on C such that two among x, y and z are zero; e.g., if $x = y = 0$, it follows from $F = 0$ that $z = 0$, which is a contradiction.

Secondly, consider a singular point $(x : y : z)$ on C such that only one among x, y and z is zero, e.g.; the case $x = 0, y \neq 0$ and $z \neq 0$. It follows from $2y^2 + sz^2 = 2z^2 + sy^2 = 0$ that $(4 - s^2)z^2 = 0$, whence $s = \pm 2$. One can check the case of $y = 0$ or $z = 0$ in the same way.

Lastly, consider a singular point $(x : y : z)$ on C such that all of x, y and z are non-zero. Using the Jacobian criterion, we obtain

$$\begin{pmatrix} 2 & r & t \\ r & 2 & s \\ t & s & 2 \end{pmatrix} \begin{pmatrix} x^2 \\ y^2 \\ z^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Here, we can calculate

$$\det \begin{pmatrix} 2 & r & t \\ r & 2 & s \\ t & s & 2 \end{pmatrix} = -2(r^2 + s^2 + t^2 - rst - 4),$$

and thus $r^2 + s^2 + t^2 - rst - 4 = 0$ since $(x^2, y^2, z^2) \neq (0, 0, 0)$.

Conversely, we show the “only if”-part. If $r = \pm 2$, we can find singular points $(x : y : 0)$ on C such that $x^2 \pm y^2 = 0$. One can check the case of $s = \pm 2$ or $t = \pm 2$ in the same way. Moreover, by a tedious calculation, we can find singular points $(x : y : z)$ on C such that

$$(4 - r^2)(rt - 2s)x^2 = (4 - r^2)(rs - 2t)y^2 = (rt - 2s)(rs - 2t)z^2$$

if $r^2 + s^2 + t^2 - rst - 4 = 0$. The proof is completed. \square

Definition 2.2. We say that a nonsingular Ciani curve of the form (2.1) is of $[a, b, c]$ -type when the triple (r, s, t) is a permutation of

$$(a, b, c), (a, -b, -c), (-a, b, -c), (-a, -b, c).$$

If both C and C' are two Ciani curves of $[a, b, c]$ -type, it is obvious that C is isomorphic to C' .

Remark 2.3. Suppose that C and C' are two Ciani curves, then the type of C is not necessarily same as that of C' even if C and C' are isomorphic. For example, the Fermat curve $x^4 + y^4 + z^4 = 0$ is isomorphic to the curve $x^4 + y^4 + 6y^2z^2 + z^4 = 0$. We find a necessary and sufficient condition for two Ciani curves C and C' to be isomorphic in [6, Proposition 2.2], but do not use it in this paper.

Example 2.4. Consider a nonsingular curve

$$C : x^4 + y^4 + z^4 + rx^2yz + sy^2z^2 = 0. \quad (2.2)$$

Note that the automorphism group of C contains D_8 as shown in [9, Theorem 3.1]. Let us confirm that this curve C is of $[a, b, a]$ -type with $a = \frac{r}{\sqrt{s+2}}$ and $b = 2 - \frac{16}{s+2}$.

By replacing $u = (y+z)/2$ and $z = (y-z)/(2\sqrt{-1})$, we obtain $y = u + \sqrt{-1}v$ and $z = u + \sqrt{-1}v$ thus

$$\begin{aligned} yz &= (u + \sqrt{-1}v)(u + \sqrt{-1}v) = u^2 + v^2, \\ y^2 + z^2 &= (u + \sqrt{-1}v)^2 + (u + \sqrt{-1}v)^2 = 2(u^2 - v^2). \end{aligned}$$

The left side of (3.2) can be transformed into

$$\begin{aligned} x^4 + y^4 + z^4 + rx^2yz + sy^2z^2 &= x^4 + (y^2 + z^2)^2 + rx^2yz + (s-2)y^2z^2 \\ &= x^4 + (s+2)u^4 + (s+2)v^4 + rx^2u^2 + 2(s-6)u^2v^2 + rv^2x^2, \end{aligned}$$

and the map

$$u \mapsto \frac{u}{\sqrt[4]{s+2}}, \quad v \mapsto \frac{v}{\sqrt[4]{s+2}}$$

transforms the curve C into

$$x^4 + u^4 + v^4 + \frac{r}{\sqrt{s+2}}x^2u^2 + \left(2 - \frac{16}{s+2}\right)u^2v^2 + \frac{r}{\sqrt{s+2}}v^2x^2.$$

Hence, the curve C is a Ciani curve of $[a, b, a]$ -type.

