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TORSION IN DIFFERENTIALS AND BERGER'S CONJECTURE

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This paper is dedicated to Jürgen Herzog, whose fundamental research in commutative algebra has inspired researchers for 50 years.

Abstract. Let $(R, \mathfrak{m}, \Bbbk)$ be an equicharacteristic one-dimensional complete local domain over an algebraically closed field $\Bbbk$ of characteristic 0. R. Berger conjectured that R is regular if and only if the universally finite module of differentials Ω_R is a torsion-free R module. We give new cases of this conjecture by extending works of Güttes ([11]) and Cortiñas, Geller and Weibel ([8]). This is obtained by constructing a new subring S of $\mathrm{Hom}_R(\mathfrak{m}, \mathfrak{m})$ and constructing enough torsion in Ω_S, enabling us to pull back a nontrivial torsion to Ω_R.

1. Introduction

This paper gives new cases of a conjecture made by R. Berger in 1963 [2]. Let $\Bbbk$ be an algebraically closed field of characteristic 0, and let $(R, \mathfrak{m}_R, \Bbbk)$ be an equicharacteristic reduced one-dimensional complete local $\Bbbk$-algebra. Berger conjectured that the universally finite module of differentials, Ω_R, is torsion-free if and only if R is regular. The case in which R is regular is easy, since in that case Ω_R is free. Hence another formulation of his conjecture is that Ω_R is torsion-free if and only if it is a free R-module.

There are many approaches to the conjecture which have been partially successful. We refer [2], [24], [13], [1], [12], [7], [26], [18], [14], [15], [27], [3], [22], [11], [16], [17], [23], [8], [9] and [21] for these approaches. A very nice summary of a majority of these results, along with the main ideas of proofs, can be found in [4].

Our generalizations have to do with how the conductor $\mathfrak{C}_R = R :_{\mathrm{Quot}(R)} \overline{R}$ of R sits inside R. We first prove that if the conductor is not in the square of the maximal ideal, then Berger's conjecture is true (Theorem 3.1). When the conductor is in the square of the maximal ideal, we construct a certain subring S of $\mathrm{Hom}_R(\mathfrak{m}_R, \mathfrak{m}_R)$. By construction, there always exists torsion in Ω_S. We show that if there are enough torsion elements in Ω_S, we can construct a nonzero torsion element in Ω_R (Theorem 4.9). One of the first cases we prove is if S is quasi-homogeneous then Berger's conjecture is true (Theorem 4.11), generalizing a result of Scheja [24].

Let x be a minimal reduction of the maximal ideal $\mathfrak{m}_R$. The next set of results depends on which power of the maximal ideal $\mathfrak{m}_R$ is contained in the conductor $\mathfrak{C}_R$ of R. We study the quantity $s(R) := \dim_{\Bbbk} \frac{(\mathfrak{C}_R, x)}{(x)}$ which we shall refer to as the *reduced type* of R. The terminology reduced type is natural due to the fact that $\frac{(\mathfrak{C}_R, x)}{(x)} \subseteq \frac{(x):\mathfrak{m}}{(x)}$ and the $\Bbbk$-dimension of the latter module is precisely the type of R.

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Our main results extend both those of Güttes ([11]), who proved that if either $\mathfrak{m}_R^4 \subseteq xR$ or R is Gorenstein and $\mathfrak{m}_R^5 \subseteq xR$, then Berger's conjecture holds, and of Cortiñas, Geller, and Weibel ([8]), who proved that if $\mathfrak{m}_R^3 \subseteq \mathfrak{C}_R$, then Berger's conjecture holds.

We summarize our extensions below, with edim denoting the embedding dimension (Theorem 5.6 and Corollary 5.9):

Theorem A. Let $(R, \mathfrak{m}_R, \Bbbk)$ be an equicharacteristic one-dimensional complete local domain over an algebraically closed field $\Bbbk$ of characteristic 0, x be a minimal reduction of $\mathfrak{m}_R$ and $\mathfrak{C}_R$ be the conductor ideal of R in its integral closure. Further, let $n = \operatorname{edim} R$ and $s(R)$ be the reduced type of R. Then Berger's conjecture is true in the following cases:

(1) $\mathfrak{m}_R^4 \subseteq (\mathfrak{C}_R, x)$ and $2 \cdot s(R) \leqslant n(n-3)$,
(2) $\mathfrak{m}_R^6 \subseteq (x)$, $n \geqslant 6$ and R is Gorenstein.

We give other generalizations which relate to the structure of the ring S. In particular, when the length of S/R is one, we obtain results similar in spirit to the case in which R is Gorenstein, without having to assume the Gorenstein property.

The structure of the article is as follows: Section 3 takes care of the case when the conductor $\mathfrak{C}_R$ is not contained in the square of the maximal ideal $\mathfrak{m}_R$ (Theorem 3.1) and some related cases. Section 4 gives the details of the construction of S and its basic properties (Lemma 4.1, Theorem 4.5 and Theorem 4.6). And finally, the main results are presented in Section 5 (Theorem 5.6, Theorem 5.7 and Corollary 5.9).

2. Setting and Preliminaries

Throughout this paper we assume that $\Bbbk$ is an algebraically closed field of characteristic 0, and $(R, \mathfrak{m}_R, \Bbbk)$ is a one-dimensional complete equicharacteristic local $\Bbbk$-algebra which is a domain with embedding dimension (denoted by edim) n, i.e. $\mu_R(\mathfrak{m}_R) = n$ where $\mu_R(M)$ denotes the minimal number of generators for any R-module M. We also denote the length of any R-module M by $\ell(M)$.

Choosing t to be a uniformizing parameter of the integral closure $\overline{R}$ of R, we may assume that $\overline{R} = \Bbbk[\![t]\!]$. It follows we can write $R = \Bbbk[\![\alpha_1 t^{a_1}, \dots, \alpha_n t^{a_n}]\!]$ where α_i's are units in $\overline{R}$ and $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$. Note also that here $\bar{R}$ is finitely generated over R (see for example [25][Theorem 4.3.4]).

We define an epimorphism

$$\Phi : P = \Bbbk[\![X_1, \dots, X_n]\!] \twoheadrightarrow R \quad (2.1)$$
$$\Phi(X_i) = \alpha_i t^{a_i} \text{ for } 1 \leqslant i \leqslant n.$$

We denote the kernel of Φ by $I = (f_1, \dots, f_m)$ and hence have the natural isomorphism $R \cong \Bbbk[\![X_1, \dots, X_n]\!]/I$. Since $\operatorname{edim} R = n$, I is contained in $\mathfrak{m}^2$ where $\mathfrak{m} = (X_1, \dots, X_n)$, the maximal ideal of $\Bbbk[\![X_1, \dots, X_n]\!]$. Such rings are called *analytic $\Bbbk$-algebras*. We will interchangeably use $\alpha_i t^{a_i}$ for x_i, the images of X_i in the quotient $\Bbbk[\![X_1, \dots, X_n]\!]/I$.

We also define the valuation $\operatorname{ord}(-)$ on $\overline{R}$ given by $\operatorname{ord}(p(t)) = a$ if $p(t) = t^a \alpha$ where α is a unit in $\Bbbk[\![t]\!]$ (see for example [25, Example 6.7.5]).

2.1. Universally Finite Module of Differentials.

Definition 2.1. Let R be an analytic one-dimensional $\Bbbk$-algebra as above, which is a domain. Let $I = (f_1, \dots, f_m)$ where $f_j \in P = \Bbbk[\![X_1, \dots, X_n]\!]$. We assume that $I \subseteq \mathfrak{m}_P^2$ where

$\mathfrak{m}_P = (X_1, \ldots, X_n)$. Then the *universally finite module of differentials over* $\Bbbk$, denoted by Ω_R, has a (minimal) presentation given as follows:

$$R^m \xrightarrow{\left[\frac{\partial f_j}{\partial x_i}\right]} R^n \to \Omega_R \to 0$$

where $\left[\frac{\partial f_j}{\partial x_i}\right]$ is the Jacobian matrix of I, with entries in R.

We refer the reader to the excellent resource [20] for more information.

Let $\tau(\Omega_R)$ denote the torsion submodule of Ω_R. The conjecture of interest in this article is the following:

Conjecture B (R. W. Berger [2])**.** Let $\Bbbk$ be a perfect field and let R be a reduced one-dimensional analytic $\Bbbk$-algebra. Then R is regular if and only if $\tau(\Omega_R) = 0$.

Although the conjecture is for reduced algebras, in this paper we only deal with the case in which R is a domain and $\Bbbk$ is algebraically closed of characteristic zero. Our techniques do not immediately seem to apply otherwise.

Remark 2.2. When $\Bbbk$ is a perfect field, it is well-known that $\operatorname{rank}_R(\Omega_R) = \dim(R) = 1$. Hence from Definition 2.1, we get that $\operatorname{rank} A = n - 1$ where $A = \left[\frac{\partial f_j}{\partial x_i}\right]$ is the Jacobian matrix of I, with entries in R.

Remark 2.3. It is clear from Definition 2.1, that $\tau(\Omega_R) = 0$ when R is regular. Thus, from now on we assume that $n \geqslant 2$.

2.2. **The Conductor.** The conductor ideal $\mathfrak{C}_R$ will be crucial for the purposes of this paper. Recall that the conductor is the largest common ideal of R and its integral closure, $\overline{R}$. It follows that $\mathfrak{C}_R = R :_Q \overline{R}$ where $Q = \operatorname{Quot}(R)$, denotes the fraction field of R. Since $\overline{R} = k[\![t]\!]$ and $\mathfrak{C}_R$ is an ideal of $\bar{R}$ as well, we have that $\mathfrak{C}_R = (t^i)_{i \geqslant c_R}$ where c_R is the least integer such that $t^{c_R - 1} \notin R$, and $t^{c_R + i} \in R$ for all $i \geqslant 0$. The number c_R is characterized as the least valuation in $\mathfrak{C}_R$. It is clear from this discussion that there cannot be any element $r \in R$, such that $v(r) = c_R - 1$. Since $\bar{R}$ is finitely generated over R ([25, Theorem 4.3.4]), the conductor ideal is a nonzero ideal of R, and it is never all of R unless R is regular.

2.3. **Computing Torsion.** We have the following commutative diagram using the functorial universal properties of the module of differentials and the associated universal derivations.

$$\begin{array}{ccc}
\Omega_R & \xrightarrow{f} & \Omega_{\overline{R}} \\
{\scriptstyle d}\uparrow & & \uparrow{\scriptstyle d} \\
R & \xhookrightarrow{i} & \overline{R}
\end{array}$$

We use the same symbols d for both the vertical maps. Since, $\operatorname{rk}(\Omega_R) = \operatorname{rk}(\Omega_{\overline{R}}) = 1$ (R and $\bar{R}$ have the same fraction field), and $\Omega_{\overline{R}}$ is free over $\overline{R}$, we get that the $\tau(\Omega_R) = \ker f$.

Also note that by commutativity of the diagram,

$$f(dx_i) = \frac{dx_i}{dt} dt.$$

Since $\Omega_{\overline{R}}$ is isomorphic to $\overline{R}$, we see that Ω_R surjects to a R-submodule $\displaystyle\sum_{i=1}^{n} R\frac{dx_i}{dt}$ of $\overline{R} = \Bbbk[\![t]\!]$. This is a fractional ideal in $\overline{R}$, so multiplying by a suitably high enough power of t, it is isomorphic to an ideal of R.

The torsion submodule $\tau(\Omega_R)$ is the kernel of the map $\Omega_R \to \Omega_{\overline{R}}$. Thus, from the above discussion, we get that $\tau(\Omega_R)$ consists of the tuples $\begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}$ such that $\displaystyle\sum_{i=1}^{n} r_i \frac{dx_i}{dt} = 0$. Evidently, $\tau(\Omega_R)$ is non-zero precisely when the tuples $\begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}$ are not in the image of the presentation matrix (Jacobian matrix of I) of Ω_R, all entries written in terms of the uniformizing parameter t. This provides one computational way of computing torsion using Macaulay 2.

Example 2.4. $R = [\![t^3, t^4, t^5]\!]$ and its defining ideal $I = (y^2 - xz, z^2 - x^2y, x^3 - yz)$ in $\Bbbk[\![x, y, z]\!]$. Consider the element

$$\tau = 4ydx - 3xdy = \begin{bmatrix} 4y \\ -3x \\ 0 \end{bmatrix} \overbrace{\begin{array}{c} dx \\ dy \\ dz \end{array}}^{\text{basis}}$$

in Ω_R. Clearly $4y\frac{dx}{dt} - 3x\frac{dy}{dt} = 4t^4(3t^2) - 3t^3(4t^3) = 0$. Now the presentation matrix of $\Omega_{R/\mathfrak{m}_R^2}$ is

$$\begin{bmatrix} 2x & y & z & 0 & 0 & 0 \\ 0 & x & 0 & 2y & z & 0 \\ 0 & 0 & x & 0 & y & 2z \end{bmatrix}.$$

Since the image of τ in $\Omega_{R/\mathfrak{m}_R^2}$ can never be written as a linear combination of the columns of the above presentation, τ is nonzero in $\Omega_{R/\mathfrak{m}_R^2}$. Thus τ is nonzero in Ω_R as well.

3. Nonzero torsion when $\mathfrak{C}_R \not\subseteq \mathfrak{m}_R^2$

Throughout this section, we assume that $\Bbbk$ is an algebraically closed field of characteristic 0, and $(R, \mathfrak{m}_R, \Bbbk)$ is a one-dimensional complete equicharacteristic local $\Bbbk$-algebra which is a domain with embedding dimension n. Choosing t to be a uniformizing parameter of the integral closure $\overline{R}$, we may assume that $\overline{R} = \Bbbk[\![t]\!]$. It follows we can write $R = \Bbbk[\![\alpha_1 t^{a_1}, \ldots, \alpha_n t^{a_n}]\!]$ where α_i's are units in $\overline{R}$ and $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$. Let $\mathfrak{C}_R$ denote the conductor ideal.

Our primary construction which appears in the sequel will make use of the condition $\mathfrak{C}_R \subseteq \mathfrak{m}_R^2$. So prior to our main construction, we settle the case $\mathfrak{C}_R \not\subseteq \mathfrak{m}_R^2$ by showing that this condition always leads to nonzero torsion in Ω_R. In this case at least one of the minimal generators $x_1, \ldots, x_n$ of the maximal ideal is in the conductor $\mathfrak{C}_R$. Thus, after a change of variables, the minimal generators in the conductor can be replaced by monomials (i.e., if $x_i = \alpha_i t^{a_i} \in \mathfrak{C}_R$, then after a change of variables, α_i can be chosen to be a unit in R). We will see in Remark 3.3, the presence of monomials will lead to nonzero torsion in Ω_R.

Theorem 3.1. If $\mathfrak{C}_R \not\subseteq \mathfrak{m}_R^2$, then the torsion $\tau(\Omega_R)$ is nonzero.

Proof. Write $R = \Bbbk[\![\alpha_1 t^{a_1}, \ldots, \alpha_n t^{a_n}]\!]$ with conductor $\mathfrak{C}_R = (t^{c_R})\overline{R}$. We first monomialize the r^{th} term as follows: by multiplying by a nonzero element of $\Bbbk$, we may assume that the constant term of the unit α_r is 1. By Hensel's lemma [10, Theorem 7.3], there exists an element $\beta \in R$ such that $\beta^{a_r} = \alpha_r$. Here we use that the characteristic of $\Bbbk$ is 0. We write $\beta = 1 + \beta_1 t + \cdots$. Consider the change of variables $s = \beta t$. Under this change of variables, notice that $\Bbbk[\![t]\!] = \Bbbk[\![s]\!]$. Now

$$s = \beta t = t + \beta_1 t^2 + \beta_2 t^3 + \cdots .$$

Note that $s^{a_r} = (\beta t)^{a_r} = \alpha_r t^{a_r} \in R$. Furthermore, $\alpha_i t^{a_i} = \alpha'_i s^{a_i}, 1 \leqslant i \neq r \leqslant n$, where $\alpha'_i \in \overline{R}$ are units. Then $R = \Bbbk[\![\alpha_1 t^{a_1}, \ldots, \alpha_n t^{a_n}]\!] = \Bbbk[\![\alpha'_1 s^{a_1}, \ldots, \alpha'_{r-1} s^{a_{r-1}}, s^{a_r}, \alpha'_{r+1} s^{a_{r+1}}, \ldots, \alpha'_n s^{a_n}]\!]$. We apply this change of variables with $r = 1$ to assume without loss of generality, for the remainder of this proof, that $R = \Bbbk[\![t^{a_1}, \alpha_2 t^{a_2}, \cdots, \alpha_n t^{a_n}]\!]$.