Next result tells us the classification of automorphism groups of Ciani curves. Let C_n (resp. D_n) be the cyclic (resp. dihedral) group of order n and S_n the symmetric group of degree n .

Proposition 2.5. The automorphism group of a Ciani curve C is either of the following 7 groups:

- (1) D_4 . In case C can be written as $[a, b, c]$ -type for some $a, b, c \in \overline{K}$.
- (2) D_8 . In case C can be written as $[a, b, a]$ -type for some $a, b \in \overline{K}$.
- (3) G_{16} . In case C can be written as $[0, b, 0]$ -type for some $b \in \overline{K}$.
- (4) S_4 . In case C can be written as $[a, a, a]$ -type for some $a \in \overline{K}$.
- (5) G_{48} . In case C can be written as $[0, 2\sqrt{-3}, 0]$ -type.
- (6) G_{96} . In case C can be written as $[0, 0, 0]$ -type.
- (7) G_{168} . In case C can be written as $\left[\frac{-3+\sqrt{-63}}{2}, \frac{-3+\sqrt{-63}}{2}, \frac{-3+\sqrt{-63}}{2}\right]$ -type.

Proof. It follows from [9, Theorem 3.1] about (1), (3), (4) and (6). It is well-known [10, Section 6-8] about (5) and (7). It is clear about (2) by Example 2.4. \square

3 Elliptic curves associated to a Ciani curve

Let $\sigma_1, \sigma_2, \sigma_3$ be automorphisms on a nonsingular Ciani curve C as below:

$$\begin{aligned}\sigma_1 &: (x : y : z) \mapsto (-x : y : z), \\ \sigma_2 &: (x : y : z) \mapsto (x : -y : z), \\ \sigma_3 &: (x : y : z) \mapsto (x : y : -z).\end{aligned}$$

Put $E_i := C/\langle \sigma_i \rangle$ for $i \in \{1, 2, 3\}$, then we obtain the equations

$$\begin{aligned}E_1 &: X^2 + y^4 + z^4 + rXy^2 + sy^2z^2 + tz^2X = 0, \\ E_2 &: x^4 + Y^2 + z^4 + rx^2Y + sYz^2 + tz^2x^2 = 0, \\ E_3 &: x^4 + y^4 + Z^2 + rx^2y^2 + sy^2Z + tZx^2 = 0\end{aligned}$$

with $X = x^2$, $Y = y^2$ and $Z = z^2$. One can easily check each E_i is a genus-1 curve, and the next lemma gives the Legendre forms of E_i . We choose α, β and γ such that

$$\alpha^2 = r^2 - 4, \quad \beta^2 = s^2 - 4, \quad \gamma^2 = t^2 - 4. \quad (3.1)$$

once and fix them throughout this paper.

Lemma 3.1. We can transform E_i into the Legendre forms $y^2 = x(x-1)(x-\lambda_i)$ with

$$\begin{aligned}\lambda_1 &= \frac{(rt - 2s) - \gamma\alpha}{(rt - 2s) + \gamma\alpha}, \\ \lambda_2 &= \frac{(sr - 2t) - \alpha\beta}{(sr - 2t) + \alpha\beta}, \\ \lambda_3 &= \frac{(ts - 2r) - \beta\gamma}{(ts - 2r) + \beta\gamma}.\end{aligned} \quad (3.2)$$

Proof. In this proof, we will show about E_1 . Firstly, we get the equation

$$\begin{aligned}\left(X + \frac{r}{2}y^2 + \frac{t}{2}z^2\right)^2 &= \left(\frac{r}{2}y^2 + \frac{t}{2}z^2\right)^2 - y^4 - sy^2z^2 - z^4 \\ &= \left(\frac{r^2}{4} - 1\right)y^4 + \left(\frac{rt}{2} - s\right)y^2z^2 + \left(\frac{t^2}{4} - 1\right)z^4.\end{aligned}$$

By replacing $u = y$ and $v = X + \frac{r}{2}y^2 + \frac{t}{2}z^2$ and multiplying $(r^2 - 4)/4$ of both sides, we obtain

$$v^2 = u^4 + \frac{2rt - 4s}{r^2 - 4}u^2z^2 + \frac{t^2 - 4}{r^2 - 4}z^4.$$

Here, the right side is transformed into

$$\begin{aligned}u^4 + \frac{2rt - 4s}{r^2 - 4}u^2z^2 + \frac{t^2 - 4}{r^2 - 4}z^4 &= \left(u^2 + \frac{\gamma}{\alpha}z^2\right)^2 - 2\frac{\gamma\alpha - (rt - 2s)}{r^2 - 4}u^2z^2 \\ &= \left(u^2 + buz + \frac{\gamma}{\alpha}z^2\right)\left(u^2 - buz + \frac{\gamma}{\alpha}z^2\right)\end{aligned}$$

with $b^2 = 2 \cdot \frac{\gamma\alpha - (rt - 2s)}{r^2 - 4}$. Put

$$d_1 = \frac{b + \sqrt{b^2 - 4 \cdot \frac{\gamma}{\alpha}}}{2}, \quad d_2 = \frac{b - \sqrt{b^2 - 4 \cdot \frac{\gamma}{\alpha}}}{2},$$

then we have the factorization

$$\left(u^2 + buz + \frac{\gamma}{\alpha}z^2\right)\left(u^2 - buz + \frac{\gamma}{\alpha}z^2\right) = (u + d_1z)(u + d_2z)(u - d_1z)(u - d_2z).$$

The map

$$u \mapsto \frac{u - d_1z}{u + d_1z} \cdot \frac{d_2 + d_1}{d_2 - d_1}$$

transforms this elliptic curve into the Legendre form:

$$v^2 = u(u - 1)\left(u - \frac{(d_2 + d_1)^2}{(d_2 - d_1)^2}\right) = u(u - 1)\left(u - \frac{b^2}{b^2 - 4 \cdot \frac{\gamma}{\alpha}}\right) = u(u - 1)(u - \lambda_1).$$

This is the desired conclusion. \square

Here, we can regard the curve

$$P : X^2 + Y^2 + Z^2 + rXY + sYZ + tZX = 0$$

as the quotient $C/\langle\sigma_1, \sigma_2\rangle$. One can confirm that P is nonsingular since C is nonsingular and the genus of P is 0. Hence, the curve P is isomorphic to the projective line \mathbb{P}^1 . For the above discussion, we obtain the following diagram:

$$\begin{array}{ccccc} & & C & & \\ & \swarrow & \downarrow & \searrow & \\ E_1 & & E_3 & & E_2 \\ & \swarrow & \downarrow & \searrow & \\ & & P & & \end{array} \quad (3.3)$$

The diagram (3.3) induces an isogeny $\text{Jac}(C) \rightarrow E_1 \times E_2 \times E_3$ by [8, Section 3] of degree 2^3 . Since the degree of the isogeny is not divided by p , then we obtain $\text{Jac}(C)[p] \cong (E_1 \times E_2 \times E_3)[p]$. Hence, a Ciani curve C is superspecial if and only if E_1, E_2 and E_3 are supersingular.

Proposition 3.2. The reverse transformation of that in Lemma 3.1 is given by

$$\begin{aligned} r &= \frac{\lambda_1\lambda_2 - \lambda_2\lambda_3 - \lambda_3\lambda_1 + 1}{\sqrt{\lambda_1\lambda_2}(1 - \lambda_3)}, \\ s &= \frac{\lambda_2\lambda_3 - \lambda_3\lambda_1 - \lambda_1\lambda_2 + 1}{\sqrt{\lambda_2\lambda_3}(1 - \lambda_1)}, \\ t &= \frac{\lambda_3\lambda_1 - \lambda_1\lambda_2 - \lambda_2\lambda_3 + 1}{\sqrt{\lambda_3\lambda_1}(1 - \lambda_2)}. \end{aligned}$$

Proof. In this proof, we will show the case of $\lambda_1, \lambda_2, \lambda_3 \neq -1$. By linear fractional transformations, one can check that

$$\frac{1 + \lambda_1}{1 - \lambda_1} = \frac{rt - 2s}{\gamma\alpha}, \quad \frac{1 + \lambda_2}{1 - \lambda_2} = \frac{sr - 2t}{\alpha\beta}, \quad \frac{1 + \lambda_3}{1 - \lambda_3} = \frac{ts - 2r}{\beta\gamma}.$$