Since $\mathfrak{C}_R$ is not contained in $\mathfrak{m}_R^2$, we must have that $a_n \geqslant c_R$. Write $\alpha_n = \alpha_{n0} + tb$, where $\alpha_{n0} \neq 0$ is in $\Bbbk$ and $b \in \overline{R}$. Then $\alpha_n t^{a_n} = \alpha_{n0} t^{a_n} + t^{a_n+1} b$, where $b \in \overline{R}$. However, since $a_n \geqslant c_R$, it follows that $t^{a_n+1} b \in \mathfrak{C}_R \subset R$. Hence, $t^{a_n} = \alpha_{n0}^{-1}(\alpha_n t^{a_n} - t^{a_n+1} b) \in R$ as well, and then $R = \Bbbk[\![t^{a_1}, \alpha_2 t^{a_2}, \ldots, \alpha_{n-1} t^{a_{n-1}}, t^{a_n}]\!]$.

We now use this particular form for R to find a nonzero torsion element in Ω_R. Namely, $a_n x_n dx_1 - a_1 x_1 dx_n \in \Omega_R$ and the exact sequence

$$0 \to \tau(\Omega_R) \to \Omega_R \xrightarrow{\phi} R\frac{dx_1}{dt} + \cdots + R\frac{dx_n}{dt} \to 0$$

where the map ϕ is the R-module map given by $\phi(dx_i) = \frac{dx_i}{dt}, 1 \leqslant i \leqslant n$. Under this map

$$\begin{aligned} \phi(a_n x_n dx_1 - a_1 x_1 dx_n) &= a_n t^{a_n} \frac{dt^{a_1}}{dt} - a_1 t^{a_1} \frac{dt^{a_n}}{dt} \\ &= (a_1 a_n t^{a_1+a_n-1} - a_n a_1 t^{a_1+a_n-1})dt = 0 \end{aligned}$$

Thus $a_n x_n dx_1 - a_1 x_1 dx_n \in \tau(\Omega_R)$. It remains to see that it is nonzero. Consider the image of this element $\overline{a_n x_n dx_1 - a_1 x_1 dx_n}$ in $\Omega_{R/\mathfrak{m}_R^2}$. As $x_1 x_n \in \mathfrak{m}_R^2$, it follows that in $\Omega_{R/\mathfrak{m}_R^2}$, $\overline{x_1 dx_n + x_n dx_1} = 0$. Hence $\overline{a_n x_n dx_1 - a_1 x_1 dx_n} = (a_1 + a_n)\overline{x_1} d(\overline{x_n})$ in $\Omega_{R/\mathfrak{m}_R^2}$. Now using [8, Proposition 2.6, Corollary 2.7], we have $(a_1 + a_n)\overline{x_1} d(\overline{x_n}) \neq 0$ in $\Omega_{R/\mathfrak{m}_R^2}$. Thus $a_n x_n dx_1 - a_1 x_1 dx_n \neq 0$ in Ω_R. □

Example 3.2. Let $R = \Bbbk[\![t^4 + t^5, t^7 + t^{10}, t^8 + t^{10}, t^9 + t^{10}]\!]$. Macaulay2 computations show that the conductor is $\mathfrak{C}_R = (t^{c_R})\overline{R} = (t^7)\overline{R}$. Since $a_2 \geqslant c_R$, we have $\tau(\Omega_R) \neq 0$ using the previous result.

Remark 3.3. If α_i, α_j are units in R for some $i \neq j$, then we can show that $\tau(\Omega_R)$ is nonzero. Assuming $i = 1, j = 2$ we can easily see that $R = \Bbbk[\![x_1, \ldots, x_n]\!] \cong \Bbbk[\![t^{a_1}, t^{a_2}, \alpha_3 t^{a_3}, \ldots, \alpha_n t^{a_n}]\!]$. Notice that $a_2 x_2 dx_1 - a_1 x_1 dx_2 \in \tau(\Omega_R)$. This torsion element is nonzero as $\overline{a_2 x_2 dx_1 - a_1 x_1 dx_2} = (a_2 - a_1)\overline{x_2 dx_1}$ is nonzero in $\Omega_{R/\mathfrak{m}_R^2}$ ([8, Corollary 2.7]).

The above result is also a generalization of [17, Corollary 3.7]. The next example illustrates the remark.

Example 3.4. Let $R = \Bbbk[\![t^8 + t^9, t^9 + t^{15}, t^{12} + t^{20}, t^{14}]\!]$, the conductor $\mathfrak{C}_R = (t^{c_R})\overline{R} = (t^{20})\overline{R}$. Thus $R \cong \Bbbk[\![t^8 + t^9, t^9 + t^{15}, t^{12}, t^{14}]\!]$ and hence R has at least one torsion element using Remark 3.3. Notice that in this case, none of the a_i's are bigger than the c_R.

4. The transform $R[\frac{\mathfrak{C}_R}{x_1}]$

Throughout this section, we again assume that $(R, \mathfrak{m}_R, \Bbbk)$ is a one-dimensional complete equicharacteristic local $\Bbbk$-algebra which is a domain with embedding dimension n. Further, $\Bbbk$ is an algebraically closed field of characteristic 0. Choosing t to be a uniformizing parameter of the integral closure $\overline{R}$, we may assume that $\overline{R} = \Bbbk[\![t]\!]$. Using the technique as in the first paragraph of the proof of Theorem 3.1, we can also write $R = \Bbbk[\![t^{a_1}, \alpha_2 t^{a_2}, \ldots, \alpha_n t^{a_n}]\!]$ where α_i's are units in $\overline{R}$ and $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$. Let $\mathfrak{C}_R$ denote the conductor ideal.

In this section we study the main construction $S = R[\frac{\mathfrak{C}_R}{x_1}]$ where $x_i = \alpha_i t^{a_i}$ with $\alpha_1 = 1$. Throughout this section, we assume that $\mathfrak{C}_R \subseteq \mathfrak{m}_R^2$.

4.1. Basics of $R[\frac{\mathfrak{C}_R}{x_1}]$.

We write $\frac{\mathfrak{C}_R}{x_1}$ to denote the set of elements of the form $\frac{c}{x_1}$ where $c \in \mathfrak{C}_R$. We note that the conductor is never inside a proper principal ideal (follows, for instance, from [21, Corollary 2.6]), so there are always elements in $\frac{\mathfrak{C}_R}{x_1}$ which are not in R itself. Recall from the introduction that we define the *reduced type* of R to be

$$s(R) := \dim_{\Bbbk} \frac{(\mathfrak{C}_R, x_1)}{(x_1)}.$$

We use the notation s whenever the underlying ring is clear.

Lemma 4.1. Let $S = R[\frac{\mathfrak{C}_R}{x_1}]$. The following statements hold.

(1) $\frac{\mathfrak{C}_R}{x_1} \subseteq \overline{R}$.
(2) $\mathfrak{m}_R(\frac{\mathfrak{C}_R}{x_1}) \subseteq \mathfrak{C}_R$. In particular, $S \subset \operatorname{Hom}_R(\mathfrak{m}, \mathfrak{m})$.
(3) Let $\mathfrak{C}_R \subseteq \mathfrak{m}_R^2$. Let $\alpha, \beta \in \frac{\mathfrak{C}_R}{x_1}$. Then $\alpha\beta \in \mathfrak{C}_R$.
(4) Let $\mathfrak{C}_S$ denote the conductor of S in $\overline{R}$. Then $\mathfrak{C}_S = \frac{\mathfrak{C}_R}{x_1}$.
(5) S/R is a vector space over $\Bbbk$ of dimension s, where s is the reduced type of R.

Proof. Since (x_1) is a minimal reduction of $\mathfrak{m}_R$, we have $\frac{\mathfrak{m}_R}{x_1} \subseteq \overline{R}$. Hence $\frac{\mathfrak{C}_R}{x_1} \subseteq \overline{R}$ proving (1).

For (2) note that $\mathfrak{m}_R\overline{R} = x_1\overline{R}$ and hence $\mathfrak{m}_R\mathfrak{C}_R = x_1\mathfrak{C}_R$. So, $\mathfrak{m}_R\frac{\mathfrak{C}_R}{x_1} \subseteq \mathfrak{C}_R \subseteq \mathfrak{m}_R$.

Write $\alpha = \frac{c}{x_1}, \beta = \frac{c'}{x_1}$ where $c, c' \in \mathfrak{C}_R$. For (3), first note that $cc' \in \mathfrak{C}_R^2 \subseteq \mathfrak{C}_R\mathfrak{m}_R^2 = \mathfrak{C}_R x_1 \mathfrak{m}_R = x_1^2\mathfrak{C}_R$ as in the proof of (2). This proves (3).

Finally, note that every valuation more than $c_R - a_1 - 1$ is present in the valuation semigroup of S. But there cannot be any element with valuation $c_R - a_1 - 1$ in S: if possible, let r be such an element. Then by (2), $rx_1 \in R$ and has valuation $c_R - 1$, a contradiction to the choice of c_R. This finishes the proof of (4).

For (5), first note that (2) clearly implies that S/R is a vector space. Now S/R is generated as a $\Bbbk$-vector space by elements of the form c/x_1 such that $c \in \mathfrak{C}_R$. Choose a basis $\overline{c_1}, \ldots, \overline{c_s}$ for the $\Bbbk$-vector space $(\mathfrak{C}_R, x_1)/(x_1)$. Now construct a $\Bbbk$-linear map $\eta : (\mathfrak{C}_R, x_1)/(x_1) \to S/R$ by mapping $\eta(\overline{c_i}) = c_i/x_1$. Suppose $\overline{c} = \sum_i k_i\overline{c_i}$ with $k_i \in \Bbbk$ be such that $\eta(\overline{c}) = 0$. Then we get $\sum_i k_i\frac{c_i}{x_1} = 0$ in S/R. It follows that $(\sum_i k_i c_i)/x_1 \in R$ and hence $\sum_i k_i c_i \in (x_1)$. This in turn implies that $\overline{c} = 0$ and thus, η is injective. It is also surjective as any element c/x_1 of S/R has a pre-image $\overline{c} \in (\mathfrak{C}_R, x_1)/(x_1)$. □

We shall see that a decrease in the valuation of the conductor can significantly help us in gaining better understanding of torsion elements of Ω_R. If we construct the ring $S = R[\frac{\mathfrak{C}_R}{x_1}]$, then Lemma 4.1 guarantees such a drop. We try to explicitly describe the ring S now. First we set up some notation.

Notation 4.2. We know that $\mathfrak{C}_R = (t^c, \ldots, t^{c+a_1-1})R$ where $c = c_R$ is the valuation (of the conductor ideal) as discussed in Section 2.2. Hence $\mathfrak{C}_R/x_1$ is generated in $\overline{R}$ (in fact it is the ideal $(t^{c-a_1})\overline{R})$ by the monomials

$$t^{c-a_1}, t^{c-a_1+1}, \cdots, t^{c-1}. \tag{4.1}$$

This is true because if $\alpha_{c-a-1}, \ldots, \alpha_{c-1}$ are arbitrary units of the integral closure $\overline{R}$, the ideal generated by $t^{c-a_1}, t^{c-a_1+1}, \cdots, t^{c-1}$ is the same as the ideal generated by $\alpha_{c-a-1}t^{c-a_1}, \ldots, \alpha_{c-1}t^{c-1}$. The new ring S is constructed by adjoining $\mathfrak{C}_R/x_1$. By Lemma 4.1(5) S/R is a $\Bbbk$ vector space of dimension s. Using [13, Proposition 2.9] we see that this is generated by those powers of t from Equation (4.1), which are not in the valuation semigroup of R. We call these powers say $b_1, \ldots, b_s$ in ascending order.

Remark 4.3. Since $(\mathfrak{C}_R, x_1)/(x_1) \subseteq ((x_1) : \mathfrak{m}_R)/(x_1)$ and the dimension of the latter quantity represents the type of R, it is clear that the reduced type s is at most the type of R. Since S never equals R, it is always at least one. The number s can also be described as $\mu(\omega_R/\omega_S)$ where ω_R, ω_S are canonical modules of R, S respectively.

Proof. For the last statement, dualize the following short exact sequence into the canonical module ω_R

$$0 \to R \to S \to \Bbbk^s \to 0$$

gives the short exact sequence:

$$0 \to \omega_S \to \omega_R \to \operatorname{Ext}^1_R(\Bbbk^s, \omega_R) \to 0.$$

Since the number of generators of ω_R is the type of R, and since $\operatorname{Ext}^1_R(\Bbbk^s, \omega_R) \cong \Bbbk^s$ by duality, the remark follows. □

When $s = 1$ we say R is of *reduced type one*. In particular, if R is Gorenstein (type of R equals one), necessarily $s = 1$. The converse is not necessarily true as the next example shows.

Example 4.4. Let $R = \Bbbk[\![t^4, t^{11}, t^{17}]\!]$. We can check R is not Gorenstein using [19]. M2 computations show that the conductor $\mathfrak{C}_R = (t^{19})R$. It follows that $S = \Bbbk[\![t^4, t^{11}, t^{15}, t^{16}, t^{17}, t^{18}]\!] = \Bbbk[\![t^4, t^{11}, t^{17}, t^{18}]\!]$ and hence R is of reduced type one.

When $\mathfrak{C}_R \subseteq \mathfrak{m}_R^2$, it follows that $\operatorname{edim} S = n + s$. Recall that canonical ideal ω_R of R exists ([6, Proposition 3.3.18]). We can also prove that $s = \mu_R\left(\frac{\omega_R}{\overline{\mathfrak{m}_R\omega_R} \cap \omega_R}\right)$, where $\overline{\mathfrak{m}_R\omega_R}$ is the integral closure of the ideal $\mathfrak{m}_R\omega_R$, thinking of the canonical module ω_R as an ideal of R. We can show that the canonical module ω_S of S is in fact $\overline{\mathfrak{m}_R\omega_R} \cap \omega_R$.

Theorem 4.5. Suppose $S = R[\frac{\mathfrak{C}_R}{x_1}]$, then a canonical module ω_S of S can be chosen to be $\omega_R \cap \overline{\mathfrak{m}_R\omega_R}$.

Proof. The first part of the proof is essentially due to [5, Lemma 3]. But we provide the proof here with more details suitable for our purposes. Let (y) be a minimal reduction of a canonical ideal ω_R of R. Thus, we have $\omega_R\overline{R} = y\overline{R}$ and $\omega_R\mathfrak{C}_R = y\mathfrak{C}_R$. Let $\omega'_R = \frac{\omega_R}{y}$. Clearly, $R \subseteq \omega'_R \subseteq \overline{R}$.