Hence we obtain the equation

$$\frac{(1 - \lambda_1)(1 - \lambda_2)(1 + \lambda_3)}{(1 + \lambda_1)(1 + \lambda_2)(1 - \lambda_3)} = \frac{(r^2 - 4)(ts - 2r)}{(rt - 2s)(sr - 2t)}.$$

Since $(r^2 - 4)(ts - 2r) + (rt - 2s)(sr - 2t) = -2r(r^2 + s^2 + t^2 - rst - 4)$, then we have

$$1 + \frac{(1 - \lambda_1)(1 - \lambda_2)(1 + \lambda_3)}{(1 + \lambda_1)(1 + \lambda_2)(1 - \lambda_3)} = -2r \cdot \frac{r^2 + s^2 + t^2 - rst - 4}{(rt - 2s)(sr - 2t)}. \quad (3.4)$$

On the other hand, note that

$$\begin{aligned} (rt - 2s)^2 - 4(r^2 + s^2 + t^2 - rst - 4) &= (r^2 - 4)(t^2 - 4), \\ (sr - 2t)^2 - 4(r^2 + s^2 + t^2 - rst - 4) &= (s^2 - 4)(r^2 - 4), \\ (ts - 2r)^2 - 4(r^2 + s^2 + t^2 - rst - 4) &= (t^2 - 4)(s^2 - 4), \end{aligned} \quad (3.5)$$

thus one can check that

$$\begin{aligned} \frac{\lambda_1}{(1 + \lambda_1)^2} &= \frac{r^2 + s^2 + t^2 - rst - 4}{(rt - 2s)^2}, \\ \frac{\lambda_2}{(1 + \lambda_2)^2} &= \frac{r^2 + s^2 + t^2 - rst - 4}{(sr - 2t)^2}, \\ \frac{\lambda_3}{(1 + \lambda_3)^2} &= \frac{r^2 + s^2 + t^2 - rst - 4}{(ts - 2r)^2}. \end{aligned}$$

Therefore, we have

$$\frac{\sqrt{\lambda_1 \lambda_2}}{(1 + \lambda_1)(1 + \lambda_2)} = \frac{r^2 + s^2 + t^2 - rst - 4}{(rt - 2s)(sr - 2t)}. \quad (3.6)$$

Using the equations (3.4) and (3.6), then

$$\frac{(1 + \lambda_1)(1 + \lambda_2)(1 - \lambda_3) + (1 - \lambda_1)(1 - \lambda_2)(1 + \lambda_3)}{(1 + \lambda_1)(1 + \lambda_2)(1 - \lambda_3)} = -2r \cdot \frac{\sqrt{\lambda_1 \lambda_2}}{(1 + \lambda_1)(1 + \lambda_2)}.$$

We may solve this with respect to r . The other two formulas can be shown in the same way. \square

4 Proof of the main theorem

In this section, we will show the theorem stated in Introduction.

Theorem 1.1. Assume that a nonsingular curve

$$C : x^4 + y^4 + z^4 + rx^2y^2 + sy^2z^2 + tz^2x^2 = 0$$

is superspecial, then r, s and t belong to \mathbb{F}_{p^2} . Moreover C is maximal or minimal over \mathbb{F}_{p^2} .

A key to the proof is the following proposition by Auer and Top [1, Proposition 2.2].

Proposition 4.1. Let $E : y^2 = x(x - 1)(x - \lambda)$ be a supersingular elliptic curve, then $\lambda \in (\mathbb{F}_{p^2})^4$. Moreover, the followings are true:

- If $p \equiv 3 \pmod{4}$, then an elliptic curve E is maximal over \mathbb{F}_{p^2} .
- If $p \equiv 1 \pmod{4}$, then an elliptic curve E is minimal over \mathbb{F}_{p^2} .

In particular, the curve C is maximal or minimal over \mathbb{F}_{p^2} .

Recall the discussions in Section 3. Let E_1, E_2 and E_3 be the following three elliptic curves:

$$E_1 : X^2 + y^4 + z^4 + rXy^2 + sy^2z^2 + tz^2X = 0,$$

$$E_2 : x^4 + Y^2 + z^4 + rx^2Y + sYz^2 + tz^2x^2 = 0,$$

$$E_3 : x^4 + y^4 + Z^2 + rx^2y^2 + sy^2Z + tZx^2 = 0.$$

Then, there exist surjectives $C \rightarrow E_i$ defined over \mathbb{F}_{p^2} .