Let $Q = \operatorname{Quot}(R)$. Now recall that $\omega_R :_Q \omega_R = R$ ([6, Proposition 3.3.11(c)] and [25, Lemma 2.4.2]). It is also well-known that $R :_Q \mathfrak{C}_R = \overline{R}$ (see for instance the proof of [21, Corollary 2.6]). Combining these facts, we obtain that

$$\begin{aligned}\omega'_R :_Q \overline{R} = y^{-1}(\omega_R :_Q \overline{R}) &= y^{-1}(\omega_R :_Q (R :_Q \mathfrak{C}_R))\\ &= y^{-1}(\omega_R :_Q ((\omega_R :_Q \omega_R) :_Q \mathfrak{C}_R))\\ &= y^{-1}(\omega_R :_Q (\omega_R :_Q \omega_R\mathfrak{C}_R))\\ &= y^{-1}\omega_R\mathfrak{C}_R\end{aligned}$$

where the last equalities follow, by applying duality to the maximal Cohen-Macaulay module $\omega_R\mathfrak{C}_R$ ([6, Theorem 3.3.10(c)]). The equality $\omega_R\mathfrak{C}_R = y\mathfrak{C}_R$ now shows that $\omega'_R :_Q \overline{R} = y^{-1}\omega_R\mathfrak{C}_R = \mathfrak{C}_R$. Since $\omega'_R, \mathfrak{C}_R$ are fractional ideals, we have $\operatorname{Hom}(\overline{R}, \omega'_R) = \mathfrak{C}_R$ [25, Lemma 2.4.2]. This implies $\operatorname{Hom}(\mathfrak{C}_R, \omega'_R) = \overline{R}$ as $\overline{R}$ is a maximal Cohen-Macaulay module. Thus $\omega'_R : \mathfrak{C}_R = \overline{R}$ (again using [25, Lemma 2.4.2]).

The canonical module $\omega_S \cong \operatorname{Hom}(S, \omega'_R) \cong \omega'_R : S$. Now let $\alpha \in \operatorname{Quot}(R)$ such that $\alpha \in \omega'_R : S$. Then $\alpha S \subseteq \omega'_R$ or $\alpha R\left[\frac{\mathfrak{C}_R}{x_1}\right] \subseteq \omega'_R$. Thus

$$\begin{aligned}\alpha R\left[\frac{\mathfrak{C}_R}{x_1}\right] \subseteq \omega'_R &\Leftrightarrow \alpha R \subseteq \omega'_R \text{ and } \alpha(\mathfrak{C}_R/x) \subseteq \omega'_R\\ &\Leftrightarrow \alpha R \subseteq \omega'_R \text{ and } \alpha/x \subseteq \omega'_R : \mathfrak{C}_R = \overline{R}\\ &\Leftrightarrow \alpha R \subseteq \omega'_R \text{ and } \alpha \in x\overline{R} = \mathfrak{m}_R\overline{R}\end{aligned}$$

Thus $\omega_S \cong \omega'_R \cap \mathfrak{m}_R\overline{R} \cong \omega'_R y \cap \mathfrak{m}_R y\overline{R} = \omega_R \cap \overline{\mathfrak{m}_R y} = \omega_R \cap \overline{\mathfrak{m}_R\omega_R}$. □

Combining the above theorem with Remark 4.3, it is easy to see that $s = \mu_R\left(\frac{\omega_R}{\overline{\mathfrak{m}_R\omega_R}\cap\omega_R}\right)$.

Theorem 4.6. Let $\mathfrak{C}_R \subseteq \mathfrak{m}_R^2$. Set $P = \Bbbk[\![X_1, ..., X_n]\!]$. Choose $S, b_1, \ldots, b_s$ as in Notation 4.2. Then there exists a presentation of S as follows:

$$S = R\left[\frac{\mathfrak{C}_R}{x_1}\right] = \frac{\Bbbk[\![X_1, \ldots, X_n, T_1, \ldots, T_s]\!]}{\ker\Phi + (X_iT_j - g_{ij}(X_1, \ldots, X_n), T_kT_l - h_{kl}(X_1, \ldots, X_n))_{\substack{1\leqslant i\leqslant n, 1\leqslant j\leqslant s\\ 1\leqslant k\leqslant l\leqslant s}}}$$

where $g_{ij}(X_1, \ldots, X_n), h_{kl}(X_1, \ldots, X_n) \in (X_1, \ldots, X_n)^2P$ for all i, j, k, l.
Moreover, $g_{ij}(x_1, \ldots, x_n), h_{kl}(x_1, \ldots, x_n) \in \mathfrak{C}_R$.

Proof. Define $\Psi : \Bbbk[\![X_1, \ldots, X_n, T_1, \ldots, T_s]\!] \twoheadrightarrow S$ where

$$\Psi(X_i) = \alpha_i t^{a_i}, 1 \leqslant i \leqslant n, \qquad \Psi(T_j) = t^{b_j}, 1 \leqslant j \leqslant s.$$

Note that $c_R - a_1 \leqslant b_1, \ldots, b_s \leqslant c_R - 1$. By same arguments as in Lemma 4.1(3) and also by reading off valuations, we see that the images of X_iT_j and T_kT_l are all in $\mathfrak{C}_R$. Hence there exists $g_{ij}(X_1, \ldots, X_n)$, $h_{kl}(X_1, \ldots, X_n)$ such that

$$X_iT_j - g_{ij}(X_1, \ldots, X_n), T_kT_l - h_{kl}(X_1, \ldots, X_n) \in \ker\Psi.$$

Moreover, since $\mathfrak{C}_R \subseteq \mathfrak{m}_R^2$, we can choose $g_{ij}(X_1, \ldots, X_n), h_{kl}(X_1, \ldots, X_n) \in (X_1, \ldots, X_n)^2P$ for all i, j, k, l as well as $g_{ij}(x_1, \ldots, x_n), h_{kl}(x_1, \ldots, x_n) \in \mathfrak{C}_R$. Set

$$J = (X_iT_j - g_{ij}(X_1, \ldots, X_n), T_kT_l - h_{kl}(X_1, \ldots, X_n))_{\substack{1\leqslant i\leqslant n, 1\leqslant j\leqslant s\\ 1\leqslant k\leqslant l\leqslant s}}.$$

By construction, elements in J do not have any purely linear terms in any X_i. By the above discussion, $J \subset \ker \Psi$.

Conversely, let $p(X_1, \ldots, X_n, T_1, \ldots T_s) \in \ker \Psi$. Modulo the ideal J, we can write

$$p(X_1, \ldots, X_n, T_1, \ldots, T_s) \equiv p'(X_1, \ldots, X_n) + \sum_{i=1}^{s} \beta_i T_i$$

where $\beta_i \in \Bbbk$ and $p'(X_1, \ldots, X_n) \in P$. Since $J \subset \ker \Psi$, we have $p'(X_1, \ldots, X_n) + \sum_{i=1}^{s} \beta_i T_i \in \ker \Psi$. Thus, $\displaystyle\sum_{i=1}^{s} \beta_i t^{b_i} = \Psi(\sum_{i=1}^{s} \beta_i T_i) = \Psi(-p'(X_1, \ldots, X_n)) \in R$. By the choice of b_i's, we immediately obtain that $\sum_{i=1}^{s} \beta_i t^{b_i} = 0$. Thus $\beta_i = 0$ for all i and hence $p'(X_1, \ldots, X_n) \in \ker \Phi$. This shows that $\ker \Psi = \ker \Phi + J$. $\square$

Remark 4.7. Using the defining ideal of the S in the previous theorem gives the following presentation of Ω_S:
(4.2)

$$\begin{bmatrix} & T_1 - \partial g_{11}/\partial x_1 & \cdots & \partial g_{n1}/\partial x_1 & T_2 - \partial g_{12}/\partial x_1 & \cdots & \partial g_{n2}/\partial x_1 & \cdots & T_n - \partial g_{1n}/\partial x_1 & \cdots & \partial g_{nn}/\partial x_1 & \partial h_{11}/\partial x_1 & \partial h_{12}/\partial x_1 & \cdots & \partial h_{1n}/\partial x_1 & \cdots & \partial h_{nn}/\partial x_1 \\ \partial f_i/\partial x_j & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & \partial g_{11}/\partial x_n & \cdots & T_1 - \partial g_{n1}/\partial x_n & \partial g_{12}/\partial x_n & \cdots & T_2 - \partial g_{n2}/\partial x_n & \cdots & \partial g_{1n}/\partial x_n & \cdots & T_n - \partial g_{nn}/\partial x_n & \partial h_{11}/\partial x_1 & \partial h_{12}/\partial x_1 & \cdots & \partial h_{1n}/\partial x_n & \cdots & \partial h_{nn}/\partial x_n \\ & x_1 & \cdots & x_n & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 2T_1 & T_2 & \cdots & 0 & \cdots & 0 \\ & 0 & \cdots & 0 & x_1 & \cdots & x_n & \cdots & 0 & \cdots & 0 & 0 & T_1 & \cdots & 0 & \cdots & 0 \\ & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & x_1 & \cdots & x_n & 0 & 0 & \cdots & 0 & \cdots & 2T_n \end{bmatrix}$$

By abuse of notation, we denote the images of T_i in S, by T_i again. Thus S is also the same as $\Bbbk[\![x_1, \ldots, x_n, T_1, \ldots, T_s]\!]$. In $S = R\left[\frac{\mathfrak{C}_R}{x_1}\right] = \Bbbk[\![t^{a_1}, \ldots, \alpha_n t^{a_n}, t^{b_1}, \ldots, t^{b_s}]\!]$, the torsion submodule $\tau(\Omega_S)$ is nonzero (Remark 3.3).

In Ω_S, notice that $\Psi(T_j) = t^{b_j}$ (refer to Theorem 4.6 for definition of Ψ). Clearly, $b_i T_i dT_j - b_j T_j dT_i \in \tau(\Omega_S), 1 \leqslant i < j \leqslant s$ as $b_i t^{b_j} dt^{b_i} - b_i t^{b_i} dt^{b_j} = 0$ in $\Omega_{\overline{R}}$. This torsion element is nonzero due to [8, Proposition 2.6]. Thus Ω_S always has nonzero torsion elements of the form $\gamma_{ij} = b_i T_i dT_j - b_j T_j dT_i, 1 \leqslant i < j \leqslant s$ (which are $\binom{s}{2}$ in number). So the torsion submodule $\tau(\Omega_S)$ has at least $\binom{s}{2}$ elements. Moreover, all these elements are $\Bbbk$-linearly independent as follows again from [8, Proposition 2.6].

Lemma 4.8. Let $\mathfrak{C}_R \subseteq \mathfrak{m}_R^2$ and construct $S = R\left[\frac{\mathfrak{C}_R}{x_1}\right] = R[T_1, \ldots, T_s]$ as in Theorem 4.6. Let $\tau(\Omega_R), \tau(\Omega_S)$ represent the torsion submodules of Ω_R, Ω_S respectively and $\gamma_{ij} = b_i T_i dT_j - b_j T_j dT_i, 1 \leqslant i < j \leqslant s$. Consider an element $\tau = \sum_i r_i dx_i + \sum_j r'_j dT_j$ in Ω_S where $r'_j, 1 \leqslant j \leqslant s$ are not units in S. Then there exist $c_{ij} \in \Bbbk$ such that

$$\tau - \sum_{i,j} c_{ij} \gamma_{ij} = \sum_i r''_i dx_i \in \Omega_S,$$

where $r''_i \in S$. In particular, if $\tau \in \tau(\Omega_S)$ then $\sum_i r''_i dx_i \in \tau(\Omega_S)$.

Proof. Since $\mathfrak{m}_S^2 = \mathfrak{m}_R^2 + \mathfrak{m}_R(\frac{\mathfrak{C}_R}{x_1}) + (\frac{\mathfrak{C}_R}{x_1})^2 = \mathfrak{m}_R^2$, and since $S/\mathfrak{m}_S = R/\mathfrak{m}_R$, we can represent the entries $r'_j, 1 \leqslant j \leqslant s$ by elements of R plus linear forms over $\Bbbk$ in $T_1, ..., T_s$. Let $r'_j = p_j(x_1, \ldots, x_n) + \sum_l k_{jl} T_l, 1 \leqslant j \leqslant s$ where $k_{jl} \in \Bbbk$. Clearly, using the Jacobian matrix (as in Remark 4.7) we can rewrite τ as $\sum_i u_i dx_i + \sum_j u'_j dT_j$ where $u'_j = \sum_l k'_{jl} T_l$ where $k'_{jl} \in \Bbbk$. We wish to eliminate the variables $T_1, \ldots, T_s$ from u'_j. Suppose $k'_{jl} \neq 0$.

The column in the Jacobian matrix corresponding to the defining equation $T_l T_j - h_{lj}(X_1, \ldots, X_n)$ of S (refer to Theorem 4.6) is of the form $\theta_{lj} = T_l dT_j + T_j dT_l - \sum_t \frac{\partial h_{lj}}{\partial X_t} dX_t$. Notice that

$\gamma_{lj} + b_j\theta_{lj} = (b_l + b_j)T_l dT_j - \sum_t b_l \frac{\partial h_{lj}}{\partial X_t} dX_t$. Since $\theta_{lj} = 0$ in Ω_S, γ_{lj} can be rewritten as $(b_l + b_j)T_l dT_j - \sum_t b_l \frac{\partial h_{lj}}{\partial x_t} dx_t$ in Ω_S. Now $\tau - \frac{k'_{jl}}{(b_l+b_j)}\gamma_{lj}$ will not have a T_l term in the $(n+j)$-th row (corresponding to dT_j). Since j, l were arbitrary, a $\Bbbk$-linear combination of τ and γ_{lj} will eliminate all the variables $T_1, \dots, T_s$ from u'_j. Since j was arbitrary, we can eliminate the variables $T_1, \dots, T_s$ from $u'_1, \dots, u'_s$ as well, to get the result. □

The following theorem is an important technical result of this article. This will serve as the main tool that will help us in pulling back nonzero torsion elements from Ω_S to Ω_R, as we shall see. We shall write $\ell(M)$ to denote the length of an R-module M.

Theorem 4.9. Let $\mathfrak{C}_R \subseteq \mathfrak{m}_R^2$ and construct $S = R\left[\frac{\mathfrak{C}_R}{x_1}\right] = R[T_1, \dots, T_s]$ as in Theorem 4.6. Let $\tau(\Omega_R), \tau(\Omega_S)$ represent the torsion submodules of Ω_R, Ω_S respectively. If $\ell(\tau(\Omega_S)) \geqslant ns + \binom{s}{2} + 1$ and all these torsion elements have non-units in the last s rows (corresponding to $dT_1, \dots, dT_s$), then a $\Bbbk$-linear combination of these torsion elements can be pulled back to a nonzero torsion element in $\tau(\Omega_R)$. In particular, $\tau(\Omega_R) \neq 0$.

Remark 4.10. Although the hypothesis of at least $ns + \binom{s}{2} + 1$ $\Bbbk$-linear torsion elements sounds very strong, in fact it is not. We can prove that there are always at least $ns + \binom{s}{2}$ $\Bbbk$-linearly independent torsion elements in Ω_S, so this theorem requires only one more new torsion element. Of course, we are only searching for one nonzero torsion element in Ω_R in any case, but the point is that the search for one extra torsion element is often better in S than in R, since S is usually a simpler ring.

The number $ns + \binom{s}{2}$ comes from the following observation: suppose some $\alpha_i = 1$. Then we can look at the torsion element $a_i x_i dT_j - b_j T_j dx_i$ for $1 \leqslant j \leqslant s$. We can apply the 'monomialization technique' as in the proof of Theorem 3.1, to generate ns such torsion elements; we always have $\binom{s}{2}$ torsion elements coming from the variables T_j's. Finally we can use [8, Proposition 2.6] to prove $\Bbbk$-linear independence among these. We defer the technical details to a future article.

Proof of Theorem 4.9. Since $\ell(\tau(\Omega_S)) \geqslant ns + \binom{s}{2} + 1$, let $\tau_1, \dots, \tau_{ns+\binom{s}{2}+1}$ denote these $\Bbbk$-linearly independent torsion elements with non units in the last s rows (corresponding to $dT_1, \dots, dT_s$). By Lemma 4.8, we can use γ_{ij} to rewrite $\tau_1, \dots, \tau_{ns+\binom{s}{2}+1}$ as $\tau'_1, \dots, \tau'_{ns+\binom{s}{2}+1}$ which have zeroes in the last s rows. Let $V = \left\{\tau'_1, \dots, \tau'_{ns+\binom{s}{2}+1}\right\}$ and $V' = V \bigcup \{\gamma_{ij} \mid 1 \leqslant i < j \leqslant s\}$. Thus we have

$$\dim_{\Bbbk}\langle V' \rangle \leqslant \dim_{\Bbbk}\langle V \rangle + \binom{s}{2} \tag{4.3}$$

where $\langle \cdot \rangle$ denotes the $\Bbbk$-linear span. Notice that $\{\tau_1, \dots, \tau_{ns+\binom{s}{2}+1}\} \subseteq \langle V' \rangle$ and hence

$$ns + \binom{s}{2} + 1 \leqslant \dim_{\Bbbk}\langle V' \rangle. \tag{4.4}$$

Combining (4.3),(4.4), we have $\dim_{\Bbbk}\langle V \rangle \geqslant ns + 1$. Thus there are at least $ns + 1$ $\Bbbk$-linearly independent elements $\rho_i, 1 \leqslant i \leqslant ns + 1$ all having the last s rows (corresponding to $dT_1, \dots, dT_s$) consisting of zeroes.