Proof of the first half of Theorem 1.1. The elliptic curves E_i are supersingular by the assumption, since the quotient of supersingular curve is supersingular. By Lemma 3.1, each elliptic curve E_i is isomorphic to $y^2 = x(x-1)(x-\lambda_i)$. By using Proposition 4.1, each λ_i is a fourth power in $(\mathbb{F}_{p^2})^\times$ and thus $\sqrt{\lambda_i} \in \mathbb{F}_{p^2}$. Hence, it follows from Proposition 3.2 that $r, s, t \in \mathbb{F}_{p^2}$. \square

Therefore, the question of whether a Ciani curve C is maximal or minimal over \mathbb{F}_{p^2} makes sense. To prove the second assertion of Theorem 1.1, we need the following four lemmas:

Lemma 4.2. We choose Δ such that $\Delta^2 = r^2 + s^2 + t^2 - rst - 4$, then the followings are true:

- (1) If C is superspecial, then $\alpha\beta, \beta\gamma$ and $\gamma\alpha$ belong to \mathbb{F}_{p^2} where α, β, γ are chosen in (3.1).
- (2) If C is superspecial, then Δ belongs to \mathbb{F}_{p^2} .

Proof. The elliptic curve E_i is supersingular by assumption, and so $\lambda_i \in (\mathbb{F}_{p^2})^4$ by Proposition 4.1.

(1) This claim holds from (3.2) and the first assertion of Theorem 1.1.

(2) We obtain $-\lambda_1 \in \mathbb{F}_{p^2}$ clearly, and so whether $\gamma\alpha + (rt - 2s)$ is a square in \mathbb{F}_{p^2} is in accord with whether $\gamma\alpha - (rt - 2s)$ is a square in \mathbb{F}_{p^2} . This means that

$$\{\gamma\alpha + (rt - 2s)\} \{\gamma\alpha - (rt - 2s)\} = -4(r^2 + s^2 + t^2 - rst - 4)$$

is a square in \mathbb{F}_{p^2} . \square

Lemma 4.3. If a Ciani curve C is superspecial, then $\frac{r^2-4}{t^2-4}, \frac{s^2-4}{r^2-4}$ and $\frac{t^2-4}{s^2-4}$ are fourth powers in \mathbb{F}_{p^2} .

Proof. One can check that

$$r^2 - 4 = \frac{D}{\lambda_1\lambda_2(1-\lambda_3)^2}, \quad s^2 - 4 = \frac{D}{\lambda_2\lambda_3(1-\lambda_1)^2}, \quad t^2 - 4 = \frac{D}{\lambda_3\lambda_1(1-\lambda_2)^2}$$

with $D = (\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 - 1)^2 - 4\lambda_1\lambda_2\lambda_3(\lambda_1 + \lambda_2 + \lambda_3 - 2)$. Hence, we have

$$\frac{r^2 - 4}{t^2 - 4} = \frac{\lambda_3(1-\lambda_2)^2}{\lambda_2(1-\lambda_3)^2}, \quad \frac{s^2 - 4}{r^2 - 4} = \frac{\lambda_1(1-\lambda_3)^2}{\lambda_3(1-\lambda_1)^2}, \quad \frac{t^2 - 4}{s^2 - 4} = \frac{\lambda_2(1-\lambda_1)^2}{\lambda_1(1-\lambda_2)^2}.$$

Here, we have $\lambda_i, 1 - \lambda_i \in (\mathbb{F}_{p^2})^4$ by Proposition 4.1, thus $\frac{r^2-4}{t^2-4}, \frac{s^2-4}{r^2-4}, \frac{t^2-4}{s^2-4}$ are fourth powers in \mathbb{F}_{p^2} . Indeed, the elliptic curve $y^2 = x(x-1)(x-\lambda_i)$ is isomorphic to $y^2 = x(x-1)(x-(1-\lambda_i))$ by the proof of [11, Proposition III.1.7]. Therefore $y^2 = x(x-1)(x-(1-\lambda_i))$ is supersingular. \square

Next, we construct other elliptic curves E'_i which are 2-isogenous to E_i . We define

$$\begin{aligned} \mu_1 &:= (rt - 2s) + 2\Delta, & \nu_1 &:= (rt - 2s) - 2\Delta, \\ \mu_2 &:= (sr - 2t) + 2\Delta, & \nu_2 &:= (sr - 2t) - 2\Delta, \\ \mu_3 &:= (ts - 2r) + 2\Delta, & \nu_3 &:= (ts - 2r) - 2\Delta, \end{aligned}$$

then we obtain $\mu_i, \nu_i \in \mathbb{F}_{p^2}$ by Lemma 4.2 (2) if a Ciani curve C is nonsingular and superspecial.