Let $B = [\rho_1 \ \dots \ \rho_{ns+1}]$ be the $(n+s) \times (ns+1)$ matrix obtained by concatenating the column vectors $\rho_i, 1 \leqslant i \leqslant ns+1$. Since the last s rows of B are zero, it is effectively an $n \times (ns+1)$ matrix.

Our goal is to pullback a $\Bbbk-$linear combination of the columns of B to Ω_R. Passing to the vector space S/R gives an $n \times (ns+1)$ matrix of linear forms in $T_1, ..., T_s$. Writing the coefficients of each linear form as its own $s \times 1$ column, we obtain an $ns \times (ns+1)$ matrix over $\Bbbk$. By elementary column operations over $\Bbbk$ it follows that we can obtain a column of zeroes. Performing the same operations on the matrix B gives us a nonzero torsion element whose last s rows are zeroes, and whose entries are in R. These necessarily also represent a torsion element in Ω_R, since they are a syzygy of $\frac{dx_1}{dt}, \dots, \frac{dx_n}{dt}$. If this torsion element were zero in Ω_R, it would also be zero in Ω_S, since the presentation of Ω_S contains the Jacobian matrix associated to R. $\square$

As an immediate application, we generalize a result of Scheja ([24]), proved by many researchers, who proved Berger's conjecture in the case R is quasi-homogeneous. If R is quasi-homogeneous, so too is S, so the next result is strictly stronger:

Theorem 4.11. Let $R = \Bbbk[\![\alpha_1 t^{a_1}, \dots, \alpha_n t^{a_n}]\!]$ with conductor $\mathfrak{C}_R$. Construct $S = R\left[\frac{\mathfrak{C}_R}{x_1}\right] = R[T_1, \dots, T_s]$ as in Theorem 4.6. If S is quasi-homogeneous, then $\tau(\Omega_R)$ is nonzero.

Proof. Since S is quasi-homogeneous, Scheja in [24] showed that $\mu(\tau(\Omega_R)) \geqslant \binom{n+s}{2}$ where $\operatorname{edim} S = n+s$. After possibly a change of generators, we have that $\Omega_S \twoheadrightarrow \mathfrak{n} = (x_1, \dots, x_n, T_1, \dots, T_s), dx_i \to x_i, dT_j \to T_j$. (This map is well-defined because it is induced from the Euler derivation, see [20, 1.5]) Let τ be a torsion element in Ω_S, and write $\tau = \sum_{i=1}^n r_i dx_i + \sum_{j=1}^s r'_j dT_j$. Since $\tau \to 0$ under the above map, we have $\sum_{i=1}^n r_i x_i + \sum_{j=1}^s r'_j T_j = 0$. Thus none of the r'_j can be units in S, else it would be a contradiction to $x_1, \dots, x_n, T_1, \dots, T_s$ being a minimal generating set for $\mathfrak{n}$. Now since Ω_S has $\binom{n+s}{2}$ $\Bbbk$-linearly independent torsion elements and none of the coefficients of dT_j in the description of the torsion elements are units, we can pull back a $\Bbbk$-linear combination of these torsion elements to a torsion element in Ω_R (by Theorem 4.9). $\square$

Example 4.12. Let $R = \Bbbk[\![t^5, t^8+t^{11}, t^9+t^{11}, t^{12}+t^{11}]\!]$. Macaulay2 computations show that the conductor $\mathfrak{C}_R = (t^{13})\overline{R}$ and $S = R\left[\frac{\mathfrak{C}_R}{x_1}\right]$ is the monomial curve $\Bbbk[\![t^5, t^8, t^9, t^{11}, t^{12}]\!]$. The embedding dimension of S equals 5 which is one more than that of R. Using the previous theorem we see that the torsion $\tau(\Omega_R) \neq 0$. In fact the deviation $d(R) = \mu(I) - \operatorname{edim} R + 1 = 8 - 4 + 1 = 5$. The defining ideal of R is of height three and is not a Gorenstein ideal.

5. Main Results

In this section we prove some of the main results of this article. In the rest of this section we assume that $\mathfrak{C}_R \subseteq \mathfrak{m}_R^2$ and $S = R\left[\frac{\mathfrak{C}_R}{x_1}\right]$ as in Theorem 4.6. Recall that the $\Bbbk$-dimension of S/R is called the reduced type of R, denoted by s. We proved that s is always at most the type of R. In particular, if R is Gorenstein, then $s = 1$.

Proposition 5.1. Suppose $S = R[\frac{\mathfrak{C}_R}{x_1}]$ and $\tau \in \Omega_S$. If $\tau = \sum r_i dx_i + \sum_{j=1}^s r_{n+j} dT_j, r_i \in S$ such that r_{n+j} is a unit in S for some j, then for all $1 \leqslant i \leqslant n$, $x_i \tau \neq 0$.

Proof. Fix $i \leqslant n$. Let $J = \langle x_i^2, x_i T_j, T_j^2, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n, T_1, \dots, T_{j-1}, T_{j+1}, \dots, T_s \rangle$. Writing $\overline{()}$ for images in $\Omega_{R/J}$, $\overline{x_i \tau} = \overline{x_i r_i} dx_i + \overline{x_i r_{n+j}} dT_j$. If r_i is not a unit in S, then

$x_ir_i \in \mathfrak{m}_S^2 \subseteq J$. Thus $\overline{x_i r_i} dx_i = 0$. If r_i is a unit then $\overline{x_i r_i} dx_i = 0$ as $x_i^2 \in J$. Thus we have $\overline{\tau} = \overline{x_i r_{n+j}} dT_j$ which is nonzero in $\Omega_{S/J}$ by [8, Proposition 2.6]. □

Corollary 5.2. Under the hypothesis of the above theorem, for every $\tau = \sum r_i dx_i + \sum_{j=1}^{s} r_{n+j} dT_j \in 0 :_{\Omega_S} x_1$, $r_{n+j} \in S$ cannot be a unit.

Proof. Follows from Proposition 5.1. □

Lemma 5.3. Let $\mathfrak{C}_R \subseteq \mathfrak{m}_R^2$ and $S = R[\frac{\mathfrak{C}}{x_1}]$ with $\operatorname{edim}(S) = n + s$ where s is the reduced type of R. Then

$$\operatorname{type}(S) \leqslant \operatorname{type}(R) + s(n-1).$$

Proof. Using Remark 4.3, we get the following short exact sequence

$$0 \to \omega_S \to \omega_R \to \Bbbk^s \to 0$$

and tensoring this sequence with $\Bbbk$, we get

$$\operatorname{Tor}_1^R(\Bbbk, \Bbbk^s) \to \omega_S \otimes_R \Bbbk \to \omega_R \otimes_R \Bbbk \to \Bbbk^s \to 0.$$

Comparing the length of the modules appearing this short exact sequence now yields, $\dim_{\Bbbk} \operatorname{Tor}_1^R(\Bbbk, \Bbbk^s) + \operatorname{type}(R) \geqslant \operatorname{type}(S) + s$ ([6, Theorem 3.3.11]), and hence

$$\operatorname{type}(S) \leqslant ns - s + \operatorname{type}(R) = \operatorname{type}(R) + s(n-1).$$

□

Proposition 5.4. Let R and S be as in Lemma 5.3. Then $\mathfrak{m}_S^k \subseteq x_1 S$ if and only if $\mathfrak{m}_R^k \subseteq (x_1, \mathfrak{C}_R)R$.