Lemma 4.4. Suppose that a Ciani curve C is superspecial. Let E'_i be the elliptic curve defined by

$$E'_i : \mu_i y^2 = x(x-1)(x - \nu_i/\mu_i).$$

Then, there exists an isogeny $E_i \rightarrow E'_i$ defined over \mathbb{F}_{p^2} of degree 2.

Proof. In this proof, we will show only about E'_1 . Firstly, the elliptic curve

$$E_1 : X^2 + y^4 + z^4 + rXy^2 + sy^2z^2 + tz^2X = 0$$

is isomorphic to

$$v^2 = (r^2 - 4)u^4 + 2(rt - 2s)u^2 + (t^2 - 4) \quad (4.1)$$

by replacing $u = y$ and $v = 2X + ru^2 + t$. The elliptic curve of the form (4.1) is 2-isogenous to

$$v^2 = (r^2 - 4)u^3 + 2(rt - 2s)u^2 + (t^2 - 4)u \quad (4.2)$$

with the morphism $(u, v) \mapsto (u^2, uv)$ defined over \mathbb{F}_{p^2} . The right hand side of (4.2) can be factorized as $(r^2 - 4)u(u - \mu'_1)(u - \nu'_1)$, where

$$\mu'_1 = \frac{(rt - 2s) + 2\Delta}{r^2 - 4}, \quad \nu'_1 = \frac{(rt - 2s) - 2\Delta}{r^2 - 4}.$$

Here, the elliptic curve $v^2 = (r^2 - 4)u(u - \mu'_1)(u - \nu'_1)$ is isomorphic to

$$E'_1 : \mu_1 y^2 = x(x-1)(x - \nu'_1/\mu'_1) = x(x-1)(x - \nu_1/\mu_1).$$

This is the desired conclusion. \square

There exists an isogeny $\text{Jac}(C) \rightarrow E'_1 \times E'_2 \times E'_3$ defined over \mathbb{F}_{p^2} , whose degree is a power of 2. In particular, a curve C is maximal (resp. minimal) if and only if E'_i are maximal (resp. minimal).

Lemma 4.5. If C is superspecial, then all μ_i are squares in \mathbb{F}_{p^2} or none of μ_i is a square in \mathbb{F}_{p^2} .

Proof. It suffices to show that products $\mu_1\mu_2, \mu_2\mu_3, \mu_3\mu_1$ are squares in \mathbb{F}_{p^2} . Recall from (3.5) that

$$\begin{aligned} \mu_1\nu_1 &= (rt - 2s)^2 - 4(r^2 + s^2 + t^2 - rst - 4) = (r^2 - 4)(t^2 - 4), \\ \mu_2\nu_2 &= (sr - 2t)^2 - 4(r^2 + s^2 + t^2 - rst - 4) = (s^2 - 4)(r^2 - 4), \\ \mu_3\nu_3 &= (ts - 2r)^2 - 4(r^2 + s^2 + t^2 - rst - 4) = (t^2 - 4)(s^2 - 4). \end{aligned}$$

Since E'_i is supersingular by assumption, thus ν_i/μ_i is a fourth power in $(\mathbb{F}_{p^2})^\times$ for all $i \in \{1, 2, 3\}$ by Proposition 4.1. Hence, the product

$$\frac{\nu_1}{\mu_1} \cdot \frac{\nu_2}{\mu_2} = \frac{\mu_1\nu_1\mu_2\nu_2}{(\mu_1\mu_2)^2} = \frac{(r^2 - 4)^2(s^2 - 4)(t^2 - 4)}{(\mu_1\mu_2)^2}$$

is a fourth power. On the other hand,

$$(r^2 - 4)^2(s^2 - 4)(t^2 - 4) = (r^2 - 4)^4 \cdot \frac{s^2 - 4}{r^2 - 4} \cdot \frac{t^2 - 4}{r^2 - 4}$$

is a fourth power by Lemma 4.3, so thus $(\mu_1\mu_2)^2 \in (\mathbb{F}_{p^2})^4$. This means that $\mu_1\mu_2$ is a square in \mathbb{F}_{p^2} . In the same way, the products $\mu_2\mu_3$ and $\mu_3\mu_1$ are also squares in \mathbb{F}_{p^2} . \square

Now, we show that a nonsingular superspecial Ciani curve C is maximal or minimal over \mathbb{F}_{p^2} .