Proof. First we observe that $x_1 S = (x_1, \mathfrak{C}_R)R$. Since $\mathfrak{m}_R \subseteq \mathfrak{m}_S$, one implication follows at once. To prove the reverse implication, assume that $\mathfrak{m}_R^k \subseteq (x_1, \mathfrak{C}_R)R$. It then suffices to prove that $\mathfrak{m}_S^k \subseteq \mathfrak{m}_R^k + \mathfrak{C}_R$. A typical term in the expansion of $\mathfrak{m}_S^k = (\mathfrak{m}_R + \frac{\mathfrak{C}}{x_1})^k S$, which is not $\mathfrak{m}_R^k$, is of the form $\mathfrak{m}_R^i (\frac{\mathfrak{C}}{x_1})^{k-i}$ for $0 \leqslant i \leqslant k-1$. Since $\mathfrak{C}_R \subseteq \mathfrak{m}_R^2$ and $k - i \geqslant 1$, this term is contained in

$$\mathfrak{m}_R^i (\mathfrak{m}_R^{2k-2i-2}) (\frac{\mathfrak{C}_R}{x_1}) \subseteq \mathfrak{C}_R$$

as $\mathfrak{m}_R \mathfrak{C}_R = x_1 \mathfrak{C}_R$. □

Lemma 5.5 ([11])**.** Let $(R, \mathfrak{m}_R, \Bbbk)$ be a one dimensional complete local reduced $\Bbbk$-algebra with $\operatorname{char}(\Bbbk) = 0$ and embedding dimension $n \geqslant 3$. Suppose y is a non-zero divisor such that $\operatorname{edim}(R/R.y) = n - 1$

a) If $\mathfrak{m}_R^4 \subseteq R.y$, then

$$\ell(0 :_{\Omega_R} y) \geqslant \frac{(n-2)(n-1)}{2}.$$

b) If $\mathfrak{m}_R^5 \subseteq R.y$, then

$$\ell(0 :_{\Omega_R} y) \geqslant \frac{(n-2)(n-1)}{2} - \operatorname{type}(R).$$

Proof. For *a*), we refer the reader to the proof of [11, Satz 4]. For (*b*), see the proof of [11, Anmerkung, Page 506-507]. □

Theorem 5.6. Suppose $\mathfrak{m}_R^4 \subseteq (\mathfrak{C}_R, x_1)$ and $n(n-3) \geqslant 2s$ where s is the reduced type of R, then $\tau(\Omega_R) \neq 0$.

Proof. We first construct $S = R\left[\frac{\mathfrak{C}_R}{x_1}\right]$ with $\operatorname{edim} S = n + s$ where $\operatorname{edim} R = n$. By Proposition 5.4, we have that $\mathfrak{m}_S^4 \subseteq x_1 S$. Since x_1 is a nonzero divisor on S, by definition of torsion submodule we have $0 :_{\Omega_S} x_1 \subseteq \tau(\Omega_S)$. Now using Lemma 5.5(a) with $R = S$ and $y = x_1$, we get that

$$\ell(\tau(\Omega_S)) \geqslant \ell(0 :_{\Omega_S} x_1) \geqslant \frac{(n+s-2)(n+s-1)}{2},$$

where the latter number is more than $ns + \binom{s}{2} + 1$ when $n(n-3) \geqslant 2s$. The result now follows from Theorem 4.9. □

Theorem 5.7. Let $S = R\left[\frac{\mathfrak{C}}{x_1}\right]$ with $\operatorname{edim} S = n + s$ and assume that

$$\operatorname{type}(R) \leqslant \frac{n^2 - 3n - 2ns}{2}.$$

If $\mathfrak{m}_R^5 \subseteq \mathfrak{C}_R + x_1 R$, then $\tau(\Omega_R) \neq 0$.

Proof. Since $\mathfrak{C}_R \subseteq \mathfrak{m}^2$, using Proposition 5.4, we immediately obtain that $\mathfrak{m}_S^5 \subseteq x_1 S$.

Since x_1 is a nonzero divisor on S, by definition of torsion submodule, $0 :_{\Omega_S} x_1 \subseteq \tau(\Omega_S)$. By Lemma 5.5(b) with $R = S$ and $y = x_1$, we get that

$$\ell(\tau(\Omega_S)) \geqslant \ell(0 :_{\Omega_S} x_1) \geqslant \frac{(n+s-2)(n+s-1)}{2} - \operatorname{type}(S).$$

Here we used that the embedding dimension of S is $n+s$, which follows from the presentation of S given in Remark 4.7. All torsion elements in $(0 :_{\Omega_S} x_1)$ have non-units in the last row by Proposition 5.1. It follows from Theorem 4.9 that it suffices to prove

$$\frac{(n+s-2)(n+s-1)}{2} - \operatorname{type}(S) \geqslant ns + \binom{s}{2} + 1.$$

Using Lemma 5.3, it suffices to prove that

$$\frac{(n+s-2)(n+s-1)}{2} - \operatorname{type}(R) - s(n-1) \geqslant ns + \binom{s}{2} + 1$$

or

$$\operatorname{type}(R) \leqslant \frac{(n+s-2)(n+s-1)}{2} - 2ns + s - \binom{s}{2} - 1,$$

which simplifies to our assumption on the type. □

Corollary 5.8. Let R be of reduced type one such that $\operatorname{type}(R) \leqslant \binom{n}{2} - 2n$. If $\mathfrak{m}_R^5 \subseteq \mathfrak{C}_R + x_1 R$, then $\tau(\Omega_R) \neq 0$.

Proof. Set $s = 1$ in the above theorem. □

Corollary 5.9. Let R be Gorenstein and $n = \operatorname{edim} R \geqslant 6$. If $\mathfrak{m}_R^6 \subseteq x_1 R$, then $\tau(\Omega_R) \neq 0$.

Proof. We may assume that $\mathfrak{C}_R \subseteq \mathfrak{m}_R^2$, because we have already shown that the torsion is nonzero in case the inclusion does not hold (Theorem 3.1). Since R is Gorenstein, R is also of reduced type one (Remark 4.3). The condition that $\mathfrak{m}_R^6 \subseteq x_1 R$ implies that $\mathfrak{m}_R^5 \subseteq (x_1 R : \mathfrak{m}_R)$. However, it is always true that $\mathfrak{C}_R + x_1 R \subseteq x_1 : \mathfrak{m}_R$, since $x_1 \mathfrak{C}_R = \mathfrak{m}_R \mathfrak{C}_R$. As the conductor can never lie inside a proper principal ideal (follows, for instance, from [21, Corollary 2.6]) and since R is Gorenstein, we obtain that $\mathfrak{C}_R + x_1 R = x_1 : \mathfrak{m}_R$. Therefore, $\mathfrak{m}_R^5 \subseteq \mathfrak{C}_R + x_1 R$. The inequality $1 = \operatorname{type}(R) \leqslant \binom{n}{2} - 2n$ is also satisfied as $n \geqslant 6$. Thus all the conditions of the previous theorem hold, and the result now follows. □

Remark 5.10. As mentioned in the above proof, it is always true that $\mathfrak{C}_R + x_1R \subseteq x_1 : \mathfrak{m}_R$, since $x_1\mathfrak{C}_R = \mathfrak{m}_R\mathfrak{C}_R$. If equality holds, then s is equal to the type of R, and the condition that $\mathfrak{m}_R^5 \subseteq \mathfrak{C}_R + x_1R$ is equivalent to the condition that $\mathfrak{m}_R^6 \subseteq x_1R$. This case of "maximal" reduced type gives a further extension of the work of Güttes. This maximality occurs if R is Gorenstein, but it can also occur in other cases. For one such example, using Macaulay2, we check that $R = \Bbbk[[t^{10}, t^{11} + t^{16}, t^{12} + t^{16}, t^{13} + t^{16}]] = k[[x, y, z, w]]$ has conductor $\mathfrak{C}_R = t^{20}\overline{R}$ and $(x) : \mathfrak{m}_R = (x, w^2, zw, yw, z^2, yz, y^2) = (x, \mathfrak{C}_R)$.

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