Proof of second half of Theorem 1.1. Suppose that a Ciani curve C is superspecial, then the elliptic curves E'_1, E'_2 and E'_3 are all supersingular.

- If $p \equiv 3 \pmod{4}$, then the elliptic curve E'_i is maximal if and only if μ_i is a square in \mathbb{F}_{p^2} , and moreover E'_i is minimal if and only if μ_i is not a square in \mathbb{F}_{p^2} by using Proposition 4.1. It follows from Proposition 4.5 that all E'_i are maximal or minimal, so the proof is done.
- If $p \equiv 1 \pmod{4}$, then the elliptic curve E'_i is minimal if and only if μ_i is a square in \mathbb{F}_{p^2} , and moreover E'_i is maximal if and only if μ_i is not a square in \mathbb{F}_{p^2} by using Proposition 4.1. It follows from Proposition 4.5 that all E'_i are minimal or maximal, so the proof is done. \square

Corollary 4.6. Suppose that a Ciani curve C is superspecial, then the followings are true:

- If $p \equiv 3 \pmod{4}$, then C is maximal if and only if μ_i for an (equivalently, every) $i \in \{1, 2, 3\}$ is a square in \mathbb{F}_{p^2} .
- If $p \equiv 1 \pmod{4}$, then C is maximal if and only if μ_i for an (equivalently, every) $i \in \{1, 2, 3\}$ is not a square in \mathbb{F}_{p^2} .

Example 4.7. Consider the Ciani curve C defined by the equation

$$C : x^4 + y^4 + z^4 + \frac{3}{\sqrt{-2}}x^2y^2 - \frac{9}{4}y^2z^2 + \frac{3}{\sqrt{-2}}z^2x^2 = 0.$$

A simple computation says $\lambda_1 = -1$ and $\lambda_2 = \lambda_3 = 2$ by using Lemma 3.1. Hence, the three elliptic curves associated to the Ciani curve C are given as below:

$$\begin{aligned} E_1 : y^2 &= x(x-1)(x+1), \\ E_2 : y^2 &= x(x-1)(x-2), \\ E_3 : y^2 &= x(x-1)(x-2). \end{aligned}$$

It follows from [11, Example V.4.5] that these elliptic curves with $j(E_i) = 1728$ are supersingular if and only if $p \equiv 3 \pmod{4}$. Therefore C is superspecial if and only if $p \equiv 3 \pmod{4}$. Moreover, the Ciani curve C is maximal over \mathbb{F}_{p^2} if and only if $p \equiv 3 \pmod{4}$ since $\mu_1 = \frac{17\sqrt{-1}}{2} \in (\mathbb{F}_{p^2})^2$.

Lastly, let us discuss the maximality of a Ciani curve C over $\mathbb{F}_{p^{2e}}$.

Corollary 4.8. Assume that a curve $C : x^4 + y^4 + z^4 + rx^2y^2 + sy^2z^2 + tz^2x^2 = 0$ is superspecial.

(1) When e is odd, then the followings are true:

- If $p \equiv 3 \pmod{4}$, then C is maximal over $\mathbb{F}_{p^{2e}}$ if and only if μ_i for an (equivalently, every) $i \in \{1, 2, 3\}$ is a square in \mathbb{F}_{p^2} .
- If $p \equiv 1 \pmod{4}$, then C is maximal over $\mathbb{F}_{p^{2e}}$ if and only if μ_i for an (equivalently, every) $i \in \{1, 2, 3\}$ is not a square in \mathbb{F}_{p^2} .

(2) When e is even, then C is not maximal (i.e. minimal) over $\mathbb{F}_{p^{2e}}$.

Proof. Let Γ be a maximal (resp. minimal) curve over \mathbb{F}_{p^2} , then Γ over $\mathbb{F}_{p^{2e}}$ is also maximal (resp. minimal) if e is odd, and is minimal if e is even. This follows immediately from the Weil conjecture (cf. [4, Appendix C, Exercise 5.7]) and the fact that Γ over \mathbb{F}_{p^2} is maximal (resp. minimal) if and only if all the eigenvalues of the Frobenius on the first étale cohomology group are $-p$ (resp. p). \square

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